Qualifying Exam

D. Zack Garza

Friday 15^{th} May, 2020

Contents

1 Question 1 (UW 2008 #1)

Let $f(x)$ be an irreducible polynomial of degree 5 over the field $\mathbb{Q}$ of rational numbers with exactly 3 real roots.

- Show that $f(x)$ is not solvable by radicals.
- Let E be the splitting field of f over $\mathbb{Q}$. Construct a Galois extension K of degree 2 over $\mathbb{Q}$ lying in E such that *no* field F strictly between K and E is Galois over $\mathbb{Q}$.

2 Question 2 (UW 2007 #1)

Let K be a field of characteristic zero and L a Galois extension of K. Let f be an irreducible polynomial in $K[x]$ of degree 7 and suppose f has no zeroes in L. Show that f is irreducible in $L[x]$.

3 Question 3 (UW 2016 #6)

Let $A = \mathbb{C}[x, y]/(y^2 - (x-1)^3 - (x-1)^2)$.

- Show that A is an integral domain and sketch the $\mathbb{R}$-points of Spec$A$.
- Find the integral closure of A. Recall that for an integral domain A with fraction field K, the integral closure of A in K is the set of all elements of K integral over A.

4 Question 4 (UW 2012 #3)

Let R be a (commutative) principal ideal domain, let M and N be finitely generated free R-modules, and let $\varphi : M \to N$ be an R-module homomorphism.

- Let K be the kernel of φ. Prove that K is a direct summand of M.
- Let C be the image of φ. Show by example (specifying R, M, N, and φ) that C need not be a direct summand of N.

5 Question 5 (UW 2015 #7)

Let G be a non-abelian group of order p^3 with p a prime.

- Determine the order of the center Z of G.
- Determine the number of inequivalent complex 1-dimensional representations of G.
- Compute the dimensions of all the inequivalent irreducible representations of G and verify that the number of such representations equals the number of conjugacy classes of G.

6 Question 6 (UW 2011 #2)

In this problem, as you apply Sylow,s Theorem, state precisely which portions you are using.

- Prove that there is no simple group of order 30.
- Suppose that G is a simple group of order 60. Determine the number of p-Sylow subgroups of G for each prime p dividing 60, then prove that G is isomorphic to the alternating group A_5.

> Note: in the second part, you needn,t show that A_5 is simple. You need only show that if there is a simple group of order 60, then it must be isomorphic to A_5.

7 Question 7 (UW 2010 #7)

Let F be a field of characteristic zero, and let K be an *algebraic* extension of F that possesses the following property: every polynomial $f \in F[x]$ has a root in K. Show that K is algebraically closed.\ **Hint:** if $K(\theta)/K$ is algebraic, consider $F(\theta)/F$ and its normal closure; primitive elements might be of help.

8 Question 8 (UGA 2019 #5)

Let R be a ring and M an R-module.

> Recall that the set of torsion elements in M is defined by
>
> $$\operatorname{Tor}(m) = \{m \in M \;\middle|\; \exists r \in R,\ r \neq 0,\ rm = 0\}.$$

(a) Prove that if R is an integral domain, then $\operatorname{Tor}(M)$ is a submodule of M .

(b) Give an example where $\operatorname{Tor}(M)$ is not a submodule of M.

(c) If R has zero-divisors, prove that every non-zero R-module has non-zero torsion elements.

9 Question 9 (UW 2011 #3)

Describe the Galois group and the intermediate fields of the cyclotomic extension $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$.

10 Question 10 (UW 2017 #1)

Let R be a Noetherian ring. Prove that $R[x]$ and $R[[x]]$ are both Noetherian. (The first part of the question is asking you to prove the Hilbert Basis Theorem, not to use it!)