SOME QUALS PROBLEMS

SANTIAGO ARANGO-PIÑEROS

1. Algebra

1.1. Groups.

(1) Classify the groups of order $182 = 2 \cdot 7 \cdot 13$.
(2) Let G be a finite group of order $p^n m$ where p is a prime and m is not divisible by p. Prove that if H is a subgroup of G of order p^k for some $k < n$, then the normalizer of H in G properly contains H.
(3) Let H be a subgroup of S_n of index n. Prove:
   a) There is an isomorphism $f : S_n \to S_n$ such that $f(H)$ is the subgroup of S_n stabilizing n. In particular, H is isomorphic to S_{n-1}.
   b) The only subgroups of S_n containing H are S_n and H.
(4) a) Prove that a group of order $351 = 3^3 \cdot 13$ cannot be simple.
   b) Prove that a group of order 33 must be cyclic.
(5) a) Let G be a group, and $Z(G)$ the center of G. Prove that if $G/Z(G)$ is cyclic, then G is abelian.
   b) Prove that a group of order p^n, where p is a prime and $n \geq 1$, has non-trivial center.
   c) Prove that a group of order p^2 must be abelian.
(6) Let G be a finite group.
   a) Prove that if $H < G$ is a proper subgroup, then G is not the union of conjugates of H.
   b) Suppose that G acts transitively on a set X with $|X| > 1$. Prove that there exists an element of G with no fixed points in X.
(7) Classify all groups of order 15 and of order 30.
(8) Count the number of p-Sylow subgroups of S_p.
(9) a) Let G be a group of order n. Suppose that for every divisor d of n, G contains at most one subgroup of order d. Show that G is clyclic.
   b) Let F be a field. Show that every finite subgroup of the group of units $F^\times$ is cyclic.

1.2. Fields and Galois Theory.

(1) Let K and L be finite fields. Show that K is contained in L if and only if $\#K = p^r$ and $\#L = p^s$ for the same prime p, and $r \leq s$.

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(2) Let K and L be finite fields with $K \subseteq L$. Prove that L is Galois over K and that $\mathrm{Gal}(L/K)$ is cyclic.
(3) Fix a field F, a separable polynomial $f \in F[x]$ of degree $n \geq 3$, and a splitting field L for f. Prove that if $[L : F] = n!$ then:
   a) f is irreducible.
   b) For each root r of f, r is the unique root of f in $F(r)$.
   c) For every root r of f, there are no proper intermediate fields $F \subset L \subset F(r)$.
(4) a) Show that $\sqrt{2+\sqrt{2}}$ is a root of $p(x) = x^2 - 4x^2 + 2 \in \mathbf{Q}[x]$.
   b) Prove that $\mathbf{Q}(\sqrt{2+\sqrt{2}})$ is a Galois extension of $\mathbf{Q}$ and find its Galois group. (Hint: note that $\sqrt{2-\sqrt{2}}$ is another root of $p(x)$).
   c) Let $f(x) = x^3 - 5$. Determine the splitting field K of $f(x)$ over $\mathbf{Q}$ and the Galois group of $f(x)$. Give an example of a proper sub-extension $\mathbf{Q} \subset L \subset K$, such that $L/\mathbf{Q}$ is Galois.

1.3. **Rings.**

(1) An integral domain R is said to be an *Euclidean domain* if there is a function $N : R \to \{n \in \mathbf{Z} \mid n \geq 0\}$ such that $N(0) = 0$ and for each $a, b \in R$ with $b \neq 0$, there exist elements $q, r \in R$ with

$$a = qb + r, \quad \text{and} \quad r = 0 \text{ or } N(r) < N(b).$$

Prove:
   a) The ring $F[[x]]$ of power series over a field F is an Euclidean domain.
   b) Every Euclidean domain is a PID.
(2) Let F be a field, and let R be the subring of $F[X]$ of polynomials with X coefficient equal to 0. Prove that R is not a UFD.
(3) R is a commutative ring with 1. Prove that if I is a maximal ideal in R, then R/I is a field. Prove that if R is a PID, then every nonzero prime ideal in R is maximal. Conclude that if R is a PID and $p \in R$ is prime, then $R/(p)$ is a field.

1.4. **Linear Algebra.**

(1) Prove that any square matrix is conjugate to its transpose matrix. (You may prove it over $\mathbf{C}$).
(2) Determine the number of conjugacy classes of 16×16 matrices with entries in $\mathbf{Q}$ and minimal polynomial $(x^2+1)^2(x^3+2)^2$.
(3) Let V be a vector space over a field F. The evaluation map $e\colon V \to (V^\vee)^\vee$ is defined by $e(v)(f) := f(v)$ for $v \in V$ and $f \in V^\vee$.
   a) Prove that e is an injection.
   b) Prove that e is an isomorphism if and only if V is finite dimensional.

(4) Let R be a principal ideal domain that is not a field, and write F for its field of fractions. Prove that F is not a finitely generated R-module.

(5) Carefully state Zorn's lemma and use it to prove that every vector space has a basis.

2. Analysis

2.1. **Complex Analysis.**

(1) Use residues to compute the integral

$$\int_0^\infty \frac{\cos x}{(x^2+1)^2} \mathrm{d}x \,.$$

(2) State and prove the Cauchy integral formula for holomorphic functions.

(3) Use the Cauchy integral formula to prove the maximal principle for analytic functions.

(4) Let f be an entire function and suppose that $|f(z)| \le A|z|^2$ for all z and some constant A. Show that f is a polynomial of degree ≤ 2.

(5) a) State the Schwarz lemma for analytic functions in the unit disc.
   b) Let $f : \mathbf{D} \to \mathbf{D}$ be an analytic map from the unit disc $\mathbf{D}$ into itself. Use the Schwarz lemma to show that for each $a \in \mathbf{D}$ we have

$$\frac{|f'(a)|}{1-|f(a)|^2} \le \frac{1}{1-|a|^2} \,.$$

(6) State the Riemann mapping theorem and prove the uniqueness part.

(7) Compute the integrals

$$\int_{|z-2|=1} \frac{e^z}{z(z-1)^2} \, \mathrm{d}z, \quad \int_0^\infty \frac{\cos 2x}{x^2+2} \, \mathrm{d}x \,.$$

(8) Let (f_n) be a sequence of holomorphic functions in a domain D. Suppose that $f_n \to f$ uniformly on each compact subset of D. Show that
   a) f is holomorphic on D.
   b) $f_n' \to f'$ uniformly on each compact subset of D.

(9) If f is a non-constant entire function, then $f(\mathbf{C})$ is dense in the plane.

(10) a) State Rouche's theorem.
   b) Let f be analytic in a neighborhood of 0, and satisfying $f'(0) \neq 0$. Use Rouche's theorem to show that there exists a neighborhood U of 0 such that f is a bijection in U.

(11) Let f be a meromorphic function in the plane such that

$$\lim_{|z|\to\infty} |f(z)| = \infty \,.$$

   a) Show that f has only finitely many poles.

b) Show that f is a rational function.

2.2. **Real Analysis.**

(1) Describe the process that extends a measure on an algebra $\mathcal{A}$ of subsets of X, to a complete measure defined on a σ-algebra $\mathcal{B}$ containing $\mathcal{A}$. State the corresponding definitions and results (without proofs).

(2) State and prove Fatou's Lemma on a general measurable space.

(3) a) State the Dominated Convergence Theorem for Lebesgue integrals.

b) Let $\{f_n\}$ be a sequence of measurable functions on a Lebesgue measurable set E which converges *in measure* to a function f on E. Suppose that for every n, $|f_n| \leq g$ with g integrable on E. Using the above theorem show that

$$\int_E |f_n - f| \longrightarrow 0 \, .$$

(4) Let $f \in L^1([0,1])$. Show that

a) The limit $\lim_{p\to 0^+} \|f\|_p$ exists.

b) If $m\{x : f(x) = 0\} > 0$, then the above limit is zero.

(5) Let f be a continuous function on $[0,1]$. Show that the following statements are equivalent.

a) f is absolutely continuous.

b) For any $\epsilon > 0$ there exists $\delta > 0$ such that $m(f(E)) < \epsilon$ for any set $E \subseteq [0,1]$ with $m(E) < \delta$.

c) $m(f(E)) = 0$ for any set $E \subseteq [0,1]$ with $m(E) = 0$.