

Title

D. Zack Garza

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1 Question 1

Let G be a finite group with n distinct conjugacy classes. Let $g_1 \cdots g_n$ be representatives of the conjugacy classes of G .

Prove that if $g_i g_j = g_j g_i$ for all i, j then G is abelian.

2 Question 2

Let G be a group of order 105 and let P, Q, R be Sylow 3, 5, 7 subgroups respectively.

- (a) Prove that at least one of Q and R is normal in G .
- (b) Prove that G has a cyclic subgroup of order 35.
- (c) Prove that both Q and R are normal in G .
- (d) Prove that if P is normal in G then G is cyclic.

3 Question 3

Let R be a ring with the property that for every $a \in R$, $a^2 = a$.

- (a) Prove that R has characteristic 2.
- (b) Prove that R is commutative.

4 Question 4

Let F be a finite field with q elements.

Let n be a positive integer relatively prime to q and let ω be a primitive n th root of unity in an extension field of F .

Let $E = F[\omega]$ and let $k = [E : F]$.

- (a) Prove that n divides $q^k - 1$.
- (b) Let m be the order of q in $\mathbb{Z}/n\mathbb{Z}$. Prove that m divides k .
- (c) Prove that $m = k$.

5 Question 5

Let R be a ring and M an R -module.

Recall that the set of torsion elements in M is defined by

$$\text{Tor}(M) = \{m \in M \mid \exists r \in R, r \neq 0, rm = 0\}.$$

- (a) Prove that if R is an integral domain, then $\text{Tor}(M)$ is a submodule of M .
- (b) Give an example where $\text{Tor}(M)$ is not a submodule of M .
- (c) If R has zero-divisors, prove that every non-zero R -module has non-zero torsion elements.