

# Combined Qual Questions

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## 1 Algebra (140 Questions)

### Question 1

Let  $G$  be a finite group with  $n$  distinct conjugacy classes. Let  $g_1 \cdots g_n$  be representatives of the conjugacy classes of  $G$ .

Prove that if  $g_i g_j = g_j g_i$  for all  $i, j$  then  $G$  is abelian.

### Question 2

Let  $G$  be a group of order 105 and let  $P, Q, R$  be Sylow 3, 5, 7 subgroups respectively.

- (a) Prove that at least one of  $Q$  and  $R$  is normal in  $G$ .
- (b) Prove that  $G$  has a cyclic subgroup of order 35.
- (c) Prove that both  $Q$  and  $R$  are normal in  $G$ .
- (d) Prove that if  $P$  is normal in  $G$  then  $G$  is cyclic.

### Question 3

Let  $R$  be a ring with the property that for every  $a \in R$ ,  $a^2 = a$ .

- (a) Prove that  $R$  has characteristic 2.
- (b) Prove that  $R$  is commutative.

### Question 4

Let  $F$  be a finite field with  $q$  elements.

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Let  $n$  be a positive integer relatively prime to  $q$  and let  $\omega$  be a primitive  $n$ th root of unity in an extension field of  $F$ .

Let  $E = F[\omega]$  and let  $k = [E : F]$ .

- (a) Prove that  $n$  divides  $q^k - 1$ .
- (b) Let  $m$  be the order of  $q$  in  $\mathbb{Z}/n\mathbb{Z}$ . Prove that  $m$  divides  $k$ .
- (c) Prove that  $m = k$ .

### Question 5

Let  $R$  be a ring and  $M$  an  $R$ -module.

Recall that the set of torsion elements in  $M$  is defined by

$$\text{Tor}(M) = \{m \in M \mid \exists r \in R, r \neq 0, rm = 0\}.$$

- (a) Prove that if  $R$  is an integral domain, then  $\text{Tor}(M)$  is a submodule of  $M$ .
- (b) Give an example where  $\text{Tor}(M)$  is not a submodule of  $M$ .
- (c) If  $R$  has zero-divisors, prove that every non-zero  $R$ -module has non-zero torsion elements.

### Question 6

Let  $R$  be a commutative ring with multiplicative identity. Assume Zorn's Lemma.

- (a) Show that

$$N = \{r \in R \mid r^n = 0 \text{ for some } n > 0\}$$

is an ideal which is contained in any prime ideal.

- (b) Let  $r$  be an element of  $R$  not in  $N$ . Let  $S$  be the collection of all proper ideals of  $R$  not containing any positive power of  $r$ . Use Zorn's Lemma to prove that there is a prime ideal in  $S$ .
- (c) Suppose that  $R$  has exactly one prime ideal  $P$ . Prove that every element  $r$  of  $R$  is either nilpotent or a unit.

### Question 7

Let  $\zeta_n$  denote a primitive  $n$ th root of 1 in  $\mathbb{Q}$ . You may assume the roots of the minimal polynomial  $p_n(x)$  of  $\zeta_n$  are exactly the primitive  $n$ th roots of 1.

Show that the field extension  $\mathbb{Q}(\zeta_n)$  over  $\mathbb{Q}$  is Galois and prove its Galois group is  $(\mathbb{Z}/n\mathbb{Z})^\times$ .

How many subfields are there of  $\mathbb{Q}(\zeta_{20})$ ?

### Question 8

Let  $\{e_1, \dots, e_n\}$  be a basis of a real vector space  $V$  and let

$$\Lambda := \left\{ \sum r_i e_i \mid r_i \in \mathbb{Z} \right\}$$

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Let  $\cdot$  be a non-degenerate ( $v \cdot w = 0$  for all  $w \in V \iff v = 0$ ) symmetric bilinear form on  $V$  such that the Gram matrix  $M = (e_i \cdot e_j)$  has integer entries.

Define the dual of  $\Lambda$  to be

$$\Lambda^\vee := \{v \in V \mid v \cdot x \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

- (a) Show that  $\Lambda \subset \Lambda^\vee$ .
- (b) Prove that  $\det M \neq 0$  and that the rows of  $M^{-1}$  span  $\Lambda^\vee$ .
- (c) Prove that  $\det M = |\Lambda^\vee / \Lambda|$ .

### Question 9

Let  $A$  be a square matrix over the complex numbers. Suppose that  $A$  is nonsingular and that  $A^{2019}$  is diagonalizable over  $\mathbb{C}$ .

Show that  $A$  is also diagonalizable over  $\mathbb{C}$ .

### Question 10

Let  $F = \mathbb{F}_p$ , where  $p$  is a prime number.

- (a) Show that if  $\pi(x) \in F[x]$  is irreducible of degree  $d$ , then  $\pi(x)$  divides  $x^{p^d} - x$ .
- (b) Show that if  $\pi(x) \in F[x]$  is an irreducible polynomial that divides  $x^{p^n} - x$ , then  $\deg \pi(x)$  divides  $n$ .

### Question 11

How many isomorphism classes are there of groups of order 45?

Describe a representative from each class.

### Question 12

For a finite group  $G$ , let  $c(G)$  denote the number of conjugacy classes of  $G$ .

- (a) Prove that if two elements of  $G$  are chosen uniformly at random, then the probability they commute is precisely

$$\frac{c(G)}{|G|}.$$

- (b) State the class equation for a finite group.
- (c) Using the class equation (or otherwise) show that the probability in part (a) is at most

$$\frac{1}{2} + \frac{1}{2[G : Z(G)]}.$$

Here, as usual,  $Z(G)$  denotes the center of  $G$ .

### Question 13

Let  $R$  be an integral domain. Recall that if  $M$  is an  $R$ -module, the *rank* of  $M$  is defined to be the maximum number of  $R$ -linearly independent elements of  $M$ .

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- (a) Prove that for any  $R$ -module  $M$ , the rank of  $\text{Tor}(M)$  is 0.
  - (b) Prove that the rank of  $M$  is equal to the rank of  $M/\text{Tor}(M)$ .
  - (c) Suppose that  $M$  is a non-principal ideal of  $R$ .
  - (d) Prove that  $M$  is torsion-free of rank 1 but not free.

**Question 14**

Let  $R$  be a commutative ring with 1.

Recall that  $x \in R$  is nilpotent iff  $x^n = 0$  for some positive integer  $n$ .

- (a) Show that every proper ideal of  $R$  is contained within a maximal ideal.
- (b) Let  $J(R)$  denote the intersection of all maximal ideals of  $R$ .  
Show that  $x \in J(R) \iff 1 + rx$  is a unit for all  $r \in R$ .
- (c) Suppose now that  $R$  is finite. Show that in this case  $J(R)$  consists precisely of the nilpotent elements in  $R$ .

**Question 15**

Let  $p$  be a prime number. Let  $A$  be a  $p \times p$  matrix over a field  $F$  with 1 in all entries except 0 on the main diagonal.

Determine the Jordan canonical form (JCF) of  $A$

- (a) When  $F = \mathbb{Q}$ ,
- (b) When  $F = \mathbb{F}_p$ .

Hint: In both cases, all eigenvalues lie in the ground field. In each case find a matrix  $P$  such that  $P^{-1}AP$  is in JCF.

**Question 16**

Let  $\zeta = e^{2\pi i/8}$ .

- (a) What is the degree of  $\mathbb{Q}(\zeta)/\mathbb{Q}$ ?
- (b) How many quadratic subfields of  $\mathbb{Q}(\zeta)$  are there?
- (c) What is the degree of  $\mathbb{Q}(\zeta, \sqrt[4]{2})$  over  $\mathbb{Q}$ ?

**Question 17**

Let  $G$  be a finite group whose order is divisible by a prime number  $p$ . Let  $P$  be a normal  $p$ -subgroup of  $G$  (so  $|P| = p^c$  for some  $c$ ).

- (a) Show that  $P$  is contained in every Sylow  $p$ -subgroup of  $G$ .
- (b) Let  $M$  be a maximal proper subgroup of  $G$ . Show that either  $P \subseteq M$  or  $|G/M| = p^b$  for some  $b \leq c$ .

**Question 18**

- (a) Suppose the group  $G$  acts on the set  $X$ . Show that the stabilizers of elements in the same orbit are conjugate.

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- (b) Let  $G$  be a finite group and let  $H$  be a proper subgroup. Show that the union of the conjugates of  $H$  is strictly smaller than  $G$ , i.e.

$$\bigcup_{g \in G} gHg^{-1} \subsetneq G$$

- (c) Suppose  $G$  is a finite group acting transitively on a set  $S$  with at least 2 elements. Show that there is an element of  $G$  with no fixed points in  $S$ .

### Question 19

Let  $F \subset K \subset L$  be finite degree field extensions. For each of the following assertions, give a proof or a counterexample.

- (a) If  $L/F$  is Galois, then so is  $K/F$ .
- (b) If  $L/F$  is Galois, then so is  $L/K$ .
- (c) If  $K/F$  and  $L/K$  are both Galois, then so is  $L/F$ .

### Question 20

Let  $V$  be a finite dimensional vector space over a field (the field is not necessarily algebraically closed).

Let  $\phi : V \rightarrow V$  be a linear transformation. Prove that there exists a decomposition of  $V$  as  $V = U \oplus W$ , where  $U$  and  $W$  are  $\phi$ -invariant subspaces of  $V$ ,  $\phi|_U$  is nilpotent, and  $\phi|_W$  is nonsingular.

### Question 21

Let  $A$  be an  $n \times n$  matrix.

- (a) Suppose that  $v$  is a column vector such that the set  $\{v, Av, \dots, A^{n-1}v\}$  is linearly independent. Show that any matrix  $B$  that commutes with  $A$  is a polynomial in  $A$ .
- (b) Show that there exists a column vector  $v$  such that the set  $\{v, Av, \dots, A^{n-1}v\}$  is linearly independent  $\iff$  the characteristic polynomial of  $A$  equals the minimal polynomial of  $A$ .

### Question 22

Let  $R$  be a commutative ring, and let  $M$  be an  $R$ -module. An  $R$ -submodule  $N$  of  $M$  is maximal if there is no  $R$ -module  $P$  with  $N \subsetneq P \subsetneq M$ .

- (a) Show that an  $R$ -submodule  $N$  of  $M$  is maximal  $\iff M/N$  is a simple  $R$ -module: i.e.,  $M/N$  is nonzero and has no proper, nonzero  $R$ -submodules.
- (b) Let  $M$  be a  $\mathbb{Z}$ -module. Show that a  $\mathbb{Z}$ -submodule  $N$  of  $M$  is maximal  $\iff \#M/N$  is a prime number.
- (c) Let  $M$  be the  $\mathbb{Z}$ -module of all roots of unity in  $\mathbb{C}$  under multiplication. Show that there is no maximal  $\mathbb{Z}$ -submodule of  $M$ .

### Question 23

Let  $R$  be a commutative ring.

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- (a) Let  $r \in R$ . Show that the map

$$\begin{aligned} r \bullet : R &\longrightarrow R \\ x &\mapsto rx. \end{aligned}$$

is an  $R$ -module endomorphism of  $R$ .

- (b) We say that  $r$  is a **zero-divisor** if  $r \bullet$  is not injective. Show that if  $r$  is a zero-divisor and  $r \neq 0$ , then the kernel and image of  $R$  each consist of zero-divisors.
- (c) Let  $n \geq 2$  be an integer. Show: if  $R$  has exactly  $n$  zero-divisors, then  $\#R \leq n^2$ .
- (d) Show that up to isomorphism there are exactly two commutative rings  $R$  with precisely 2 zero-divisors.

You may use without proof the following fact: every ring of order 4 is isomorphic to exactly one of the following:

$$\frac{\mathbb{Z}}{4\mathbb{Z}}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2 + t + 1)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2 - t)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2)}.$$

#### Question 24

- (a) Use the Class Equation (equivalently, the conjugation action of a group on itself) to prove that any  $p$ -group (a group whose order is a positive power of a prime integer  $p$ ) has a nontrivial center.
- (b) Prove that any group of order  $p^2$  (where  $p$  is prime) is abelian.
- (c) Prove that any group of order  $5^2 \cdot 7^2$  is abelian.
- (d) Write down exactly one representative in each isomorphism class of groups of order  $5^2 \cdot 7^2$ .

#### Question 25

Let  $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$ .

- (a) Find the splitting field  $K$  of  $f$ , and compute  $[K : \mathbb{Q}]$ .
- (b) Find the Galois group  $G$  of  $f$ , both as an explicit group of automorphisms, and as a familiar abstract group to which it is isomorphic.
- (c) Exhibit explicitly the correspondence between subgroups of  $G$  and intermediate fields between  $\mathbb{Q}$  and  $k$ .

#### Question 26

Let  $K$  be a Galois extension of  $\mathbb{Q}$  with Galois group  $G$ , and let  $E_1, E_2$  be intermediate fields of  $K$  which are the splitting fields of irreducible  $f_i(x) \in \mathbb{Q}[x]$ .

Let  $E = E_1 E_2 \subset K$ .

Let  $H_i = \text{Gal}(K/E_i)$  and  $H = \text{Gal}(K/E)$ .

- (a) Show that  $H = H_1 \cap H_2$ .
- (b) Show that  $H_1 H_2$  is a subgroup of  $G$ .

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(c) Show that

$$\text{Gal}(K/(E_1 \cap E_2)) = H_1 H_2.$$

### Question 27

Let

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -3 \\ 1 & 2 & -4 \end{bmatrix} \in M_3(\mathbb{C})$$

- (a) Find the Jordan canonical form  $J$  of  $A$ .  
(b) Find an invertible matrix  $P$  such that  $P^{-1}AP = J$ .

You should not need to compute  $P^{-1}$ .

### Question 28

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} x & u \\ -y & -v \end{pmatrix}$$

over a commutative ring  $R$ , where  $b$  and  $x$  are units of  $R$ . Prove that

$$MN = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \implies MN = 0.$$

### Question 29

Let

$$M = \{(w, x, y, z) \in \mathbb{Z}^4 \mid w + x + y + z \in 2\mathbb{Z}\},$$

and

$$N = \{(w, x, y, z) \in \mathbb{Z}^4 \mid 4 \mid (w - x), 4 \mid (x - y), 4 \mid (y - z)\}.$$

- (a) Show that  $N$  is a  $\mathbb{Z}$ -submodule of  $M$ .  
(b) Find vectors  $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$  and integers  $d_1, d_2, d_3, d_4$  such that

$$\{u_1, u_2, u_3, u_4\}$$

is a free basis for  $M$ , and

$$\{d_1 u_1, d_2 u_2, d_3 u_3, d_4 u_4\}$$

is a free basis for  $N$ .

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- (c) Use the previous part to describe  $M/N$  as a direct sum of cyclic  $\mathbb{Z}$ -modules.

**Question 30**

Let  $R$  be a PID and  $M$  be an  $R$ -module. Let  $p$  be a prime element of  $R$ . The module  $M$  is called  $\langle p \rangle$ -primary if for every  $m \in M$  there exists  $k > 0$  such that  $p^k m = 0$ .

- (a) Suppose  $M$  is  $\langle p \rangle$ -primary. Show that if  $m \in M$  and  $t \in R$ ,  $t \notin \langle p \rangle$ , then there exists  $a \in R$  such that  $atm = m$ .
- (b) A submodule  $S$  of  $M$  is said to be *pure* if  $S \cap rM = rS$  for all  $r \in R$ . Show that if  $M$  is  $\langle p \rangle$ -primary, then  $S$  is pure if and only if  $S \cap p^k M = p^k S$  for all  $k \geq 0$ .

**Question 31**

Let  $R = C[0, 1]$  be the ring of continuous real-valued functions on the interval  $[0, 1]$ . Let  $I$  be an ideal of  $R$ .

- (a) Show that if  $f \in I$ ,  $a \in [0, 1]$  are such that  $f(a) \neq 0$ , then there exists  $g \in I$  such that  $g(x) \geq 0$  for all  $x \in [0, 1]$ , and  $g(x) > 0$  for all  $x$  in some open neighborhood of  $a$ .
- (b) If  $I \neq R$ , show that the set  $Z(I) = \{x \in [0, 1] \mid f(x) = 0 \text{ for all } f \in I\}$  is nonempty.
- (c) Show that if  $I$  is maximal, then there exists  $x_0 \in [0, 1]$  such that  $I = \{f \in R \mid f(x_0) = 0\}$ .

**Question 32**

Suppose the group  $G$  acts on the set  $A$ . Assume this action is faithful (recall that this means that the kernel of the homomorphism from  $G$  to  $\text{Sym}(A)$  which gives the action is trivial) and transitive (for all  $a, b$  in  $A$ , there exists  $g$  in  $G$  such that  $g \cdot a = b$ ).

- (a) For  $a \in A$ , let  $G_a$  denote the stabilizer of  $a$  in  $G$ . Prove that for any  $a \in A$ ,

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \{1\}.$$

- (b) Suppose that  $G$  is abelian. Prove that  $|G| = |A|$ . Deduce that every abelian transitive subgroup of  $S_n$  has order  $n$ .

**Question 33**

- (a) Classify the abelian groups of order 36.

For the rest of the problem, assume that  $G$  is a non-abelian group of order 36.

You may assume that the only subgroup of order 12 in  $S_4$  is  $A_4$  and that  $A_4$  has no subgroup of order 6.

- (b) Prove that if the 2-Sylow subgroup of  $G$  is normal,  $G$  has a normal subgroup  $N$  such that  $G/N$  is isomorphic to  $A_4$ .
- (c) Show that if  $G$  has a normal subgroup  $N$  such that  $G/N$  is isomorphic to  $A_4$  and a subgroup  $H$  isomorphic to  $A_4$  it must be the direct product of  $N$  and  $H$ .
- (d) Show that the dihedral group of order 36 is a non-abelian group of order 36 whose Sylow-2 subgroup is not normal.



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**Question 34**

Let  $F$  be a field. Let  $f(x)$  be an irreducible polynomial in  $F[x]$  of degree  $n$  and let  $g(x)$  be any polynomial in  $F[x]$ . Let  $p(x)$  be an irreducible factor (of degree  $m$ ) of the polynomial  $f(g(x))$ .

Prove that  $n$  divides  $m$ . Use this to prove that if  $r$  is an integer which is not a perfect square, and  $n$  is a positive integer then every irreducible factor of  $x^{2n} - r$  over  $\mathbb{Q}[x]$  has even degree.

**Question 35**

- (a) Let  $f(x)$  be an irreducible polynomial of degree 4 in  $\mathbb{Q}[x]$  whose splitting field  $K$  over  $\mathbb{Q}$  has Galois group  $G = S_4$ .

Let  $\theta$  be a root of  $f(x)$ . Prove that  $\mathbb{Q}[\theta]$  is an extension of  $\mathbb{Q}$  of degree 4 and that there are no intermediate fields between  $\mathbb{Q}$  and  $\mathbb{Q}[\theta]$ .

- (b) Prove that if  $K$  is a Galois extension of  $\mathbb{Q}$  of degree 4, then there is an intermediate subfield between  $K$  and  $\mathbb{Q}$ .

**Question 36**

A ring  $R$  is called *simple* if its only two-sided ideals are 0 and  $R$ .

- (a) Suppose  $R$  is a commutative ring with 1. Prove  $R$  is simple if and only if  $R$  is a field.  
(b) Let  $k$  be a field. Show the ring  $M_n(k)$ ,  $n \times n$  matrices with entries in  $k$ , is a simple ring.

**Question 37**

For a ring  $R$ , let  $U(R)$  denote the multiplicative group of units in  $R$ . Recall that in an integral domain  $R$ ,  $r \in R$  is called *irreducible* if  $r$  is not a unit in  $R$ , and the only divisors of  $r$  have the form  $ru$  with  $u$  a unit in  $R$ .

We call a non-zero, non-unit  $r \in R$  *prime* in  $R$  if  $r \mid ab \implies r \mid a$  or  $r \mid b$ . Consider the ring  $R = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ .

- (a) Prove  $R$  is an integral domain.  
(b) Show  $U(R) = \{\pm 1\}$ .  
(c) Show  $3$ ,  $2 + \sqrt{-5}$ , and  $2 - \sqrt{-5}$  are irreducible in  $R$ .  
(d) Show  $3$  is not prime in  $R$ .  
(e) Conclude  $R$  is not a PID.

**Question 38**

Let  $F$  be a field and let  $V$  and  $W$  be vector spaces over  $F$ .

Make  $V$  and  $W$  into  $F[x]$ -modules via linear operators  $T$  on  $V$  and  $S$  on  $W$  by defining  $X \cdot v = T(v)$  for all  $v \in V$  and  $X \cdot w = S(w)$  for all  $w \in W$ .

Denote the resulting  $F[x]$ -modules by  $V_T$  and  $W_S$  respectively.

- (a) Show that an  $F[x]$ -module homomorphism from  $V_T$  to  $W_S$  consists of an  $F$ -linear transformation  $R : V \rightarrow W$  such that  $RT = SR$ .

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**Question 39**

Classify the groups of order  $182 = 2 \cdot 7 \cdot 13$ .

**Question 40**

Let  $G$  be a finite group of order  $p^n m$  where  $p$  is a prime and  $m$  is not divisible by  $p$ . Prove that if  $H$  is a subgroup of  $G$  of order  $p^k$  for some  $k < n$ , then the normalizer of  $H$  in  $G$  properly contains  $H$ .

**Question 41**

Let  $H$  be a subgroup of  $S_n$  of index  $n$ . Prove:

1. There is an isomorphism  $f : S_n \longrightarrow S_n$  such that  $f(H)$  is the subgroup of  $S_n$  stabilizing  $n$ . In particular,  $H$  is isomorphic to  $S_{n-1}$ .
2. The only subgroups of  $S_n$  containing  $H$  are  $S_n$  and  $H$ .

**Question 42**

- Prove that a group of order  $351 = 3^3 \cdot 13$  cannot be simple.
- Prove that a group of order 33 must be cyclic.

**Question 43**

1. Let  $G$  be a group, and  $Z(G)$  the center of  $G$ . Prove that if  $G/Z(G)$  is cyclic, then  $G$  is abelian.
2. Prove that a group of order  $p^n$ , where  $p$  is a prime and  $n \geq 1$ , has non-trivial center.
3. Prove that a group of order  $p^2$  must be abelian.

**Question 44**

Let  $G$  be a finite group.

1. Prove that if  $H < G$  is a proper subgroup, then  $G$  is not the union of conjugates of  $H$ .
2. Suppose that  $G$  acts transitively on a set  $X$  with  $|X| > 1$ . Prove that there exists an element of  $G$  with no fixed points in  $X$ .

**Question 45**

Classify all groups of order 15 and of order 30.

**Question 46**

Count the number of  $p$ -Sylow subgroups of  $S_p$ .

**Question 47**

1. Let  $G$  be a group of order  $n$ . Suppose that for every divisor  $d$  of  $n$ ,  $G$  contains at most one subgroup of order  $d$ . Show that  $G$  is cyclic.
2. Let  $F$  be a field. Show that every finite subgroup of the group of units  $F^\times$  is cyclic.

**Question 48**

Let  $K$  and  $L$  be finite fields. Show that  $K$  is contained in  $L$  if and only if  $\#K = p^r$  and  $\#L = p^s$  for the same prime  $p$ , and  $r \leq s$ .

**Question 49**

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Let  $K$  and  $L$  be finite fields with  $K \subseteq L$ . Prove that  $L$  is Galois over  $K$  and that  $\text{Gal}(L/K)$  is cyclic.

**Question 50**

Fix a field  $F$ , a separable polynomial  $f \in F[x]$  of degree  $n \geq 3$ , and a splitting field  $L$  for  $f$ . Prove that if  $[L : F] = n!$  then:

1.  $f$  is irreducible.
2. For each root  $r$  of  $f$ ,  $r$  is the unique root of  $f$  in  $F(r)$ .
3. For every root  $r$  of  $f$ , there are no proper intermediate fields  $F \subset L \subset F(r)$ .

**Question 51**

1. Show that  $\sqrt{2 + \sqrt{2}}$  is a root of  $p(x) = x^2 - 4x^2 + 2 \in \mathbb{Q}[x]$ .
2. Prove that  $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$  is a Galois extension of  $\mathbb{Q}$  and find its Galois group. (Hint: note that  $\sqrt{2 - \sqrt{2}}$  is another root of  $p(x)$ ).
3. Let  $f(x) = x^3 - 5$ . Determine the splitting field  $K$  of  $f(x)$  over  $\mathbb{Q}$  and the Galois group of  $f(x)$ . Give an example of a proper sub-extension  $\mathbb{Q} \subset L \subset K$ , such that  $L/\mathbb{Q}$  is Galois.

**Question 52**

An integral domain  $R$  is said to be an *Euclidean domain* if there is a function  $N : R \rightarrow \{n \in \mathbb{Z} \mid n \geq 0\}$  such that  $N(0) = 0$  and for each  $a, b \in R$  with  $b \neq 0$ , there exist elements  $q, r \in R$  with

$$a = qb + r, \quad \text{and} \quad r = 0 \text{ or } N(r) < N(b).$$

Prove:

1. The ring  $F[[x]]$  of power series over a field  $F$  is an Euclidean domain.
2. Every Euclidean domain is a PID.

**Question 53**

Let  $F$  be a field, and let  $R$  be the subring of  $F[X]$  of polynomials with  $X$  coefficient equal to 0. Prove that  $R$  is not a UFD.

**Question 54**

$R$  is a commutative ring with 1. Prove that if  $I$  is a maximal ideal in  $R$ , then  $R/I$  is a field. Prove that if  $R$  is a PID, then every nonzero prime ideal in  $R$  is maximal. Conclude that if  $R$  is a PID and  $p \in R$  is prime, then  $R/(p)$  is a field.

**Question 55**

Prove that any square matrix is conjugate to its transpose matrix. (You may prove it over  $\mathbb{C}$ ).

**Question 56**

Determine the number of conjugacy classes of  $16 \times 16$  matrices with entries in  $\mathbb{Q}$  and minimal polynomial  $(x^2 + 1)^2(x^3 + 2)^2$ .

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**Question 57**

Let  $V$  be a vector space over a field  $F$ . The evaluation map  $e: V \longrightarrow (V^\vee)^\vee$  is defined by  $e(v)(f) := f(v)$  for  $v \in V$  and  $f \in V^\vee$ .

1. Prove that  $e$  is an injection.
2. Prove that  $e$  is an isomorphism if and only if  $V$  is finite dimensional.

**Question 58**

Let  $R$  be a principal ideal domain that is not a field, and write  $F$  for its field of fractions. Prove that  $F$  is not a finitely generated  $R$ -module.

**Question 59**

Carefully state Zorn's lemma and use it to prove that every vector space has a basis.

**Question 60**

Show that no finite group is the union of conjugates of a proper subgroup.

**Question 61**

Classify all groups of order 18 up to isomorphism.

**Question 62**

Let  $\alpha, \beta$  denote the unique positive real 5<sup>th</sup> root of 7 and 4<sup>th</sup> root of 5, respectively. Determine the degree of  $\mathbb{Q}(\alpha, \beta)$  over  $\mathbb{Q}$ .

**Question 63**

Show that the field extension  $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2 + \sqrt{2}})$  is Galois and determine its Galois group.

**Question 64**

Let  $M$  be a square matrix over a field  $K$ . Use a suitable canonical form to show that  $M$  is similar to its transpose  $M^T$ .

**Question 65**

Let  $G$  be a finite group and  $\pi_0, \pi_1$  be two irreducible representations of  $G$ . Prove or disprove the following assertion:  $\pi_0$  and  $\pi_1$  are equivalent if and only if  $\det \pi_0(g) = \det \pi_1(g)$  for all  $g \in G$ .

**Question 66**

Let  $R$  be a Noetherian ring. Prove that  $R[x]$  and  $R[[x]]$  are both Noetherian. (The first part of the question is asking you to prove the Hilbert Basis Theorem, not to use it!)

**Question 67**

Classify (with proof) all fields with finitely many elements.

**Question 68**

Suppose  $A$  is a commutative ring and  $M$  is a finitely presented module. Given any surjection  $\phi: A^n \rightarrow M$  from a finite free  $A$ -module, show that  $\ker \phi$  is finitely generated.

**Question 69**

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Classify all groups of order 57.

**Question 70**

Show that a finite simple group cannot have a 2-dimensional irreducible representation over  $\mathbb{C}$ .

Hint: the determinant might prove useful.

**Question 71**

Let  $G$  be a finite simple group. Assume that every proper subgroup of  $G$  is abelian. Prove that then  $G$  is cyclic of prime order.

**Question 72**

Let  $a \in \mathbb{N}$ ,  $a > 0$ . Compute the Galois group of the splitting field of the polynomial  $x^5 - 5a^4x + a$  over  $\mathbb{Q}$ .

**Question 73**

Recall that an inner automorphism of a group is an automorphism given by conjugation by an element of the group. An outer automorphism is an automorphism that is not inner.

- Prove that  $S_5$  has a subgroup of order 20.
- Use the subgroup from (a) to construct a degree 6 permutation representation of  $S_5$  (i.e., an embedding  $S_5 \hookrightarrow S_6$  as a transitive permutation group on 6 letters).
- Conclude that  $S_6$  has an outer automorphism.

**Question 74**

Let  $A$  be a commutative ring and  $M$  a finitely generated  $A$ -module. Define

$$\text{Ann}(M) = \{a \in A : am = 0 \text{ for all } m \in M\}.$$

Show that for a prime ideal  $\mathfrak{p} \subset A$ , the following are equivalent:

- $\text{Ann}(M) \not\subset \mathfrak{p}$
- The localization of  $M$  at the prime ideal  $\mathfrak{p}$  is 0.
- $M \otimes_A k(\mathfrak{p}) = 0$ , where  $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$  is the residue field of  $A$  at  $\mathfrak{p}$ .

**Question 75**

Let  $A = \mathbb{C}[x, y]/(y^2 - (x - 1)^3 - (x - 1)^2)$ .

- Show that  $A$  is an integral domain and sketch the  $\mathbb{R}$ -points of  $\text{Spec} A$ .
- Find the integral closure of  $A$ . Recall that for an integral domain  $A$  with fraction field  $K$ , the integral closure of  $A$  in  $K$  is the set of all elements of  $K$  integral over  $A$ .

**Question 76**

Let  $R = k[x, y]$  where  $k$  is a field, and let  $I = (x, y)R$ .

- Show that

$$0 \longrightarrow R \xrightarrow{\phi} R \oplus R \xrightarrow{\psi} R \longrightarrow k \longrightarrow 0$$

where  $\phi(a) = (-ya, xa)$ ,  $\psi((a, b)) = xa + yb$  for  $a, b \in R$ , is a projective resolution of the  $R$ -module  $k \simeq R/I$ .

- 
- Show that  $I$  is not a flat  $R$ -module by computing  $\text{Tor}_i^R(I, k)$

**Question 77**

- Find an irreducible polynomial of degree 5 over the field  $\mathbb{Z}/2$  of two elements and use it to construct a field of order 32 as a quotient of the polynomial ring  $\mathbb{Z}/2[x]$ .
- Using the polynomial found in part (a), find a  $5 \times 5$  matrix  $M$  over  $\mathbb{Z}/2$  of order 31, so that  $M^{31} = I$  but  $M \neq I$ .

**Question 78**

Find the minimal polynomial of  $\sqrt{2} + \sqrt{3}$  over  $\mathbb{Q}$ . Justify your answer.

**Question 79**

- Let  $R$  be a commutative ring with no nonzero nilpotent elements. Show that the only units in the polynomial ring  $R[x]$  are the units of  $R$ , regarded as constant polynomials.
- Find all units in the polynomial ring  $\mathbb{Z}_4[x]$ .

**Question 80**

Let  $p, q$  be two distinct primes. Prove that there is at most one non-abelian group of order  $pq$  and describe the pairs  $(p, q)$  such that there is no non-abelian group of order  $pq$ .

**Question 81**

- Let  $L$  be a Galois extension of a field  $K$  of degree 4. What is the minimum number of subfields there could be strictly between  $K$  and  $L$ ? What is the maximum number of such subfields? Give examples where these bounds are attained.
- How do these numbers change if we assume only that  $L$  is separable (but not necessarily Galois) over  $K$ ?

**Question 82**

Let  $R$  be a commutative algebra over  $\mathbb{C}$ . A derivation of  $R$  is a  $\mathbb{C}$ -linear map  $D : R \rightarrow R$  such that (i)  $D(1) = 0$  and (ii)  $D(ab) = D(a)b + aD(b)$  for all  $a, b \in R$ .

- Describe all derivations of the polynomial ring  $\mathbb{C}[x]$ .
- Let  $A$  be the subring (or  $\mathbb{C}$ -subalgebra) of  $\text{End}_{\mathbb{C}}(\mathbb{C}[x])$  generated by all derivations of  $\mathbb{C}[x]$  and the left multiplications by  $x$ . Prove that  $\mathbb{C}[x]$  is a simple left  $A$ -module. > Note that the inclusion  $A \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}[x])$  defines a natural left  $A$ -module structure on  $\mathbb{C}[x]$ .

**Question 83**

Let  $G$  be a non-abelian group of order  $p^3$  with  $p$  a prime.

- Determine the order of the center  $Z$  of  $G$ .
- Determine the number of inequivalent complex 1-dimensional representations of  $G$ .
- Compute the dimensions of all the inequivalent irreducible representations of  $G$  and verify that the number of such representations equals the number of conjugacy classes of  $G$ .

**Question 84**

- 
- Let  $G$  be a group (not necessarily finite) that contains a subgroup of index  $n$ . Show that  $G$  contains a *normal* subgroup  $N$  such that  $n \leq [G : N] \leq n!$
  - Use part (a) to show that there is no simple group of order 36.

### Question 85

Let  $p$  be a prime, let  $\mathbb{F}_p$  be the  $p$ -element field, and let  $K = \mathbb{F}_p(t)$  be the field of rational functions in  $t$  with coefficients in  $\mathbb{F}_p$ . Consider the polynomial  $f(x) = x^p - t \in K[x]$ .

- Show that  $f$  does not have a root in  $K$ .
- Let  $E$  be the splitting field of  $f$  over  $K$ . Find the factorization of  $f$  over  $E$ .
- Conclude that  $f$  is irreducible over  $K$ .

### Question 86

Recall that a ring  $A$  is called *graded* if it admits a direct sum decomposition  $A = \bigoplus_{n=0}^{\infty} A_n$  as abelian groups, with the property that  $A_i A_j \subseteq A_{i+j}$  for all  $i, j \geq 0$ . Prove that a graded commutative ring  $A = \bigoplus_{n=0}^{\infty} A_n$  is Noetherian if and only if  $A_0$  is Noetherian and  $A$  is finitely generated as an algebra over  $A_0$ .

### Question 87

Let  $R$  be a ring with the property that  $a^2 = a$  for all  $a \in R$ .

- Compute the Jacobson radical of  $R$ .
- What is the characteristic of  $R$ ?
- Prove that  $R$  is commutative.
- Prove that if  $R$  is finite, then  $R$  is isomorphic (as a ring) to  $(\mathbb{Z}/2\mathbb{Z})^d$  for some  $d$ .

### Question 88

Let  $\overline{\mathbb{F}_p}$  denote the algebraic closure of  $\mathbb{F}_p$ . Show that the Galois group  $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$  has no non-trivial finite subgroups.

### Question 89

Let  $C_p$  denote the cyclic group of order  $p$ .

- Show that  $C_p$  has two irreducible representations over  $\mathbb{Q}$  (up to isomorphism), one of dimension 1 and one of dimension  $p - 1$ .
- Let  $G$  be a finite group, and let  $\rho : G \rightarrow \text{GL}_n(\mathbb{Q})$  be a representation of  $G$  over  $\mathbb{Q}$ . Let  $\rho_{\mathbb{C}} : G \rightarrow \text{GL}_n(\mathbb{C})$  denote  $\rho$  followed by the inclusion  $\text{GL}_n(\mathbb{Q}) \rightarrow \text{GL}_n(\mathbb{C})$ . Thus  $\rho_{\mathbb{C}}$  is a representation of  $G$  over  $\mathbb{C}$ , called the *complexification* of  $\rho$ . We say that an irreducible representation  $\rho$  of  $G$  is *absolutely irreducible* if its complexification remains irreducible over  $\mathbb{C}$ . Now suppose  $G$  is abelian and that every representation of  $G$  over  $\mathbb{Q}$  is absolutely irreducible. Show that  $G \cong (C_2)^k$  for some  $k$  (i.e., is a product of cyclic groups of order 2).

### Question 90

Let  $G$  be a finite group and  $\mathbb{Z}[G]$  the integral group algebra. Let  $\mathcal{Z}$  be the center of  $\mathbb{Z}[G]$ . For each conjugacy class  $C \subseteq G$ , let  $P_C = \sum_{g \in C} g$ .

- Show that the elements  $P_C$  form a  $\mathbb{Z}$ -basis for  $\mathcal{Z}$ . Hence  $\mathcal{Z} \cong \mathbb{Z}^d$  as an abelian group, where  $d$  is the number of conjugacy classes in  $G$ .
- Show that if a ring  $R$  is isomorphic to  $\mathbb{Z}^d$  as an abelian group, then every element in  $R$  satisfies a monic integral polynomial.

**Hint:** Let  $\{v_1, \dots, v_d\}$  be a basis of  $R$  and for a fixed non-zero  $r \in R$ , write  $rv_i = \sum_j a_{ij}v_j$ . Use the Hamilton-Cayley theorem.

- Let  $\pi : G \rightarrow \text{GL}(V)$  be an irreducible representation of  $G$  (over  $\mathbb{C}$ ). Show that  $\pi(P_C)$  acts on  $V$  as multiplication by the scalar

$$\frac{|C|\chi_\pi(C)}{\dim V},$$

where  $\chi_\pi(C)$  is the value of the character  $\chi_\pi$  on any element of  $C$ .

- Conclude that  $|C|\chi_\pi(C)/\dim V$  is an algebraic integer.

### Question 91

- Suppose that  $G$  is a finitely generated group. Let  $n$  be a positive integer. Prove that  $G$  has only finitely many subgroups of index  $n$ .
- Let  $p$  be a prime number. If  $G$  is any finitely-generated abelian group, let  $t_p(G)$  denote the number of subgroups of  $G$  of index  $p$ . Determine the possible values of  $t_p(G)$  as  $G$  varies over all finitely-generated abelian groups.

### Question 92

Suppose that  $G$  is a finite group of order 2013. Prove that  $G$  has a normal subgroup  $N$  of index 3 and that  $N$  is a cyclic group. Furthermore, prove that the center of  $G$  has order divisible by 11. (You will need the factorization  $2013 = 3 \cdot 11 \cdot 61$ .)

### Question 93

This question concerns an extension  $K$  of  $\mathbb{Q}$  such that  $[K : \mathbb{Q}] = 8$ . Assume that  $K/\mathbb{Q}$  is Galois and let  $G = \text{Gal}(K/\mathbb{Q})$ . Furthermore, assume that  $G$  is non-abelian.

- Prove that  $K$  has a unique subfield  $F$  such that  $F/\mathbb{Q}$  is Galois and  $[F : \mathbb{Q}] = 4$ .
- Prove that  $F$  has the form  $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$  where  $d_1, d_2$  are non-zero integers.
- Suppose that  $G$  is the quaternionic group. Prove that  $d_1$  and  $d_2$  are positive integers.

### Question 94

This question concerns the polynomial ring  $R = \mathbb{Z}[x, y]$  and the ideal  $I = (5, x^2 + 2)$  in  $R$ .

- Prove that  $I$  is a prime ideal of  $R$  and that  $R/I$  is a PID.
- Give an explicit example of a maximal ideal of  $R$  which contains  $I$ . (Give a set of generators for such an ideal.)
- Show that there are infinitely many distinct maximal ideals in  $R$  which contain  $I$ .



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**Question 95**

Classify all groups of order 2012 up to isomorphism.

Hint: 503 is prime.

**Question 96**

For any positive integer  $n$ , let  $G_n$  be the group generated by  $a$  and  $b$  subject to the following three relations:

$$a^2 = 1, \quad b^2 = 1, \quad \text{and} \quad (ab)^n = 1.$$

- Find the order of the group  $G_n$

**Question 97**

Determine the Galois groups of the following polynomials over  $\mathbb{Q}$ .

- $f(x) = x^4 + 4x^2 + 1$
- $f(x) = x^4 + 4x^2 - 5$ .

**Question 98**

Let  $R$  be a (commutative) principal ideal domain, let  $M$  and  $N$  be finitely generated free  $R$ -modules, and let  $\varphi : M \rightarrow N$  be an  $R$ -module homomorphism.

- Let  $K$  be the kernel of  $\varphi$ . Prove that  $K$  is a direct summand of  $M$ .
- Let  $C$  be the image of  $\varphi$ . Show by example (specifying  $R$ ,  $M$ ,  $N$ , and  $\varphi$ ) that  $C$  need not be a direct summand of  $N$ .

**Question 99**

In this problem, as you apply Sylow's Theorem, state precisely which portions you are using.

- Prove that there is no simple group of order 30.
- Suppose that  $G$  is a simple group of order 60. Determine the number of  $p$ -Sylow subgroups of  $G$  for each prime  $p$  dividing 60, then prove that  $G$  is isomorphic to the alternating group  $A_5$ .

Note: in the second part, you needn't show that  $A_5$  is simple. You need only show that if there is a simple group of order 60, then it must be isomorphic to  $A_5$ .

**Question 100**

Describe the Galois group and the intermediate fields of the cyclotomic extension  $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$ .

**Question 101**

Let

$$R = \mathbb{Z}[x]/(x^2 + x + 1).$$

- Answer the following questions with suitable justification.
  - Is  $R$  a Noetherian ring?
  - Is  $R$  an Artinian ring?
- Prove that  $R$  is an integrally closed domain.

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**Question 102**

Let  $R$  be a commutative ring. Recall that an element  $r$  of  $R$  is *nilpotent* if  $r^n = 0$  for some positive integer  $n$  and that the *nilradical* of  $R$  is the set  $N(R)$  of nilpotent elements.

- Prove that

$$N(R) = \cap_{P \text{ prime}} P.$$

Hint: given a non-nilpotent element  $r$  of  $R$ , you may wish to construct a prime ideal that does not contain  $r$  or its powers.

- Given a positive integer  $m$ , determine the nilradical of  $\mathbb{Z}/(m)$ .
- Determine the nilradical of  $\mathbb{C}[x, y]/(y^2 - x^3)$ .
- Let  $p(x, y)$  be a polynomial in  $\mathbb{C}[x, y]$  such that for any complex number  $a$ ,  $p(a, a^{3/2}) = 0$ . Prove that  $p(x, y)$  is divisible by  $y^2 - x^3$ .

**Question 103**

Given a finite group  $G$ , recall that its *regular representation* is the representation on the complex group algebra  $\mathbb{C}[G]$  induced by left multiplication of  $G$  on itself and its *adjoint representation* is the representation on the complex group algebra  $\mathbb{C}[G]$  induced by conjugation of  $G$  on itself.

- Let  $G = \mathrm{GL}_2(\mathbb{F}_2)$ . Describe the number and dimensions of the irreducible representations of  $G$ . Then describe the decomposition of its regular representation as a direct sum of irreducible representations.
- Let  $G$  be a group of order 12. Show that its adjoint representation is reducible; that is, there is an  $H$ -invariant subspace of  $\mathbb{C}[H]$  besides 0 and  $\mathbb{C}[H]$ .

**Question 104**

Let  $R$  be a commutative integral domain. Show that the following are equivalent:

- $R$  is a field;
- $R$  is a semi-simple ring;
- Any  $R$ -module is projective.

**Question 105**

Let  $p$  be a positive prime number,  $\mathbb{F}_p$  the field with  $p$  elements, and let  $G = \mathrm{GL}_2(\mathbb{F}_p)$ .

- Compute the order of  $G$ ,  $|G|$ .
- Write down an explicit isomorphism from  $\mathbb{Z}/p\mathbb{Z}$  to

$$U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p \right\}.$$

- How many subgroups of order  $p$  does  $G$  have?

Hint: compute  $gug^{-1}$  for  $g \in G$  and  $u \in U$ ; use this to find the size of the normalizer of  $U$  in  $G$ .

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**Question 106**

- Give definitions of the following terms:
  - (i) a finite length (left) module, (ii) a composition series for a module, and (iii) the length of a module,
- Let  $l(M)$  denote the length of a module  $M$ . Prove that if

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \rightarrow 0.$$

is an exact sequence of modules of finite length, then

$$\sum_{i=1}^n (-1)^i l(M_i) = 0.$$

**Question 107**

Let  $\mathbb{F}$  be a field of characteristic  $p$ , and  $G$  a group of order  $p^n$ . Let  $R = \mathbb{F}[G]$  be the group ring (group algebra) of  $G$  over  $\mathbb{F}$ , and let  $u := \sum_{x \in G} x$  (so  $u$  is an element of  $R$ ).

- Prove that  $u$  lies in the center of  $R$ .
- Verify that  $Ru$  is a 2-sided ideal of  $R$ .
- Show there exists a positive integer  $k$  such that  $u^k = 0$ . Conclude that for such a  $k$ ,  $(Ru)^k = 0$ .
- Show that  $R$  is **not** a semi-simple ring.

**Warning:** Please use the definition of a semi-simple ring; do **not** use the result that a finite length ring fails to be semisimple if and only if it has a non-zero nilpotent ideal.

**Question 108**

Let  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$  (where  $a_n \neq 0$ ) and let  $R = \mathbb{Z}[x]/(f)$ . Prove that  $R$  is a finitely generated module over  $\mathbb{Z}$  if and only if  $a_n = \pm 1$ .

**Question 109**

Consider the ring

$$S = C[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}.$$

with the usual operations of addition and multiplication of functions.

- What are the invertible elements of  $S$ ?
- For  $a \in [0, 1]$ , define  $I_a = \{f \in S : f(a) = 0\}$ . Show that  $I_a$  is a maximal ideal of  $S$ .
- Show that the elements of any proper ideal of  $S$  have a common zero, i.e., if  $I$  is a proper ideal of  $S$ , then there exists  $a \in [0, 1]$  such that  $f(a) = 0$  for all  $f \in I$ . Conclude that every maximal ideal of  $S$  is of the form  $I_a$  for some  $a \in [0, 1]$ .

**Hint:** As  $[0, 1]$  is compact, every open cover of  $[0, 1]$  contains a finite subcover.

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**Question 110**

Let  $F$  be a field of characteristic zero, and let  $K$  be an *algebraic* extension of  $F$  that possesses the following property: every polynomial  $f \in F[x]$  has a root in  $K$ . Show that  $K$  is algebraically closed.

**Hint:** if  $K(\theta)/K$  is algebraic, consider  $F(\theta)/F$  and its normal closure; primitive elements might be of help.

**Question 111**

Let  $G$  be the unique non-abelian group of order 21.

- Describe all 1-dimensional complex representations of  $G$ .
- How many (non-isomorphic) irreducible complex representations does  $G$  have and what are their dimensions?
- Determine the character table of  $G$ .

**Question 112**

- Classify all groups of order  $2009 = 7^2 \times 41$ .
- Suppose that  $G$  is a group of order 2009. How many intermediate groups are there—that is, how many groups  $H$  are there with  $1 \subsetneq H \subsetneq G$ , where both inclusions are proper? (There may be several cases to consider.)

**Question 113**

Let  $K$  be a field. A discrete valuation on  $K$  is a function  $\nu : K \setminus \{0\} \rightarrow \mathbb{Z}$  such that

- $\nu(ab) = \nu(a) + \nu(b)$
- $\nu$  is surjective
- $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$  for  $a, b \in K \setminus \{0\}$  with  $a + b \neq 0$ .

Let  $R := \{x \in K \setminus \{0\} : \nu(x) \geq 0\} \cup \{0\}$ . Then  $R$  is called the valuation ring of  $\nu$ .

Prove the following:

- $R$  is a subring of  $K$  containing the 1 in  $K$ .
- for all  $x \in K \setminus \{0\}$ , either  $x$  or  $x^{-1}$  is in  $R$ .
- $x$  is a unit of  $R$  if and only if  $\nu(x) = 0$ .
- Let  $p$  be a prime number,  $K = \mathbb{Q}$ , and  $\nu_p : \mathbb{Q} \setminus \{0\} \rightarrow \mathbb{Z}$  be the function defined by  $\nu_p(\frac{a}{b}) = n$  where  $\frac{a}{b} = p^n \frac{c}{d}$  and  $p$  does not divide  $c$  and  $d$ . Prove that the corresponding valuation ring  $R$  is the ring of all rational numbers whose denominators are relatively prime to  $p$ .

**Question 114**

Let  $F$  be a field of characteristic not equal to 2.

- Prove that any extension  $K$  of  $F$  of degree 2 is of the form  $F(\sqrt{D})$  where  $D \in F$  is not a square in  $F$  and, conversely, that each such extension has degree 2 over  $F$ .

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- Let  $D_1, D_2 \in F$  neither of which is a square in  $F$ . Prove that  $[F(\sqrt{D_1}, \sqrt{D_2}) : F] = 4$  if  $D_1 D_2$  is not a square in  $F$  and is of degree 2 otherwise.

### Question 115

Let  $F$  be a field and  $p(x) \in F[x]$  an irreducible polynomial.

- Prove that there exists a field extension  $K$  of  $F$  in which  $p(x)$  has a root.
- Determine the dimension of  $K$  as a vector space over  $F$  and exhibit a vector space basis for  $K$ .
- If  $\theta \in K$  denotes a root of  $p(x)$ , express  $\theta^{-1}$  in terms of the basis found in part (b).
- Suppose  $p(x) = x^3 + 9x + 6$ . Show  $p(x)$  is irreducible over  $\mathbb{Q}$ . If  $\theta$  is a root of  $p(x)$ , compute the inverse of  $(1 + \theta)$  in  $\mathbb{Q}(\theta)$ .

### Question 116

Fix a ring  $R$ , an  $R$ -module  $M$ , and an  $R$ -module homomorphism  $f : M \rightarrow M$ .

- If  $M$  satisfies the descending chain condition on submodules, show that if  $f$  is injective, then  $f$  is surjective.

Hint: note that if  $f$  is injective, so are  $f \circ f$ ,  $f \circ f \circ f$ , etc.

- Give an example of a ring  $R$ , an  $R$ -module  $M$ , and an injective  $R$ -module homomorphism  $f : M \rightarrow M$  which is not surjective.
- If  $M$  satisfies the ascending chain condition on submodules, show that if  $f$  is surjective, then  $f$  is injective.
- Give an example of a ring  $R$ , and  $R$ -module  $M$ , and a surjective  $R$ -module homomorphism  $f : M \rightarrow M$  which is not injective.

### Question 117

Let  $G$  be a finite group,  $k$  an algebraically closed field, and  $V$  an irreducible  $k$ -linear representation of  $G$ .

- Show that  $\text{hom}_{kG}(V, V)$  is a division algebra with  $k$  in its center.
- Show that  $V$  is finite-dimensional over  $k$ , and conclude that  $\text{hom}_{kG}(V, V)$  is also finite dimensional.
- Show the inclusion  $k \hookrightarrow \text{hom}_{kG}(V, V)$  found in (a) is an isomorphism. (For  $f \in \text{hom}_{kG}(V, V)$ , view  $f$  as a linear transformation and consider  $f - \alpha I$ , where  $\alpha$  is an eigenvalue of  $f$ ).

### Question 118

Let  $f(x)$  be an irreducible polynomial of degree 5 over the field  $\mathbb{Q}$  of rational numbers with exactly 3 real roots.

- Show that  $f(x)$  is not solvable by radicals.
- Let  $E$  be the splitting field of  $f$  over  $\mathbb{Q}$ . Construct a Galois extension  $K$  of degree 2 over  $\mathbb{Q}$  lying in  $E$  such that no field  $F$  strictly between  $K$  and  $E$  is Galois over  $\mathbb{Q}$ .

### Question 119

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Let  $F$  be a finite field. Show for any positive integer  $n$  that there are irreducible polynomials of degree  $n$  in  $F[x]$ .

**Question 120**

Show that the order of the group  $\mathrm{GL}_n(\mathbb{F}_q)$  of invertible  $n \times n$  matrices over the field  $\mathbb{F}_q$  of  $q$  elements is given by  $(q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$ .

**Question 121**

- Let  $R$  be a commutative principal ideal domain. Show that any  $R$ -module  $M$  generated by two elements takes the form  $R/(a) \oplus R/(b)$  for some  $a, b \in R$ . What more can you say about  $a$  and  $b$ ?
- Give a necessary and sufficient condition for two direct sums as in part (a) to be isomorphic as  $R$ -modules.

**Question 122**

Let  $G$  be the subgroup of  $\mathrm{GL}_3(\mathbb{C})$  generated by the three matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where  $i^2 = -1$ . Here  $\mathbb{C}$  denotes the complex field.

- Compute the order of  $G$ .
- Find a matrix in  $G$  of largest possible order (as an element of  $G$ ) and compute this order.
- Compute the number of elements in  $G$  with this largest order.

**Question 123**

- Let  $G$  be a group of (finite) order  $n$ . Show that any irreducible left module over the group algebra  $\mathbb{C}G$  has complex dimension at least  $\sqrt{n}$ .
- Give an example of a group  $G$  of order  $n \geq 5$  and an irreducible left module over  $\mathbb{C}G$  of complex dimension  $\lfloor \sqrt{n} \rfloor$ , the greatest integer to  $\sqrt{n}$ .

**Question 124**

Use the rational canonical form to show that any square matrix  $M$  over a field  $k$  is similar to its transpose  $M^t$ , recalling that  $p(M) = 0$  for some  $p \in k[t]$  if and only if  $p(M^t) = 0$ .

**Question 125**

Let  $K$  be a field of characteristic zero and  $L$  a Galois extension of  $K$ . Let  $f$  be an irreducible polynomial in  $K[x]$  of degree 7 and suppose  $f$  has no zeroes in  $L$ . Show that  $f$  is irreducible in  $L[x]$ .

**Question 126**

Let  $K$  be a field of characteristic zero and  $f \in K[x]$  an irreducible polynomial of degree  $n$ . Let  $L$  be a splitting field for  $f$ . Let  $G$  be the group of automorphisms of  $L$  which act trivially on  $K$ .

- Show that  $G$  embeds in the symmetric group  $S_n$ .

- For each  $n$ , give an example of a field  $K$  and polynomial  $f$  such that  $G = S_n$ .
- What are the possible groups  $G$  when  $n = 3$ . Justify your answer.

### Question 127

Show there are exactly two groups of order 21 up to isomorphism.

### Question 128

Let  $K$  be the field  $\mathbb{Q}(z)$  of rational functions in a variable  $z$  with coefficients in the rational field  $\mathbb{Q}$ . Let  $n$  be a positive integer. Consider the polynomial  $x^n - z \in K[x]$ .

- Show that the polynomial  $x^n - z$  is irreducible over  $K$ .
- Describe the splitting field of  $x^n - z$  over  $K$ .
- Determine the Galois group of the splitting field of  $x^5 - z$  over the field  $K$ .

### Question 129

- Let  $p < q < r$  be prime integers. Show that a group of order  $pqr$  cannot be simple.
- Consider groups of orders  $2^2 \cdot 3 \cdot p$  where  $p$  has the values 5, 7, and 11. For each of those values of  $p$ , either display a simple group of order  $2^2 \cdot 3 \cdot p$ , or show that there cannot be a simple group of that order.

### Question 130

Let  $K/F$  be a finite Galois extension and let  $n = [K : F]$ . There is a theorem (often referred to as the “normal basis theorem”) which states that there exists an irreducible polynomial  $f(x) \in F[x]$  whose roots form a basis for  $K$  as a vector space over  $F$ . You may assume that theorem in this problem.

- Let  $G = \text{Gal}(K/F)$ . The action of  $G$  on  $K$  makes  $K$  into a finite-dimensional representation space for  $G$  over  $F$ . Prove that  $K$  is isomorphic to the regular representation for  $G$  over  $F$ .

The regular representation is defined by letting  $G$  act on the group algebra  $F[G]$  by multiplication on the left.

- Suppose that the Galois group  $G$  is cyclic and that  $F$  contains a primitive  $n^{\text{th}}$  root of unity. Show that there exists an injective homomorphism  $\chi : G \rightarrow F^\times$ .
- Show that  $K$  contains a non-zero element  $a$  with the following property:

$$g(a) = \chi(g) \cdot a.$$

for all  $g \in G$ .

- If  $a$  has the property stated in (c), show that  $K = F(a)$  and that  $a^n \in F^\times$ .

### Question 131

Let  $G$  be the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

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with entries in the finite field  $\mathbb{F}_p$  of  $p$  element, where  $p$  is a prime.

- Prove that  $G$  is non-abelian.
- Suppose  $p$  is odd. Prove that  $g^p = I_3$  for all  $g \in G$ .
- Suppose that  $p = 2$ . It is known that there are exactly two non-abelian groups of order 8, up to isomorphism: the dihedral group  $D_8$  and the quaternionic group. Assuming this fact without proof, determine which of these groups  $G$  is isomorphic to.

### Question 132

There are five nonisomorphic groups of order 8. For each of those groups  $G$ , find the smallest positive integer  $n$  such that there is an injective homomorphism  $\varphi : G \rightarrow S_n$ .

### Question 133

For any group  $G$  we define  $\Omega(G)$  to be the image of the group homomorphism  $\rho : G \rightarrow \text{Aut}(G)$  where  $\rho$  maps  $g \in G$  to the conjugation automorphism  $x \mapsto gxg^{-1}$ . Starting with a group  $G_0$ , we define  $G_1 = \Omega(G_0)$  and  $G_{i+1} = \Omega(G_i)$  for all  $i \geq 0$ . If  $G_0$  is of order  $p^e$  for a prime  $p$  and integer  $e \geq 2$ , prove that  $G_{e-1}$  is the trivial group.

### Question 134

Let  $\mathbb{F}_2$  be the field with two elements.

- What is the order of  $\text{GL}_3(\mathbb{F}_2)$ ?
- Use the fact that  $\text{GL}_3(\mathbb{F}_2)$  is a simple group (which you should not prove) to find the number of elements of order 7 in  $\text{GL}_3(\mathbb{F}_2)$ .

### Question 135

Let  $G$  be a finite abelian group. Let  $f : \mathbb{Z}^m \rightarrow G$  be a surjection of abelian groups. We may think of  $f$  as a homomorphism of  $\mathbb{Z}$ -modules. Let  $K$  be the kernel of  $f$ .

- Prove that  $K$  is isomorphic to  $\mathbb{Z}^m$ .
- We can therefore write the inclusion map  $K \rightarrow \mathbb{Z}^m$  as  $\mathbb{Z}^m \rightarrow \mathbb{Z}^m$  and represent it by an  $m \times m$  integer matrix  $A$ . Prove that  $|\det A| = |G|$ .

### Question 136

Let  $R = C([0, 1])$  be the ring of all continuous real-valued functions on the closed interval  $[0, 1]$ , and for each  $c \in [0, 1]$ , denote by  $M_c$  the set of all functions  $f \in R$  such that  $f(c) = 0$ .

- Prove that  $g \in R$  is a unit if and only if  $g(c) \neq 0$  for all  $c \in [0, 1]$ .
- Prove that for each  $c \in [0, 1]$ ,  $M_c$  is a maximal ideal of  $R$ .
- Prove that if  $M$  is a maximal ideal of  $T$ , then  $M = M_c$  for some  $c \in [0, 1]$ .

Hint: compactness of  $[0, 1]$  may be relevant.

### Question 137

Let  $R$  and  $S$  be commutative rings, and  $f : R \rightarrow S$  a ring homomorphism.



- Show that if  $I$  is a prime ideal of  $S$ , then

$$f^{-1}(I) = \{r \in R : f(r) \in I\}$$

is a prime ideal of  $R$ .

- Let  $N$  be the set of nilpotent elements of  $R$ :

$$N = \{r \in R : r^m = 0 \text{ for some } m \geq 1\}..$$

$N$  is called the *nilradical* of  $R$ . Prove that it is an ideal which is contained in every prime ideal.

- Part (a) lets us define a function

$$f^* : \{\text{prime ideals of } S\} \rightarrow \{\text{prime ideals of } R\}.I \quad \mapsto f^{-1}(I)..$$

Let  $N$  be the nilradical of  $R$ . Show that if  $S = R/N$  and  $f : R \rightarrow R/N$  is the quotient map, then  $f^*$  is a bijection

### Question 138

Consider the polynomial  $f(x) = x^{10} + x^5 + 1 \in \mathbb{Q}[x]$  with splitting field  $K$  over  $\mathbb{Q}$ .

- Determine whether  $f(x)$  is irreducible over  $\mathbb{Q}$  and find  $[K : \mathbb{Q}]$ .
- Determine the structure of the Galois group  $\text{Gal}(K/\mathbb{Q})$ .

### Question 139

For each prime number  $p$  and each positive integer  $n$ , how many elements  $\alpha$  are there in  $\mathbb{F}_{p^n}$  such that  $F_p(\alpha) = F_{p^6}$ ?

### Question 140

Assume that  $K$  is a cyclic group,  $H$  is an arbitrary group, and  $\varphi_1$  and  $\varphi_2$  are homomorphisms from  $K$  into  $\text{Aut}(H)$  such that  $\varphi_1(K)$  and  $\varphi_2(K)$  are conjugate subgroups of  $\text{Aut}(H)$ .

Prove by constructing an explicit isomorphism that  $H \rtimes_{\varphi_1} K \cong H \rtimes_{\varphi_2} K$ .

Suppose  $\sigma_{\varphi_1}(K)\sigma^{-1} = \varphi_2(K)$  so that for some  $a \in \mathbb{Z}$  we have  $\sigma\varphi_1(k)\sigma^{-1} = \varphi_2(k)^a$  for all  $k \in K$ . Show that the map  $\psi : H \rtimes_{\varphi_1} K \rightarrow H \rtimes_{\varphi_2} K$  defined by  $\psi((h, k)) = (\sigma(h), k^a)$  is a homomorphism. Show  $\psi$  is bijective by constructing a 2-sided inverse.

## 2 Real Analysis (73 Questions)

### Question 1

Let  $C([0, 1])$  denote the space of all continuous real-valued functions on  $[0, 1]$ .

- Prove that  $C([0, 1])$  is complete under the uniform norm  $\|f\|_u := \sup_{x \in [0, 1]} |f(x)|$ .
- Prove that  $C([0, 1])$  is not complete under the  $L^1$ -norm  $\|f\|_1 = \int_0^1 |f(x)| dx$ .

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**Question 2**

Let  $\mathcal{B}$  denote the set of all Borel subsets of  $\mathbb{R}$  and  $\mu : \mathcal{B} \rightarrow [0, \infty)$  denote a finite Borel measure on  $\mathbb{R}$ .

- a. Prove that if  $\{F_k\}$  is a sequence of Borel sets for which  $F_k \supseteq F_{k+1}$  for all  $k$ , then

$$\lim_{k \rightarrow \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

- b. Suppose  $\mu$  has the property that  $\mu(E) = 0$  for every  $E \in \mathcal{B}$  with Lebesgue measure  $m(E) = 0$ . Prove that for every  $\varepsilon > 0$  there exists  $\delta > 0$  so that if  $E \in \mathcal{B}$  with  $m(E) < \delta$ , then  $\mu(E) < \varepsilon$ .

**Question 3**

Let  $\{f_k\}$  be any sequence of functions in  $L^2([0, 1])$  satisfying  $\|f_k\|_2 \leq M$  for all  $k \in \mathbb{N}$ .

Prove that if  $f_k \rightarrow f$  almost everywhere, then  $f \in L^2([0, 1])$  with  $\|f\|_2 \leq M$  and

$$\lim_{k \rightarrow \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

Hint: Try using Fatou's Lemma to show that  $\|f\|_2 \leq M$  and then try applying Egorov's Theorem.

**Question 4**

Let  $f$  be a non-negative function on  $\mathbb{R}^n$  and  $\mathcal{A} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$ .

Prove the validity of the following two statements:

- a.  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n \iff \mathcal{A}$  is a Lebesgue measurable subset of  $\mathbb{R}^{n+1}$   
b. If  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n$ , then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt$$

**Question 5**

- a. Show that  $L^2([0, 1]) \subseteq L^1([0, 1])$  and argue that  $L^2([0, 1])$  in fact forms a dense subset of  $L^1([0, 1])$ .  
b. Let  $\Lambda$  be a continuous linear functional on  $L^1([0, 1])$ .

Prove the Riesz Representation Theorem for  $L^1([0, 1])$  by following the steps below:

- i. Establish the existence of a function  $g \in L^2([0, 1])$  which represents  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x)g(x)dx \text{ for all } f \in L^2([0, 1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for  $L^2([0, 1])$ .

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- ii. Argue that the  $g$  obtained above must in fact belong to  $L^\infty([0, 1])$  and represent  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x) \overline{g(x)} dx \quad \text{for all } f \in L^1([0, 1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

### Question 6

Let  $\{a_n\}_{n=1}^\infty$  be a sequence of real numbers.

- a. Prove that if  $\lim_{n \rightarrow \infty} a_n = 0$ , then  $\lim_{n \rightarrow \infty} a_1 + \cdots + a_n = 0$ .

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

- b. Prove that if  $\sum_{n=1}^\infty \frac{a_n}{n}$  converges, then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

### Question 7

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \leq \frac{1}{n}$$

for all  $x \neq 0$  and positive integers  $n$ .

Hint: Consider  $\int_0^1 \cos(tx) dt$

### Question 8

Let  $(X, \mathcal{B}, \mu)$  be a measure space with  $\mu(X) = 1$  and  $\{B_n\}_{n=1}^\infty$  be a sequence of  $\mathcal{B}$ -measurable subsets of  $X$ , and

$$B := \left\{ x \in X \mid x \in B_n \text{ for infinitely many } n \right\}.$$

- a. Argue that  $B$  is also a  $\mathcal{B}$ -measurable subset of  $X$ .
- b. Prove that if  $\sum_{n=1}^\infty \mu(B_n) < \infty$  then  $\mu(B) = 0$ .
- c. Prove that if  $\sum_{n=1}^\infty \mu(B_n) = \infty$  **and** the sequence of set complements  $\{B_n^c\}_{n=1}^\infty$  satisfies

$$\mu \left( \bigcap_{n=k}^K B_n^c \right) = \prod_{n=k}^K (1 - \mu(B_n))$$

for all positive integers  $k$  and  $K$  with  $k < K$ , then  $\mu(B) = 1$ .

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Hint: Use the fact that  $1 - x \leq e^{-x}$  for all  $x$ .

**Question 9**

Let  $\{u_n\}_{n=1}^\infty$  be an orthonormal sequence in a Hilbert space  $\mathcal{H}$ .

- a. Prove that for every  $x \in \mathcal{H}$  one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

- b. Prove that for any sequence  $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$  there exists an element  $x \in \mathcal{H}$  such that

$$a_n = \langle x, u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

**Question 10**

- a. Show that if  $f$  is continuous with compact support on  $\mathbb{R}$ , then

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

- b. Let  $f \in L^1(\mathbb{R})$  and for each  $h > 0$  let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

- c. Prove that  $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$  for all  $h > 0$ .  
ii. Prove that  $\mathcal{A}_h f \rightarrow f$  in  $L^1(\mathbb{R})$  as  $h \rightarrow 0^+$ .

**Question 11**

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that  $m(E) = 0$ .

**Question 12**

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

- a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on  $[0, \infty)$ ?

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b. Compute

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx$$

**Question 13**

Let  $f$  be a non-negative measurable function on  $[0, 1]$ .

Show that

$$\lim_{p \rightarrow \infty} \left( \int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty.$$

**Question 14**

Let  $f \in L^2([0, 1])$  and suppose

$$\int_{[0,1]} f(x) x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that  $f = 0$  almost everywhere.

**Question 15**

Suppose that

- $f_n, f \in L^1$ ,
- $f_n \rightarrow f$  almost everywhere, and
- $\int |f_n| \rightarrow \int |f|$ .

Show that  $\int f_n \rightarrow \int f$

**Question 16**

Let  $f(x) = \frac{1}{x}$ . Show that  $f$  is uniformly continuous on  $(1, \infty)$  but not on  $(0, \infty)$ .

**Question 17**

Let  $E \subset \mathbb{R}$  be a Lebesgue measurable set. Show that there is a Borel set  $B \subset E$  such that  $m(E \setminus B) = 0$ .

**Question 18**

Suppose  $f(x)$  and  $xf(x)$  are integrable on  $\mathbb{R}$ . Define  $F$  by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = - \int_{-\infty}^{\infty} xf(x) \sin(xt) dx.$$

**Question 19**

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Let  $f \in L^1([0, 1])$ . Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) |\sin nx| \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

Hint: Begin with the case that  $f$  is the characteristic function of an interval.

**Question 20**

Let  $f \geq 0$  be a measurable function on  $\mathbb{R}$ . Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\}) \, dt$$

**Question 21**

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{dx}{(1 + \frac{x}{n})^n \sqrt[n]{x}}$$

**Question 22**

Let  $K$  be the set of numbers in  $[0, 1]$  whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with  $399\ldots$ . For example,  $0.8754 = 0.8753999\ldots$ .

Show that  $K$  is a compact, nowhere dense set without isolated points, and find the Lebesgue measure  $m(K)$ .

**Question 23**

a. Let  $\mu$  be a measure on a measurable space  $(X, \mathcal{M})$  and  $f$  a positive measurable function.

Define a measure  $\lambda$  by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for  $g$  any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

b. Let  $E \subset \mathbb{R}$  be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that  $m(E) = 0$ .

**Question 24**

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

- 
- a.  $\sum_{n=1}^{\infty} |f_n|$  is not in  $L^1([0, \infty), m)$

Hint:  $f_n(x)$  has a root  $x_n$ .

- b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

**Question 25**

Let  $f(x, y)$  on  $[-1, 1]^2$  be defined by

$$f(x, y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if  $f$  is integrable.

**Question 26**

Let  $f, g \in L^2(\mathbb{R})$ . Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

defines a uniformly continuous function  $h$  on  $\mathbb{R}$ .

**Question 27**

Show that the space  $C^1([a, b])$  is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)|.$$

**Question 28**

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which  $f$  does and does not converge uniformly.

**Question 29**

Let  $f(x) = x^2$  and  $E \subset [0, \infty) := \mathbb{R}^+$ .

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

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$$\begin{aligned}\phi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E)\end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of  $[0, \infty)$  to itself.

### Question 30

Let

$$S = \text{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \mid a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form  $(a, b)$ .

Show that for every  $f \in L^1(\mathbb{R})$ , there exists a sequence of functions  $\{f_n\} \subset S$  such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$$

### Question 31

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that  $f_n \rightarrow 0$  pointwise but not uniformly on  $[0, 1]$ .

Hint: Consider the maximum of  $f_n$ .

- 2.

$$\lim_{n \rightarrow \infty} \int_0^1 n(1-x)^n \sin x dx = 0$$

### Question 32

Let  $\phi$  be a compactly supported smooth function that vanishes outside of an interval  $[-N, N]$  such that  $\int_{\mathbb{R}} \phi(x) dx = 1$ .

For  $f \in L^1(\mathbb{R})$ , define

$$K_j(x) := j\phi(jx), \quad f * K_j(x) := \int_{\mathbb{R}} f(x-y)K_j(y) dy$$

and prove the following:

1. Each  $f * K_j$  is smooth and compactly supported.
- 2.

$$\lim_{j \rightarrow \infty} \|f * K_j - f\|_1 = 0$$



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Hint:

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

**Question 33**

Let  $X$  be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that  $(C^0(\mathbb{R}), \|\cdot\|)$  (the space of continuous functions  $f : X \rightarrow \mathbb{R}$ ) is complete.

**Question 34**

For  $n \in \mathbb{N}$ , define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that  $e_n < E_n$ , and prove Bernoulli's inequality:

$$(1+x)^n \geq 1+nx \quad \text{for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence  $e_n$  is increasing.
2. The sequence  $E_n$  is decreasing.
3.  $2 < e_n < E_n < 4$ .
4.  $\lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} E_n$ .

**Question 35**

Let  $0 < \lambda < 1$  and construct a Cantor set  $C_\lambda$  by successively removing middle intervals of length  $\lambda$ .

Prove that  $m(C_\lambda) = 0$ .

**Question 36**

Let  $f$  be Lebesgue measurable on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  be measurable such that

$$0 < A = \int_E f(x) dx < \infty.$$

Show that for every  $0 < t < 1$ , there exists a measurable set  $E_t \subset E$  such that

$$\int_{E_t} f(x) dx = tA.$$

**Question 37**

Let  $E \subset \mathbb{R}$  be measurable with  $m(E) < \infty$ . Define

$$f(x) = m(E \cap (E+x)).$$

Show that

1.  $f \in L^1(\mathbb{R})$ .
2.  $f$  is uniformly continuous.
3.  $\lim_{|x| \rightarrow \infty} f(x) = 0$

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y) \chi_E(y-x)$$

### Question 38

Let  $(X, \mathcal{M}, \mu)$  be a measure space. For  $f \in L^1(\mu)$  and  $\lambda > 0$ , define

$$\phi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that  $\phi, \psi$  are Borel measurable and

$$\int_X |f| \, d\mu = \int_0^\infty [\phi(\lambda) + \psi(\lambda)] \, d\lambda$$

### Question 39

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x) e^x dx \right| \mid f \in L^2([0, 1], m), \|f\|_2 \leq 1 \right\}$$

### Question 40

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that  $f$  converges to a differentiable function on  $(1, \infty)$  and that

$$f'(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n^x} \right)'.$$

Hint:

$$\left( \frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

### Question 41

Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be measurable with

$$\int_a^b f(x) \, dx = \int_a^b g(x) \, dx.$$

Show that either

- 
1.  $f(x) = g(x)$  almost everywhere, or
  2. There exists a measurable set  $E \subset [a, b]$  such that

$$\int_E f(x) \, dx > \int_E g(x) \, dx$$

**Question 42**

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\lim_{x \rightarrow 0} \int_{\mathbb{R}} |f(y-x) - f(y)| \, dy = 0$$

**Question 43**

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $\{E_n\} \subset \mathcal{M}$  satisfies

$$\lim_{n \rightarrow \infty} \mu(X \setminus E_n) = 0.$$

Define

$$G := \left\{ x \in X \mid x \in E_n \text{ for only finitely many } n \right\}.$$

Show that  $G \in \mathcal{M}$  and  $\mu(G) = 0$ .

**Question 44**

Let  $\phi \in L^\infty(\mathbb{R})$ . Show that the following limit exists and satisfies the equality

$$\lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} = \|\phi\|_\infty.$$

**Question 45**

Let  $f, g \in L^2(\mathbb{R})$ . Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x)g(x+n) \, dx = 0$$

**Question 46**

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $f : X \rightarrow Y$ , and  $x_0 \in X$ .

Prove that the following statements are equivalent:

1. For every  $\varepsilon > 0$   $\exists \delta > 0$  such that  $\rho(f(x), f(x_0)) < \varepsilon$  whenever  $d(x, x_0) < \delta$ .
2. The sequence  $\{f(x_n)\}_{n=1}^\infty \rightarrow f(x_0)$  for every sequence  $\{x_n\} \rightarrow x_0$  in  $X$ .

**Question 47**

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be continuous with period 1. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n\alpha) = \int_0^1 f(t) \, dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

---

Hint: show this first for the functions  $f(t) = e^{2\pi ikt}$  for  $k \in \mathbb{Z}$ .

### Question 48

Let  $\mu$  be a finite Borel measure on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  Borel. Prove that the following statements are equivalent:

1.  $\forall \varepsilon > 0$  there exists  $G$  open and  $F$  closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a  $V \in G_\delta$  and  $H \in F_\sigma$  such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

### Question 49

Define

$$f(x, y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that  $f \in L^1(\mathbb{R}^2)$ .

### Question 50

Let  $\mathcal{H}$  be a Hilbert space.

1. Let  $x \in \mathcal{H}$  and  $\{u_n\}_{n=1}^N$  be an orthonormal set. Prove that the best approximation to  $x$  in  $\mathcal{H}$  by an element in  $\text{span}_{\mathbb{C}}\{u_n\}$  is given by

$$\hat{x} := \sum_{n=1}^N \langle x, u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of  $\mathcal{H}$  are always closed.

### Question 51

Let  $f \in L^1(\mathbb{R})$  and  $g$  be a bounded measurable function on  $\mathbb{R}$ .

1. Show that the convolution  $f * g$  is well-defined, bounded, and uniformly continuous on  $\mathbb{R}$ .
2. Prove that one further assumes that  $g \in C^1(\mathbb{R})$  with bounded derivative, then  $f * g \in C^1(\mathbb{R})$  and

$$\frac{d}{dx}(f * g) = f * \left( \frac{d}{dx}g \right)$$

### Question 52

Define

$$f(x) = c_0 + c_1x^1 + c_2x^2 + \dots + c_nx^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number  $x_m$  such that  $f(x_m) \leq f(x)$  for all  $x \in \mathbb{R}$ .

### Question 53

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be Lebesgue measurable.

- 
1. Show that there is a sequence of simple functions  $s_n(x)$  such that  $s_n(x) \rightarrow f(x)$  for all  $x \in \mathbb{R}$ .
  2. Show that there is a Borel measurable function  $g$  such that  $g = f$  almost everywhere.

**Question 54**

Compute the following limit:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

**Question 55**

Let  $f : [1, \infty) \rightarrow \mathbb{R}$  such that  $f(1) = 1$  and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \rightarrow \infty} f(x) \leq 1 + \frac{\pi}{4}$$

**Question 56**

Let  $f, g \in L^1(\mathbb{R})$  be Borel measurable.

1. Show that
  - The function

$$F(x, y) := f(x - y)g(y)$$

is Borel measurable on  $\mathbb{R}^2$ , and

- For almost every  $y \in \mathbb{R}$ ,

$$F_y(x) := f(x - y)g(y)$$

is integrable with respect to  $y$ .

2. Show that  $f * g \in L^1(\mathbb{R})$  and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

**Question 57**

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. Show that

$$\sup \left\{ \|fg\|_1 \mid g \in L^1[0, 1], \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

**Question 58**

1. Give an example of a continuous  $f \in L^1(\mathbb{R})$  such that  $f(x) \not\rightarrow 0$  as  $|x| \rightarrow \infty$ .
2. Show that if  $f$  is *uniformly* continuous, then

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

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**Question 59**

Let  $\{a_n\}$  be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that  $\sum a_n^2 < \infty$ .

Note: Assume  $a_n, b_n$  are all non-negative.

**Question 60**

Let  $f : \mathbb{R} \longrightarrow \mathbb{R}$  and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \rightarrow x} f(y)$$

Prove that  $f$  is Borel measurable.

**Question 61**

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $f$  is a measurable function on  $X$ . Show that

$$\lim_{n \rightarrow \infty} \int_X f^n d\mu = \begin{cases} \infty \\ \mu(f^{-1}(1)), \end{cases} \quad \text{or}$$

and characterize the collection of functions of each type.

**Question 62**

Let  $f, g \in L^1([0, 1])$  and for all  $x \in [0, 1]$  define

$$F(x) := \int_0^x f(y) dy \quad \text{and} \quad G(x) := \int_0^x g(y) dy.$$

Prove that

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx$$

**Question 63**

Let  $\{f_n\}$  be a sequence of continuous functions such that  $\sum f_n$  converges uniformly.

Prove that  $\sum f_n$  is also continuous.

**Question 64**

Let  $I$  be an index set and  $\alpha : I \longrightarrow (0, \infty)$ .

1. Show that

$$\sum_{i \in I} \alpha(i) < \infty \implies I \text{ is countable.}$$

- 
2. Suppose  $I = \mathbb{Q}$  and  $\sum_{q \in \mathbb{Q}} a(q) < \infty$ . Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that  $f$  is continuous at  $x \iff x \notin \mathbb{Q}$ .

**Question 65**

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } m(E) < \delta \implies \int_E |f(x)| dx < \varepsilon$$

**Question 66**

Let  $g \in L^\infty([0, 1])$ . Prove that

$$\int_{[0,1]} f(x)g(x)dx = 0 \quad \text{for all continuous } f : [0, 1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

**Question 67**

1. Let  $f \in C_c^0(\mathbb{R}^n)$ , and show

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| dx = 0.$$

2. Extend the above result to  $f \in L^1(\mathbb{R}^n)$  and show that

$$f \in L^1(\mathbb{R}^n), g \in L^\infty(\mathbb{R}^n) \implies f * g \text{ is bounded and uniformly continuous.}$$

**Question 68**

Let  $1 \leq p, q \leq \infty$  be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q=1} \left| \int f(x)g(x)dx \right|$$

**Question 69**

Describe the process that extends a measure on an algebra  $\mathcal{A}$  of subsets of  $X$ , to a complete measure defined on a  $\sigma$ -algebra  $\mathcal{B}$  containing  $\mathcal{A}$ . State the corresponding definitions and results (without proofs).

**Question 70**

State and prove Fatou's Lemma on a general measurable space.

**Question 71**

1. State the Dominated Convergence Theorem for Lebesgue integrals.

- 
2. Let  $\{f_n\}$  be a sequence of measurable functions on a Lebesgue measurable set  $E$  which converges *in measure* to a function  $f$  on  $E$ . Suppose that for every  $n$ ,  $|f_n| \leq g$  with  $g$  integrable on  $E$ . Using the above theorem show that

$$\int_E |f_n - f| \longrightarrow 0.$$

### Question 72

Let  $f \in L^1([0, 1])$ . Show that

1. The limit  $\lim_{p \rightarrow 0^+} \|f\|_p$  exists.
2. If  $m\{x : f(x) = 0\} > 0$ , then the above limit is zero.

### Question 73

Let  $f$  be a continuous function on  $[0, 1]$ . Show that the following statements are equivalent.

1.  $f$  is absolutely continuous.
2. For any  $\epsilon > 0$  there exists  $\delta > 0$  such that  $m(f(E)) < \epsilon$  for any set  $E \subseteq [0, 1]$  with  $m(E) < \delta$ .
3.  $m(f(E)) = 0$  for any set  $E \subseteq [0, 1]$  with  $m(E) = 0$ .

## 3 Complex Analysis (119 Questions)

### Question 1

- (1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$ . Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1).

### Question 2

Let  $f$  be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\} \quad \text{where} \quad 1 \leq \theta \leq 2\pi.$$

If there exists  $k$  such that  $\lim_{z \rightarrow \infty} z f(z) = k$  for  $z$  in the region  $D$ . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where  $L$  is the part of the circle  $|z| = R'$  which lies in the region  $D$ .

### Question 3

Suppose that  $f$  is an analytic function in the region  $D$  which contains the point  $a$ . Let

$$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter.}$$



- 
- (1) Let  $K \subset D$  be a circle with the center at point  $a$  and also we assume that  $f(z) \neq 0$  for  $z \in K$ . Prove that the function  $F$  has one and only one zero  $z = w$  on the closed disc  $\bar{K}$  whose boundary is the circle  $K$  if  $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$ .
- (2) Let  $G(z)$  be an analytic function on the disk  $\bar{K}$ . Apply the residue theorem to prove that  $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$ , where  $w$  is the zero from (1).
- (3) If  $z \in K$ , prove that the function  $\frac{1}{F(z)}$  can be represented as a convergent series with respect to  $q$ :  $\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}$ .

#### Question 4

Evaluate

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx.$$

#### Question 5

Let  $f = u + iv$  be differentiable (i.e.  $f'(z)$  exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

#### Question 6

Show that  $\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis,  $0 < a < n$ . Here  $n$  is a positive integer.

#### Question 7

For  $s > 0$ , the **gamma function** is defined by  $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$ .

1. Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.
2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need  $\Gamma(1-s) = t \int_0^{\infty} e^{-vt} (vt)^{-s} dv$  for  $t > 0$ .

#### Question 8

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

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is a polynomial of degree  $n$ , then it has  $n$  zeros in  $\mathbb{C}$ .

### Question 9

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that

- (i)  $f$  is a polynomial and
- (ii) the degree of  $f$  is at least  $N$ .

### Question 10

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an injective analytic (also called *univalent*) function. Show that there exist complex numbers  $a \neq 0$  and  $b$  such that  $f(z) = az + b$ .

### Question 11

Let  $g$  be analytic for  $|z| \leq 1$  and  $|g(z)| < 1$  for  $|z| = 1$ .

1. Show that  $g$  has a unique fixed point in  $|z| < 1$ .
2. What happens if we replace  $|g(z)| < 1$  with  $|g(z)| \leq 1$  for  $|z| = 1$ ? Give an example if (a) is not true or give a proof if (a) is still true.
3. What happens if we simply assume that  $f$  is analytic for  $|z| < 1$  and  $|f(z)| < 1$  for  $|z| < 1$ ? Suppose that  $f(z) \neq z$ . Can  $f$  have more than one fixed point in  $|z| < 1$ ?

Hint: The map  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  may be useful.

### Question 12

Find a conformal map from  $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$  to the unit disk  $\Delta = \{z : |z| < 1\}$ .

### Question 13

Let  $f(z)$  be entire and assume values of  $f(z)$  lie outside a *bounded* open set  $\Omega$ . Show without using Picard's theorems that  $f(z)$  is a constant.

### Question 14

- (1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$ . Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1).

### Question 15

Let  $f(z)$  be entire and assume that  $f(z) \leq M|z|^2$  outside some disk for some constant  $M$ . Show that  $f(z)$  is a polynomial in  $z$  of degree  $\leq 2$ .

### Question 16

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Let  $a_n(z)$  be an analytic sequence in a domain  $D$  such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ . Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ .

**Question 17**

Let  $f(z)$  be analytic in an open set  $\Omega$  except possibly at a point  $z_0$  inside  $\Omega$ . Show that if  $f(z)$  is bounded in near  $z_0$ , then  $\int_{\Delta} f(z)dz = 0$  for all triangles  $\Delta$  in  $\Omega$ .

**Question 18**

Assume  $f$  is continuous in the region:  $0 < |z - a| \leq R$ ,  $0 \leq \arg(z - a) \leq \beta_0$  ( $0 < \beta_0 \leq 2\pi$ ) and the limit  $\lim_{z \rightarrow a} (z - a)f(z) = A$  exists. Show that

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z)dz = iA\beta_0 ,$$

where

$$\gamma_r := \{z \mid z = a + re^{it}, 0 \leq t \leq \beta_0\}.$$

**Question 19**

Show that  $f(z) = z^2$  is uniformly continuous in any open disk  $|z| < R$ , where  $R > 0$  is fixed, but it is not uniformly continuous on  $\mathbb{C}$ .

**Question 20**

(1) Show that the function  $u = u(x, y)$  given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on  $D = \{(x, y) \mid x^2 + y^2 < 1\}$  of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points  $(x, y) \in D$  such that  $\limsup_{n \rightarrow \infty} |u(x, y)| = \infty$ .

**Question 21**

(1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$ . Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(2) Deduce Liouville's theorem from (1).

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**Question 22**

Let  $f$  be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists  $k$  such that  $\lim_{z \rightarrow \infty} zf(z) = k$  for  $z$  in the region  $D$ . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where  $L$  is the part of the circle  $|z| = R'$  which lies in the region  $D$ .

**Question 23**

Suppose that  $f$  is an analytic function in the region  $D$  which contains the point  $a$ . Let

$$F(z) = z - a - qf(z), \quad \text{where} \quad q \text{ is a complex parameter.}$$

- (1) Let  $K \subset D$  be a circle with the center at point  $a$  and also we assume that  $f(z) \neq 0$  for  $z \in K$ . Prove that the function  $F$  has one and only one zero  $z = w$  on the closed disc  $\bar{K}$  whose boundary is the circle  $K$  if  $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$ .
- (2) Let  $G(z)$  be an analytic function on the disk  $\bar{K}$ . Apply the residue theorem to prove that  $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$ , where  $w$  is the zero from (1).
- (3) If  $z \in K$ , prove that the function  $\frac{1}{F(z)}$  can be represented as a convergent series with respect to  $q$ : 
$$\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z - a)^{n+1}}.$$

**Question 24**

Evaluate  $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$ .

**Question 25**

Let  $f = u + iv$  be differentiable (i.e.  $f'(z)$  exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

**Question 26**

Show that  $\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis,  $0 < a < n$ . Here  $n$  is a positive integer.

**Question 27**

For  $s > 0$ , the **gamma function** is defined by  $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$ .

1. Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.
2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need  $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$  for  $t > 0$ .

### Question 28

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that

- (i)  $f$  is a polynomial and
- (ii) the degree of  $f$  is at least  $N$ .

### Question 29

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an injective analytic (also called univalent) function. Show that there exist complex numbers  $a \neq 0$  and  $b$  such that  $f(z) = az + b$ .

### Question 30

Let  $g$  be analytic for  $|z| \leq 1$  and  $|g(z)| < 1$  for  $|z| = 1$ .

- Show that  $g$  has a unique fixed point in  $|z| < 1$ .
- What happens if we replace  $|g(z)| < 1$  with  $|g(z)| \leq 1$  for  $|z| = 1$ ? Give an example if (a) is not true or give a proof if (a) is still true.
- What happens if we simply assume that  $f$  is analytic for  $|z| < 1$  and  $|f(z)| < 1$  for  $|z| < 1$ ? Suppose that  $f(z) \neq z$ . Can  $f$  have more than one fixed point in  $|z| < 1$ ?

Hint: The map  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  may be useful.

### Question 31

Find a conformal map from  $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$  to the unit disk  $\Delta = \{z : |z| < 1\}$ .

### Question 32

Let  $f(z)$  be entire and assume values of  $f(z)$  lie outside a *bounded* open set  $\Omega$ . Show without using Picard's theorems that  $f(z)$  is a constant.

### Question 33

- (1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$ . Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

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(2) Deduce Liouville's theorem from (1).

**Question 34**

Let  $f(z)$  be entire and assume that  $f(z) \leq M|z|^2$  outside some disk for some constant  $M$ . Show that  $f(z)$  is a polynomial in  $z$  of degree  $\leq 2$ .

**Question 35**

Let  $a_n(z)$  be an analytic sequence in a domain  $D$  such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ . Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ .

**Question 36**

Let  $f(z)$  be analytic in an open set  $\Omega$  except possibly at a point  $z_0$  inside  $\Omega$ . Show that if  $f(z)$  is bounded in near  $z_0$ , then  $\int_{\Delta} f(z)dz = 0$  for all triangles  $\Delta$  in  $\Omega$ .

**Question 37**

Assume  $f$  is continuous in the region:  $0 < |z - a| \leq R$ ,  $0 \leq \arg(z - a) \leq \beta_0$  ( $0 < \beta_0 \leq 2\pi$ ) and the limit  $\lim_{z \rightarrow a} (z - a)f(z) = A$  exists. Show that

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z)dz = iA\beta_0,$$

where

$$\gamma_r := \{z \mid z = a + re^{it}, 0 \leq t \leq \beta_0\}.$$

**Question 38**

Show that  $f(z) = z^2$  is uniformly continuous in any open disk  $|z| < R$ , where  $R > 0$  is fixed, but it is not uniformly continuous on  $\mathbb{C}$ .

(1) Show that the function  $u = u(x, y)$  given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on  $D = \{(x, y) \mid x^2 + y^2 < 1\}$  of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

**Question 39**

This question provides some insight into Cauchy's theorem. Solve the problem without using Cauchy's theorem.

1. Evaluate the integral  $\int_{\gamma} z^n dz$  for all integers  $n$ . Here  $\gamma$  is any circle centered at the origin with the positive (counterclockwise) orientation.

- 
2. Same question as (a), but with  $\gamma$  any circle not containing the origin.
3. Show that if  $|a| < r < |b|$ , then  $\int_{\gamma} \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}$ . Here  $\gamma$  denotes the circle centered at the origin, of radius  $r$ , with the positive orientation.

**Question 40**

- (1) Assume the infinite series  $\sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$  and let  $f(z)$  be the limit. Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1).

Liouville's theorem: If  $f(z)$  is entire and bounded, then  $f$  is constant.

**Question 41**

Let  $f$  be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\} \quad \text{where } 0 \leq \theta \leq 2\pi.$$

If there exists  $k$  such that  $\lim_{z \rightarrow \infty} zf(z) = k$  for  $z$  in the region  $D$ . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where  $L$  is the part of the circle  $|z| = R'$  which lies in the region  $D$ .

**Question 42**

Evaluate  $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$ .

**Question 43**

Let  $f = u + iv$  be differentiable (i.e.  $f'(z)$  exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

**Question 44**

Show that  $\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis,  $0 < a < n$ . Here  $n$  is a positive integer.

**Question 45**

For  $s > 0$ , the **gamma function** is defined by  $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$ .

- Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.

- Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need  $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$  for  $t > 0$ .

#### Question 46

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that

- (i)  $f$  is a polynomial and
- (ii) the degree of  $f$  is at least  $N$ .

#### Question 47

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an injective analytic (also called univalent) function. Show that there exist complex numbers  $a \neq 0$  and  $b$  such that  $f(z) = az + b$ .

#### Question 48

Let  $g$  be analytic for  $|z| \leq 1$  and  $|g(z)| < 1$  for  $|z| = 1$ .

- Show that  $g$  has a unique fixed point in  $|z| < 1$ .
- What happens if we replace  $|g(z)| < 1$  with  $|g(z)| \leq 1$  for  $|z| = 1$ ? Give an example if (a) is not true or give a proof if (a) is still true.
- What happens if we simply assume that  $f$  is analytic for  $|z| < 1$  and  $|f(z)| < 1$  for  $|z| < 1$ ? Suppose that  $f(z) \neq z$ . Can  $f$  have more than one fixed point in  $|z| < 1$ ?

Hint: The map  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  may be useful.

#### Question 49

Find a conformal map from  $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$  to the unit disk  $\Delta = \{z : |z| < 1\}$ .

#### Question 50

Let  $a_n \neq 0$  and assume that  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$ . Show that  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ . In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

#### Question 51

- (a) Let  $z, w$  be complex numbers, such that  $\bar{z}w \neq 1$ . Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$



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(b) Prove that for fixed  $w$  in the unit disk  $\mathbb{D}$ , the mapping

$$F : z \mapsto \frac{w - z}{1 - \bar{w}z}$$

satisfies the following conditions:

- (c)  $F$  maps  $\mathbb{D}$  to itself and is holomorphic.
- (ii)  $F$  interchanges 0 and  $w$ , namely,  $F(0) = w$  and  $F(w) = 0$ .
- (iii)  $|F(z)| = 1$  if  $|z| = 1$ .
- (iv)  $F : \mathbb{D} \mapsto \mathbb{D}$  is bijective.

Hint: Calculate  $F \circ F$ .

### Question 52

Use  $n$ -th roots of unity (i.e. solutions of  $z^n - 1 = 0$ ) to show that

$$2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \cdots \sin \frac{(n-1)\pi}{n} = n .$$

Hint:  $1 - \cos 2\theta = 2 \sin^2 \theta$ ,  $\sin 2\theta = 2 \sin \theta \cos \theta$ .

### Question 53

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is a holomorphic function in the region  $r > 0$ ,  $-\pi < \theta < \pi$ . Also show that  $\log z$  defined above is not continuous in  $r > 0$ .

### Question 54

Assume  $f$  is continuous in the region:  $x \geq x_0$ ,  $0 \leq y \leq b$  and the limit

$$\lim_{x \rightarrow +\infty} f(x + iy) = A$$

exists uniformly with respect to  $y$  (independent of  $y$ ).

Show that

$$\lim_{x \rightarrow +\infty} \int_{\gamma_x} f(z) dz = iAb ,$$

where  $\gamma_x := \{z \mid z = x + it, 0 \leq t \leq b\}$ .

### Question 55

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(Cauchy's formula for "exterior" region) Let  $\gamma$  be piecewise smooth simple closed curve with interior  $\Omega_1$  and exterior  $\Omega_2$ . Assume  $f'(z)$  exists in an open set containing  $\gamma$  and  $\Omega_2$  and  $\lim_{z \rightarrow \infty} f(z) = A$ . Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

### Question 56

Let  $f(z)$  be bounded and analytic in  $\mathbb{C}$ . Let  $a \neq b$  be any fixed complex numbers. Show that the following limit exists

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that  $f(z)$  must be a constant (Liouville's theorem).

### Question 57

Prove by *justifying all steps* that for all  $\xi \in \mathbb{C}$  we have  $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx$ .

Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of  $\xi$ .

### Question 58

Suppose that  $f$  is holomorphic in an open set containing the closed unit disc, except for a pole at  $z_0$  on the unit circle. Let  $f(z) = \sum_{n=1}^{\infty} c_n z^n$  denote the power series in the open disc. Show that

(1)  $c_n \neq 0$  for all large enough  $n$ 's, and

(2)  $\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = z_0$ .

### Question 59

Let  $f(z)$  be a non-constant analytic function in  $|z| > 0$  such that  $f(z_n) = 0$  for infinite many points  $z_n$  with  $\lim_{n \rightarrow \infty} z_n = 0$ . Show that  $z = 0$  is an essential singularity for  $f(z)$ . (An example of such a function is  $f(z) = \sin(1/z)$ .)

### Question 60

Let  $f$  be entire and suppose that  $\lim_{z \rightarrow \infty} f(z) = \infty$ . Show that  $f$  is a polynomial.

### Question 61

Expand the following functions into Laurent series in the indicated regions:

(a)  $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$ ,  $2 < |z| < 3$ ,  $3 < |z| < +\infty$ .

(b)  $f(z) = \sin \frac{z}{1-z}$ ,  $0 < |z-1| < +\infty$

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**Question 62**

Assume  $f(z)$  is analytic in region  $D$  and  $\Gamma$  is a rectifiable curve in  $D$  with interior in  $D$ . Prove that if  $f(z)$  is real for all  $z \in \Gamma$ , then  $f(z)$  is a constant.

**Question 63**

Find the number of roots of  $z^4 - 6z + 3 = 0$  in  $|z| < 1$  and  $1 < |z| < 2$  respectively.

**Question 64**

Prove that  $z^4 + 2z^3 - 2z + 10 = 0$  has exactly one root in each open quadrant.

**Question 65**

(1) Let  $f(z) \in H(\mathbb{D})$ ,  $\operatorname{Re}(f(z)) > 0$ ,  $f(0) = a > 0$ . Show that

$$\left| \frac{f(z) - a}{f(z) + a} \right| \leq |z|, \quad |f'(0)| \leq 2a.$$

(2) Show that the above is still true if  $\operatorname{Re}(f(z)) > 0$  is replaced with  $\operatorname{Re}(f(z)) \geq 0$ .

**Question 66**

Assume  $f(z)$  is analytic in  $\mathbb{D}$  and  $f(0) = 0$  and is not a rotation (i.e.  $f(z) \neq e^{i\theta}z$ ). Show that  $\sum_{n=1}^{\infty} f^n(z)$  converges uniformly to an analytic function on compact subsets of  $\mathbb{D}$ , where  $f^{n+1}(z) = f(f^n(z))$ .

**Question 67**

Let  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  be analytic and one-to-one in  $|z| < 1$ . For  $0 < r < 1$ , let  $D_r$  be the disk  $|z| < r$ . Show that the area of  $f(D_r)$  is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n |c_n|^2 r^{2n}.$$

(Note that in general the area of  $f(D_1)$  is infinite.)

**Question 68**

Let  $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$  be analytic and one-to-one in  $r_0 < |z| < R_0$ . For  $r_0 < r < R < R_0$ , let  $D(r, R)$  be the annulus  $r < |z| < R$ . Show that the area of  $f(D(r, R))$  is finite and is given by

$$S = \pi \sum_{n=-\infty}^{\infty} n |c_n|^2 (R^{2n} - r^{2n}).$$

**Question 69**

Let  $a_n(z)$  be an analytic sequence in a domain  $D$  such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ . Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ .

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**Question 70**

Let  $f_n, f$  be analytic functions on the unit disk  $\mathbb{D}$ . Show that the following are equivalent.

- (i)  $f_n(z)$  converges to  $f(z)$  uniformly on compact subsets in  $\mathbb{D}$ .
- (ii)  $\int_{|z|=r} |f_n(z) - f(z)| |dz|$  converges to 0 if  $0 < r < 1$ .

**Question 71**

Let  $f$  and  $g$  be non-zero analytic functions on a region  $\Omega$ . Assume  $|f(z)| = |g(z)|$  for all  $z$  in  $\Omega$ . Show that  $f(z) = e^{i\theta} g(z)$  in  $\Omega$  for some  $0 \leq \theta < 2\pi$ .

**Question 72**

Suppose  $f$  is analytic in an open set containing the unit disc  $\mathbb{D}$  and  $|f(z)| = 1$  when  $|z|=1$ . Show that either  $f(z) = e^{i\theta}$  for some  $\theta \in \mathbb{R}$  or there are finite number of  $z_k \in \mathbb{D}$ ,  $k \leq n$  and  $\theta \in \mathbb{R}$  such that  $f(z) = e^{i\theta} \prod_{k=1}^n \frac{z - z_k}{1 - \bar{z}_k z}$ .

Also cf. Stein et al, 1.4.7, 3.8.17

**Question 73**

- (1) Let  $p(z)$  be a polynomial,  $R > 0$  any positive number, and  $m \geq 1$  an integer. Let

$$M_R = \sup\{|z^m p(z) - 1| : |z| = R\}.$$

Show that  $M_R > 1$ .

- (2) Let  $m \geq 1$  be an integer and  $K = \{z \in \mathbb{C} : r \leq |z| \leq R\}$  where  $r < R$ . Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number  $\varepsilon_0 > 0$  such that for each polynomial  $p(z)$ ,

$$\sup\{|p(z) - z^{-m}| : z \in K\} \geq \varepsilon_0.$$

**Question 74**

Let  $f(z) = \frac{1}{z} + \frac{1}{z^2 - 1}$ . Find all the Laurent series of  $f$  and describe the largest annuli in which these series are valid.

**Question 75**

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that  $|f(z)| \leq A|z|^N$  for  $|z| \geq R$ . Show that

- (i)  $f$  is a polynomial and
- (ii) the degree of  $f$  is at most  $N$ .

**Question 76**

- (1) Explicitly write down an example of a non-zero analytic function in  $|z| < 1$  which has infinitely zeros in  $|z| < 1$ .
- (2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

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**Question 77**

- (1) Assume  $u$  is harmonic on open set  $O$  and  $z_n$  is a sequence in  $O$  such that  $u(z_n) = 0$  and  $\lim z_n \in O$ . Prove or disprove that  $u$  is identically zero. What if  $O$  is a region?
- (2) Assume  $u$  is harmonic on open set  $O$  and  $u(z) = 0$  on a disc in  $O$ . Prove or disprove that  $u$  is identically zero. What if  $O$  is a region?
- (3) Formulate and prove a Schwarz reflection principle for harmonic functions

cf. Theorem 5.6 on p.60 of Stein et al.

Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

**Question 78**

Let  $f$  be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any  $s < r$ , there exists a constant  $c > 0$  such that

$$\|f\|_{(\infty,s)} \leq c \|f\|_{(1,r)},$$

where  $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$  and  $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$ .

Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

**Question 79**

- (1) Let  $f$  be analytic in  $\Omega : 0 < |z - a| < r$  except at a sequence of poles  $a_n \in \Omega$  with  $\lim_{n \rightarrow \infty} a_n = a$ . Show that for any  $w \in \mathbb{C}$ , there exists a sequence  $z_n \in \Omega$  such that  $\lim_{n \rightarrow \infty} f(z_n) = w$ .
- (2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

**Question 80**

Compute the following integrals.

- (i)  $\int_0^\infty \frac{1}{(1+x^n)^2} dx$ ,  $n \geq 1$  (ii)  $\int_0^\infty \frac{\cos x}{(x^2+a^2)^2} dx$ ,  $a \in \mathbb{R}$  (iii)  $\int_0^\pi \frac{1}{a + \sin \theta} d\theta$ ,  $a > 1$
- (iv)  $\int_0^{\frac{\pi}{2}} \frac{d\theta}{a + \sin^2 \theta}$ ,  $a > 0$ . (v)  $\int_{|z|=2} \frac{1}{(z^5-1)(z-3)} dz$  (vi)  $\int_{-\infty}^\infty \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix\xi} dx$ ,  $0 < a < 1$ ,  $\xi \in \mathbb{R}$  (vii)  $\int_{|z|=1} \cot^2 z dz$ .

**Question 81**

Compute the following integrals.

- (i)  $\int_0^\infty \frac{\sin x}{x} dx$  (ii)  $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$  (iii)  $\int_0^\infty \frac{x^{a-1}}{(1+x)^2} dx$ ,  $0 < a < 2$
- (i)  $\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx$ ,  $a, b > 0$  (ii)  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$ ,  $0 < a < n$

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(iii)  $\int_0^\infty \frac{\log x}{1+x^n} dx, n \geq 2$  (iv)  $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$  (v)  $\int_0^\pi \log |1 - a \sin \theta| d\theta, a \in \mathbb{C}$

**Question 82**

Let  $0 < r < 1$ . Show that polynomials  $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$  have no zeros in  $|z| < r$  for all sufficiently large  $n$ 's.

**Question 83**

Let  $f$  be an analytic function on a region  $\Omega$ . Show that  $f$  is a constant if there is a simple closed curve  $\gamma$  in  $\Omega$  such that its image  $f(\gamma)$  is contained in the real axis.

**Question 84**

(1) Show that  $\frac{\pi^2}{\sin^2 \pi z}$  and  $g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$  have the same principal part at each integer point.

(2) Show that  $h(z) = \frac{\pi^2}{\sin^2 \pi z} - g(z)$  is bounded on  $\mathbb{C}$  and conclude that  $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ .

**Question 85**

Let  $f(z)$  be an analytic function on  $\mathbb{C} \setminus \{z_0\}$ , where  $z_0$  is a fixed point. Assume that  $f(z)$  is bijective from  $\mathbb{C} \setminus \{z_0\}$  onto its image, and that  $f(z)$  is bounded outside  $D_r(z_0)$ , where  $r$  is some fixed positive number. Show that there exist  $a, b, c, d \in \mathbb{C}$  with  $ad - bc \neq 0, c \neq 0$  such that  $f(z) = \frac{az + b}{cz + d}$ .

**Question 86**

Assume  $f(z)$  is analytic in  $\mathbb{D} : |z| < 1$  and  $f(0) = 0$  and is not a rotation (i.e.  $f(z) \neq e^{i\theta}z$ ). Show that  $\sum_{n=1}^{\infty} f^n(z)$  converges uniformly to an analytic function on compact subsets of  $\mathbb{D}$ , where  $f^{n+1}(z) = f(f^n(z))$ .

**Question 87**

Let  $f$  be a non-constant analytic function on  $\mathbb{D}$  with  $f(\mathbb{D}) \subseteq \mathbb{D}$ . Use  $\psi_a(f(z))$  (where  $a = f(0)$ ,  $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$ ) to prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

**Question 88**

Find a conformal map

1. from  $\{z : |z - 1/2| > 1/2, \operatorname{Re}(z) > 0\}$  to  $\mathbb{H}$
2. from  $\{z : |z - 1/2| > 1/2, |z| < 1\}$  to  $\mathbb{D}$
3. from the intersection of the disk  $|z + i| < \sqrt{2}$  with  $\mathbb{H}$  to  $\mathbb{D}$ .
4. from  $\mathbb{D} \setminus [a, 1)$  to  $\mathbb{D} \setminus [0, 1)$  ( $0 < a < 1$ ). Short solution possible using Blaschke factor
5. from  $\{z : |z| < 1, \operatorname{Re}(z) > 0\} \setminus (0, 1/2]$  to  $\mathbb{H}$ .

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**Question 89**

Let  $C$  and  $C'$  be two circles and let  $z_1 \in C$ ,  $z_2 \notin C$ ,  $z'_1 \in C'$ ,  $z'_2 \notin C'$ . Show that there is a unique fractional linear transformation  $f$  with  $f(C) = C'$  and  $f(z_1) = z'_1$ ,  $f(z_2) = z'_2$ .

**Question 90**

Assume  $f_n \in H(\Omega)$  is a sequence of holomorphic functions on the region  $\Omega$  that are uniformly bounded on compact subsets and  $f \in H(\Omega)$  is such that the set  $\{z \in \Omega : \lim_{n \rightarrow \infty} f_n(z) = f(z)\}$  has a limit point in  $\Omega$ . Show that  $f_n$  converges to  $f$  uniformly on compact subsets of  $\Omega$ .

**Question 91**

Let  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  with  $|\alpha| < 1$  and  $\mathbb{D} = \{z : |z| < 1\}$ . Prove that

- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^2 dx dy = 1.$
- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$

**Question 92**

Prove that  $f(z) = -\frac{1}{2} \left( z + \frac{1}{z} \right)$  is a conformal map from half disc  $\{z = x + iy : |z| < 1, y > 0\}$  to upper half plane  $\mathbb{H} = \{z = x + iy : y > 0\}$ .

**Question 93**

Let  $\Omega$  be a simply connected open set and let  $\gamma$  be a simple closed contour in  $\Omega$  and enclosing a bounded region  $U$  anticlockwise. Let  $f : \Omega \rightarrow \mathbb{C}$  be a holomorphic function and  $|f(z)| \leq M$  for all  $z \in \gamma$ . Prove that  $|f(z)| \leq M$  for all  $z \in U$ .

**Question 94**

Compute the following integrals.

- (i)  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx, 0 < a < n$
- (ii)  $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$

**Question 95**

Let  $0 < r < 1$ . Show that polynomials  $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$  have no zeros in  $|z| < r$  for all sufficiently large  $n$ 's.

**Question 96**

Let  $f$  be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any  $s < r$ , there exists a constant  $c > 0$  such that

$$\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)},$$

where  $\|f\|_{(\infty, s)} = \sup_{z \in D_s(z_0)} |f(z)|$  and  $\|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)| dx dy.$

**Question 97**

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Let  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  with  $|\alpha| < 1$  and  $\mathbb{D} = \{z : |z| < 1\}$ . Prove that

- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^2 dx dy = 1.$
- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$

**Question 98**

Prove that  $f(z) = -\frac{1}{2} \left( z + \frac{1}{z} \right)$  is a conformal map from half disc  $\{z = x + iy : |z| < 1, y > 0\}$  to upper half plane  $\mathbb{H} = \{z = x + iy : y > 0\}$ .

**Question 99**

Let  $\Omega$  be a simply connected open set and let  $\gamma$  be a simple closed contour in  $\Omega$  and enclosing a bounded region  $U$  anticlockwise. Let  $f : \Omega \rightarrow \mathbb{C}$  be a holomorphic function and  $|f(z)| \leq M$  for all  $z \in \gamma$ . Prove that  $|f(z)| \leq M$  for all  $z \in U$ .

**Question 100**

Compute the following integrals.

- (i)  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx, 0 < a < n$
- (ii)  $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$

**Question 101**

Let  $f$  be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any  $s < r$ , there exists a constant  $c > 0$  such that

$$\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)},$$

where  $\|f\|_{(\infty, s)} = \sup_{z \in D_s(z_0)} |f(z)|$  and  $\|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)| dx dy$ .

**Question 102**

Let  $u(x, y)$  be harmonic and have continuous partial derivatives of order three in an open disc of radius  $R > 0$ .

- (a) Let two points  $(a, b), (x, y)$  in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x, y) = \int_{a, b}^{x, y} \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right).$$

- (b)

- (i) Prove that  $u(x, y) + iv(x, y)$  is an analytic function in this disc.
- (ii) Prove that  $v(x, y)$  is harmonic in this disc.



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**Question 103**

- (a)  $f(z) = u(x, y) + iv(x, y)$  be analytic in a domain  $D \subset \mathbb{C}$ . Let  $z_0 = (x_0, y_0)$  be a point in  $D$  which is in the intersection of the curves  $u(x, y) = c_1$  and  $v(x, y) = c_2$ , where  $c_1$  and  $c_2$  are constants. Suppose that  $f'(z_0) \neq 0$ . Prove that the lines tangent to these curves at  $z_0$  are perpendicular.
- (b) Let  $f(z) = z^2$  be defined in  $\mathbb{C}$ .
- (c) Describe the level curves of  $\operatorname{Re}(f)$  and of  $\operatorname{Im}(f)$ .
- (ii) What are the angles of intersections between the level curves  $\operatorname{Re}(f) = 0$  and  $\operatorname{Im}(f)$ ? Is your answer in agreement with part a) of this question?

**Question 104**

- (a) Let  $f : D \rightarrow \mathbb{C}$  be a continuous function, where  $D \subset \mathbb{C}$  is a domain. Let  $\alpha : [a, b] \rightarrow D$  be a smooth curve. Give a precise definition of the *complex line integral*

$$\int_{\alpha} f.$$

- (b) Assume that there exists a constant  $M$  such that  $|f(\tau)| \leq M$  for all  $\tau \in \operatorname{Image}(\alpha)$ . Prove that

$$\left| \int_{\alpha} f \right| \leq M \times \operatorname{length}(\alpha).$$

- (c) Let  $C_R$  be the circle  $|z| = R$ , described in the counterclockwise direction, where  $R > 1$ . Provide an upper bound for  $\left| \int_{C_R} \frac{\log(z)}{z^2} \right|$  which depends *only* on  $R$  and other constants.

**Question 105**

- (a) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire function. Assume the existence of a non-negative integer  $m$ , and of positive constants  $L$  and  $R$ , such that for all  $z$  with  $|z| > R$  the inequality

$$|f(z)| \leq L|z|^m$$

holds. Prove that  $f$  is a polynomial of degree  $\leq m$ .

- (b) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire function. Suppose that there exists a real number  $M$  such that for all  $z \in \mathbb{C}$

$$\operatorname{Re}(f) \leq M.$$

Prove that  $f$  must be a constant.

**Question 106**

Prove that all the roots of the complex polynomial

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles  $|z| = 1$  and  $|z| = 2$ .

**Question 107**

- 
- (a) Let  $F$  be an analytic function inside and on a simple closed curve  $C$ , except for a pole of order  $m \geq 1$  at  $z = a$  inside  $C$ . Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \rightarrow a} \frac{d^{m-1}}{d\tau^{m-1}} ((\tau - a)^m F(\tau)).$$

**Question 108**

Find the conformal map that takes the upper half-plane conformally onto the half-strip  $\{w = x + iy : -\pi/2 < x < \pi/2, y > 0\}$ .

**Question 109**

Compute the integral  $\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx$  where  $\cosh z = \frac{e^z + e^{-z}}{2}$ .

**Question 110**

Use residues to compute the integral

$$\int_0^{\infty} \frac{\cos x}{(x^2 + 1)^2} dx$$

**Question 111**

State and prove the Cauchy integral formula for holomorphic functions.

**Question 112**

Let  $f$  be an entire function and suppose that  $|f(z)| \leq A|z|^2$  for all  $z$  and some constant  $A$ . Show that  $f$  is a polynomial of degree  $\leq 2$ .

**Question 113**

1. State the Schwarz lemma for analytic functions in the unit disc.
2. Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be an analytic map from the unit disc  $\mathbb{D}$  into itself. Use the Schwarz lemma to show that for each  $a \in \mathbb{D}$  we have

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}$$

**Question 114**

State the Riemann mapping theorem and prove the uniqueness part.

**Question 115**

Compute the integrals

$$\int_{|z-2|=1} \frac{e^z}{z(z-1)^2} dz, \quad \int_0^{\infty} \frac{\cos 2x}{x^2 + 2} dx$$

**Question 116**

Let  $(f_n)$  be a sequence of holomorphic functions in a domain  $D$ . Suppose that  $f_n \rightarrow f$  uniformly on each compact subset of  $D$ . Show that

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- $f$  is holomorphic on  $D$ .
  - $f'_n \rightarrow f'$  uniformly on each compact subset of  $D$ .

**Question 117**

If  $f$  is a non-constant entire function, then  $f(\mathbb{C})$  is dense in the plane.

**Question 118**

1. State Rouché's theorem.
2. Let  $f$  be analytic in a neighborhood of 0, and satisfying  $f'(0) \neq 0$ . Use Rouché's theorem to show that there exists a neighborhood  $U$  of 0 such that  $f$  is a bijection in  $U$ .

**Question 119**

Let  $f$  be a meromorphic function in the plane such that

$$\lim_{|z| \rightarrow \infty} |f(z)| = \infty$$

1. Show that  $f$  has only finitely many poles.
2. Show that  $f$  is a rational function.

## 4 Topology (1 Questions)

**Question 1**

Something something  $G$ .