Homework 1
Asset Allocation with Stock Portfolios and Bills

1. (20%) **Three-Asset Portfolio**

Let $R_{pt} = w_1 R_{1t} + w_2 R_{2t} + w_3 R_{3t}$,

where R_{pt} is the return of portfolio at time t

w_i is the weight of investment in asset (company) i

R_{it} is the return of asset (company) i at time t

The expected return of the portfolio is $E(R_{pt}) = \mu_p = w_1\mu_1 + w_2\mu_2 + w_3\mu_3$ and the variance of the portfolio is

$$Var(R_{pt}) = \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{12} + 2w_2w_3\sigma_{23} + 2w_1w_3\sigma_{13}$$

where μ_i is the mean of asset (company) i

σ_i^2 is the variance of asset (company) i

σ_{ij} is the covariance between asset i and j

(1) (8%) Please show **in detail** that the expected return of the portfolio $E(R_{pt}) = \mu_p = W^T\mu$ and the variance of the portfolio $Var(R_{pt}) = \sigma_p^2 = W^T\Sigma W$

$$\text{where } \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}; W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}; \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

(2) (4%) Given the variance-covariance matrix

$$\Sigma = \begin{bmatrix} 24 & -10 & 25 \\ -10 & 50 & 32 \\ 25 & 32 & 12 \end{bmatrix}$$

Calculate the variance of an equally weighted portfolio. Please verify your answers by Matlab.

(3) (8%) **Matrix derivative with respect to W:** Please show **in detail** that

$$\frac{\partial\sigma_p^2}{\partial \mathrm{W}} \equiv \begin{pmatrix} \dfrac{\partial\sigma_p^2}{\partial \mathrm{w}_1} \\ \dfrac{\partial\sigma_p^2}{\partial \mathrm{w}_2} \\ \dfrac{\partial\sigma_p^2}{\partial \mathrm{w}_3} \end{pmatrix} = \frac{\partial(W^T\Sigma W)}{\partial \mathrm{W}} = 2\Sigma W$$

2. Covariance between Two Portfolios

(8%) Suppose we want to express the covariance between two portfolios, *p* and *q*, using matrix notation. Let $W_p^T = [w_{1p} \quad w_{2p}]$ be the weights held in portfolio p. For example, one might construct portfolio *p* by holding 50% of his/her wealth in asset 1 and the remaining 50% in asset 2. Similarly, let $W_q^T = [w_{1q} \quad w_{2q}]$ be the weights used to construct portfolio *q*.

Please show **in detail** that the covariance between portfolios *p* and *q* can be expressed as the following matrix form:

$$Cov(R_{pt}, R_{qt}) \equiv \sigma_{pq} = E[(R_{pt} - E(R_p))(R_{qt} - E(R_q))] = W_p^T \Sigma W_q$$

where $W_p = \begin{bmatrix} w_{1p} \\ w_{2p} \end{bmatrix}; W_q = \begin{bmatrix} w_{1q} \\ w_{2q} \end{bmatrix}; \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

3. (8%) Please derive Equation (7.13) of the textbook by maximizing the slope, $\frac{\mu_p - r_f}{\sigma_p}$, of CAL. See footnote 6 for solution procedure.

4. (64%) You can find the dateset containing the annual returns of "Small/Value" and "Small/Growth" portfolios at the Professor Kenneth French's Web site http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

(Go to Kenneth French's Web site, find **6 Portfolios Formed on Size and Book-to-Market (2 x 3) Details**; Download the CSV file **(6_Portfolios_2x3.CSV)** and use **(Average Equal Weighted Returns -- Annual**).

Copy the 20 years data (from year 1999 to 2018) into a new spreadsheet. Next analyze the risk-return tradeoff that would have characterized portfolios constructed from "Small/Value" and "Small/Growth" over this period of time.

(6%) (1) What were the average rate of return, standard deviation, skewness, and kurtosis of their annual returns?

(6%) (2) What were the covariance and the correlation coefficient of their annual returns?

(8%) (3) Draw the portfolio opportunity set of "Small/Value" and "Small/Growth" portfolios. Please consider weights in "Small/Value" portfolio starting at -1 and incrementing by .10 up to a weight of 3.0.

(6%) (4) Redo (3) by setting correlation coefficients to -1 and +1. Please plot the expected returns and standard deviations on the same chart in problem (3)

Use data from 1999 to 2018.

(6%) (5) What was the average return and standard deviation of the minimum-variance combination of "Small/Value" and "Small/Growth" portfolios?

Assume T-bill has annual expected return 1.76% and **standard deviation 0**.

(6%) (6) Find the optimal risky portfolio (P) and its expected return and standard deviation.

(10%) (7) Find the slope of the CAL generated by T-bill and portfolio P. Draw the new portfolio opportunity set and plot it on the same chart in problem (3).

Assume the utility function of the textbook $U = E(r) - \frac{1}{2} A\sigma^2$.

(8%) (8) If an investor has a coefficient of risk aversion A=4, calculate the composition of the complete portfolio of T-bill, "Small/Value" and "Small/Growth" portfolios. Please plot the expected return and standard deviation of this complete portfolio as well as the indifference curve on the same chart in problem (3).

(8%) (9) If another investor has a coefficient of risk aversion A=1, calculate the composition of the complete portfolio of T-bill, "Small/Value" and "Small/Growth" portfolios. Please plot the expected return and standard deviation of this complete portfolio as well as the indifference curve on the same chart in problem (3).