

Machine Learning

Homework 3

Due on May 20, 2020

1. Let $f : R^n \rightarrow R$ be a *strictly convex* function *i.e.*,

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y),$$

and $0 < \lambda < 1$. If x^* is a local minimizer then x^* is the unique global minimizer. (15 %)

2. Let

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{bmatrix} \quad \text{and} \quad \mathbf{h}(\mathbf{x}) = \begin{bmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \\ \vdots \\ h_k(\mathbf{x}) \end{bmatrix}$$

where $g_i : R^n \rightarrow R$ be a *convex* function for all $i = 1, 2, \dots, m$, and $h_j : R^n \rightarrow R$ be a *linear* function for all $j = 1, 2, \dots, k$.

Consider $\mathcal{F} = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \mathbf{h}(\mathbf{x}) = \mathbf{0}\} \subset R^n$. Prove that \mathcal{F} is a *convex* set. (15 %)

3. Prove that for any matrix $B \in R^{m \times n}$, either the system (I)

$$B\mathbf{x} < \mathbf{0}$$

or the system (II)

$$B^T \alpha = \mathbf{0}, \alpha \geq \mathbf{0} \text{ and } \alpha \neq \mathbf{0}$$

has a solution but *never both*. (20 %)

Hint 1: $B\mathbf{x} < \mathbf{0}$ if and only if $B\mathbf{x} + \mathbf{1}z \leq \mathbf{0}, z > 0$.

Hint 2: Use Farkas' Lemma with a suitable $b \in R^{n+1}$ and $A \in R^{m \times (n+1)}$

4. Prove that for any matrix $A \in R^{m \times n}$ and matrix $\mathbf{b} \in R^m$,

$$A\mathbf{x} = \mathbf{b} \quad \text{if and only if} \quad A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{1}^T A\mathbf{x} \geq \mathbf{1}^T \mathbf{b}$$

(15 %)

5. Write down the *dual* problem for

$$\max_{\mathbf{x}} p^T \mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b}$$

(15 %)

6. Find an approximate solution using MATLAB to the following system by minimizing $\|Ax - b\|_p$ for $p = 1, 2, \infty$. Write down both the approximate solution, and the value of the $\|Ax - b\|_p$. Draw the solution points in R^2 and the four equations being solved.

$$\begin{array}{rclcl} x_1 & + & 2x_2 & = & 2 \\ 2x_1 & - & x_2 & = & -2 \\ x_1 & + & x_2 & = & 1 \\ 4x_1 & - & x_2 & = & 1 \\ x_1 & - & x_2 & = & -2 \\ x_1 & - & 2x_2 & = & -1 \end{array}$$

(20 %)