University of Waterloo
Mechatronics Engineering

Sensors and Instrumentation

MTE 220

Lecture Notes

by

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Operational Issues

Why not the easy way out?

The Butterfly...... One day a small opening appeared on a cocoon. A man sat and watched for the butterfly for several hours as it struggled to force its body through that little hole. Then it seemed to stop making any progress. It appeared as if it had gotten as far as it could and it could go no further. So the man decided to help the butterfly, he took a pair of scissors and snipped off the remaining bit of the cocoon. The butterfly then emerged easily. But it had a swollen body and small, shriveled wings. The man continued to watch the butterfly because he expected that, at any moment, the wings would enlarge and expand to be able to support the body, which would contract in time. Neither happened! In fact, the butterfly spent the rest of its life crawling around with a swollen body and shriveled wings. It never was able to fly. What the man in his kindness and haste did not understand was that the restricting cocoon and the struggle required for the butterfly to get through the tiny opening were God's way of forcing fluid from the body of the butterfly into its wings so that it would be ready for flight once it achieved its freedom from the cocoon. Sometimes struggles are exactly what we need in our life. If God allowed us to go through our life without any obstacles, it would cripple us. We would not be as strong as what we could have been. We could never fly.

Faculty of Engineering Course Responsibilities

Refer to

http://uwaterloo.ca/engineering/current-undergraduate-students/academic-support/course-responsibilities

for the various policies and procedures associated with courses offered in the Faculty of Engineering.

In Case of Absence

- Course attendance is not marked in MTE 220
- You must obtain a medical certificate if you miss an assignment deadline or an examination for health reasons
- Medical certificates can be obtained from Health Services

  http://www.healthservices.uwaterloo.ca/Health_Services/verification.html

- If you must miss a laboratory session or an examination to participate in a co-op interview, it is your responsibility to inform your lab instructor / teaching assistants prior to the absence
- When in doubt, consult your instructor / lab instructor / teaching assistants

University Rules and Policies

For descriptions of academic programs and rules, refer to the Online University Calendar

http://ugradcalendar.uwaterloo.ca

- Policy 71

  http://www.adm.uwaterloo.ca/infosec/Policies/policy71.htm

  provides some examples of academic offences: infringing unreasonably on work of others, violation of safety regulations, cheating, impersonating another student, plagiarism, obtaining materials by improper means, falsifying academic records, oral or written misrepresentations.

Course overview and how it fits into the Mechatronics program

calendar description

Review of circuit theory; input-output relationships, transfer functions and frequency response of linear systems; operational amplifiers, operational amplifier circuits using negative or positive feedback; diodes, operational amplifier circuits using diodes; analog signal detection, conditioning and conversion systems; transducers and sensors, difference and instrumentation amplifiers, active filters.

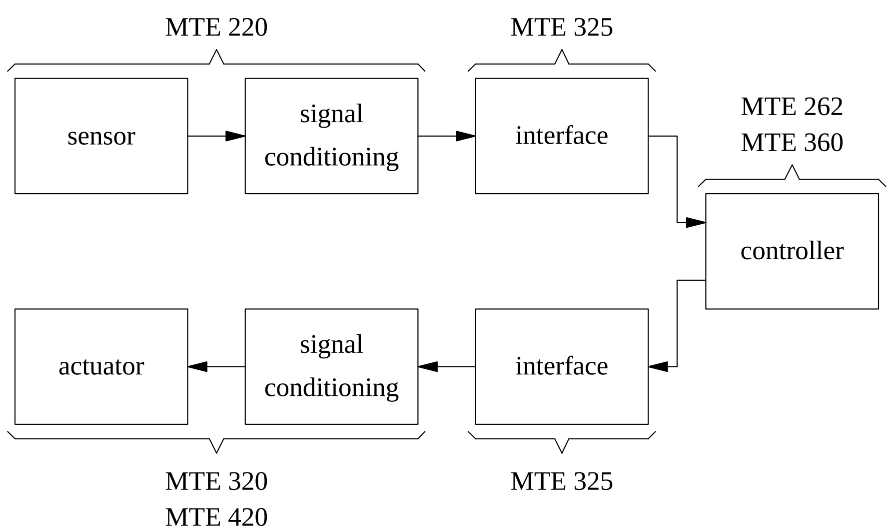
course learning objectives

Students will be able to design and build signal conditioning circuits for robot sensors and actuators.

course outline

(1) linear circuits, (2) operational amplifier circuits, (3) electronic devices, (4) filters, (5) sensors & process control, (6) analog signal conditioning, (7) thermal sensors, (8) mechanical sensors, (9) optical sensors

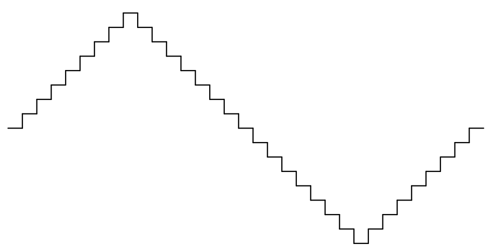
mechatronics system



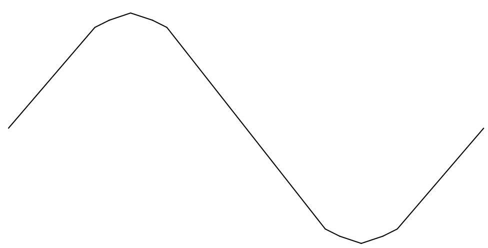
types of sensors (transducers)

proximity: mechanical, electrical, magnetic
encoders: digital sensors
motion: infrared
temperature: thermocouples
chemical: pH meters

discrete value sensor output



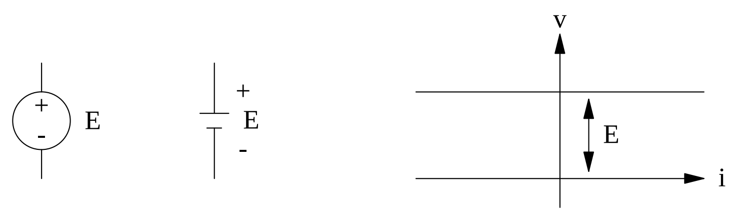
continuous value sensor output



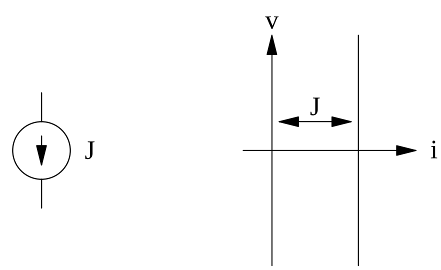
Linear Circuit Fundamentals

linear circuit components

- electric current
  - flow of electrons
  - measured in amperes, denoted by A
- voltage
  - a force which causes electrons to flow
  - measured in volts, denoted by V
  - by definition, positive current flows from “+” to “-”
- independent voltage source

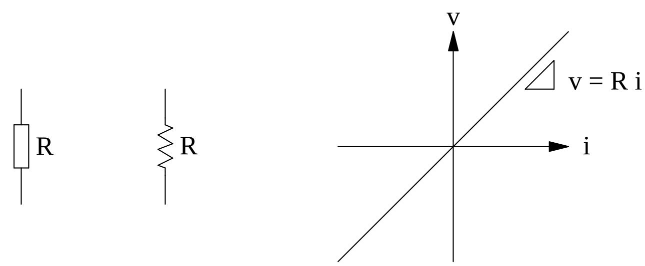


  - specified voltage but arbitrary current
  - parallel connect: identical voltages only
- independent current source



  - specified current but arbitrary voltage
  - series connected: identical currents only

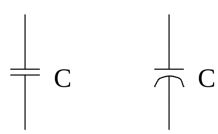
- resistor



  - element which resists the flow of current, denoted by R
  - measured in ohms, denoted by Ω

- conductance
  - inverse of resistance, denoted by G
  - $G = \dfrac{1}{R}$ so $i = G\,v$
  - measured in siemens, denoted by S

- capacitor

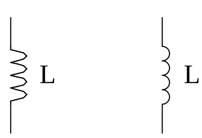


$$q(t) = C\,v(t)$$

$$i(t) = \frac{d\,q(t)}{dt} = C\,\frac{d\,v(t)}{dt}$$

  - element that stores energy in the form of charge
  - formed by two conductors separated by an insulator
  - Charge is denoted by Q when independent of time, or q(t) when a function of time; measured in coulombs, denoted by C

- inductor

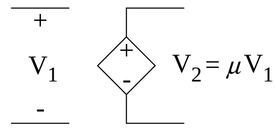


$$\Phi = L\,i(t)$$

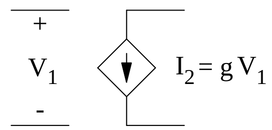
$$v(t) = \frac{d\,\Phi(t)}{dt} = L\,\frac{d\,i(t)}{dt}$$

  - element that stores energy in the form of flux
  - formed by coil of insulated wire, possibly placed into a core of magnetic material
  - flux is denoted by Φ
    measured in webers, denoted by Wb

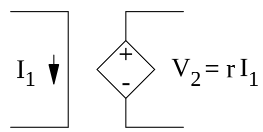
- voltage-controlled voltage source (VV)
  - voltage-to-voltage transducer (VVT)



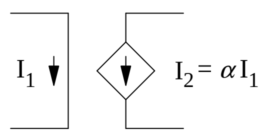
  - an ideal voltage amplifier
  - μ is unitless
- voltage-controlled current source (VC)
  - voltage-to-current transducer (VCT)



  - also referred to as a transconductance
  - g is measured in siemens
- current-controlled voltage source (CV)
  - current-to-voltage transducer (CVT)

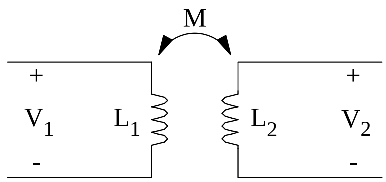


  - also referred to as a transresistance
  - r is measured in ohms
- current-controlled current source (CC)
  - current-to-current transducer (CCT)



  - an ideal current amplifier
  - α is unitless

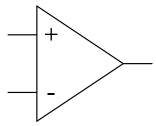
- transformer



  - contains two (or more) coils (inductors) whose magnetic fields interact

$$v_1 = L_1 \frac{d\, i_1(t)}{dt} \pm M \frac{d\, i_2(t)}{dt}$$
$$v_2 = L_2 \frac{d\, i_2(t)}{dt} \pm M \frac{d\, i_1(t)}{dt}$$

- operational amplifier (opamp)



  - a voltage amplifier with infinite gain in the ideal case
  - actual gain is finite and frequency dependent

power and energy

- energy is the amount of work a source can deliver
  - it is denoted by W if constant or w(t) if time varying, measured in joules, denoted by J
- power is the change in stored energy
  - it is denoted by P if constant or p(t) if time varying, measured in watts, denoted by W

$$p(t) = \frac{d\,w(t)}{dt}$$

$$w(t) = \int_{t_1}^{t_2} p(t)\,dt$$

  - note $1\ W = 1\ \frac{J}{s}$
  - power, voltage and current in a resistive circuit are related by

$$p(t) = v(t) \cdot i(t)$$

$$p(t) = \frac{v^2(t)}{R} = i^2(t) \cdot R$$

$$p(t) = v^2(t) \cdot G = \frac{i^2(t)}{G}$$

Kirchhoff's laws

- Kirchhoff current law (KCL)
  sum of currents leaving a node is zero
  - assume positive current is flowing away from a node
  - a node is a connection of several elments/components
  - used for **nodal** analysis/formulation
- Kirchhoff voltage law (KVL)
  sum of voltage drops around a loop is zero
  - recall positive current flows

    from "+" to "-"
  - a loop is a connection of elments/components that we can walk around and get back to the starting point
  - used for **mesh** analysis/formulation

impedance for $v(t) = V_o e^{st}$ and $i(t) = I_o e^{st}$

- if v(t) & i(t) are exponential functions

$$i(t) = I_o e^{st} \;\rightarrow\; \frac{d\,i(t)}{dt} = s\,I_o\,e^{st} = s\,i(t)$$

$$v(t) = V_o e^{st} \;\rightarrow\; \frac{d\,v(t)}{dt} = s\,V_o\,e^{st} = s\,v(t)$$

$$v_L(t) = L\,\frac{d\,i_L(t)}{dt} = s\,L\,i_L(t) \;\rightarrow\; Z_L = \frac{v_L(t)}{i_L(t)} = s\,L$$

$$Y_L = \frac{i_L(t)}{v_L(t)} = \frac{1}{s\,L}$$

$$i_C(t) = C\,\frac{d\,v_C(t)}{dt} = s\,C\,v_C(t) \;\rightarrow\; Y_C = \frac{i_C(t)}{v_C(t)} = s\,C$$

$$Z_C = \frac{v_C(t)}{i_C(t)} = \frac{1}{s\,C}$$

sinusoidal v(t) and i(t)

- recall

$$\frac{d}{dx}(\sin(u)) = \cos(u)\,\frac{d\,u}{dx} \quad ; \quad \frac{d}{dx}(\cos(u)) = -\sin(u)\,\frac{d\,u}{dx}$$

$$\cos(\omega\,t) = \sin(\omega\,t + \frac{\pi}{2}) \quad ; \quad \sin(\omega\,t) = \cos(\omega\,t - \frac{\pi}{2})$$

- if v(t) & i(t) are sinusoidal functions

$$v(t) = V_p\cos(\omega\,t) \;\rightarrow\; \frac{d\,v(t)}{dt} = -\,\omega\,V_p\sin(\omega\,t) = -\,\omega\,V_p\cos(\omega\,t - \frac{\pi}{2}) = \omega\,V_p\cos(\omega\,t + \frac{\pi}{2})$$

$$i(t) = I_p\cos(\omega\,t) \;\rightarrow\; \frac{d\,i(t)}{dt} = -\,\omega\,I_p\sin(\omega\,t) = -\,\omega\,I_p\cos(\omega\,t - \frac{\pi}{2}) = \omega\,I_p\cos(\omega\,t + \frac{\pi}{2})$$

- recall

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad ; \quad \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad ; \quad e^{j\theta} = \cos(\theta) + j\sin(\theta) \quad ; \quad e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\vec{w} = a + jb = M(\cos(\theta) + j\sin(\theta)) = M e^{j\theta} = M \underline{/\,\theta} \quad ; \quad \text{where} \quad M = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

giving us

$$w(t) = M e^{j(\omega t + \theta)}$$

- recall

$$\mathrm{Re}\{e^{j(\omega t + \theta)}\} = \cos(\omega t + \theta) \quad ; \quad \mathrm{Im}\{e^{j(\omega t + \theta)}\} = \sin(\omega t + \theta)$$

- let

$$v(t) = V_p \cos(\omega t + \theta) = \sqrt{2}\, V_{rms} \cos(\omega t + \theta) = \mathrm{Re}\{V_p e^{j(\omega t + \theta)}\} = \mathrm{Re}\{(V_{rms} e^{j\theta})(\sqrt{2}\, e^{j\omega t})\}$$

- phasor

$$\vec{V} = V_{rms} e^{j\theta} = V_{rms} \underline{/\,\theta}$$

impedance for phasors v(t) = Re{$V_p e^{j(\omega t + \theta)}$}

Since

$$\mathrm{Re}\{(\sqrt{2}\, e^{j\omega t})\}$$

is common to all waveforms for a single tone linear network, we set it aside and only use

$$\mathrm{Re}\{(V_{rms} e^{j\theta})\}$$

for calculation purposes.

$$\vec{V} = V_{rms} e^{j\theta} = V_{rms} \underline{/\,\theta}$$

- voltage source

$$v(t) = V_p \cos(\omega t + \theta) \quad \rightarrow \quad \vec{V} = \frac{V_p}{\sqrt{2}} \underline{/\,\theta}$$

- current source

$$i(t) = I_p \cos(\omega t + \theta) \quad \rightarrow \quad \vec{I} = \frac{I_p}{\sqrt{2}} \underline{/\,\theta}$$

- inductor

$$v_L(t) = L \frac{d\, i_L(t)}{dt} = \sqrt{2}\, \omega\, L\, I_L \cos(\omega\, t + \frac{\pi}{2}) = \sqrt{2}\, V_L \cos(\omega\, t + \frac{\pi}{2})$$

$$Z_L = \frac{V_L \underline{/ \pi/2}}{I_L \underline{/ 0}} = \frac{\omega\, L\, I_L \underline{/ \pi/2}}{I_L \underline{/ 0}} = \omega\, L \underline{/ \pi/2} = j\, \omega\, L$$

$$Y_L = \frac{1}{Z_L} = \frac{1}{j\, \omega\, L}$$

- capacitor

$$i_C(t) = C \frac{d\, v_C(t)}{dt} = \sqrt{2}\, \omega\, C\, V_C \cos(\omega\, t + \frac{\pi}{2}) = \sqrt{2}\, I_C \cos(\omega\, t + \frac{\pi}{2})$$

$$Y_C = \frac{I_C \underline{/ \pi/2}}{V_C \underline{/ 0}} = \frac{\omega\, C\, V_C \underline{/ \pi/2}}{I_L \underline{/ 0}} = \omega\, C \underline{/ \pi/2} = j\, \omega\, C$$

$$Z_C = \frac{1}{Y_C} = \frac{1}{j\, \omega\, C}$$

- resistor

$$v_R(t) = R\, i_R(t) = \sqrt{2}\, V_R \cos(\omega\, t)$$

$$Z_R = \frac{v_R(t)}{i_R(t)} = \frac{V_R \underline{/ 0}}{I_R \underline{/ 0}} = \frac{R\, I_R \underline{/ 0}}{I_R \underline{/ 0}} = R$$

$$Y_R = \frac{1}{Z_R} = \frac{1}{R} = G$$

time invariant vs. time varying

- time invariant: R(t) = constant
- time varying: R(t) = f(t) : varies with time

linear vs. nonlinear

- linear: v = R*i
- nonlinear $v = R * i^2$

passive vs. active elements

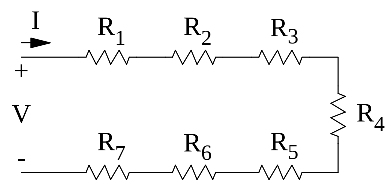
- passive: net power consumed from network
- active: net power supplied to network

parallel/series connected resistor circuits

- effective resistance of series connected resistors is

$$R_{effective} = \sum_{i=1}^{n} R_i$$

$$G_{effective} = \frac{1}{\sum_{i=1}^{n} \frac{1}{G_i}}$$

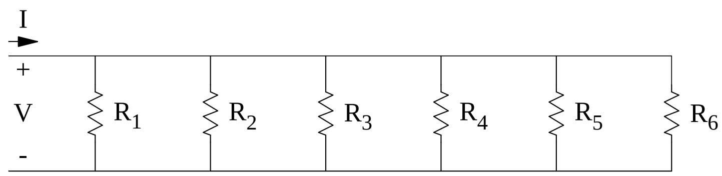


$$R_{effective} = \frac{V}{I} = \sum_{i=1}^{7} R_i$$

- effective resistance of parallel connected resistors is

$$G_{effective} = \sum_{i=1}^{n} G_i$$

$$R_{effective} = \frac{1}{\sum_{i=1}^{n} \frac{1}{R_i}}$$

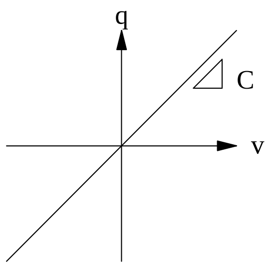


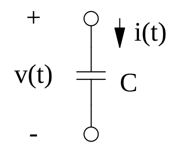
$$G_{effective} = \frac{I}{V} = \sum_{i=1}^{6} G_i = \sum_{i=1}^{6} \frac{1}{R_i}$$

$$R_{effective} = \frac{V}{I} = \frac{1}{G_{effective}}$$

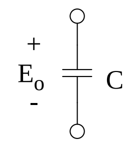
capacitors

- stores energy in the form of charge
- formed by two conductors separated by an insulator
- Charge is denoted by Q when independent of time, or q(t) when a function of time; measured in Coulombs, denoted by C





voltage & current



initial conditions

- linear capacitor charge and current

$$i(t) = \frac{d\,q(t)}{dt} \quad \rightarrow \quad q(t) = \int_{-\infty}^{t} i(\tau)\,d\tau = \int_{-\infty}^{0} i(\tau)\,d\tau + \int_{0}^{t} i(\tau)\,d\tau = q_o + \int_{0}^{t} i(\tau)\,d\tau$$

for the case of a linear capacitor

$$q(t) = C\,v(t) \quad \rightarrow \quad i(t) = C\,\frac{d\,v(t)}{dt}$$

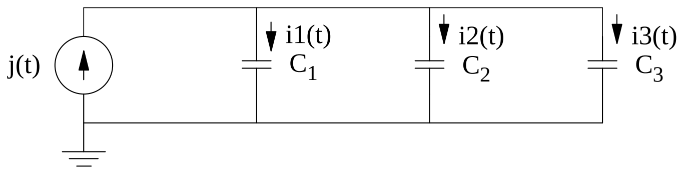
- linear capacitor voltage

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau = \frac{1}{C}\int_{-\infty}^{0} i(\tau)\,d\tau + \frac{1}{C}\int_{0}^{t} i(\tau)\,d\tau = E_o + \frac{1}{C}\int_{0}^{t} i(\tau)\,d\tau$$

- stored energy
  given $p(t) = v(t)\,i(t)$ the energy stored in a capacitor is

$$w_C = \int_{-\infty}^{t} p(\tau)\,d\tau = C\int_{-\infty}^{t} v(\tau)\,\frac{d\,v(\tau)}{d\tau}\,d\,\tau = C\int_{-\infty}^{t} v(\tau)\,d\,v(\tau) = \frac{1}{2}\,C\,v^2(t)$$

- parallel connected capacitors

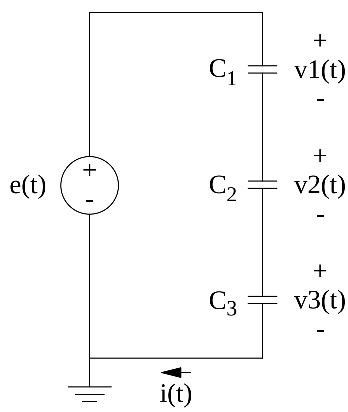


$$j(t) = i_1(t) + i_2(t) + i_3(t) = C_1 \frac{d\,v(t)}{dt} + C_2 \frac{d\,v(t)}{dt} + C_3 \frac{d\,v(t)}{dt} = (C_1 + C_2 + C_3) \frac{d\,v(t)}{dt} = C_{parallel} \frac{d\,v(t)}{dt}$$

for n capacitors connected in parallel

$$C_{parallel} = \sum_{i=1}^{n} C_i$$

- series connected capacitors



assuming the capacitors are initially discharged

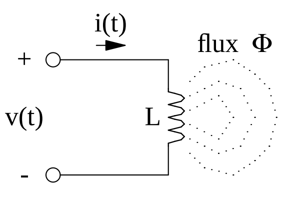
$$e(t) = v_1(t) + v_2(t) + v_3(t) = \frac{1}{C_1} \int_0^t i(\tau)\,d\tau + \frac{1}{C_2} \int_0^t i(\tau)\,d\tau + \frac{1}{C_3} \int_0^t i(\tau)\,d\tau = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right) \int_0^t i(\tau)\,d\tau = \left(\frac{1}{C_{series}} \right) \int_0^t i(\tau)\,d\tau$$

for n capacitors connected in series

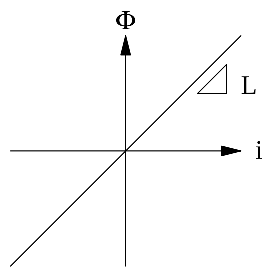
$$\frac{1}{C_{series}} = \sum_{i=1}^{n} \frac{1}{C_i}$$

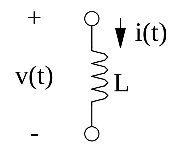
inductors

- stores energy in the form of flux
- formed by coil of insulated wire, possibly placed into a core of magnetic material
- flux is denoted by Φ
  measured in Webers, denoted by Wb

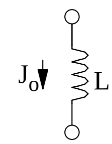


flux





voltage & current



initial conditions

- linear inductor flux and voltage

$$v(t) = \frac{d\,\Phi(t)}{dt} = L\,\frac{d\,i(t)}{dt} \;..\quad \rightarrow \quad \Phi(t) = \int_{-\infty}^{t} v(\tau)\,d\tau = \int_{-\infty}^{0} v(\tau)\,d\tau + \int_{0}^{t} v(\tau)\,d\tau = \Phi_o + \int_{0}^{t} v(\tau)\,d\tau$$

  for the case of a linear inductor

$$\Phi(t) = L\,i(t) \quad ; \quad v(t) = L\,\frac{d\,i(t)}{dt}$$

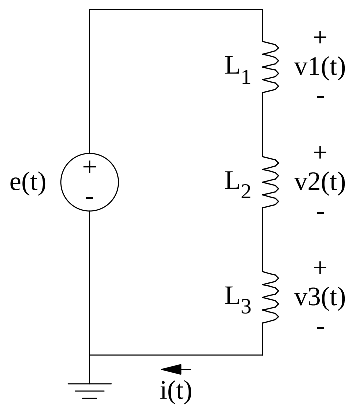
- linear inductor current

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau = \frac{1}{L}\int_{-\infty}^{0} v(\tau)\,d\tau + \frac{1}{L}\int_{0}^{t} v(\tau)\,d\tau = J_o + \frac{1}{L}\int_{0}^{t} v(\tau)\,d\tau$$

- stored energy
  given $p(t) = v(t)\,i(t)$
  the energy stored in an inductor is

$$w_C = \int_{-\infty}^{t} p(\tau)\,d\tau = L\int_{-\infty}^{t} i(\tau)\,\frac{d\,i(\tau)}{d\tau}\,d\,\tau = L\int_{-\infty}^{t} i(\tau)\,d\,i(\tau) = \frac{1}{2}\,L\,i^2(t)$$

- series connected inductors

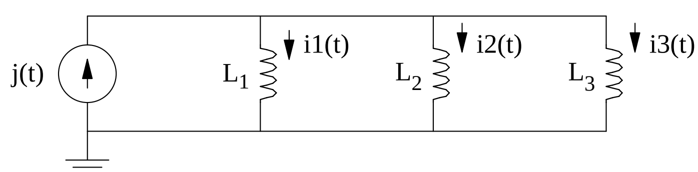


$$e(t) = v_1(t) + v_2(t) + v_3(t) = L_1 \frac{d\,i(t)}{dt} + L_2 \frac{d\,i(t)}{dt} + L_3 \frac{d\,i(t)}{dt} = (L_1 + L_2 + L_3)\frac{d\,i(t)}{dt} = L_{series}\frac{d\,i(t)}{dt}$$

for n inductors connected in series

$$L_{series} = \sum_{i=1}^{n} L_i$$

- parallel connected inductors



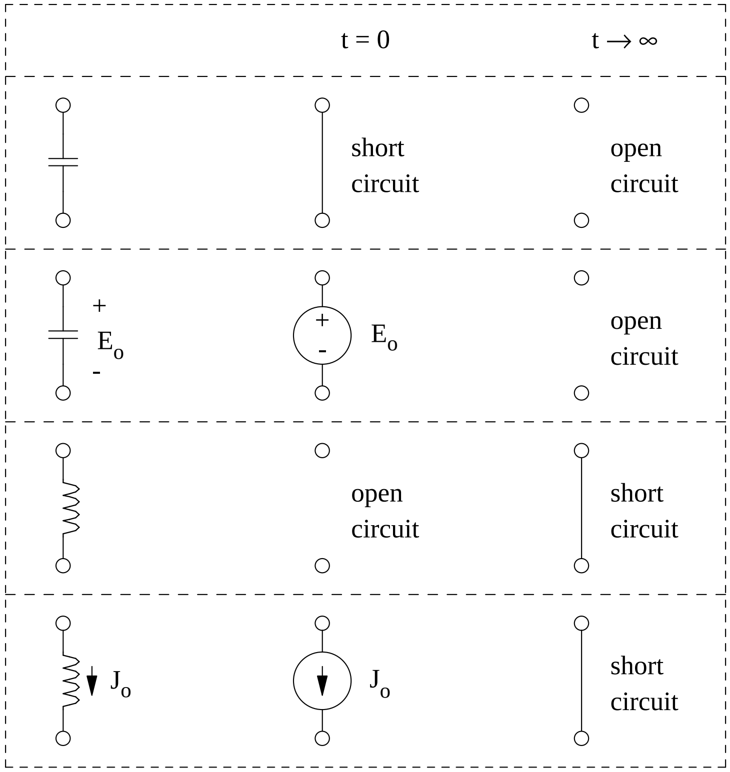
assuming the inductors are initially discharged

$$j(t) = i_1(t) + i_2(t) + i_3(t) = \frac{1}{L_1}\int_0^t v(\tau)\,d\tau + \frac{1}{L_2}\int_0^t v(\tau)\,d\tau + \frac{1}{L_3}\int_0^t v(\tau)\,d\tau = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_0^t v(\tau)\,d\tau = \left(\frac{1}{L_{parallel}}\right)\int_0^t v(\tau)\,d\tau$$

for n inductors connected in parallel

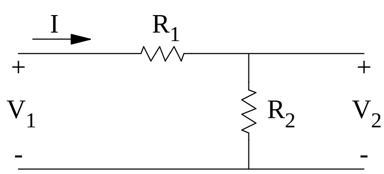
$$\frac{1}{L_{parallel}} = \sum_{i=1}^{n} \frac{1}{L_i}$$

initial and final values



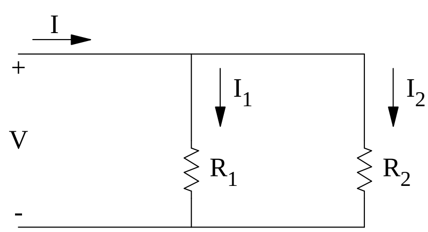
voltage and current dividers

voltage divider



$$V_2 = R_2 I = R_2 \frac{V_1}{R_1 + R_2} = \frac{R_2}{R_1 + R_2} V_1$$

current divider

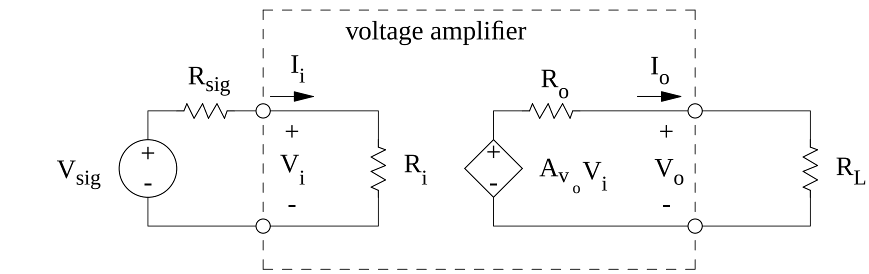


$$I_2 = \frac{V}{R_2} = \frac{\left(\dfrac{I}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}\right)}{R_2} = \frac{\left(\dfrac{R_1 \; R_2}{R_1 + R_2}\right) I}{R_2} = \frac{R_1}{R_1 + R_2} I$$

amplifier types and parameters

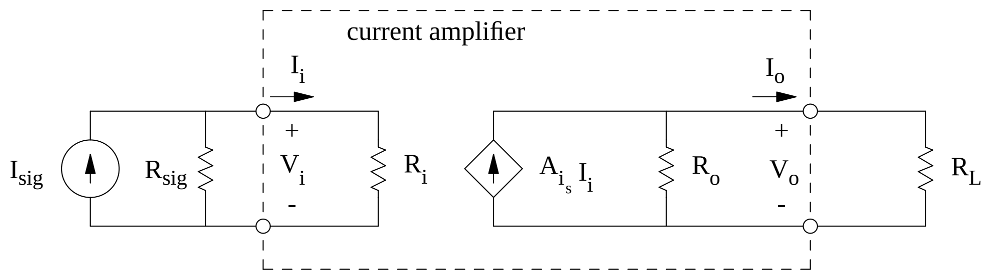
$$A_v = \frac{v_o}{v_i} \quad ; \quad G_v = \frac{v_o}{v_{sig}} \quad ; \quad A_i = \frac{i_o}{i_i} \quad ; \quad A_p = A_v\,A_i$$

$$R_{in} = \frac{v_i}{i_i} \quad ; \quad R_i = \left.\frac{v_i}{i_i}\right|_{R_L = \infty} \quad ; \quad R_{out} = \left.\frac{v_x}{i_x}\right|_{v_{sig} = 0} \quad ; \quad R_o = \left.\frac{v_x}{i_x}\right|_{v_i = 0}$$



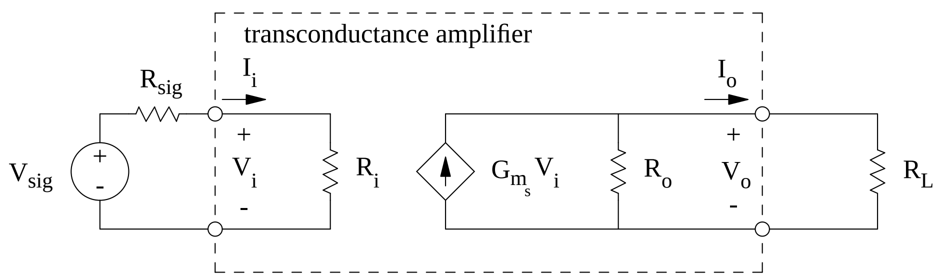
$$V_o = \frac{R_L}{R_L + R_o}\,A_{v_o}\,\frac{R_i}{R_i + R_{sig}}\,V_{sig}$$

$$A_{v_o} = \left.\frac{V_o}{V_i}\right|_{R_L = \infty}$$



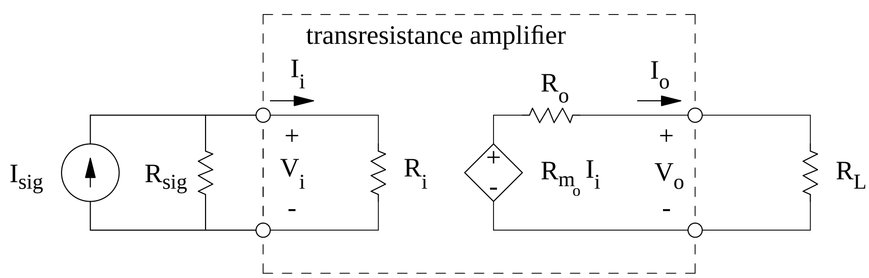
$$I_o = \frac{R_o}{R_L + R_o}\,A_{i_s}\,\frac{R_{sig}}{R_i + R_{sig}}\,I_{sig}$$

$$A_{i_s} = \left.\frac{I_o}{I_i}\right|_{R_L = 0}$$



$$I_o = \frac{R_o}{R_L + R_o}\,G_{m_s}\,\frac{R_i}{R_i + R_{sig}}\,V_{sig}$$

$$G_{m_s} = \left.\frac{I_o}{V_i}\right|_{R_L = 0}$$



$$V_o = \frac{R_L}{R_L + R_o}\,R_{m_o}\,\frac{R_{sig}}{R_i + R_{sig}}\,I_{sig}$$

$$R_{m_o} = \left.\frac{V_o}{I_i}\right|_{R_L = \infty}$$

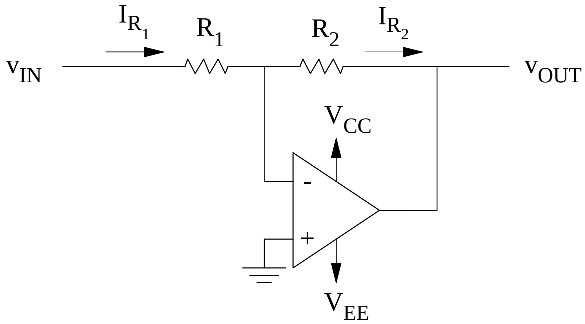
Opamp Circuits

inverting amplifier

$$v_- \approx v_+ \approx 0 \quad \rightarrow \quad i_{R_1} = i_{R_2} = \frac{v_{IN}}{R_1}$$

$$v_{OUT} = -I_{R_2} R_2 = -\frac{v_{IN}}{R_1} R_2 = -\frac{R_2}{R_1} v_{IN}$$

$$A_v = \frac{v_{OUT}}{v_{IN}} = -\frac{R_2}{R_1}$$



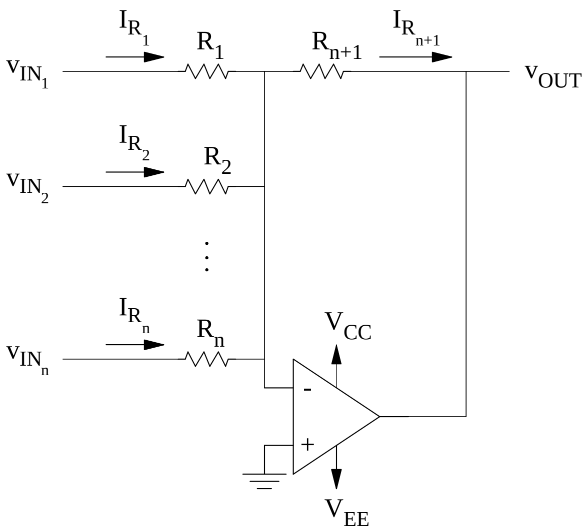
summer

$$v_- \approx v_+ \approx 0$$

$$i_{R_1} = \frac{v_{IN_1}}{R_1} \quad ; \quad i_{R_2} = \frac{v_{IN_2}}{R_2} \quad \cdots \quad i_{R_n} = \frac{v_{IN_n}}{R_n}$$

$$i_{R_{n+1}} = \sum_{k=1}^{k=n} i_{R_k} = \sum_{k=1}^{k=n} \frac{v_{IN_k}}{R_k}$$

$$v_{OUT} = -I_{R_{n+1}} R_{n+1} = -\sum_{k=1}^{k=n} \frac{v_{IN_k}}{R_k} R_{n+1} = -\sum_{k=1}^{k=n} \frac{R_{n+1}}{R_k} v_{IN_k}$$

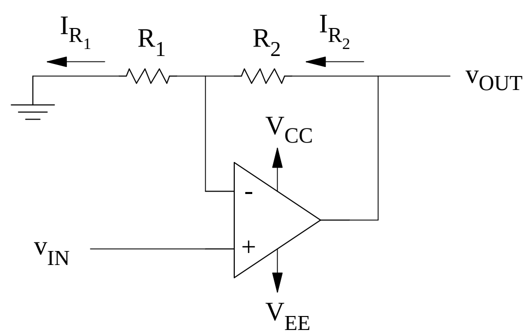


noninverting amplifier

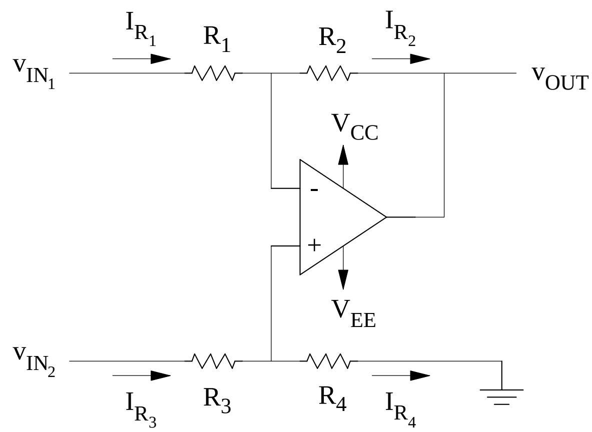
$$v_- \approx v_+ = v_{IN} \quad \rightarrow \quad i_{R_2} = i_{R_1} = \frac{v_{IN}}{R_1}$$

$$v_{OUT} = (R_2 + R_1)\, i_{R_1} = (R_2 + R_1)\frac{v_{IN}}{R_1} = \frac{R_2 + R_1}{R_1} v_{IN}$$

$$A_v = \frac{v_{OUT}}{v_{IN}} = \frac{R_2 + R_1}{R_1} = 1 + \frac{R_2}{R_1}$$



difference amplifier



$$v_- \approx v_+ = \frac{R_4}{R_3 + R_4} v_{IN_2}$$

$$i_{R_2} = i_{R_1} = \frac{v_{IN_1} - v_+}{R_1} = \frac{v_{IN_1}}{R_1} - \frac{v_+}{R_1} = \frac{v_{IN_1}}{R_1} - \frac{\frac{R_4}{R_3 + R_4} v_{IN_2}}{R_1}$$

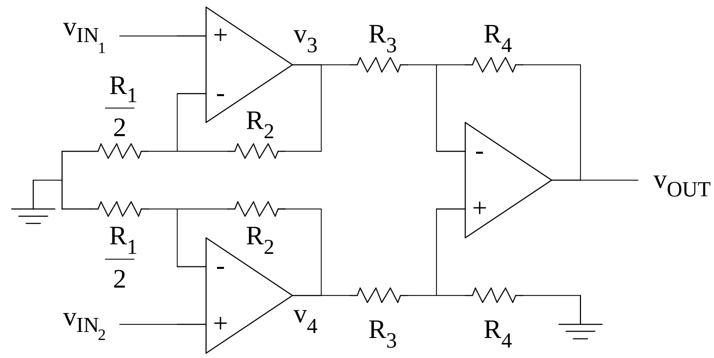
$$v_{OUT} = v_- - \left(R_2\, i_{R_2} \right)$$

$$= \frac{R_4}{R_3 + R_4} v_{IN_2} - \frac{v_{IN_1} - \frac{R_4}{R_3 + R_4} v_{IN_2}}{R_1} R_2$$

$$= \left(\left(\frac{R_1 + R_2}{R_1} \right)\left(\frac{R_4}{R_3 + R_4} \right)\right) v_{IN_2} - \left(\frac{R_2}{R_1} \right) v_{IN_1}$$

$$R_4 = R_2 \quad \text{and} \quad R_3 = R_1 \quad \rightarrow \quad v_{OUT} = \frac{R_2}{R_1}\left(v_{IN_2} - v_{IN_1} \right)$$

instrumentation amplifier



$$v_{OUT} = \frac{R_3 + R_4}{R_3} \frac{R_4}{R_3 + R_4} v_4 - \frac{R_4}{R_3} v_3 = \frac{R_4}{R_3} \left(v_4 - v_3 \right)$$

$$v_4 = v_{IN_2} + \frac{\frac{v_{IN_2} - v_{IN_1}}{2}}{\frac{R_1}{2}} R_2 \quad ; \quad v_3 = v_{IN_1} - \frac{\frac{v_{IN_2} - v_{IN_1}}{2}}{\frac{R_1}{2}} R_2$$

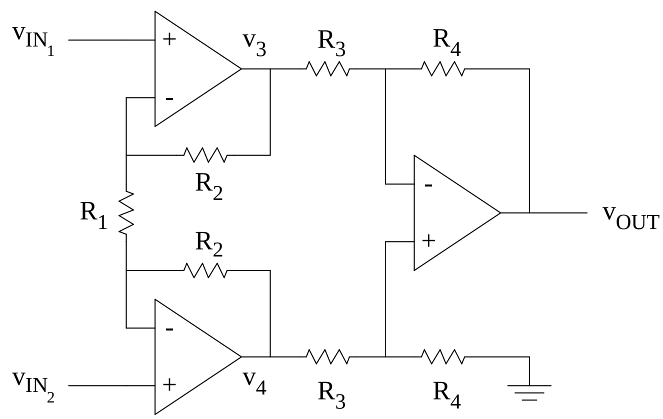
$$v_4 - v_3 = \left(v_{IN_2} + \frac{v_{IN_2} - v_{IN_1}}{R_1} R_2 \right) - \left(v_{IN_1} - \frac{v_{IN_2} - v_{IN_1}}{R_1} R_2 \right)$$

$$= \left(v_{IN_2} - v_{IN_1} \right) + \frac{2 R_2}{R_1}\left(v_{IN_2} - v_{IN_1} \right)$$

$$= \left(1 + \frac{2 R_2}{R_1} \right)\left(v_{IN_2} - v_{IN_1} \right)$$

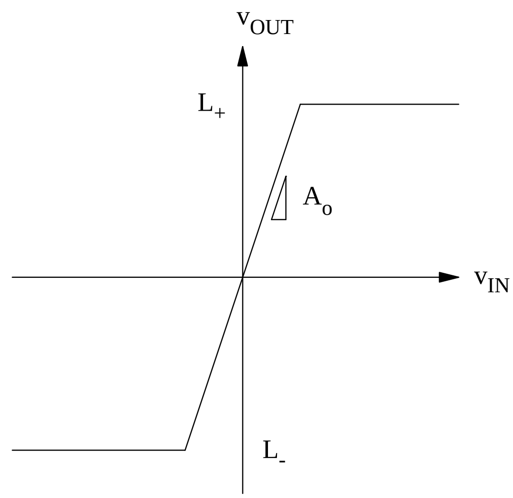
$$v_{OUT} = \frac{R_4}{R_3}\left(1 + \frac{2 R_2}{R_1} \right)\left(v_{IN_2} - v_{IN_1} \right)$$

R_1 is the gain control resistor, all others require matching.

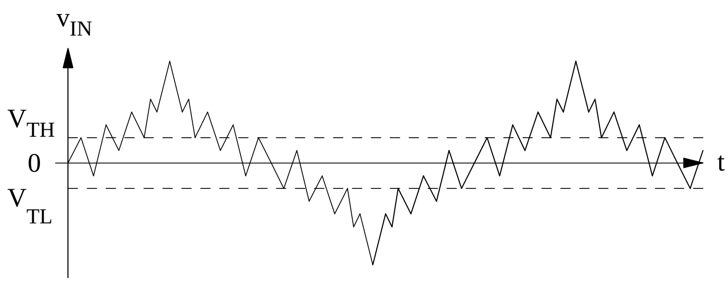


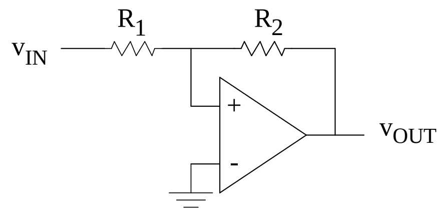
nonideal opamp parameters

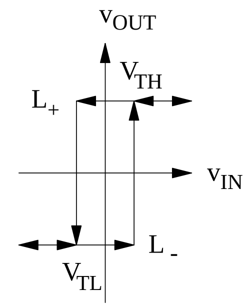
V_{OS}	input offset voltage	
I_B	input bias current	
I_{OS}	input offset current	
A_o	DC open loop voltage gain	
f_t	unity gain frequency	
L_+	maximum output voltage	
L_-	minimum output voltage	
SR	slew rate	$\max\left(\left\lvert \frac{d\,v_{OUT}}{dt} \right\rvert\right)$



zero reference Schmidt trigger







$v_+ > v_- \quad \rightarrow \quad v_{OUT} = L_+ \quad ; \quad v_+ < v_- \quad \rightarrow \quad v_{OUT} = L_- \quad ; \quad v_+ = \frac{R_2}{R_1 + R_2} v_{IN} + \frac{R_1}{R_1 + R_2} v_{OUT}$

if $v_{OUT} = L_+$ to switch state we need $v_+ = 0$

$$0 = \frac{R_2}{R_1 + R_2} v_{IN} + \frac{R_1}{R_1 + R_2} L_+$$

$$-\frac{R_2}{R_1 + R_2} v_{IN} = \frac{R_1}{R_1 + R_2} L_+$$

$$-R_2\, v_{IN} = R_1\, L_+$$

$$v_{IN} = -\left(\frac{R_1}{R_2}\right) L_+ = V_{TL}$$

if $v_{OUT} = L_-$ to switch state we need $v_+ = 0$

$$0 = \frac{R_2}{R_1 + R_2} v_{IN} + \frac{R_1}{R_1 + R_2} L_-$$

$$-\frac{R_2}{R_1 + R_2} v_{IN} = \frac{R_1}{R_1 + R_2} L_-$$

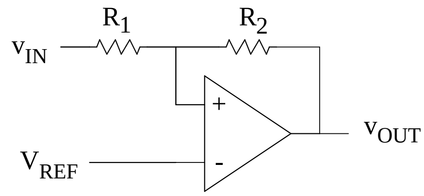
$$-R_2\, v_{IN} = R_1\, L_-$$

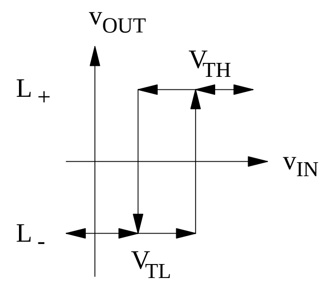
$$v_{IN} = -\left(\frac{R_1}{R_2}\right) L_- = V_{TH}$$

nonzero reference Schmidt trigger

$$v_+ = V_{REF} = \frac{R_2}{R_1 + R_2}\, v_{IN} + \frac{R_1}{R_1 + R_2}\, v_{OUT}$$

$$V_{REF} - \frac{R_1}{R_1 + R_2}\, v_{OUT} = \frac{R_2}{R_1 + R_2}\, v_{IN}$$





$$v_{IN} = \frac{V_{REF} - \dfrac{R_1}{R_1 + R_2}\, v_{OUT}}{\dfrac{R_2}{R_1 + R_2}} = \frac{R_1 + R_2}{R_2}\, V_{REF} - \frac{R_1}{R_2}\, v_{OUT}$$

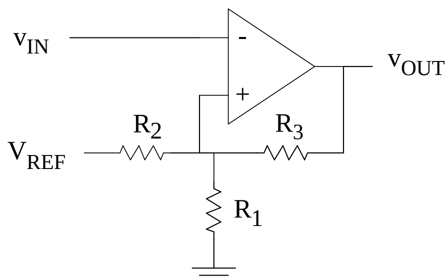
$$V_{TH} = \frac{R_1 + R_2}{R_2}\, V_{REF} - \frac{R_1}{R_2}\, L_- \quad ; \quad V_{TL} = \frac{R_1 + R_2}{R_2}\, V_{REF} - \frac{R_1}{R_2}\, L_+$$

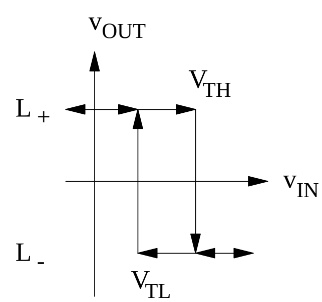
nonzero reference inverting Schmidt trigger

$$v_+ = \frac{R_1 \parallel R_3}{R_2 + (R_1 \parallel R_3)}\, V_{REF} + \frac{R_1 \parallel R_2}{R_3 + (R_1 \parallel R_2)}\, v_{OUT}$$

$$= \frac{\dfrac{R_1 R_3}{R_1 + R_3}}{R_2 + \dfrac{R_1 R_3}{R_1 + R_3}}\, V_{REF} + \frac{\dfrac{R_1 R_2}{R_1 + R_2}}{R_3 + \dfrac{R_1 R_2}{R_1 + R_2}}\, v_{OUT}$$

$$= \frac{R_1 R_3}{(R_1 + R_3) R_2 + R_1 R_3}\, V_{REF} + \frac{R_1 R_2}{(R_1 + R_2) R_3 + R_1 R_2}\, v_{OUT}$$





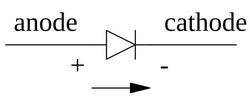
$$v_{OUT} = \begin{cases} L_+ & \text{for} \quad v_{IN} < v_+ \\ L_- & \text{for} \quad v_{IN} > v_+ \end{cases}$$

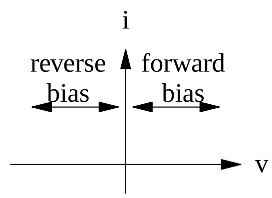
$$V_{TH} = \frac{R_1 R_3}{(R_1 + R_3) R_2 + R_1 R_3}\, V_{REF} + \frac{R_1 R_2}{(R_1 + R_2) R_3 + R_1 R_2}\, L_+ = \frac{R_1 (R_3 V_{REF} + R_2 L_+)}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

$$V_{TL} = \frac{R_1 R_3}{(R_1 + R_3) R_2 + R_1 R_3}\, V_{REF} + \frac{R_1 R_2}{(R_1 + R_2) R_3 + R_1 R_2}\, L_- = \frac{R_1 (R_3 V_{REF} + R_2 L_-)}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

Diode Circuits

ideal diode model



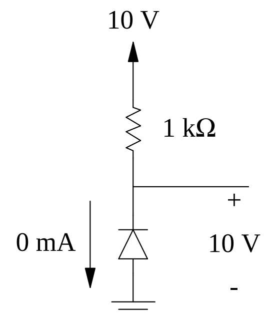


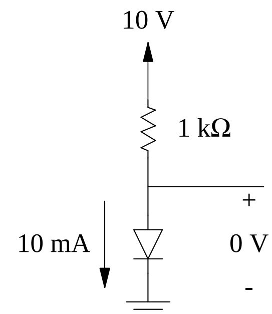
ideal diode

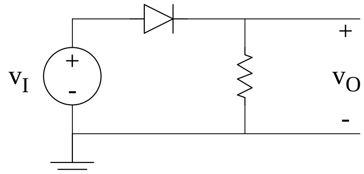
$i_D = 0$ for $v_D < 0$

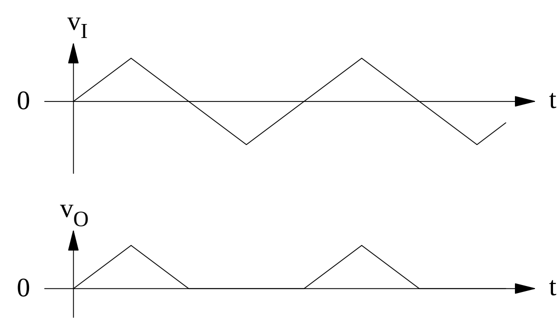
$v_D = 0$ for $i_D > 0$

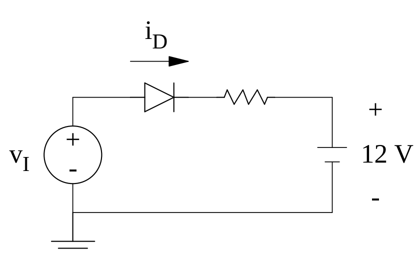
for ideal diode circuits start by assuming the diode current is zero and solve for the diode voltage, if the diode voltage is negative then the assumption was correct, otherwise set the diode voltage to zero and solve for the diode current.

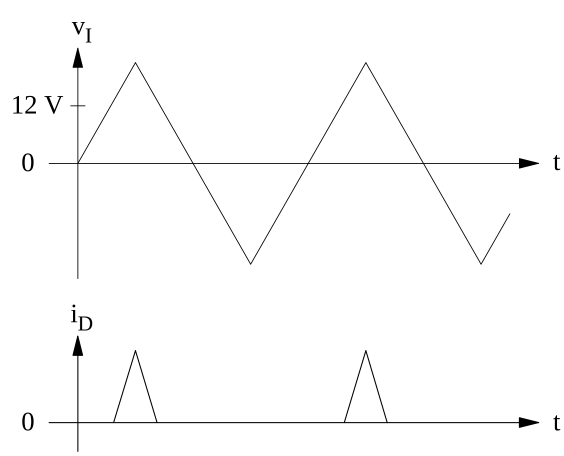




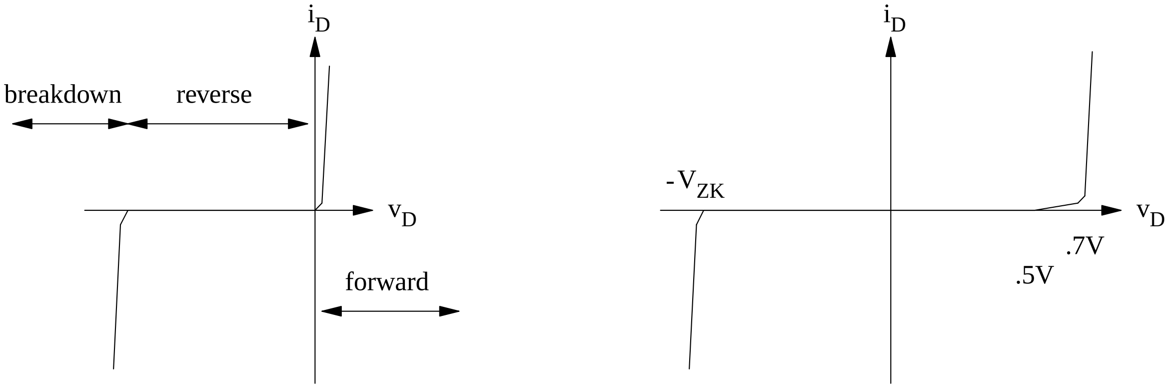








nonideal diode model



forward bias

$$i_D = I_S \left(e^{\frac{v_D}{n V_T}} - 1 \right) \quad ; \quad V_T = \frac{k\,T}{q} \quad ;$$

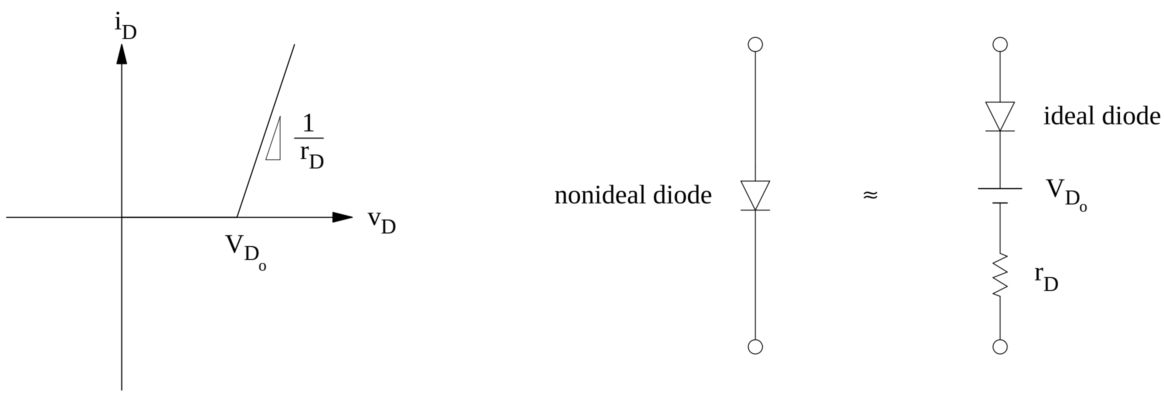
k = Boltzmann's constant = 1.38×10^{-23} joules/kelvin
T = temperature in kelvin
q = electron charge = 1.60×10^{-19} coulomb

$$i_D \approx I_S\, e^{\frac{v_D}{n V_T}} \quad \rightarrow \quad v_D = n\, V_T \ln\left(\frac{i_D}{I_S} \right)$$

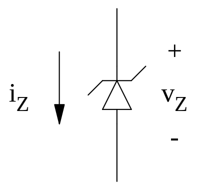
reverse bias
$i_D \approx -I_S$

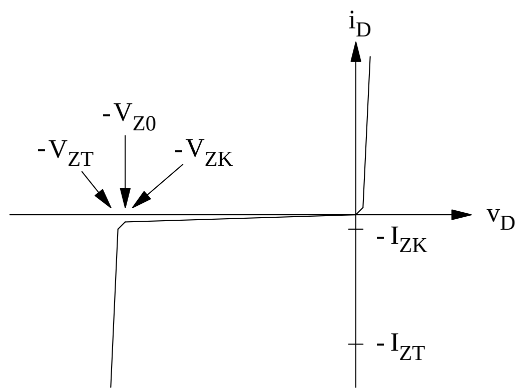
breakdown
complex function dependent on whether the diode is in zener or avalanche breakdown

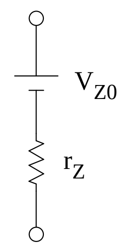
piecewise linear diode model

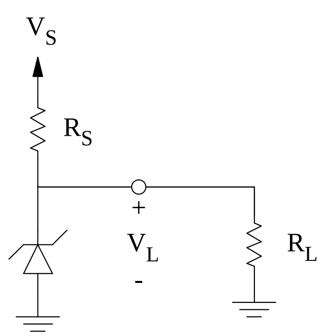


zener diode









$$V_L = V_Z$$

$$I_{R_S} = \frac{V_S - V_Z}{R_S}$$

special diodes

Schottky-Barrier Diode (SBD)
  turn on voltage lower (0.3 V to 0.5 V), switchs off faster

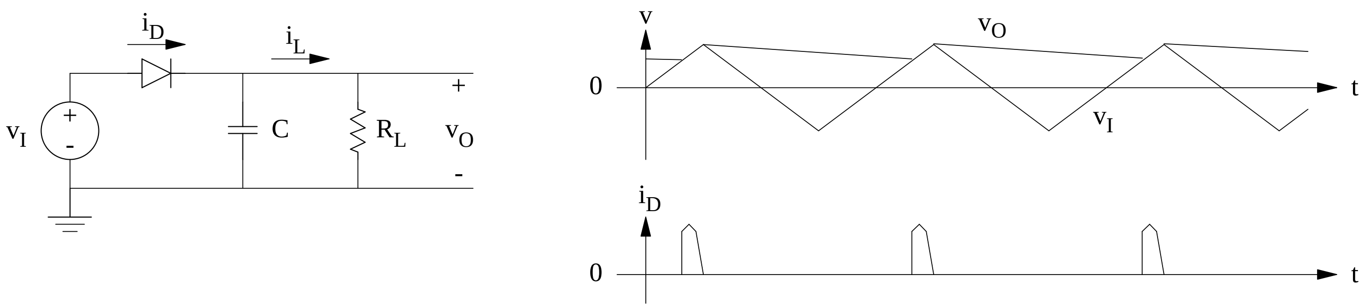
Varactors
  reverse bias diode as a voltage controlled capacitor

Photodiode
  I_S is a function of light resulting in diode current being a function of light shining on the diode junction material

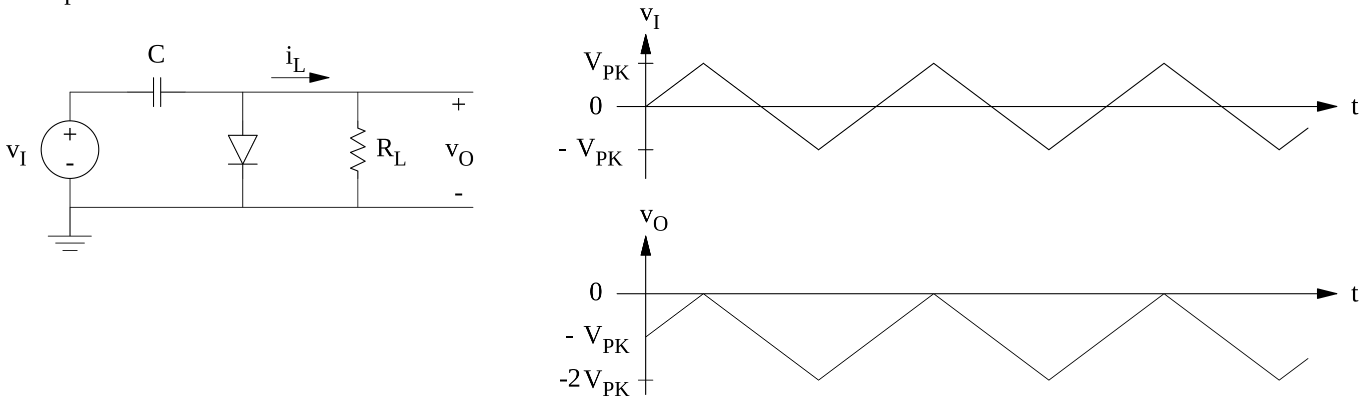
Light-Emitting Diode (LED)
  converts current into light

diode signal conditioning circuits

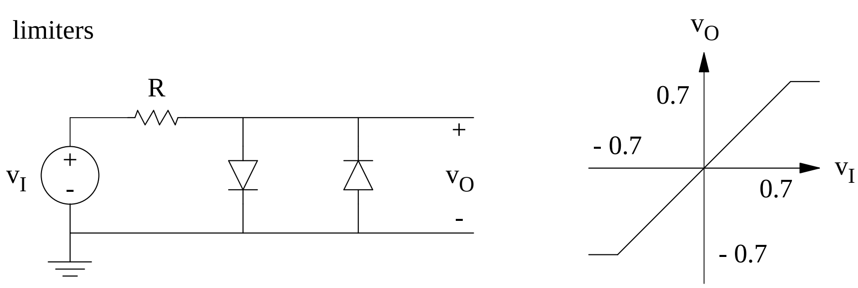
peak detector

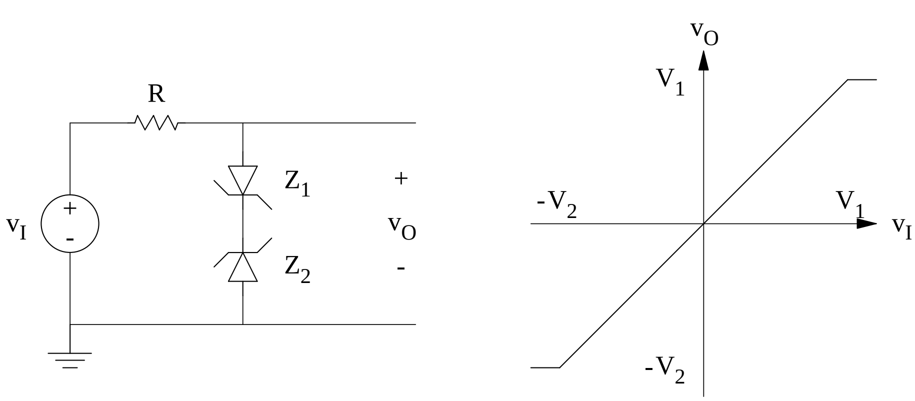


clamp



limiters





$$V_1 = 0.7 + V_{Z_2}$$

$$V_2 = 0.7 + V_{Z_1}$$

BJT

terminals

B is the base, C is the collector, E is the emitter

forward active region

$v_{BE} \geq 0.6$ and $v_{BC} < 0.4$

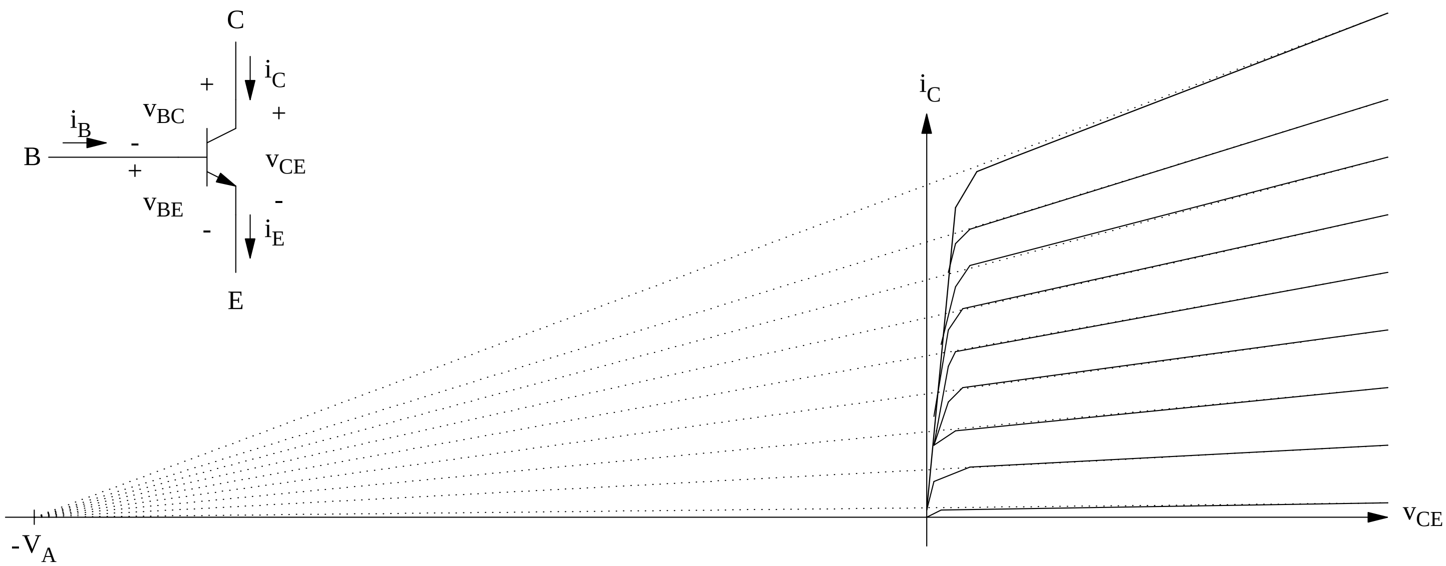
$$i_C = I_S\, e^{\frac{v_{BE}}{V_T}} \left(1 + \frac{v_{CE}}{V_A}\right) \quad ; \quad i_B = \frac{1}{\beta}\, i_C \quad ; \quad i_E = i_C + i_B$$

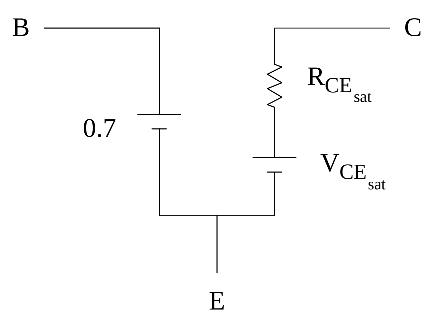
saturation region

$v_{BE} \geq 0.6$ and $v_{BC} \geq 0.4$ resulting in $v_{CE} = 0.2\,\mathrm{V}$

cut-off

$v_{BE} < 0.6$ results in $i_C = 0$

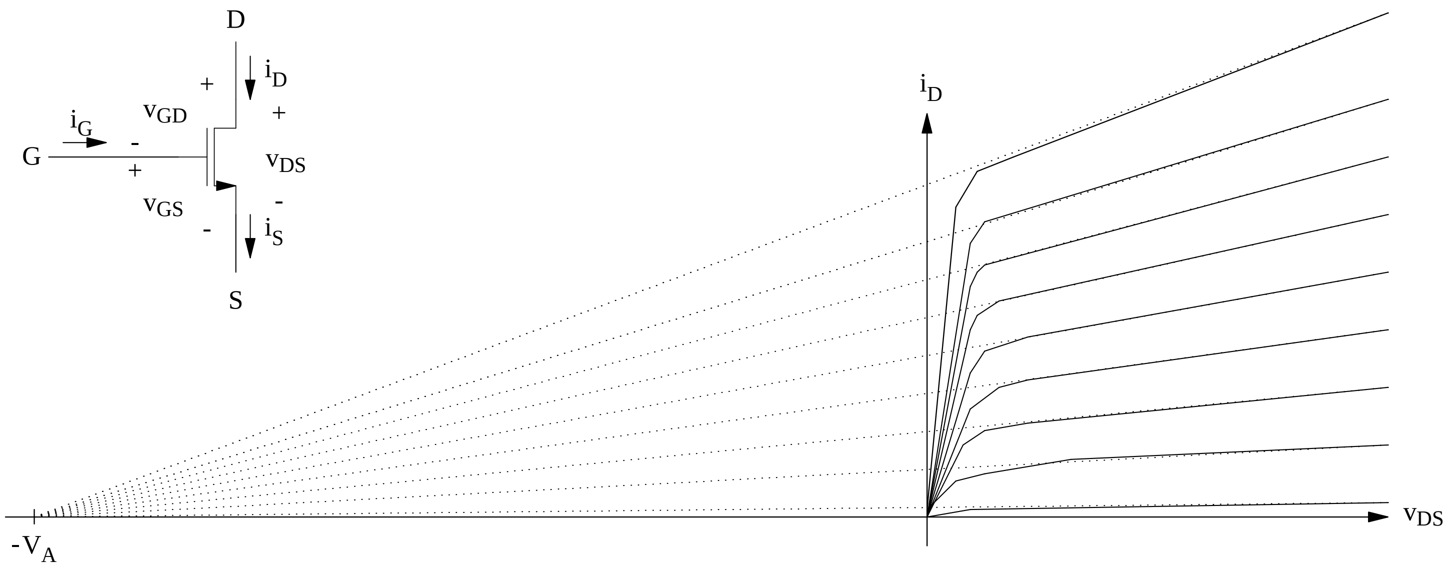




applications

voltage-to-current transducer
current-to-current transducer
photon-to-current transducer

MOSFET



cut-off

$v_{GS} \le V_t \quad \rightarrow \quad i_D = 0$

triode $v_{GS} > V_t$ and $v_{GD} > V_t \quad \rightarrow \quad i_D = k_n\left((v_{GS} - V_t)\, v_{DS} - \frac{1}{2} v_{DS}^2 \right)$

saturation

$v_{GS} > V_t$ and $v_{GD} \le V_t \quad \rightarrow \quad i_D = \frac{1}{2} k_n \,(v_{GS} - V_t)^2\, v_{DS}$

applications

voltage-to-voltage transducer
voltage-controlled-resistor

Filters

Laplace transform basics

Laplace operator

$$L(\ f(t)) = F(s)$$

$$L^{-1}(\ F(s)) = f(t)$$

$$L\left(\frac{d\,f(t)}{dt}\right) = s\,F(s) - f(0)$$

$$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}\,F(s)$$

special functions of time

$$\delta(t) = \text{dirac impulse}\ ;\ L(\ \delta(t)) = 1$$

$$u(t) = \text{unit step}\ ;\ L(\ u(t)) = \frac{1}{s}$$

$$t = \text{ramp}\ ;\ L(\ t) = \frac{1}{s^2}$$

$$\text{sine}\ ;\ L(\ \sin(\omega\,t)) = \frac{\omega}{s^2 + \omega^2}$$

$$\text{cosine}\ ;\ L(\ \cos(\omega\,t)) = \frac{s}{s^2 + \omega^2}$$

$$\text{exponential}\ ;\ L\left(e^{-a\,t}\right) = \frac{1}{s + a}$$

$$L\left(e^{-a\,t}\sin(\omega\,t)\right) = \frac{\omega}{(s+a)^2 + \omega^2}$$

$$L\left(e^{-a\,t}\cos(\omega\,t)\right) = \frac{s+a}{(s+a)^2 + \omega^2}$$

function scaling and shifting

$$\text{time scaling}\ ;\ L(\ f(k\,t)) = \frac{1}{k}\,F(\frac{s}{k})$$

$$\text{value scaling}\ ;\ L(\ k\,f(t)) = k\,F(s)$$

$$\text{time shifting}\ ;\ L(\ f(t - T)) = e^{-s\,T}\,F(s)$$

pulse: sum of time-shifted unit steps

pulse is

$$f(t) = u(t) - u(t - T)$$

which transforms to

$$L(\ u(t) - u(t - T)) = \frac{1}{s} - \frac{1}{s}\,e^{-s\,T}$$

partial fractions

$$F(s) = K_o\,\frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

$$= \sum_{i=1}^{N}\frac{K_i}{s - p_i} \qquad \text{for } N > M$$

$$= K_o + \sum_{i=1}^{N}\frac{K_i}{s - p_i} \qquad \text{for } N = M$$

$$L^{-1}(\ F(s)) = \left(\sum_{i=1}^{N} K_i\,e^{-p_i\,t}\right)u(t) \qquad \text{for } N > M$$

$$= \left(K_o\,\delta(t) + \sum_{i=1}^{N} K_i\,e^{-p_i\,t}\right)u(t) \qquad \text{for } N = M$$

partial fraction using residue method

$$K_i = \left((s - p_i) F(s) \right) \Big|_{s = p_i}$$

example

$$F(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)(s+5)} = \frac{K_1}{s+1} + \frac{K_2}{s+3} + \frac{K_3}{s+5}$$

$$K_1 = ((s+1) F(s)) \Big|_{s=-1} = \frac{(-1+2)(-1+4)}{(-1+3)(-1+5)} = \frac{(1)(3)}{(2)(4)} = \frac{3}{8}$$

$$K_2 = ((s+3) F(s)) \Big|_{s=-3} = \frac{(-3+2)(-3+4)}{(-3+1)(-3+5)} = \frac{(-1)(1)}{(-2)(2)} = \frac{1}{4}$$

$$K_3 = ((s+5) F(s)) \Big|_{s=-5} = \frac{(-5+2)(-5+4)}{(-5+1)(-5+3)} = \frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}$$

resulting in

$$L^{-1}(F(s)) = f(t) = \left(\frac{3}{8} e^{-t} + \frac{1}{4} e^{-3t} + \frac{3}{8} e^{-5t} \right) u(t)$$

frequency domain

generalised frequency:

$$s = \sigma + j\omega \quad ; \quad e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} \cdot e^{j\omega t} = e^{\sigma t} \{ \cos(\omega t) + j \sin(\omega t) \}$$

physical frequency

$$s = j\omega$$

impedances and admittances

the differential equations for L and C transform to

$$i(t) = C\,\frac{d\,v(t)}{dt} \quad \rightarrow \quad I(s) = s\,C\,V(s)$$

or just

$$i = C\,\frac{d\,v}{dt} \quad \rightarrow \quad I = s\,C\,V$$

$$v = L\,\frac{d\,i}{dt} \quad \rightarrow \quad V = s\,L\,I$$

the integral equations for L and C transform to

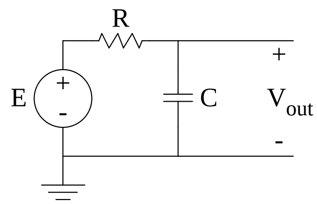
$$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau \quad \rightarrow \quad V = \frac{1}{s\,C}\,I$$

$$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau \quad \rightarrow \quad I = \frac{1}{s\,L}\,V$$

element	admittance Y	impedance Z
resistor	G	R
capacitor	$s\,C$	$\frac{1}{s\,C}$
inductor	$\frac{1}{s\,L}$	$s\,L$

RC networks

Given a series connected R-C network



a KVL gives us $\left(R + \frac{1}{sC}\right) I(s) = E \quad \rightarrow \quad I(s) = \frac{sC}{sCR+1} E$

$$V_{out}(s) = Z_C(s)\, I(s) = \frac{1}{sCR+1} E = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} E = \frac{1}{s\tau + 1} E$$

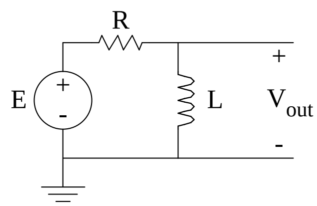
substituting $E = 1$ and $s = j\omega$ gives $V_{out}(j\omega) = \frac{1}{j\omega\tau + 1}$

the absolute value (or magnitude) is $\left| V_{out} \right| = \frac{1}{\left(\omega^2 \tau^2 + 1\right)^{\frac{1}{2}}}$

the phase angle (or phase) is $\phi = -\arctan(\omega\tau)$

RL networks

Given a series connected R-L network



a KVL gives us $(R + sL)\, I(s) = E \quad \rightarrow \quad I(s) = \frac{1}{sL+R} E$

$$V_{out}(s) = Z_L\, I(s) = \frac{sL}{sL+R} E = \frac{s}{s + \frac{R}{L}} E = \frac{s\tau}{s\tau + 1} E$$

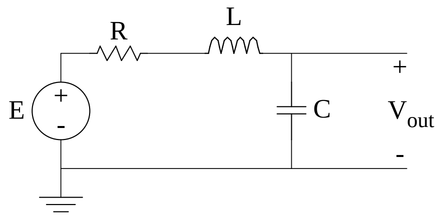
substituting $E = 1$ and $s = j\omega$ gives $V_{out}(j\omega) = \frac{j\omega\tau}{j\omega\tau + 1}$

the absolute value (or magnitude) is $\left| V_{out} \right| = \frac{\left(\omega^2 \tau^2\right)^{\frac{1}{2}}}{\left(\omega^2 \tau^2 + 1\right)^{\frac{1}{2}}}$

the phase angle (or phase) is $\phi = \frac{\pi}{2} - \arctan(\omega\tau)$

RLC networks

Given a series connected RLC network



a KVL gives us $\left(R + s\,L + \frac{1}{s\,C}\right) I(s) = E$

$$I(s) = \left(\frac{1}{R + s\,L + \frac{1}{s\,C}}\right) E = \frac{s\,C}{s^2\,L\,C + s\,C\,R + 1}\,E$$

$$= \frac{s\,\frac{1}{L}}{s^2 + s\,\frac{R}{L} + \frac{1}{L\,C}}\,E$$

from which we get

$$V_{out}(s) = Z_C\,I(s) = \frac{1}{s^2\,L\,C + s\,C\,R + 1}\,E = \frac{\frac{1}{L\,C}}{s^2 + s\,\frac{R}{L} + \frac{1}{L\,C}}\,E = \frac{\omega_o^2}{s^2 + s\,\frac{\omega_o}{Q_o} + \omega_o^2}\,E$$

from which we see that $\omega_o = \frac{1}{(L\,C)^{\frac{1}{2}}}$ and $\frac{\omega_o}{Q_o} = \frac{R}{L}$ so $Q_o = \frac{\omega_o}{\left(\frac{R}{L}\right)} = \omega_o\,\frac{L}{R} = \frac{1}{(L\,C)^{\frac{1}{2}}}\,\frac{L}{R} = \frac{1}{R}\left(\frac{L}{C}\right)^{\frac{1}{2}}$

at the resonant frequency, ω_o, we have $V_l(\omega_o) + V_c(\omega_o) = 0$

as seen by $s\,L\,I(s) = -\frac{1}{s\,C}\,I(s) \;\rightarrow\; s^2 = -\frac{1}{L\,C} \;\rightarrow\; (j\,\omega_o)^2 = -\omega_o^2 = -\frac{1}{L\,C} \;\rightarrow\; \omega_o = \frac{1}{(L\,C)^{\frac{1}{2}}}$

so $V_r(\omega_o) = E$ and $I(\omega_o) = I_r(\omega_o) = \frac{E}{R}$

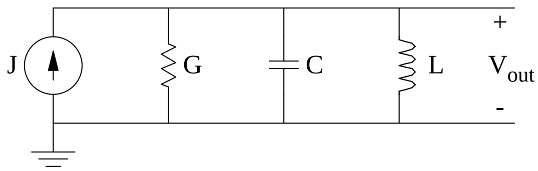
Q is the quality factor of an energy storage circuit. $Q = 2\,\pi\,\frac{\text{(maximum energy stored per period)}}{\text{(energy dissipated per period)}}$

Q_o is the tuned circuit quality factor at the resonant frequency.

At resonance in a series tuned circuit $V_c(s) = \frac{I(s)}{s\,C}$ substituting $s = j\,\omega_o$

$$|\;V_c\;| = \frac{\left(\frac{E}{R}\right)}{\omega_o\,C} = \frac{\left(\frac{1}{R}\right)}{\left(\frac{1}{(L\,C)^{\frac{1}{2}}}\,C\right)}\,E = \frac{\left(\frac{1}{R}\right)}{\left(\frac{C}{L}\right)^{\frac{1}{2}}}\,E = \frac{1}{R}\left(\frac{L}{C}\right)^{\frac{1}{2}}\,E = Q_o\,E$$

Given a parallel connected RLC network



a KCL gives us $\left(G + sC + \dfrac{1}{sL} \right) V_{out}(s) = J$

$$V_{out}(s) = \frac{1}{G + sC + \dfrac{1}{sL}} J = \frac{sL}{s^2 LC + sLG + 1} J$$

$$= \frac{s\dfrac{1}{C}}{s^2 + s\dfrac{1}{RC} + \dfrac{1}{LC}} J = \left(\frac{1}{C}\right)\left(\frac{s}{s^2 + s\dfrac{\omega_o}{Q_o} + \omega_o^2}\right) J$$

from which we see again that $\omega_o = \dfrac{1}{(LC)^{\frac{1}{2}}}$ and $\dfrac{\omega_o}{Q_o} = \dfrac{G}{C}$ so

$$Q_o = \frac{\omega_o}{\left(\dfrac{G}{C}\right)} = \omega_o \frac{C}{G} = \frac{1}{(LC)^{\frac{1}{2}}} \frac{C}{G} = \frac{1}{G}\left(\frac{C}{L}\right)^{\frac{1}{2}}$$

at the resonant frequency, ω_o, we have $I_l(\omega_o) + I_c(\omega_o) = 0$

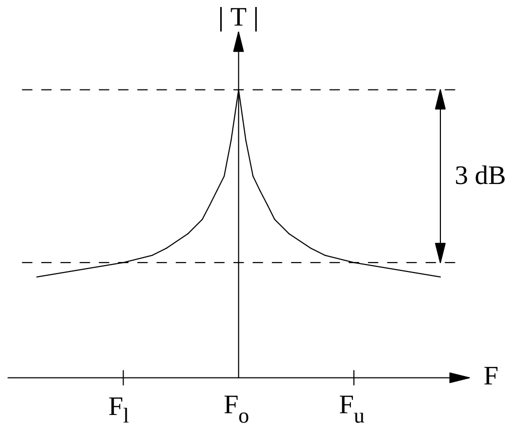
as seen by $sC\,V_{out}(s) = -\dfrac{1}{sL} V_{out}$ $\quad s^2 = -\dfrac{1}{LC} \quad \rightarrow \quad (j\omega_o)^2 = -\omega_o^2 = -\dfrac{1}{LC} \quad \rightarrow \quad \omega_o = \dfrac{1}{(LC)^{\frac{1}{2}}}$

so $I_r(\omega_o) = J$ and $V_{out}(\omega_o) = V_r(\omega_o) = \dfrac{J}{G}$

at resonance in a parallel tuned circuit $I_l = \dfrac{V_{out}}{sL}$ substituting $s = j\omega_o$

$$|\ I_l\ | = \frac{\left(\dfrac{J}{G}\right)}{\omega_o L} = \frac{\left(\dfrac{1}{G}\right)}{\left(\dfrac{1}{(LC)^{\frac{1}{2}}} L\right)} J = \frac{\left(\dfrac{1}{G}\right)}{\left(\dfrac{L}{C}\right)^{\frac{1}{2}}} J = \frac{1}{G}\left(\frac{C}{L}\right)^{\frac{1}{2}} J = Q_o J$$

returning to $V_{out}(s) = \left(\dfrac{1}{C}\right)\left(\dfrac{s}{s^2 + s\dfrac{\omega_o}{Q_o} + \omega_o^2}\right) J$ we see that we have a bandpass transfer function



recalling that

$$20\log(\sqrt{2}) = 3\,\mathrm{dB}$$

we see that

$$\frac{|\ V_{out}(j\,\omega_o)|}{|\ V_{out}(j\,\omega_l)|} = \frac{|\ V_{out}(j\,\omega_o)|}{|\ V_{out}(j\,\omega_u)|} = \sqrt{2}$$

Note that

$$V_{out}(j\,\omega_o) = \left(\frac{1}{C}\right)\left(\frac{j\,\omega_o}{(j\,\omega_o)^2 + j\,\omega_o\,\dfrac{\omega_o}{Q_o} + \omega_o^2}\right)J = \left(\frac{1}{C}\right)\left(\frac{j\,\omega_o}{-\omega_o^2 + j\,\omega_o\,\dfrac{\omega_o}{Q_o} + \omega_o^2}\right)J = \left(\frac{1}{C}\right)\left(\frac{Q_o}{\omega_o}\right)J$$

After a number of algebraic operations, we get

$$\omega_l = -\frac{\omega_o}{2\,Q_o} + \omega_o\left(\frac{1}{4\,Q_o^2} + 1\right)^{\frac{1}{2}} \quad \text{and} \quad \omega_u = \frac{\omega_o}{2\,Q_o} + \omega_o\left(\frac{1}{4\,Q_o^2} + 1\right)^{\frac{1}{2}}$$

$$3\,\mathrm{dB}\ \ \text{bandwidth} = \Delta\,\omega = \omega_u - \omega_l = \frac{\omega_o}{Q_o} \quad \text{and} \quad Q_o = \frac{\omega_o}{\Delta\,\omega} = \frac{f_o}{\Delta\,f}$$

network functions

(1) network must be linear and all Cs & Ls have zero initial conditions.
(2) connect source (E or J) to the input
(3) solve for desired output & divide by source
(4) avoid 1/s terms in network functions
(5) for frequency domain response: use phasor for E or J and set $s = j\,\omega$
(6) for time domain response: use Laplace transform of E or J and inverse Laplace transform network function

network function $= \dfrac{\text{output}}{\text{input}}$

voltage transfer function: $T_v = \dfrac{V_{out}}{E}$

current transfer function: $T_i = \dfrac{I_{out}}{J}$

transfer impedance: $Z_{tr} = \dfrac{V_{out}}{J}$

transfer admittance: $Y_{tr} = \dfrac{I_{out}}{E}$

input impedance: $Z_{in} = \dfrac{V_{in}}{J}$

input admittance: $Y_{in} = \dfrac{I_{in}}{E}$

poles and zeros

$$F(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{m} A_i s^i}{\sum_{i=0}^{n} B_i s^i} = K_0 \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)} \quad \text{where} \quad K_0 = \frac{A_m}{B_n} \quad \text{for practical circuits} \quad z_i = \alpha_i + j\,\beta_i \quad \text{and} \quad p_i = \gamma_i + j\,\zeta_i$$

stability

$$F(s) = \sum_{i=0}^{n} \frac{K_i}{s - p_i} \quad \rightarrow \quad f(t) = \sum_{i=0}^{n} K_i\, e^{t\,p_i}$$

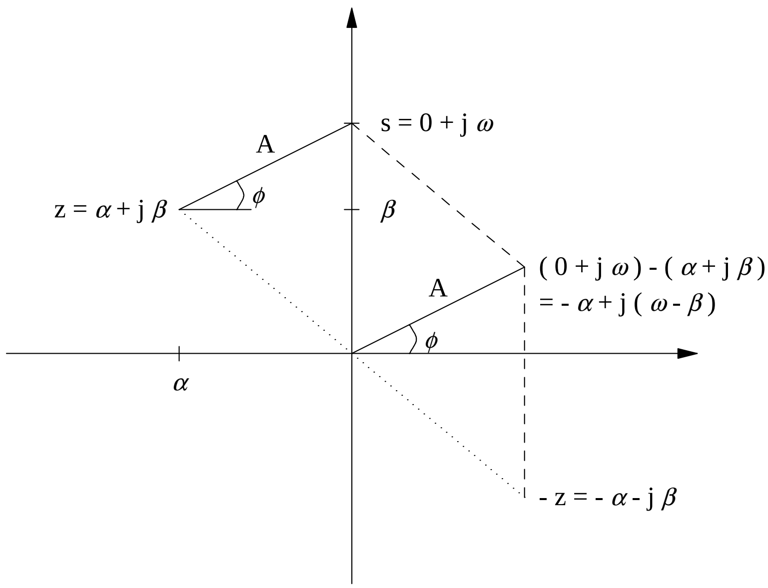
if $Re(p_i) > 0$ then f(t) unstable. ; if $Re(p_i) = 0$ then f(t) is an oscillator. ; if $Re(p_i) < 0$ then f(t) stable.

frequency response from poles & zeros

recall that $$F(s) = K_0 \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

note that s, z_i, and p_i are complex numbers which can be viewed as vectors. $(s - z) = (0 + j\,\omega) - (\alpha + j\,\beta) = A\,e^{j\,\phi}$

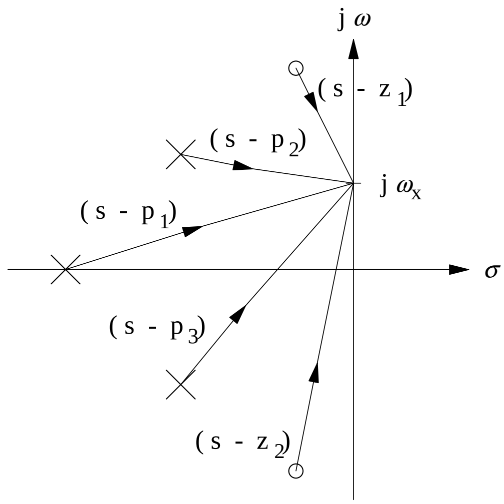
$$\text{where} \quad A = \left((\omega - \beta)^2 + \alpha^2 \right)^{\frac{1}{2}} \qquad \phi = \arctan\left(\frac{\omega - \beta}{-\alpha} \right)$$



$$|F(j\omega)| = |K_0| \frac{\prod_{i=1}^{m} A_{z,i}}{\prod_{i=1}^{n} A_{p,i}} = \left(K_0^2 \frac{\prod_{i=1}^{m}\left((\omega - \beta_i)^2 + \alpha_i^2 \right)}{\prod_{i=1}^{n}\left((\omega - \zeta_i)^2 + \gamma_i^2 \right)} \right)^{\frac{1}{2}}$$

$$20\log(|F(j\omega)|) = 20\log(|K_0|) + \sum_{i=1}^{m} 20\log(|j\omega - z_i|) - \sum_{i=1}^{n} 20\log(|j\omega - p_i|)$$

$$\begin{aligned}
\phi(j\omega) &= \phi(K_0) + \sum_{i=1}^{m} \phi_{z,i} - \sum_{i=1}^{n} \phi_{p,i} \\
&= \phi(K_0) + \sum_{i=1}^{m} \arctan\left(\frac{\omega - \beta_i}{-\alpha_i} \right) - \sum_{i=1}^{n} \arctan\left(\frac{\omega - \zeta_i}{-\gamma_i} \right) \\
&= \phi(K_0) + \sum_{i=1}^{m} \arctan\left(\frac{\beta_i - \omega}{\alpha_i} \right) - \sum_{i=1}^{n} \arctan\left(\frac{\zeta_i - \omega}{\gamma_i} \right)
\end{aligned}$$



s - plane

$$F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)(s - p_3)} \quad ; \quad z_2 = \overline{z_1} \quad \text{and} \quad p_3 = \overline{p_2}$$

Bode plots

(1) the dB magnitude plot is the summation of the dB magnitude plots of the individual K_0, $(s - z_i)$ and $\frac{1}{s - p_i}$ terms.

(2) the phase plot is the summation of the phase plots of the individual K_0, $(s - z_i)$ and $\frac{1}{s - p_i}$ terms.

these terms are one of:

(1) $T(s) = K_0 \quad ; \quad T(j\omega) = K_0$

(2) $T(s) = \tau s = \dfrac{s}{\omega_o} \quad ; \quad T(j\omega) = \dfrac{j\omega}{\omega_o}$

(3) $T(s) = \dfrac{1}{\tau s} = \dfrac{\omega_o}{s} \quad ; \quad T(j\omega) = \dfrac{\omega_o}{j\omega} = -j\dfrac{\omega_o}{\omega}$

(4) $T(s) = (\tau s + 1) = \left(\dfrac{s}{\omega_o} + 1\right) \quad ; \quad T(j\omega) = \left(j\dfrac{\omega}{\omega_o} + 1\right)$

(5) $T(s) = \dfrac{1}{\tau s + 1} = \dfrac{1}{\dfrac{s}{\omega_o} + 1} \quad ; \quad T(j\omega) = \dfrac{1}{j\dfrac{\omega}{\omega_o} + 1}$

(6) $T(s) = \dfrac{1}{\omega_o^2}\left(s^2 + \dfrac{\omega_o}{Q_o}s + \omega_o^2\right) = \left(\dfrac{s^2}{\omega_o^2} + \dfrac{s}{\omega_o Q_o} + 1\right) \quad ; \quad T(j\omega) = \left(\dfrac{-\omega^2}{\omega_o^2} + \dfrac{j\omega}{\omega_o Q_o} + 1\right)$

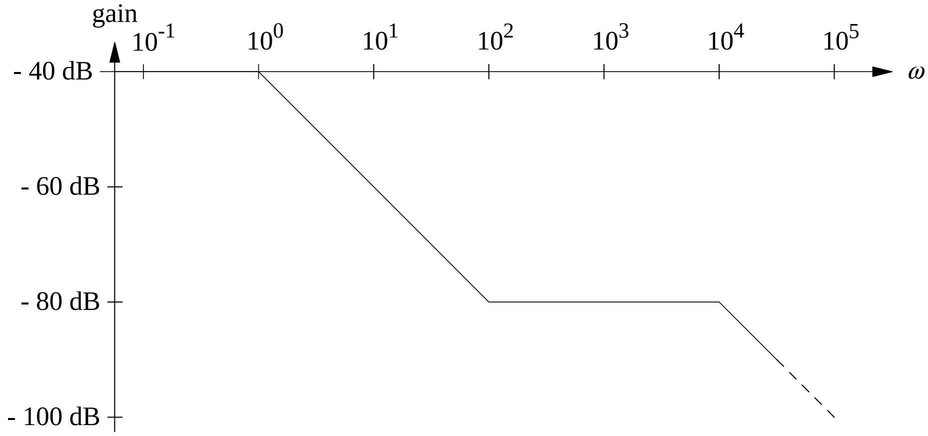
(7) $T(s) = \omega_o^2\left(\dfrac{1}{s^2 + \dfrac{\omega_o}{Q_o}s + \omega_o^2}\right) = \dfrac{1}{\left(\dfrac{s^2}{\omega_o^2} + \dfrac{s}{\omega_o Q_o} + 1\right)} \quad ; \quad T(j\omega) = \dfrac{1}{\left(\dfrac{-\omega^2}{\omega_o^2} + \dfrac{j\omega}{\omega_o Q_o} + 1\right)}$

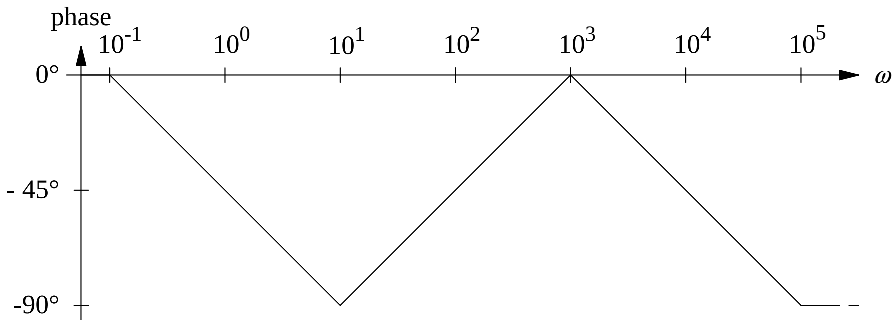
example 1

$$T(s) = \frac{(s+10^2)}{(s+1)(s+10^4)} = \frac{10^2}{10^4}\,\frac{\left(\frac{s}{10^2}+1\right)}{(s+1)\left(\frac{s}{10^4}+1\right)}$$

$$|T(j0)| = \frac{10^2}{10^4} = 10^{-2} \;\rightarrow\; -40\,\text{dB} \quad ; \quad \phi(T(j0)) = 0°$$

$$|T(j\infty)| = 0 \;\rightarrow\; -\infty\,\text{dB} \quad ; \quad \phi(T(j\infty)) = -90°$$





example 2

$$T(s) = \frac{10^5\,s}{(s+10^2)(s+10^5)} = \frac{\frac{s}{10^2}}{\left(\frac{s}{10^2}+1\right)\left(\frac{s}{10^5}+1\right)}$$

$$|T(j0)| = 0 \;\rightarrow\; -\infty\,\text{dB} \quad ; \quad \phi(T(j0)) = +90°$$

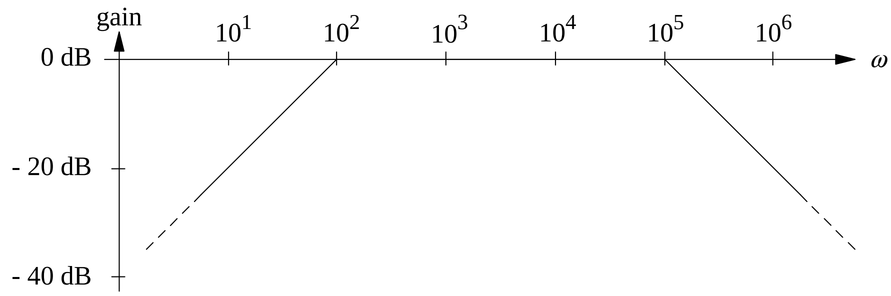
$$|T(j\infty)| = 0 \;\rightarrow\; -\infty\,\text{dB} \quad ; \quad \phi(T(j\infty)) = -90°$$

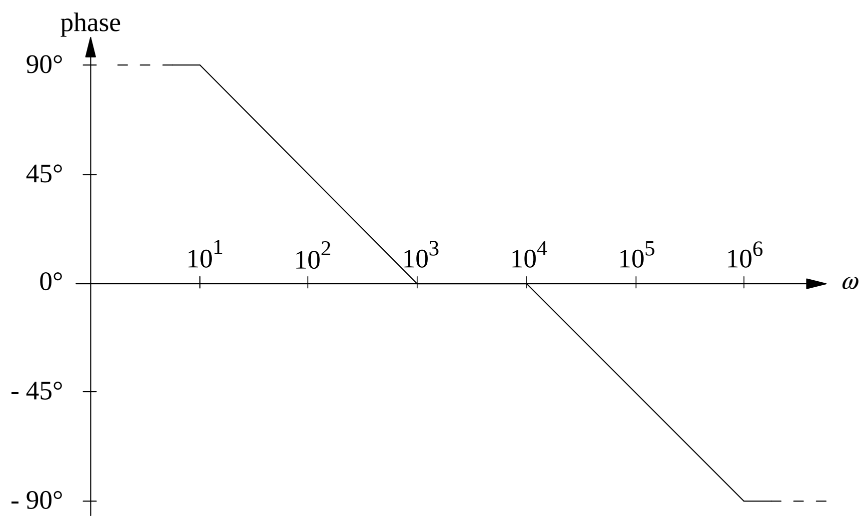
pick a point 3 decades before the pass band

$|T(j10^{-1})| = |j10^{-3}| \;\rightarrow\; -60\,\text{dB}$

gain rising at 20 dB / decade, therefore the gain at the first corner is

$-60\,\text{dB} + (3\,\text{decades} \times 20\,\text{dB/decade}) = 0\,\text{dB}$





As an aside

$$T(s) = \frac{10^5\,s}{(s+10^2)(s+10^5)} = \frac{10^5\,s}{s^2 + (10^5+10^2)\,s + (10^5\,10^2)} \quad\rightarrow\quad \omega_o^2 = 10^5\,10^2 \quad\rightarrow\quad \omega_o = 3162\,\text{Rad/s}$$

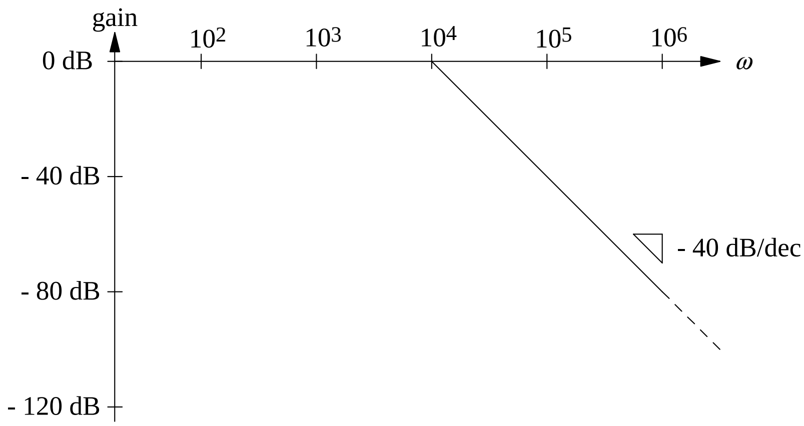
$$\frac{\omega_o}{Q_0} = 10^5 + 10^2 \quad\rightarrow\quad Q_0 = \frac{\omega_o}{10^5+10^2} = \frac{(10^7)^{\frac{1}{2}}}{10^5+10^2} = 0.03159 \quad ; \quad Q_0 = \frac{f_o}{BW} = \frac{\sqrt{f_{lp}\,f_{hp}}}{f_{lp}-f_{hp}}$$

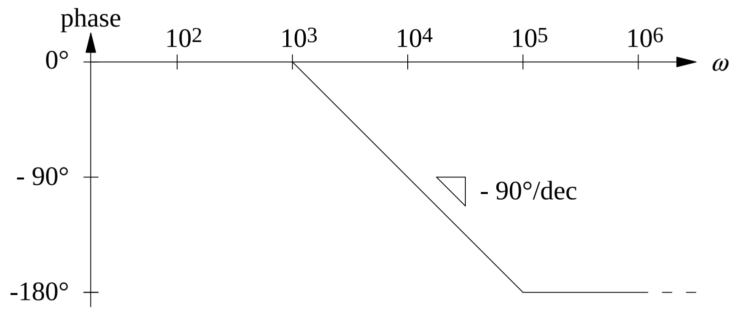
example 3

$$T(s) = \frac{(10^4)^2}{(s+10^4)^2} = \frac{1}{\left(\frac{s}{10^4}+1\right)^2}$$

$$|T(j0)| = 1 \;\rightarrow\; 0\,\text{dB} \quad ; \quad \phi(T(j0)) = 0°$$

$$|T(j\infty)| = 0 \;\rightarrow\; -\infty\,\text{dB} \quad ; \quad \phi(T(j\infty)) = -180°$$





As an aside $\omega_o = 10^4$ and $Q_o = 0.5$

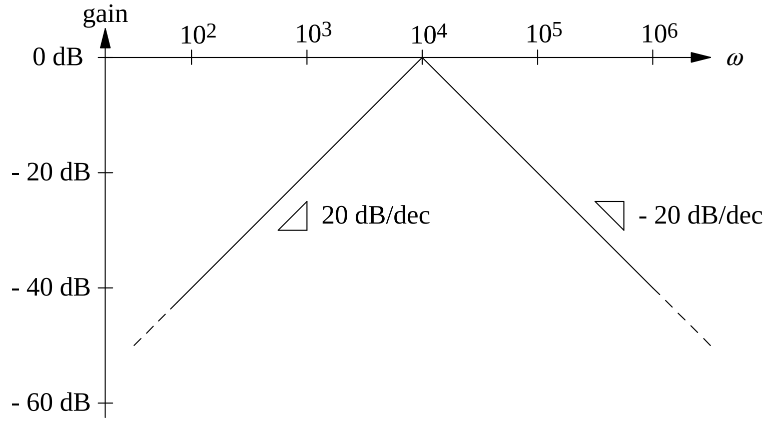
example 4

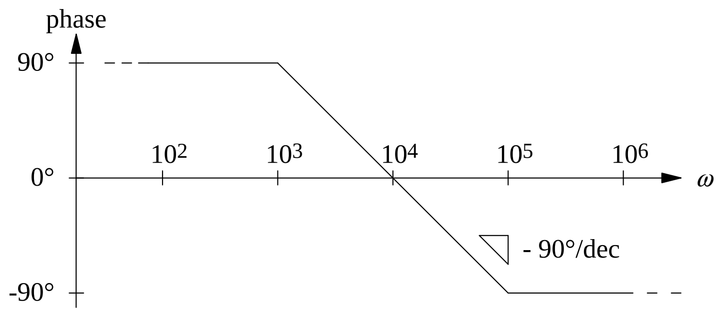
$$T(s) = \frac{10^4\, s}{(s + 10^4)^2} = \frac{\dfrac{s}{10^4}}{\left(\dfrac{s}{10^4} + 1\right)^2}$$

$$|T(j0)| = 0 \quad \rightarrow \quad -\infty\,\mathrm{dB} \quad ; \quad \phi(T(j0)) = +90°$$

$$|T(j\infty)| = 0 \quad \rightarrow \quad -\infty\,\mathrm{dB} \quad ; \quad \phi(T(j\infty)) = -90°$$

$|T(j10)| = |j10^{-3}| \quad \rightarrow \quad -60\,\mathrm{dB}$ the gain at the first corner is $-60\,\mathrm{dB} + (3\,\mathrm{decades} \times 20\,\mathrm{dB/decade}) = 0\,\mathrm{dB}$





As an aside $\omega_o = 10^4$ and $Q_o = 0.5$

Bode plot for quadratic terms with $Q_o > .5$

dB plot

start with the $Q_o = 0.5$ case, sketch resonant peak on top, resonant peak at $\omega_o \left(\frac{1}{4Q_o^2} + 1 \right)^{\frac{1}{2}}$,

resonant peak is $20\log(Q_o)$ higher/lower than $Q_o = 0.5$ case, 3 dB bandwidth of peak is $\frac{\omega_o}{Q_o}$

phase plot

starting/ending phase same as $Q_o = 0.5$ case, starting phase $\pm$ 45° at lower 3 dB frequency of resonant peak, starting phase $\pm$ 90° at resonant peak, starting phase $\pm$ 135° at upper 3 dB frequency of resonant peak

upper/lower 3 dB frequencies

lower 3 dB freq is $\omega_l = -\frac{\omega_o}{2Q_o} + \omega_o \left(\frac{1}{4Q_o^2} + 1 \right)^{\frac{1}{2}}$

upper 3 dB freq is $\omega_u = \frac{\omega_o}{2Q_o} + \omega_o \left(\frac{1}{4Q_o^2} + 1 \right)^{\frac{1}{2}}$

example 5

$$T(s) = \frac{(10^4)^2}{s^2 + 10^3 s + (10^4)^2} = \frac{1}{\frac{s^2}{(10^4)^2} + \frac{10^3}{(10^4)^2} s + 1} = \frac{1}{10^{-8} s^2 + 10^{-5} s + 1}$$

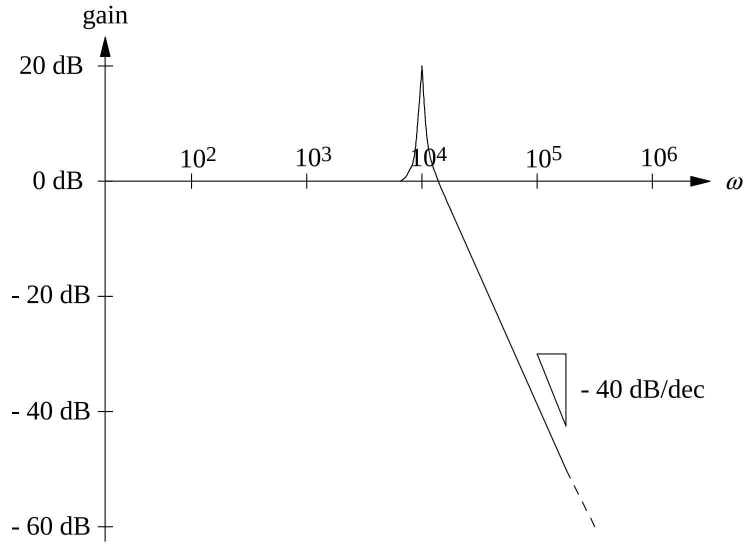
$$|T(j0)| = 1 \;\rightarrow\; 0\,\text{dB} \quad ; \quad \phi(T(j0)) = 0°$$

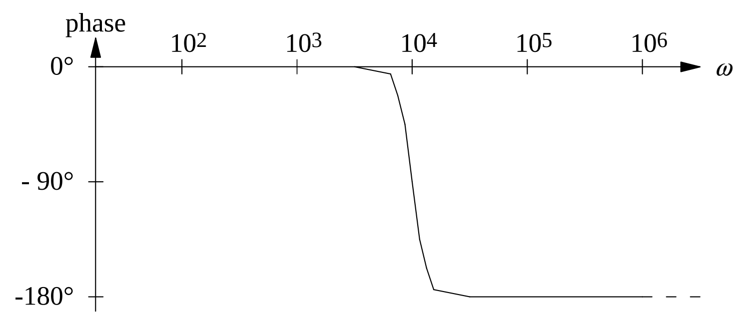
$$|T(j\infty)| = 0 \;\rightarrow\; -\infty\,\text{dB} \quad ; \quad \phi(T(j\infty)) = -180°$$

$$(\omega_o)^2 = (10^4)^2 \;\rightarrow\; \omega_o = 10^4 \text{ Rad/s} \quad ; \quad \frac{\omega_o}{Q_o} = 10^3 \;\rightarrow\; Q_o = \frac{\omega_o}{10^3} = \frac{10^4}{10^3} = 10$$

from example 3 we know that the peak for the $Q_o = 0.5$ case is 0 dB
resonant peak is $0\,\text{dB} + 20\log(Q_o) = 0\,\text{dB} + 20\log(10) = 20\,\text{dB}$

3 dB bandwidth is $\dfrac{\omega_o}{Q_o} = \dfrac{10^4}{10} = 10^3$ Rad/s





example 6

$$T(s) = \frac{10^4 s}{s^2 + 10^3 s + (10^4)^2} = \frac{\frac{s}{10^4}}{\frac{s^2}{(10^4)^2} + \frac{10^3}{(10^4)^2} s + 1} = \frac{10^{-4} s}{10^{-8} s^2 + 10^{-5} s + 1}$$

$$|T(j0)| = 0 \rightarrow -\infty \, dB \quad ; \quad \phi(T(j0)) = +90°$$

$$|T(j\infty)| = 0 \rightarrow -\infty \, dB \quad ; \quad \phi(T(j\infty)) = -90°$$

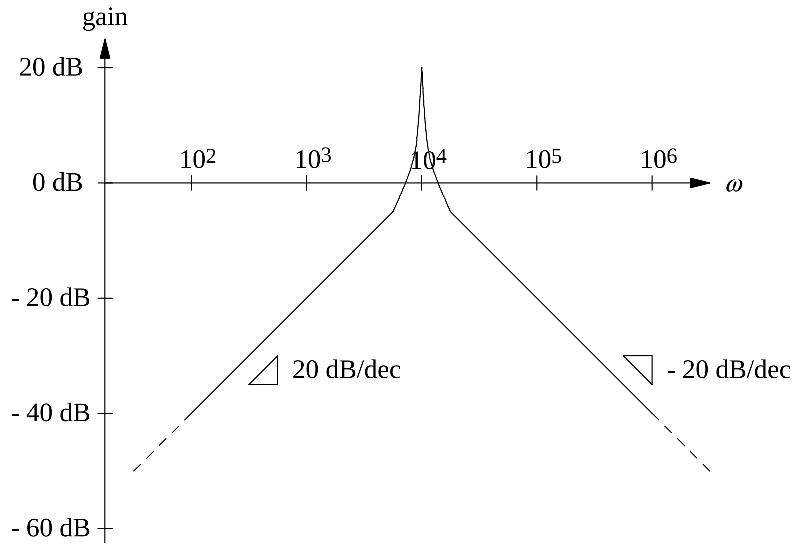
$$|T(j10)| = |j10^{-3}| \rightarrow -60 \, dB$$

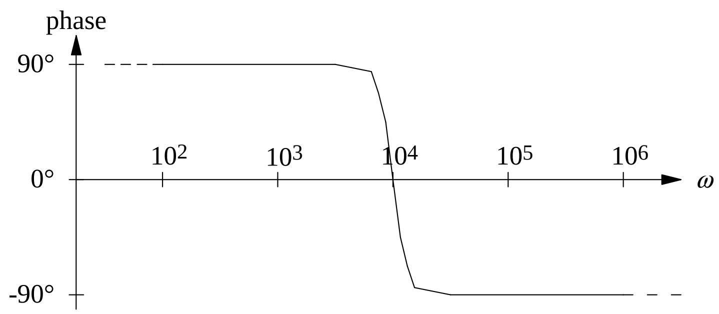
the gain at the first corner is $-60 \, dB + (3 \, decades \times 20 \, dB/decade) = 0 \, dB$

$$(\omega_o)^2 = (10^4)^2 \rightarrow \omega_o = 10^4 \text{ Rad/s} \quad ; \quad \frac{\omega_o}{Q_o} = 10^3 \rightarrow Q_o = \frac{\omega_o}{10^3} = \frac{10^4}{10^3} = 10$$

from example 4 we know that the peak for the $Q_o = 0.5$ case is 0 dB so the resonant peak is

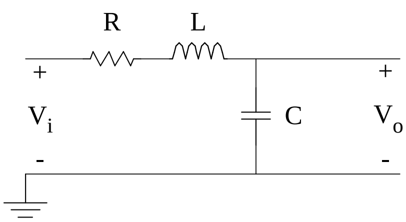
$0 \, dB + 20 \log(Q_o) = 0 \, dB + 20 \log(10) = 20 \, dB$ and 3 dB bandwidth is $\frac{\omega_o}{Q_o} = \frac{10^4}{10} = 10^3$ Rad/s





RLC passive filters

low-pass filter

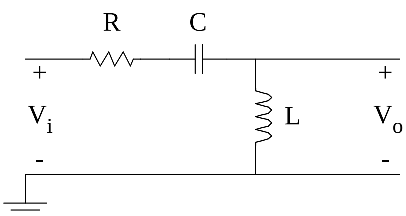


$$A_v = \frac{V_o}{V_i} = \frac{\frac{1}{L\,C}}{s^2 + \frac{R}{L}\,s + \frac{1}{L\,C}}$$

$$\omega_o = \frac{1}{\sqrt{L\,C}}$$

$$Q_o = \frac{1}{R}\sqrt{\frac{L}{C}}$$

high-pass filter

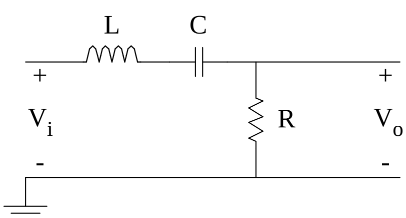


$$A_v = \frac{V_o}{V_i} = \frac{s^2}{s^2 + \frac{R}{L}\,s + \frac{1}{L\,C}}$$

$$\omega_o = \frac{1}{\sqrt{L\,C}}$$

$$Q_o = \frac{1}{R}\sqrt{\frac{L}{C}}$$

band-pass filter

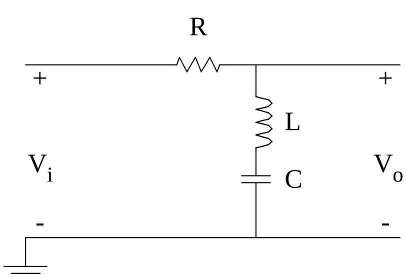


$$A_v = \frac{V_o}{V_i} = \frac{\frac{R}{L}\,s}{s^2 + \frac{R}{L}\,s + \frac{1}{L\,C}}$$

$$\omega_o = \frac{1}{\sqrt{L\,C}}$$

$$Q_o = \frac{1}{R}\sqrt{\frac{L}{C}}$$

band-stop filter



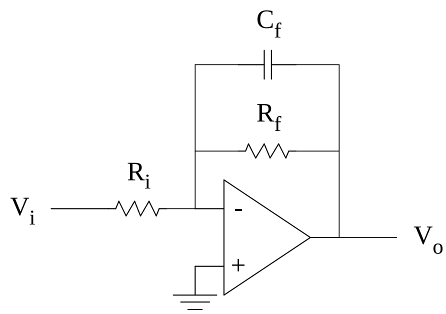
$$A_v = \frac{V_o}{V_i} = \frac{s^2 + \frac{1}{L\,C}}{s^2 + \frac{R}{L}\,s + \frac{1}{L\,C}}$$

$$\omega_o = \frac{1}{\sqrt{L\,C}}$$

$$Q_o = \frac{1}{R}\sqrt{\frac{L}{C}}$$

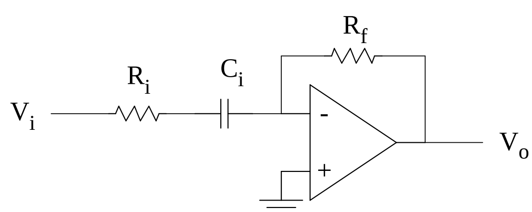
single opamp active filters (bandwidth ≥ 2 * ω_o)

low-pass filter



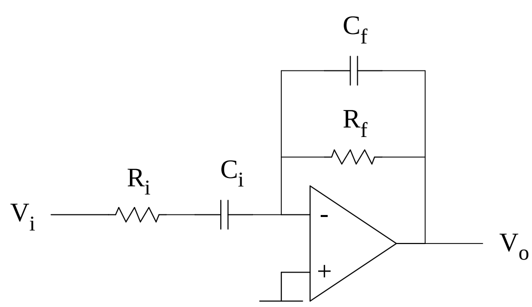
$$A_v = \frac{V_o}{V_i} = \frac{-\frac{R_f}{R_i}}{s R_f C_f + 1}$$

high-pass filter



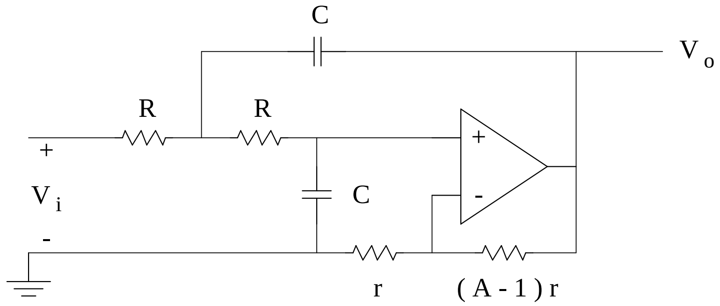
$$A_v = \frac{V_o}{V_i} = \frac{-\frac{R_f}{R_i}\, s R_i C_i}{s R_i C_i + 1}$$

band-pass filter



$$A_v = \frac{V_o}{V_i} = \frac{-\frac{R_f}{R_i}\, s R_i C_i}{\left(s R_i C_i + 1\right)\left(s R_f C_f + 1\right)}$$

Sallen-Key single opamp biquadratic filters

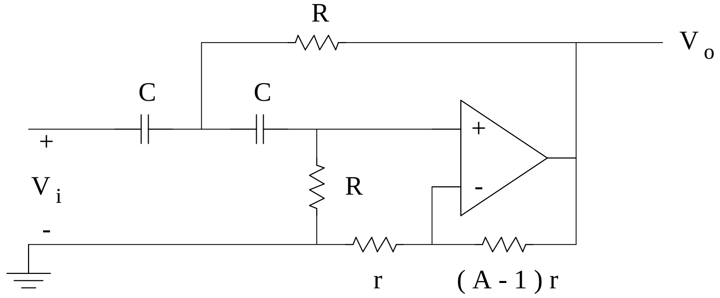


$$\omega_o = \frac{1}{R\,C}$$

$$A_v = \frac{V_o}{V_i} = A \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q_o}\, s + \omega_o^2}$$

$$Q_o = \frac{1}{3 - A}$$

$$2\,R = r \,\|\, (\,(A\text{-}1)\, r\,)$$

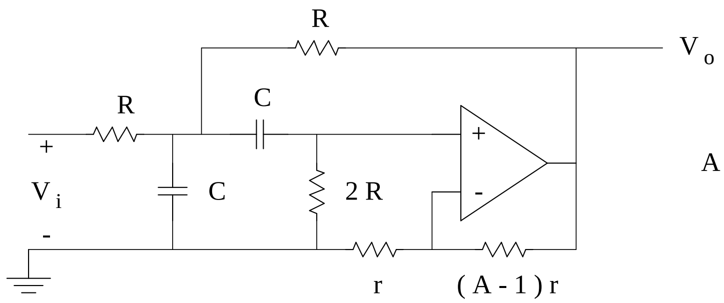


$$\omega_o = \frac{1}{R\,C}$$

$$A_v = \frac{V_o}{V_i} = A \frac{s^2}{s^2 + \frac{\omega_o}{Q_o}\, s + \omega_o^2}$$

$$Q_o = \frac{1}{3 - A}$$

$$R = r \,\|\, (\,(A\text{-}1)\, r\,)$$

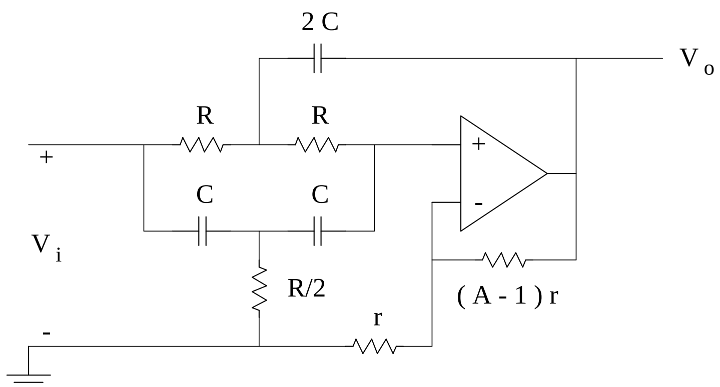


$$\omega_o = \frac{1}{R\,C}$$

$$A_v = \frac{V_o}{V_i} = A\,Q_o \frac{\frac{\omega_o}{Q_o}\, s}{s^2 + \frac{\omega_o}{Q_o}\, s + \omega_o^2}$$

$$Q_o = \frac{1}{3 - A}$$

$$2\,R = r \,\|\, (\,(A\text{-}1)\, r\,)$$



$$\omega_o = \frac{1}{R\,C}$$

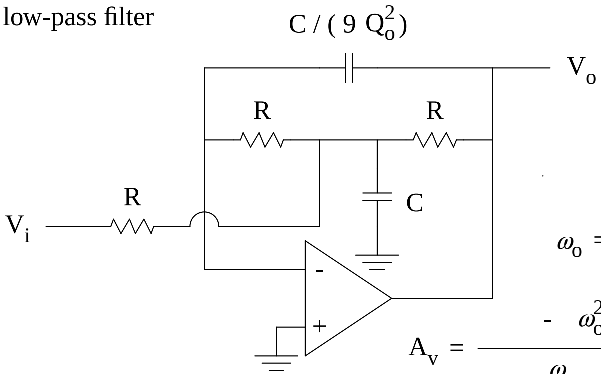
$$A_v = \frac{V_o}{V_i} = A \frac{s^2 + \omega_o^2}{s^2 + \frac{\omega_o}{Q_o}\, s + \omega_o^2}$$

$$Q_o = \frac{1}{4 - 2\,A}$$

$$2\,R = r \,\|\, (\,(A\text{-}1)\, r\,)$$

other single opamp biquadratic filters

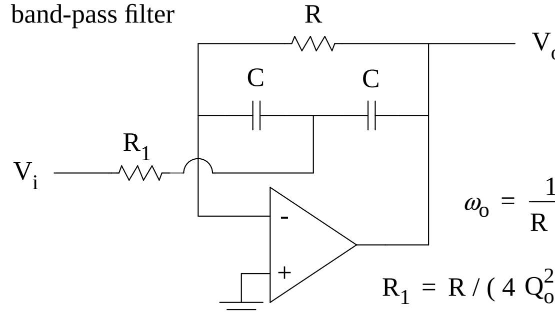
low-pass filter



$$\omega_o = \frac{1}{R\,C}$$

$$A_v = \frac{-\ \omega_o^2}{s^2 + \frac{\omega_o}{Q_o}\,s + \omega_o^2}$$

band-pass filter

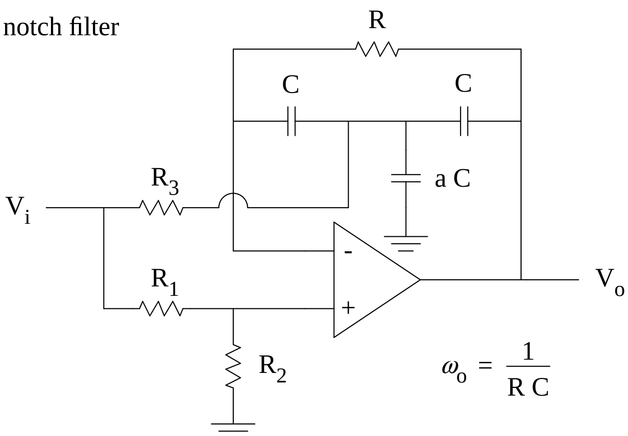


$$\omega_o = \frac{1}{R\,C}$$

$$R_1 = R / (4\ Q_o^2)$$

$$A_v = \frac{V_o}{V_i} = \frac{-\,2\ \omega_o\,Q_o\,s}{s^2 + \frac{\omega_o}{Q_o}\,s + \omega_o^2}$$

notch filter



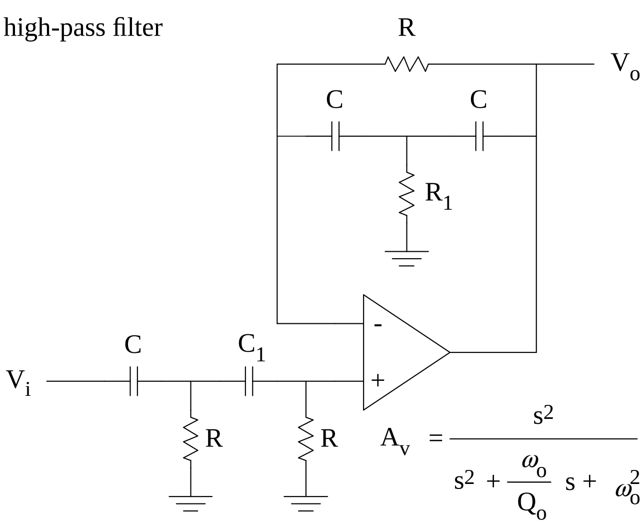
$$\omega_o = \frac{1}{R\,C}$$

$$a = \frac{\omega_o^2}{\omega_n^2} - 1 \qquad R_3 = \frac{R}{(2 + a)^2\,Q_o^2}$$

$$\frac{R_2}{R_1 + R_2} = \frac{(2 + a)\,Q_o^2}{(2 + a)\,Q_o^2 + 1} = \alpha$$

$$A_v = \frac{V_o}{V_i} = \alpha\,\frac{\omega_o^2}{\omega_n^2}\,\frac{s^2 + \omega_n^2}{s^2 + \frac{\omega_o}{Q_o}\,s + \omega_o^2}$$

high-pass filter

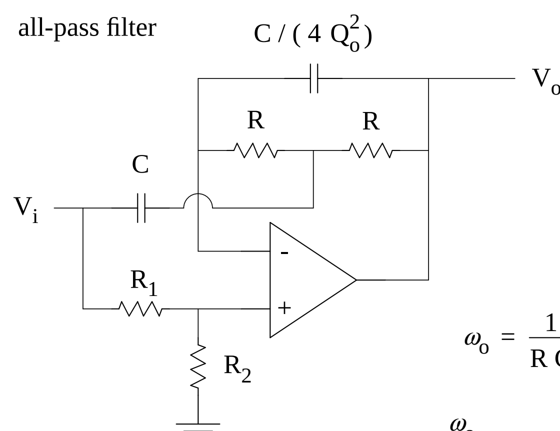


$$A_v = \frac{s^2}{s^2 + \frac{\omega_o}{Q_o}\,s + \omega_o^2}$$

$$\omega_o = \frac{1}{R\,C} \qquad C_1 = C / (4\ Q_o^2)$$

$$R_1 = R / (4\ Q_o^2)$$

all-pass filter

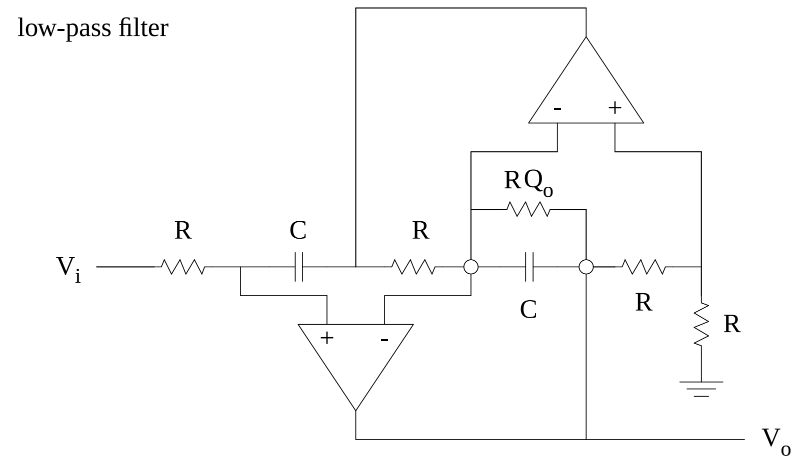


$$\omega_o = \frac{1}{R\,C}$$

$$A_v = \frac{V_o}{V_i} = \alpha\,\frac{s^2 - \frac{\omega_o}{Q_o}\,s + \omega_o^2}{s^2 + \frac{\omega_o}{Q_o}\,s + \omega_o^2}$$

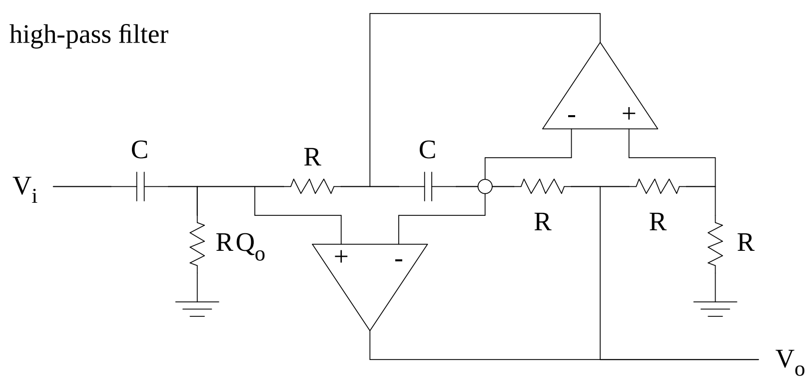
$$\frac{R_2}{R_1 + R_2} = \frac{Q_o^2}{Q_o^2 + 1} = \alpha$$

two opamp biquadratic filters



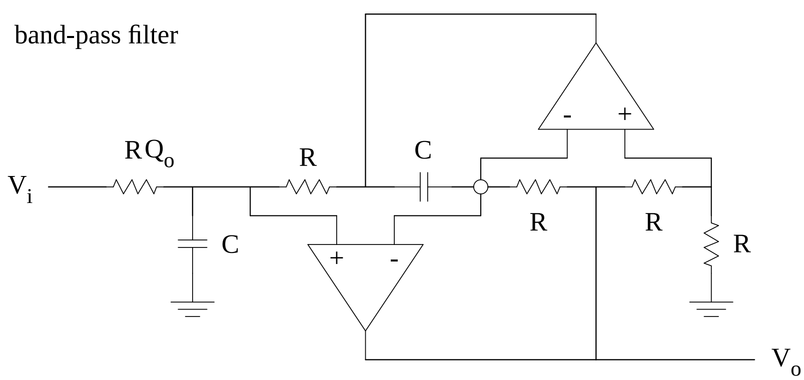
$$A_v = \frac{V_o}{V_i} = 2\ \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q_o}\, s + \omega_o^2}$$

$$\omega_o = \frac{1}{R\,C}$$



$$A_v = \frac{V_o}{V_i} = 2\ \frac{s^2}{s^2 + \frac{\omega_o}{Q_o}\, s + \omega_o^2}$$

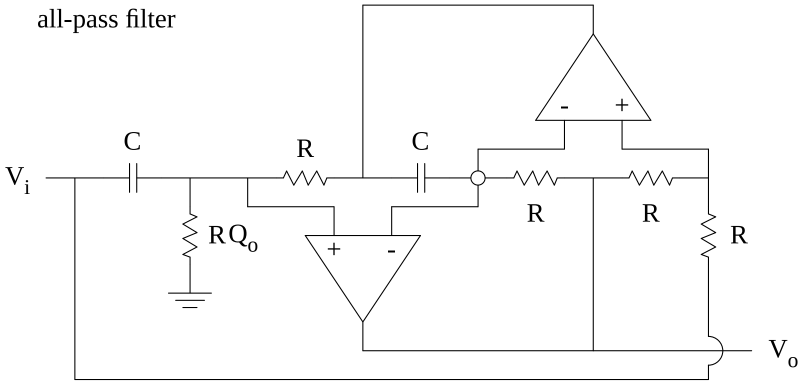
$$\omega_o = \frac{1}{R\,C}$$



$$A_v = \frac{V_o}{V_i} = 2\ \frac{\frac{\omega_o}{Q_o}\, s}{s^2 + \frac{\omega_o}{Q_o}\, s + \omega_o^2}$$

$$\omega_o = \frac{1}{R\,C}$$

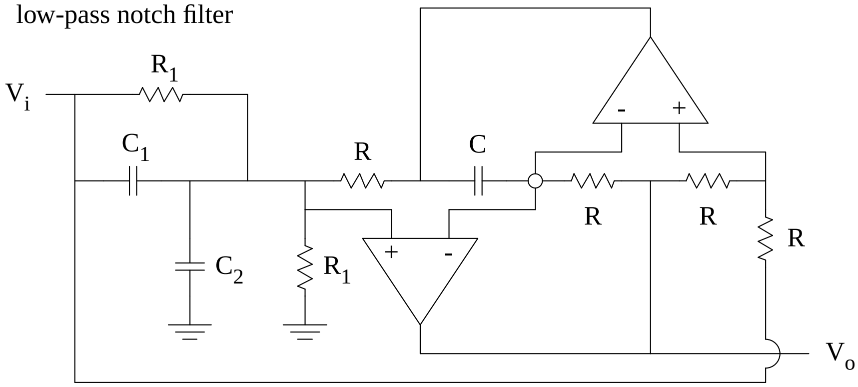
all-pass filter



$$A_v = \frac{V_o}{V_i} = \frac{s^2 - \frac{\omega_o}{Q_o}\,s + \omega_o^2}{s^2 + \frac{\omega_o}{Q_o}\,s + \omega_o^2}$$

$$\omega_o = \frac{1}{R\,C}$$

low-pass notch filter



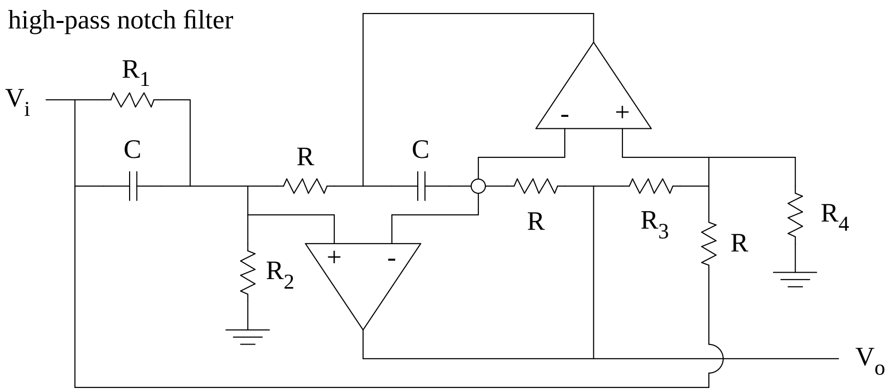
$$A_v = \frac{V_o}{V_i} = \left(\frac{\omega_o}{\omega_n}\right)^2 \frac{s^2 + \omega_n^2}{s^2 + \frac{\omega_o}{Q_o}\,s + \omega_o^2}$$

$$\alpha = \frac{\omega_n}{\omega_o} \geq 1 \qquad R_1 = 2RQ_o$$

$$\omega_o = \frac{1}{R\,C} \qquad C_1 = \frac{\alpha^2 + 1}{2\,\alpha^2}\;C$$

$$C_2 = \frac{\alpha^2 - 1}{2\,\alpha^2}\;C$$

high-pass notch filter



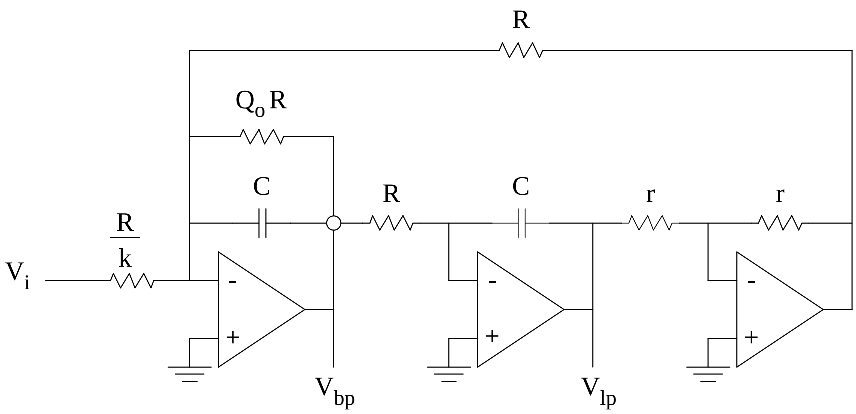
$$A_v = \frac{V_o}{V_i} = 2\,\frac{2 - \alpha^2}{3 - \alpha^2}\;\frac{s^2 + \omega_n^2}{s^2 + \frac{\omega_o}{Q_o}\,s + \omega_o^2}$$

$$\alpha = \frac{\omega_n}{\omega_o} \leq 1 \qquad R_1 = \frac{\alpha^2 + 1}{\alpha^2}\;Q_o R$$

$$\omega_o = \frac{1}{R\,C} \qquad R_2 = (\alpha^2 + 1)\;Q_o R$$

$$R_3 = \frac{2\,\alpha^2}{1 + \alpha^2}\;R \qquad R_4 = \frac{2\,\alpha^2}{1 - \alpha^2}\;R$$

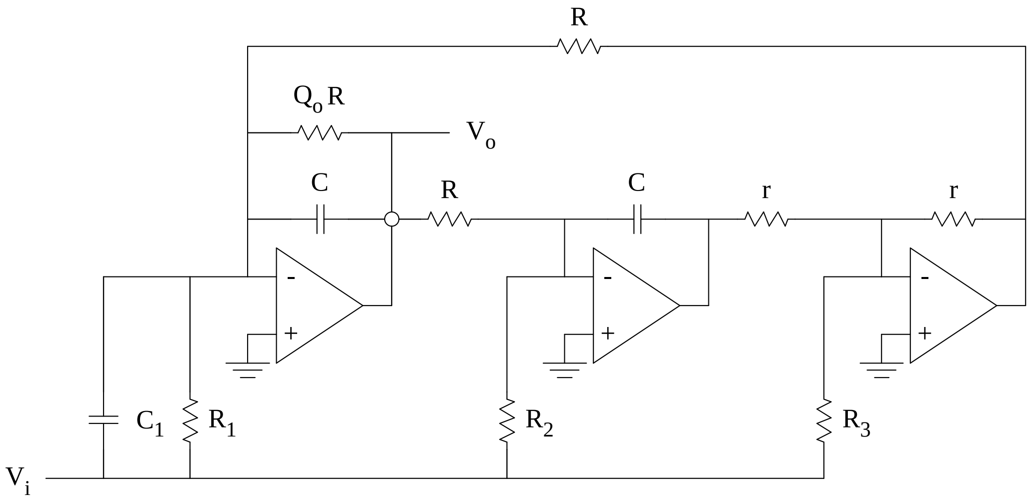
three opamp biquadratic filters



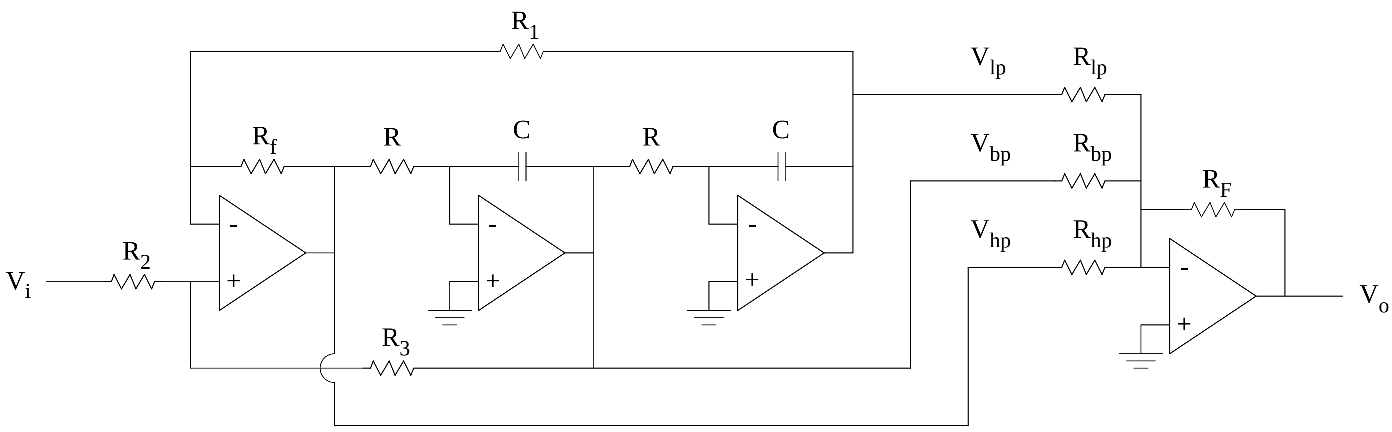
$$A_{lp} = \frac{V_{lp}}{V_i} = k \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$A_{bp} = \frac{V_{bp}}{V_i} = k \frac{\omega_o s}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$\omega_o = \frac{1}{R C}$$



$$A_v = \frac{V_o}{V_i} = \frac{\frac{C_1}{C} s^2 + \frac{1}{C}\left(\frac{1}{R_1} - \frac{r}{R_3 R}\right) s + \frac{1}{R_2 R C^2}}{s^2 + \frac{1}{Q_o R C} s + \frac{1}{C^2 R^2}}$$



$$A_{hp} = \frac{V_{hp}}{V_i} = k \frac{s^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2} \qquad A_{bp} = \frac{V_{bp}}{V_i} = k \frac{\frac{\omega_o}{Q_o} s}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2} \qquad A_{lp} = \frac{V_{lp}}{V_i} = k \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$k = 2 - \frac{1}{Q_o} \qquad \omega_o = \frac{1}{R\,C} \qquad \frac{R_f}{R_1} = 1 \qquad \frac{R_3}{R_2} = 2\,Q_o - 1 \qquad V_o = -\left(\frac{R_F}{R_{lp}} V_{lp} + \frac{R_F}{R_{bp}} V_{bp} + \frac{R_F}{R_{hp}} V_{hp} \right)$$

Sensors and Process Control

analog data representation

measurement errors
    difference between the actual value and the measured value

measurement accuracy

measured variable ± accuracy
percentage error of full scale
percentage of instrument span
percentage of actual reading

system accuracy
    take into account all the individual errors

sensitivity
    change in output signal for a change in input signal

hysteresis
    the relationship between the input and output signals depends on whether the input signal is increasing or decreasing

reproducibility
    same output sequence each time the input sequence repeats

resolution
    minimum measurable value of the input signal

linearity
    for a given input, one unique output value $c_m = m \cdot c + c_o$

sensor time response

first-order response
    sensor response is dominated by a first-order differential equation

second-order response
    sensor response is dominated by a second-order differential equation

sensors and transducers

transducer
    converts physical measured energy (radiant, mechanical, thermal, electrical, magnetic, chemical) into electrical energy

passive sensors
    passive sensors modify an applied voltage or current

active sensors
    active sensors produce an output signal (voltage or current) directly

physical effects

resistance $R = \rho \frac{L}{A}$

capacitance $C = \varepsilon_o \varepsilon_r \frac{A}{d}$

inductance $L = \mu_o \frac{N^2 A}{d}$

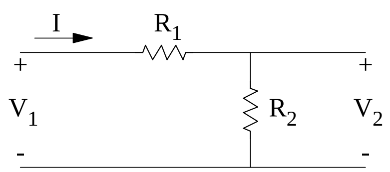
Analog Signal Conditioning

analog signal conditioning terms

change signal level (amplify or attenuate)
change signal bias or offset value
linearization: straight line approximation of the sensor response
conversions: e.g. change in R seen as a change in V
signal transmission: transport signals
filtering
impedance match between sensor and signal conditioning circuit and/or load
sensor loading by signal conditioning circuit
signal conditioning circuit loaded by process controller input

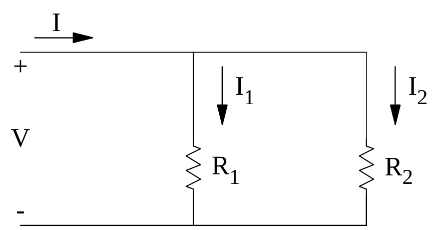
passive circuits

voltage divider



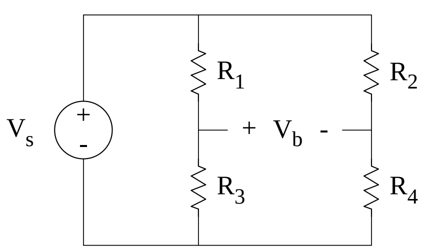
$$V_2 = R_2 I = R_2 \frac{V_1}{R_1 + R_2} = \frac{R_2}{R_1 + R_2} V_1$$

current divider



$$I_2 = \frac{V}{R_2} = \frac{\left(\dfrac{I}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}\right)}{R_2} = \frac{\left(\dfrac{R_1 R_2}{R_1 + R_2}\right) I}{R_2} = \frac{R_1}{R_1 + R_2} I$$

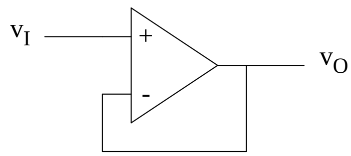
bridge



$$V_b = \frac{R_3 R_2 - R_1 R_4}{\left(R_1 + R_3\right)\left(R_2 + R_4\right)} V_s$$

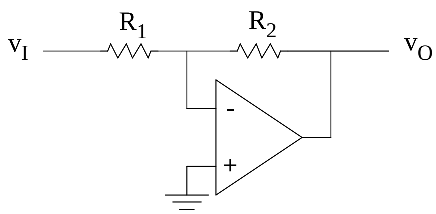
operational amplifier circuits

voltage follower



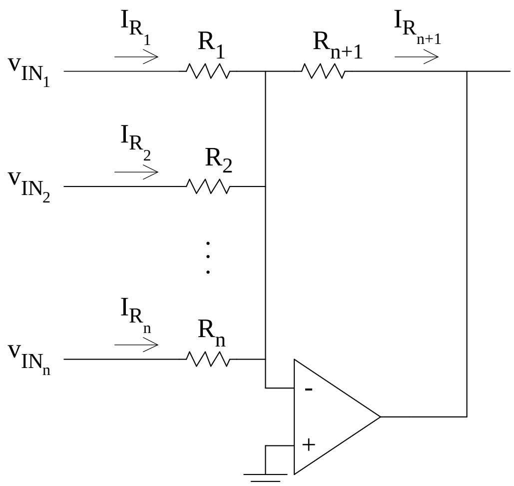
$$A_v = \frac{v_O}{v_I} = 1$$

inverting amplifier



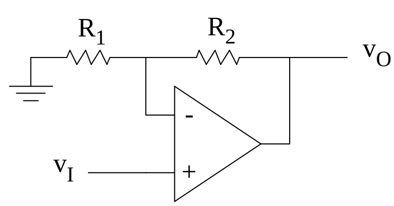
$$A_v = \frac{v_O}{v_I} = -\frac{R_2}{R_1}$$

summer



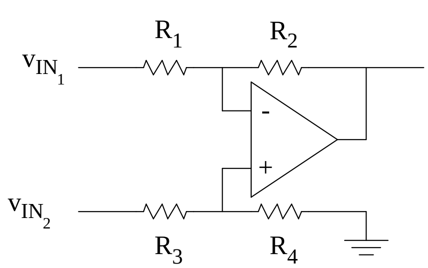
$$v_{OUT} = -\sum_{k=1}^{k=n} \frac{R_{n+1}}{R_k} v_{IN_k}$$

noninverting amplifier



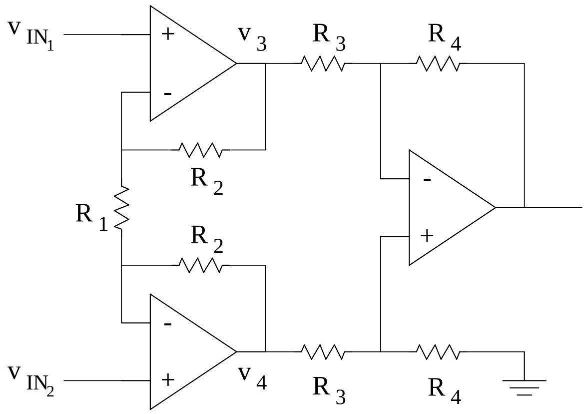
$$A_v = \frac{v_O}{v_I} = \frac{R_2 + R_1}{R_1}$$

differential amplifier



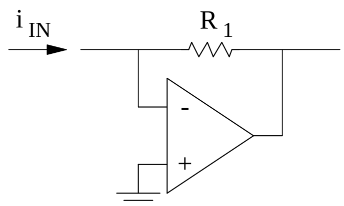
$$v_{OUT} = \left(\left(\frac{R_1 + R_2}{R_1}\right)\left(\frac{R_4}{R_3 + R_4}\right)\right) v_{IN_2} - \left(\frac{R_2}{R_1}\right) v_{IN_1}$$

instrumentation amplifier



$$v_{OUT} = \frac{R_4}{R_3}\left(1 + \frac{2R_2}{R_1}\right)\left(v_{IN_2} - v_{IN_1}\right)$$

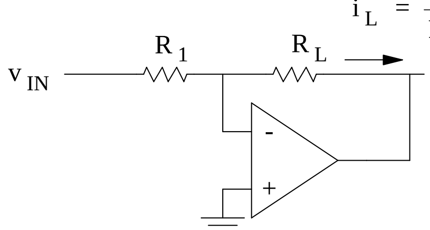
current-to-voltage converter



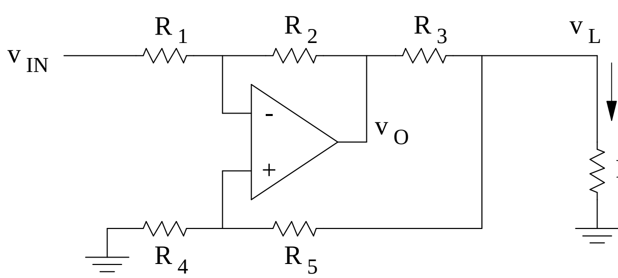
$$v_{OUT} = -R_1 i_{IN}$$

voltage-to-current converter with floating load

$$i_L = \frac{v_{IN}}{R_1}$$



voltage-to-current converter with non-floating load



$$i_L = -\frac{R_2}{R_1 R_3} v_{IN}$$

$$R_1\left(R_4 + R_5\right) = R_4\left(R_1 + R_2\right)$$

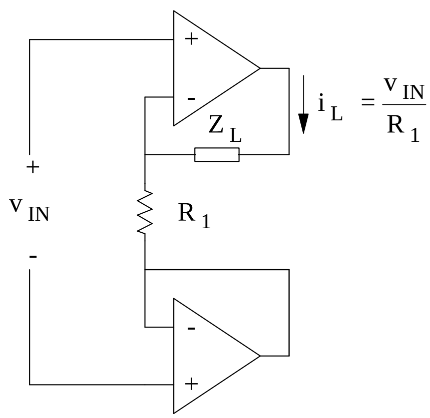
$$\left(R_4 + R_5\right) \gg R_L$$

$$\text{if} \quad (R_4 + R_5) \gg R_L \quad \text{then} \quad v_O \approx -\frac{R_2}{R_1} v_{IN} + \left(1 + \frac{R_2}{R_1}\right)\left(\frac{R_4}{R_4 + R_5}\right) v_L$$

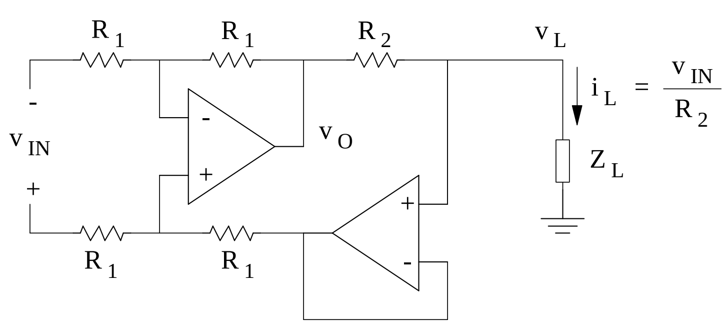
$$\text{if} \quad \left(1 + \frac{R_2}{R_1}\right)\left(\frac{R_4}{R_4 + R_5}\right) = \left(\frac{R_1 + R_2}{R_1}\right)\left(\frac{R_4}{R_4 + R_5}\right) = 1 \quad \text{then} \quad v_O = -\frac{R_2}{R_1} v_{IN} + v_L$$

$$\text{then} \quad i_L \approx i_{R_3} = \frac{v_O - v_L}{R_3} = \frac{\left(-\frac{R_2}{R_1} v_{IN} + v_L\right) - v_L}{R_3} = -\frac{R_2}{R_1 R_3} v_{IN}$$

voltage-to-current converter with floating load

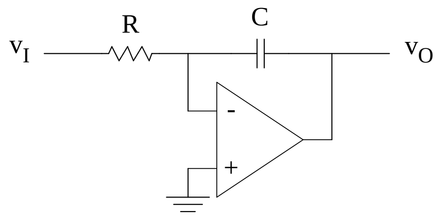


voltage-to-current converter with non-floating load

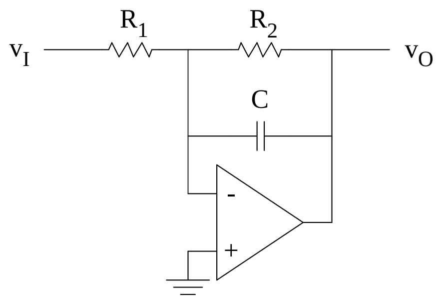


$$v_O = v_{IN} + v_L \quad \rightarrow \quad i_L = \frac{v_O - v_L}{R_2} = \frac{(v_{IN} + v_L) - v_L}{R_2} = \frac{v_{IN}}{R_2}$$

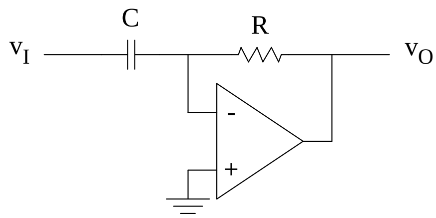
intergrator



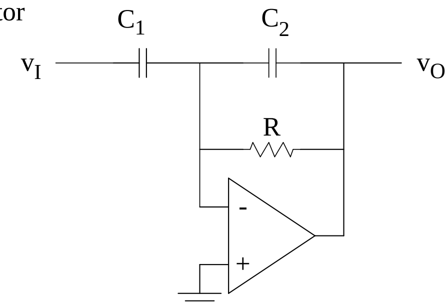
compensated integrator



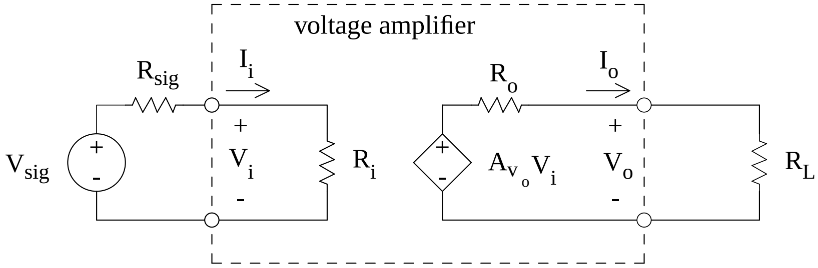
differentiator



compensated differentiator

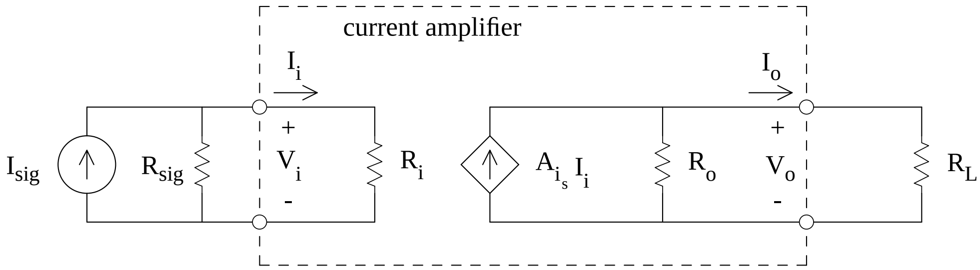


amplifier loading



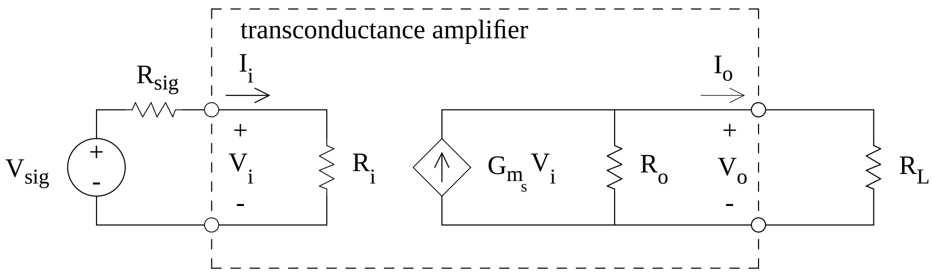
$$V_o = \frac{R_L}{R_L + R_o} A_{v_o} \frac{R_i}{R_i + R_{sig}} V_{sig}$$

$$A_{v_o} = \left. \frac{V_o}{V_i} \right|_{R_L = \infty}$$



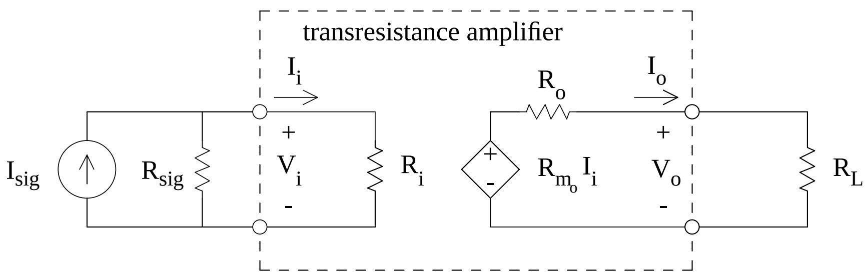
$$I_o = \frac{R_o}{R_L + R_o} A_{i_s} \frac{R_{sig}}{R_i + R_{sig}} I_{sig}$$

$$A_{i_s} = \left. \frac{I_o}{I_i} \right|_{R_L = 0}$$



$$I_o = \frac{R_o}{R_L + R_o} G_{m_s} \frac{R_i}{R_i + R_{sig}} V_{sig}$$

$$G_{m_s} = \left. \frac{I_o}{V_i} \right|_{R_L = 0}$$

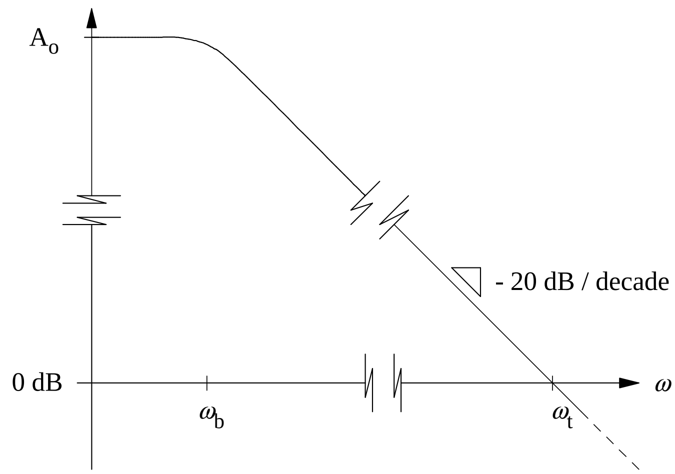


$$V_o = \frac{R_L}{R_L + R_o} R_{m_o} \frac{R_{sig}}{R_i + R_{sig}} I_{sig}$$

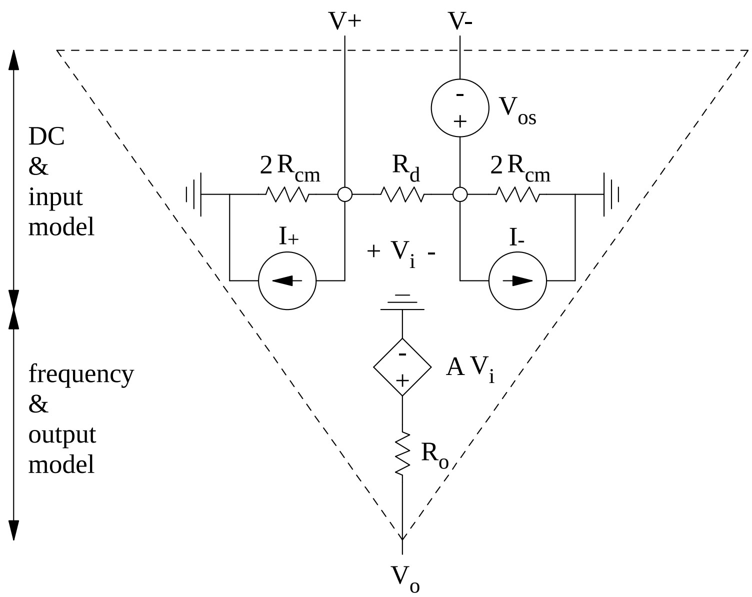
$$R_{m_o} = \left. \frac{V_o}{I_i} \right|_{R_L = \infty}$$

nonideal frequency dependent operational amplifier model

opamp data book values using ±15 V supplies			
description	values		
	min	typical	max
input offset voltage		2 mV	3 mV
input bias current		80 nA	500 nA
input offset current		20 nA	200 nA
differential input resistance	300 kΩ	2 MΩ	
output resistance		75 Ω	150 Ω
output swing	±12 V	±14 V	
slew rate	$0.3\,\frac{V}{\mu s}$	$0.5\,\frac{V}{\mu s}$	
unity gain frequency	0.4 MHz	1.0 MHz	
DC voltage gain	90 dB	106 dB	



typical opamp open loop gain function

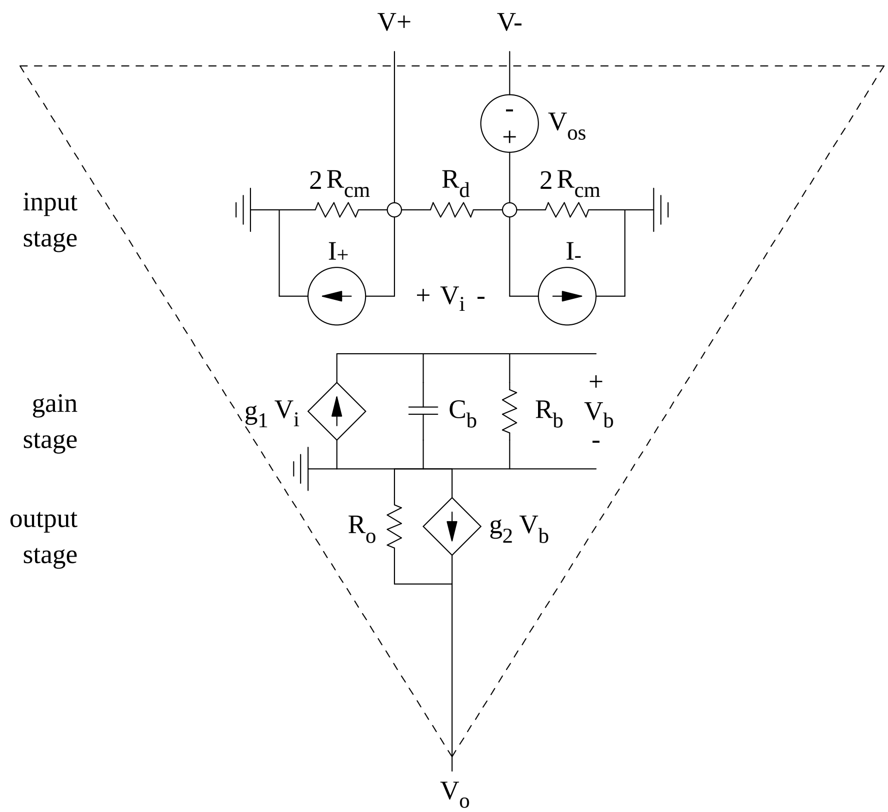


model using frequency depenent VVT

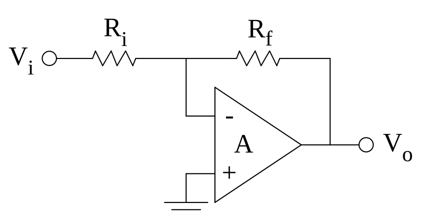
$$I_+ = I_B + 0.5\,I_{OS}$$

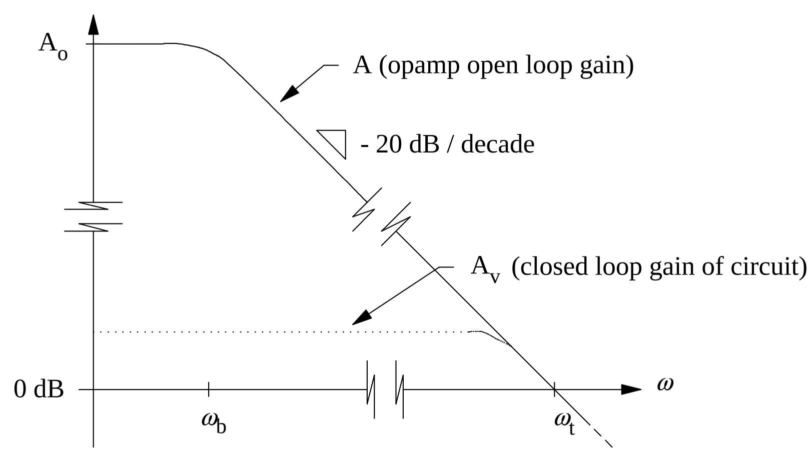
$$I_- = I_B - 0.5\,I_{OS}$$

$$A = \frac{A_o}{\frac{s}{\omega_b} + 1}$$



model using frequency independent components





opamp open and closed loop gains

Note $V_o = -A\,V_-$.

Define β to be the fraction of the opamp's output fed back to the opamp's input. For the above circuit

$$\beta = \frac{R_i}{R_i + R_f} = \frac{1}{\dfrac{R_i + R_f}{R_i}} = \frac{1}{1 + \dfrac{R_f}{R_i}}$$

The gain around the feed-forward,feed-back loop is $-A\,\beta$.

For the above circuit start with a KCL at the opamps' inverting input terminal

$$\frac{V_- - V_i}{R_i} + \frac{V_- - V_o}{R_f} = 0$$

$$\frac{V_i}{R_i} = \left(\frac{1}{R_i} + \frac{1}{R_f}\right)V_- - \frac{V_o}{R_f} = \left(\frac{1}{R_i} + \frac{1}{R_f}\right)\left(-\frac{V_o}{A}\right) - \frac{V_o}{R_f} = -\left(\left(\frac{1}{R_i} + \frac{1}{R_f}\right)\frac{1}{A} + \frac{1}{R_f}\right)V_o$$

$$A_v = \frac{V_o}{V_i} = \frac{-\dfrac{1}{R_i}}{\left(\dfrac{1}{R_i} + \dfrac{1}{R_f}\right)\dfrac{1}{A} + \dfrac{1}{R_f}} = \frac{-\dfrac{R_f}{R_i}}{\left(\dfrac{R_f}{R_i} + 1\right)\dfrac{1}{A} + 1} = \frac{-\dfrac{R_f}{R_i}}{1 + \dfrac{1}{A\,\beta}} = \frac{\text{ideal\_gain}}{1 + \dfrac{1}{A\,\beta}}$$

Given

$$A = \frac{A_o}{\dfrac{s}{\omega_b} + 1} \qquad \rightarrow \qquad A\,\Bigg|_{s = j\,\omega \gg j\,\omega_b} \approx \frac{A_o}{\dfrac{s}{\omega_b}} = \frac{A_o\,\omega_b}{s} = \frac{\omega_t}{s}$$

then

$$A_v = \frac{-\dfrac{R_f}{R_i}}{1 + \dfrac{1}{A\,\beta}} = \frac{-\dfrac{R_f}{R_i}}{1 + \dfrac{1}{\dfrac{A_o}{\dfrac{s}{\omega_b} + 1}\,\beta}} = \frac{-\dfrac{R_f}{R_i}}{1 + \dfrac{\dfrac{s}{\omega_b} + 1}{A_o\,\beta}} = \frac{-\dfrac{R_f}{R_i}}{\dfrac{s}{\omega_b\,A_o\,\beta} + \left(1 + \dfrac{1}{A_o\,\beta}\right)} \approx \frac{-\dfrac{R_f}{R_i}}{\dfrac{s}{\omega_b\,A_o\,\beta} + 1} = \frac{-\dfrac{R_f}{R_i}}{\dfrac{s}{\omega_{-3dB}} + 1}$$

The frequency dependent finite gain has an impact on the maximum stage gain in a multi-stage amplifier. There are three different ways the specifications may be supplied:

(1) step response: maximum settling time will be specified

(2) periodic response: maximum phase delay specification

(3) filtering function: bandwidth specification

Given the opamp's open loop voltage gain is dropping off at 20 dB/decade, the approximation $BW_{stage} = \frac{f_t}{A_{v_{stage}}}$ holds.

If the maximum settling time is specified, then $\tau \approx \frac{t_{st}}{5}$ resulting in the multi-stage over-all bandwidth requirement of

$$f_{spec} = \frac{1}{2\pi\tau} = \frac{5}{2\pi t_{st}} = \frac{f_t}{A_v} \quad \rightarrow \quad A_v = \frac{f_t}{f_{spec}} = \frac{f_t}{\left(\frac{5}{2\pi t_{st}}\right)} = \frac{f_t\, 2\pi t_{st}}{5}$$

Basically, the settling time specification drives the maximum overall voltage gain independent of the number of stages.

If the maximum phase delay is specified for a given frequency f_{spec}, then

$$f_{spec} = \tan\left(\frac{\text{total\_phase}}{\text{\#\_of\_stage}}\right)\frac{f_t}{A_{v_{stage}}} \quad \rightarrow \quad A_{v_{stage}} = \tan\left(\frac{\text{total\_phase}}{\text{\#\_of\_stage}}\right)\frac{f_t}{f_{spec}}$$

For example, given the spec at most 1° phase delay at 100 Hz given an opamp $f_t = 1$ MHz

# of stages	$A_{v_{stage}}$	$n \cdot A_{v_{stage}}$
2	$\tan\left(\frac{1°}{2}\right)\frac{10^6}{10^2} = 87.3$	$2 \times 87.3 = 7.62k$
3	$\tan\left(\frac{1°}{3}\right)\frac{10^6}{10^2} = 58.2$	$3 \times 58.2 = 197k$
4	$\tan\left(\frac{1°}{4}\right)\frac{10^6}{10^2} = 43.6$	$4 \times 43.6 = 3.61M$
5	$\tan\left(\frac{1°}{5}\right)\frac{10^6}{10^2} = 34.9$	$5 \times 34.9 = 51.8M$

short-circuit time-constant method

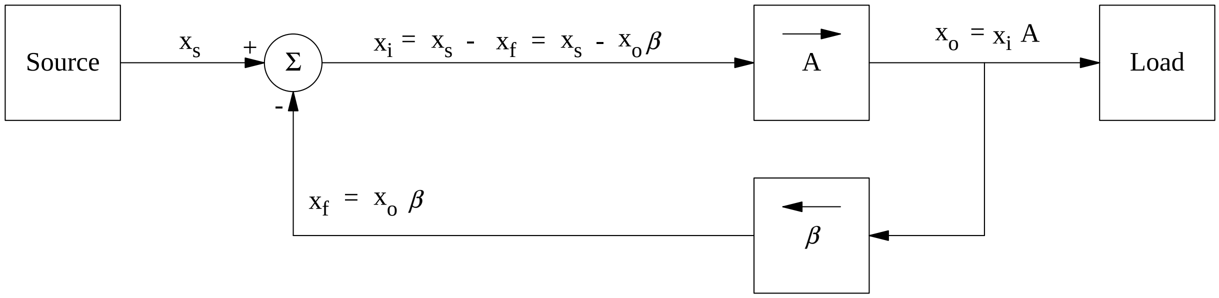
When solving for the approximate $-3\,\text{dB}$ high-pass frequency of a large/complex network,

+ replace all high-pass capacitors with a short-circuit
+ then for each high-pass capacitor,
  set it to its value and solve the resulting $-3\,\text{dB}$ frequency
  set it back to a short-circuit
+ $f_{hp} \approx \sum_{i=1}^{i=n} f_i$

open-circuit time-constant method

When solving for the approximate $-3\,\text{dB}$ low-pass frequency of a large/complex network,

+ replace all low-pass capacitors with an open-circuit
+ then for each low-pass capacitor,
  set it to its value and solve the resulting time constant τ
  set it back to an open-circuit
+ $\tau_{lp} \approx \sum_{i=1}^{i=n} \tau_i$



$$x_o = A\,x_i = A\,(x_s - \beta\,x_o) \quad\rightarrow\quad (1 + A\beta)\,x_o = A\,x_s \quad\rightarrow\quad A_f \equiv \frac{x_o}{x_s} = \frac{A}{1 + A\beta} = \frac{A}{1 + A\beta}\,\frac{\frac{1}{A\beta}}{\frac{1}{A\beta}} = \frac{\frac{1}{\beta}}{1 + \frac{1}{A\beta}} \approx \frac{1}{\beta}$$

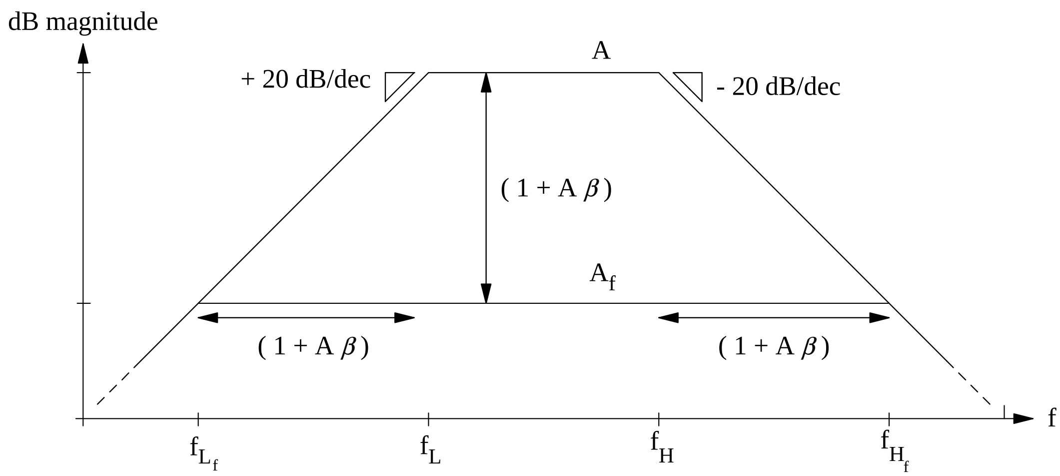
$$x_f = x_o\,\beta = \left(\frac{A}{1 + A\beta}\,x_s\right)\beta = \frac{A\beta}{1 + A\beta}\,x_s$$

$$x_i = x_s - x_f = x_s - \frac{A\beta}{1 + A\beta}\,x_s = \frac{(1 + A\beta)\,x_s}{1 + A\beta} - \frac{A\beta}{1 + A\beta}\,x_s = \frac{1}{1 + A\beta}\,x_s \quad\rightarrow\quad x_s = (1 + A\beta)\,x_i$$

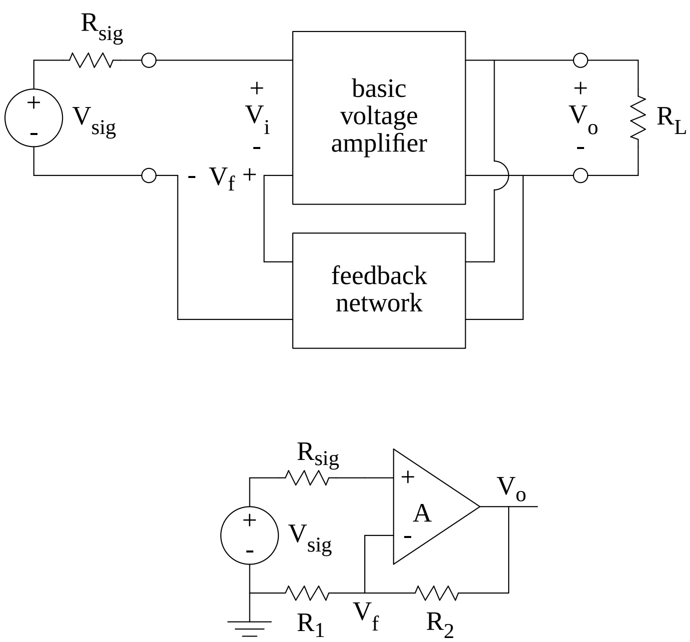
gain desensitivity

$$\frac{d\,A_f}{d\,A} = \frac{1}{(1 + A\beta)^2} \quad;\quad d\,A_f = \frac{d\,A}{(1 + A\beta)^2} \quad;\quad \frac{d\,A_f}{A_f} = \frac{d\,A}{(1 + A\beta)^2}\,\frac{1 + A\beta}{A} = \frac{1}{1 + A\beta}\,\frac{d\,A}{A}$$

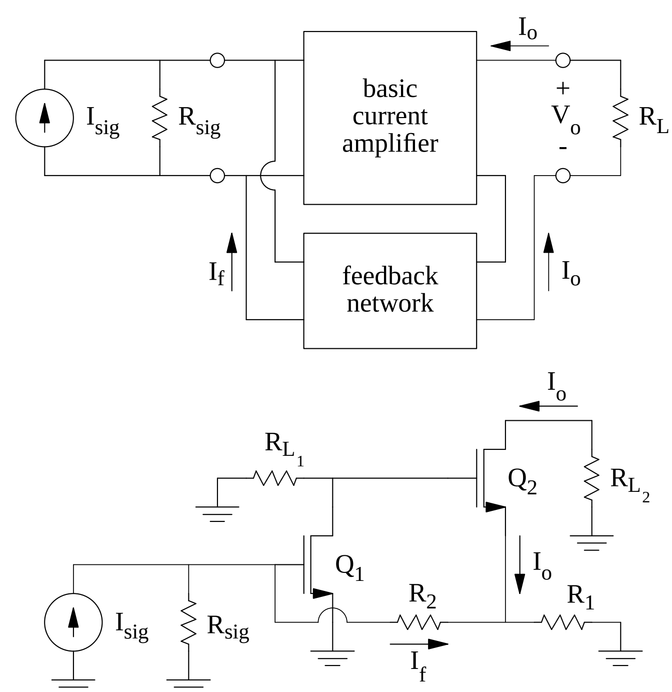
bandwidth extension



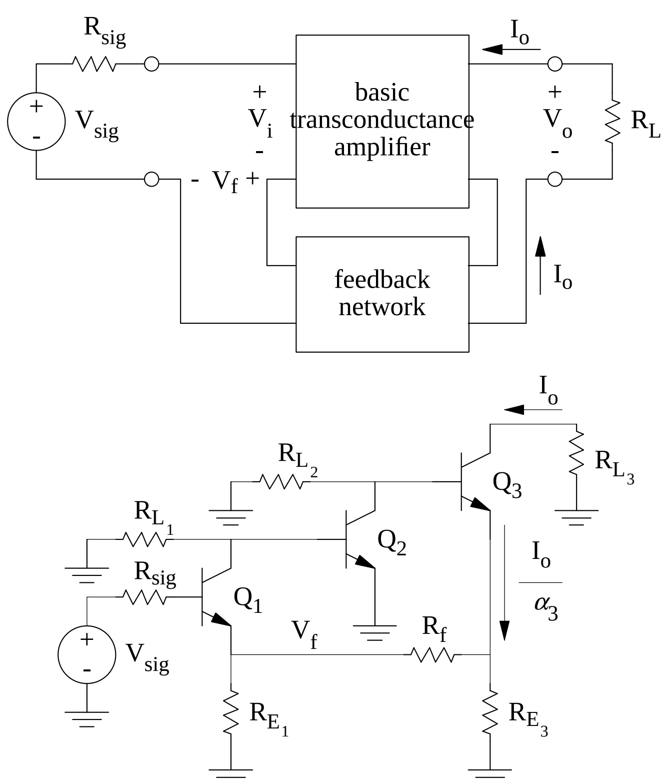
$$f_{L_f} = \frac{f_L}{1 + A\beta} \quad;\quad f_{H_f} = f_H\,(1 + A\beta)$$



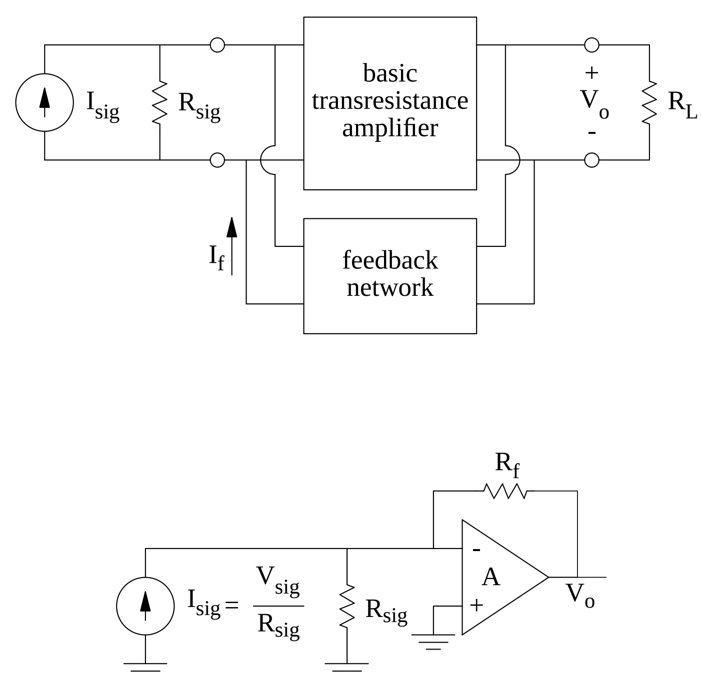
series-shunt topology



shunt-series topology

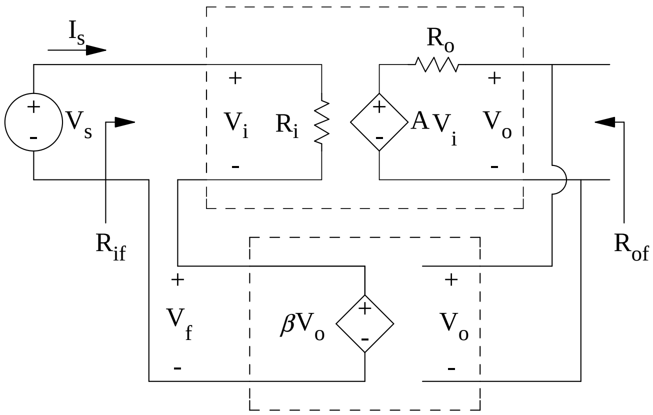


series-series topology



shunt-shunt topology

voltage mixing, voltage sampling

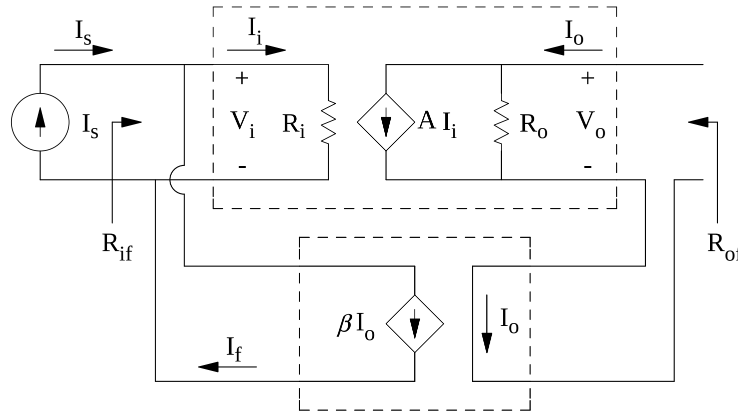


$$R_{if} = \frac{V_s}{I_s} = \frac{V_s}{V_i / R_i} = \frac{V_s}{V_i} R_i = \frac{(1 + A\beta) V_i}{V_i} R_i = (1 + A\beta) R_i$$

$$V_s = 0 \quad \rightarrow \quad V_i = -\beta V_o$$

$$R_{of} = \frac{V_o}{I_o} = \frac{V_o}{\dfrac{V_o - A V_i}{R_o}} = \frac{V_o}{\dfrac{V_o - A(-\beta V_o)}{R_o}} = \frac{R_o}{1 + A\beta}$$

current mixing, current sampling

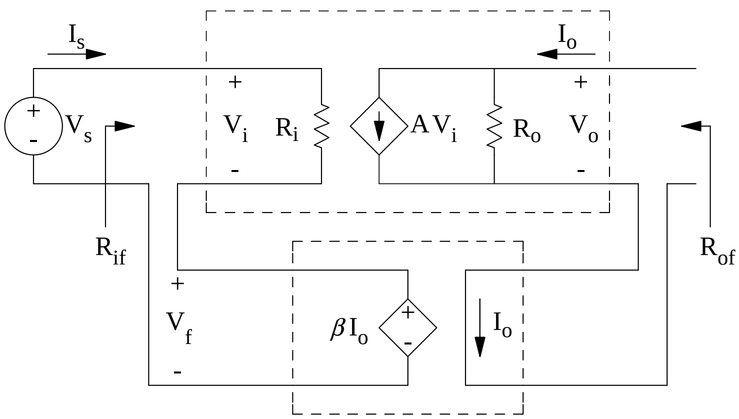


$$R_{if} = \frac{V_s}{I_s} = \frac{V_i}{I_s} = \frac{I_i R_i}{I_s} = \frac{I_i R_i}{(1 + A\beta) I_i} = \frac{R_i}{1 + A\beta}$$

$$I_s = 0 \quad \rightarrow \quad I_i = -\beta I_o$$

$$R_{of} = \frac{V_o}{I_o} = \frac{(I_o - A I_i) R_o}{I_o} = \frac{I_o - A(-\beta I_o)}{I_o} R_o = (1 + A\beta) R_o$$

voltage mixing, current sampling



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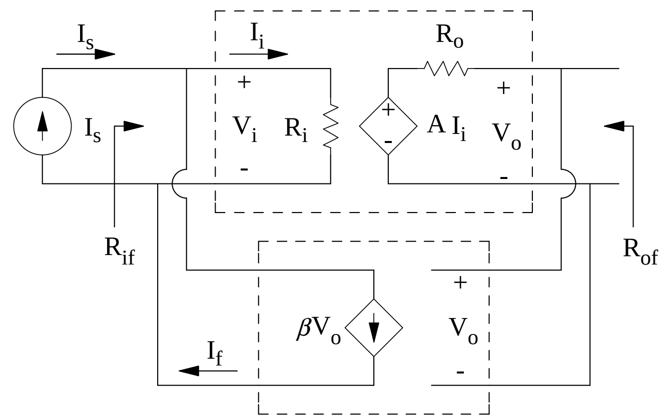
current mixing, voltage sampling



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summary for feedback systems

general

$$A_f \equiv \frac{x_o}{x_s} = \frac{A}{1 + A\,\beta} = \frac{\frac{1}{\beta}}{1 + \frac{1}{A\,\beta}} \approx \frac{1}{\beta}$$

bandwidth increases by $1 + A\,\beta$ as gain decreases by $1 + A\,\beta$.

voltage mixing, voltage sampling

$$R_{if} = (1 + A\,\beta)\,R_i$$

$$R_{of} = \frac{1}{1 + A\,\beta}\,R_o$$

current mixing, current sampling

$$R_{if} = \frac{1}{1 + A\,\beta}\,R_i$$

$$R_{of} = (1 + A\,\beta)\,R_o$$

voltage mixing, current sampling

$$R_{if} = (1 + A\,\beta)\,R_i$$

$$R_{of} = (1 + A\,\beta)\,R_o$$

current mixing, voltage sampling

$$R_{if} = \frac{1}{1 + A\,\beta}\,R_i$$

$$R_{of} = \frac{1}{1 + A\,\beta}\,R_o$$

Fourier Series

given

$$f(t) = f(t + nT) \qquad n = \pm 1, \pm 2, \pm 3, \cdots$$
$$= a_o + \sum_{n=1}^{N} a_n \cos(n\,\omega_o\,t) + \sum_{n=1}^{N} b_n \sin(n\,\omega_o\,t)$$
$$= c_o + \sum_{n=1}^{N} c_n \cos(n\,\omega_o\,t + \theta_n)$$

then

symmetry	Fourier coefficients
odd function $f(t) = -f(-t)$	$a_n = 0$ for all n $b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\,\omega_o\,t)\,dt$
even function $f(t) = f(-t)$	$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\,\omega_o\,t)\,dt$ $b_n = 0$ for all n
half-wave symmetry $f(t) = -f\left(t + \frac{T}{a}\right)$	$a_0 = 0$ $a_n = 0$ for even n $b_n = 0$ for even n $a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\,\omega_o\,t)\,dt$ for odd n $b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\,\omega_o\,t)\,dt$ for odd n

square wave with peak A and period T $\quad f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)\,\omega_0\,t)$

full-wave rectified with peak A and period T $\quad f(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(n\,\omega_0\,t)$

saw-tooth wave with peak A and period T $\quad f(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\,\omega_o\,t)}{n}$

triangular wave with peak A and period T $\quad f(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)\,\omega_0\,t)$

analog signal-conditioning design guidelines

define the measurement objective
- parameter
  - What is the nature of the measured variable: pressure, temperature, flow, level, voltage, current, resistance, ... ?
- input range
  - What is the range of the measurement?
- accuracy
  - What is the required measurement accuracy and/or resolution?
- linearity
  - Is the measured output signal to be linear?
- measurement noise
  - What is the noise level and frequency spectrum of the measurement environment?

select a sensor
- parameter
  - What is the relationship between the sensor output and the measured variable: resistance, voltage, current, ... ?
- transfer function
  - What is the relationship between the sensor output and the measured variable?
- time response
  - What is the time response of the sensor: first-order time constant, second-order damping, frequency, ... ?
- sensor output range
  - What is the range of the sensor parameter output for the given measurement range?
- power requirement
  - What is the power specification of the sensor: resistive dissipation maximum, current draw, ... ?

design the analog signal-conditioning
- parameter
  - What is the nature of the desired output: voltage, current, frequency, ... ?
- circuit output range
  - What is the desired range of the output parameter?
- input impedance
  - What input impedance should the signal-conditioning circuit present to the sensor?
- output impedance
  - What output impedance should the signal-conditioning circuit present to the output load circuit?

analog signal-conditioning tips

(a) If the input is a resistance change and a bridge or driver must be used, be sure to consider both the effect of output voltage nonlinearity with resistance and the effect of current through the resistive sensor.

(b) For the opamp portion of the design, the easiest design approach is to develop an equation for the output versus input. From this equation, it will be clear what types of circuits may be used. This equation represents the static transfer function of the signal-conditioning.

(c) Always consider any possible loading of voltage sources by the signal-conditioning. Such loading is a direct error in the measurement system.

Thermal Sensors

metal temperature versus temperature devices

energy bands

- metal
  - The conduction and valence bands overlap so there are free electrons to conduct current independent of temperature.
- semiconductor
  - There is a small energy gap between the conduction and valence bands so a small amount of energy is required for electrons to make the jump and conduct current.
- insulator
  - There is a very large energy gap between the conduction and valence bands so a very large amount of energy is required for electrons to make the jump and conduct current.

resistance-temperature detector (RTD)

$$R(T) = \rho \frac{l}{A}(T)$$

where

R = resistance to electric current (Ω)
ρ = resistivity ($\Omega \cdot m$)
l = length (m)
A = cross-sectional area (m^2)
T = temperature which is constant

temperature versus temperature approximation

ρ is a nonlinear function of temperature for most metals.
over a small enough temperature range we can linearize ρ vs T

$$R(T) = R(T_o)\left[1 + \alpha_o \cdot \Delta T\right]$$

where

$R(T)$ = approximation of R at temperature T
$R(T_o)$ = R at temperature T_o
$\Delta T = T - T_o$
α_o = slope of R vs T curve at T_o

also use a quadratic approximation

$$R(T) = R(T_o)\left[1 + \alpha_1 \cdot \Delta T + \alpha_2 \cdot (\Delta T)^2\right]$$

where values of α_o, α_1, and α_2 are available in tables for a given metal

resistance-temperature detectors (RTD)

sensitivity $\frac{\partial \rho}{\partial T}$
response time is 0.5 s to 5 s for metal to stabilize at new T and is a function of thermal conductivity
RTD is a length of wire possibly a coil of wire to increase sensitivity
use a bridge with compensation lines to detect ΔR
bridge results in I^2R losses heating up the RTD changing R known as self-heating, as such to correct for this

$$\Delta T = \frac{P}{P_D}$$

where

ΔT = temperature rise due to self-heating (°C)
P = power dissipated in RTD (W)

P_D = dissipation constant (W/°C)

RTD have a range of -180°C to 300°C

thermistors (semiconductors)

RTD resistance increases with temperature
thermistors resistance decreases with temperature
thermistor resistance is a highly nonlinear function of temperature
sensitivity: 10% per °C
construction: bulk semiconductor material doped to adjust resistance range
range

melting: less than 300°C
package: thermistor package material limits temperature range
nonlinear: -80°C $R > 3\,M\Omega$

response time: 0.5 s to 10 s or higher depending on the package material
signal conditioning: must keep power dissipation low
self-heating: 1 mW/°C to 10 mW/°C

thermocouples (TC)

RTD and themistors are passive devices
thermocouples are active devices, emf varies with temperature
thermoelectric devices with emf approximately linear in T
based on electron movement vs temperature dependent on metal type
sensitivity: 6 μV/°C to 50 μV/°C
range: -150°C to 1765°C
response time:

10 s to 20 s for large industrial units
10 ms to 20 ms for small-gauge wire units

signal conditioning:

voltage amplification required as measured voltage ˜ 50 mV
noise issues given high impedance → differential amplifier required

reference compensation

controlled temperature reference block
reference compensation circuits
software reference correction

noise issues

low voltage, high impedance, ... in noisy industrial environment
use twisted pair in a grounded metal or foil sheath
measurement junction is 0 V (ground)
instrumentation amplifier with high common-mode rejection

other thermal sensors

bimetal strips

metals expand differing amounts for the same temperature
bimetallic sensor as an on/off switch

gas thermometers

if volume constant, pressure varies with temperature vapour-pressure thermometers

converts temperature measurement into pressure measurement

liquid-expansion thermometers

use liquid expansion with temperature in a small column to measure temperature

$$V(T) = V(T_o)\left[\,1 + \beta \cdot \Delta T\,\right]$$

where

$V(T)$ = volume at temperature T
$V(T_o)$ = volume at temperature T_o
$\Delta T = T - T_o$
β = volume thermal expansion coefficient

solid-state temperature sensors

range: -50°C to 150°C
response time: 1 s to 5 s
dissipation constant: 2 mW/°C to 20 mW/°C
accurate to 1°C
application; reference temperature sensor for thermocouples

thermal sensor system design consideration

1. identify the nature of the measurement
2. identify the required output signal
3. select an appropriate sensor
4. design the required signal conditioning

Mechanical Sensors

dispacement, location, position sensors

potentiometric sensors

linear position translated into resistance and then into a voltage
problems with: wear, electronic noise, limited resolution

capacitive sensors

$$C = \varepsilon_o \, \varepsilon_r \, \frac{A}{d}$$

where

ε_o = permittivity of free space = 8.85 pF/m
ε_r = relative permittivity of dielectric (dielectric constant K)
A = plate common area
d = plate separation

can vary C by

1. vary d
2. vary A
3. vary ε_r

use an impedance bridge to measure ΔC

inductive sensors

$$L = \mu_o \, \mu_r \, \frac{N^2 A}{d}$$

where

μ_o = permeability of free space
μ_r = relative permeability of core
N = number of turns
A = cross sectional area
d = distance

example: eddy current inspection instrument

variable-reluctance sensors

LVDT: linear variable differential transformer

level sensors

mechanical level sensors

float and linkage to a variable R or LVDT

electrical level sensors

use liquid level to vary capacitor dielectric

ultrasonic level sensors

transmit and receive the reflection
use length of time from transmit to receive as a gauge of distance to surface level

strain sensors

terms

deformation: change of shape/dimension due to force

stress: applied force

strain: resulting deformation

tensile stress-strain

$$\text{tensile stress} = \frac{F}{A}$$

where

F = applied force in N

A = cross-sectional area of sample in m^2

tensile stress: elongate or pull apart

$$\text{tensile strain} = \frac{\Delta l}{l}$$

where

Δl = change in length in m

l = original length in m

compression stress-strain

$$\text{compression stress} = \frac{F}{A}$$

where

F = applied force in N

A = cross-sectional area of sample in m^2

compression stress: compress or push together

$$\text{compression strain} = \frac{\Delta l}{l}$$

where

Δl = change in length in m

l = original length in m

shear stress-strain

$$\text{shear stress} = \frac{F}{A}$$

where

F = applied force in N

A = cross-sectional area of sample in m^2

shear stress: push/pull perpendicular

$$\text{shear strain} = \frac{\Delta x}{l}$$

where

Δx = deformation in m

l = original width of sample in m

stress-strain curve

$$E = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta l/l}$$

where E is the modulus of elasticity for tensile or compression in N/m^2

$$M = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta x/l}$$

where M is the modulus of elasticity for shear in N/m^2

strain gauge principles

$$R_o = \rho \frac{l_o}{A_o}$$

where

R_o = sample resistance in Ω
ρ = sample resistivity in $\Omega \cdot m$
l_o = length in m
A_o = cross-sectional area in m^2

$V = l_o\, A_o$ is volume in m^3
$V = l_o\, A_o = (\, l_o + \Delta l)\,(\, A_o - \Delta A)$

$$R = \rho \frac{l_o + \Delta l}{A_o - \Delta A} \approx \rho \frac{l_o}{A_o}\left(1 + 2\frac{\Delta l}{l_o} \right)$$

$$\Delta R \approx 2\, R_o \frac{\Delta l}{l_o}$$

example: $R_{nom} = 120\,\Omega$, $\frac{\Delta l}{l_o} = \frac{1\,mm}{m}$, $\Delta R = 0.24\,\Omega$

temperature effects

$$R(\,T\,) = R(\,T_o\,)\left[\, 1 + \alpha_o \cdot \Delta T \right]$$

$$\Delta R_T = R_o \cdot \alpha_o \cdot \Delta T$$

where

ΔR_T = resistance change due to ΔT
α_o = 0.004/°C
ΔT = change in temperature
$R(\,T_o\,)$ = nominal R at reference temperature T

example: $R_{nom} = 120\,\Omega$, α_o = 0.004/°C, ΔT = 1°C, $\Delta R_T = 0.48\,\Omega$
ΔR_T can be double ΔR due to strain

metal strain gauges

$$\text{Gauge Factor} = GF = \frac{\Delta R/R}{\text{strain}} = \frac{\Delta R/R}{\Delta l/l} = \frac{\text{fractional change in resistance}}{\text{fractional change in length}}$$

GF tends to be around 2 but could go as high as 10
R_{nom} = { 60, 120, 240, 350, 500, 1000 } Ω, typically $R_{nom} = 120\,\Omega$
use active and dummy strain gauge for temperature compensation bridge measurement

one-arm bridge

$$\Delta V = -\frac{V_s}{4}\, GF \frac{\Delta l}{l}$$

two-arm bridge

$$\Delta V = -\frac{V_s}{2}\, GF \frac{\Delta l}{l}$$

four-arm bridge

$$\Delta V = -\frac{V_s}{1}\, GF \frac{\Delta l}{l}$$

semiconductor strain gauges

higher (negative) gauge factor but highly nonlinear signal conditioning uses a bridge but still must correct nonlinearity

motion sensors

$$v(t) = \frac{d\,x(t)}{dt} \quad\rightarrow\quad a(t) = \frac{d\,v(t)}{dt} = \frac{d^2\,x(t)}{dt^2} \quad\rightarrow\quad v(t) = v(0) + \int_0^t a(t)\,dt \quad\rightarrow\quad x(t) = x(0) + \int_0^t v(t)\,dt$$

$g \approx 9.8$ m/s = acceleration of gravity

spring-mass system

$$m \cdot a = k \cdot \Delta x \quad\rightarrow\quad a = \frac{k}{m} \cdot \Delta x$$

where

- k = spring constant in N/m
- Δx = spring extension in m
- m = mass in kg
- a = acceleration in m/s^2

natural frequency and damping

$$f_N = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

where

- f_N = natural frequency in Hz
- k = spring constant in N/m
- m = seismic mass in kg

giving rise to

$$X_T(t) = X_o\, e^{-\alpha t} \sin(2\pi \cdot f_N \cdot t)$$

where

- $X_T(t)$ = transient mass position
- α = damping coefficient

vibration effects

$$a(t) = -\,\omega^2 \cdot X_o \cdot \sin(\omega t) \quad ; \quad \Delta X = -\frac{m \cdot X_o}{k}\,\omega^2 \sin(\omega t) \quad ; \quad \text{valid up to } \frac{f_N}{2.5}$$

accelerometers

- potentiometric: spring mass attached to wiper arm of potentiometer (30 Hz)
- LVDT: LVDT core itself is the seismic mass (80 Hz)
- variable reluctance: geophone (100 Hz)
- piezoelectric: spring mass attached to piezoelectric crystal (5 kHz) but output in millivolt range

pressure sensors

terms

- pressure = force per unit area $\frac{F}{A}\left[\frac{N}{m^2}\right]$
- statics pressure: fluid is not moving
- dynamic pressure: fluid is in motion
- pascal (Pa) $1\,\text{Pa} = 1\,\frac{N}{m^2}$
- 1 psi ≈ 6.895 kPa

atmosphere (atm) $1\,\text{atm} = 101.325\,\text{kPa} \approx 14.7\,\text{psi}$

gauge pressure

$$p_g = p_{abs} - p_{at}$$

where

p_g - gauge pressure
p_{abs} = absolute pressure
p_{at} = atmospheric pressure

head pressure

$$p = \rho \cdot g \cdot h$$

where

p = pressure in Pa
ρ = density in $\frac{kg}{m^3}$
g = acceleration due to gravity $(9.8\,\frac{m}{s^2})$
h = depth in liquid in m

or

$$p = \rho_w \cdot h$$

where

p = pressure in $\frac{lb}{ft^2}$
ρ_w = weight density in $\frac{lb}{ft^3}$
h = depth in ft

sensor types for $p > 1$ atmosphere

diaphragm pressure sensors

$$F = \left(p_2 - p_1 \right) \cdot A$$

Bellows pressure sensors

Bourdon tube

solid-state pressure sensors

sensitivity 10 to 100 $\frac{mV}{kPa}$
response time $10\,\text{ms}$ (10% to 90% response time)
linear voltage vs pressure
3-terminals: DC power, ground, sensor output voltage

sensor types for $p < 1$ atmosphere

Pirani gauge

thermocouple

ionization gauge

range: $10^{-3}\,\text{atm}$ to $10^{-13}\,\text{atm}$

flow sensors

sensor types

solid flow

$$Q = \frac{W\,R}{L}$$

where

Q = flow in $\frac{\text{kg}}{\text{min}}$ or $\frac{\text{lb}}{\text{min}}$
W = weight of material on section L in kg or lb
R = conveyor speed in $\frac{\text{m}}{\text{min}}$ of $\frac{\text{ft}}{\text{min}}$
L = length of weighing platform in m or ft

liquid flow

$$V = \frac{Q}{A}$$

where

V = flow velocity
Q = volume flow rate
A = cross-sectional area

and

$$F = \rho \cdot Q$$

where

F = mass or weight flow rate
ρ = mass density or weight density
Q = volume flow rate

restriction flow sensors

$$Q = k\sqrt{\Delta p}$$

where

Q = volume flow rate
K = a constant for the pipe and liquid type
Δp = drop in pressure across the restriction

pitot tube
obstruction flow sensors
magnetic flow meter

Optical Sensors

terms

speed of propagation

$$c = \lambda f$$

where

$c = 2.998 \cdot 10^8$ m/s $\approx 3 \cdot 10^8$ m/s in a vacuum
λ = wave length in m
f = frequency in Hz

index of refraction

$$n = \frac{c}{v}$$

where

v = velocity of EM in a given material in m/s

photon

$$W_p = h \cdot f = \frac{h \cdot c}{\lambda}$$

where

Planck's constant $h = 6.63 \cdot 10^{-34}$ J $\cdot$ s
W_p = photon energy in J
$1\text{ eV} = 1.602 \cdot 10^{-19}$ J

light intensity

$$I = \frac{P}{A}$$

where

I = intensity in $\frac{W}{m^2}$
P = power
A = beam cross-sectional area in m^2

divergence

θ angle the beam diverges

maximum divergence

$$I = \frac{P}{4 \pi R^2}$$

where

R = radius of sphere

photodetectors

photoconductive detectors

$$E_p = \frac{h \cdot c}{\lambda_{max}} = \Delta W_g \quad \rightarrow \quad \lambda_{max} = \frac{h \cdot c}{\Delta W_g}$$

where

λ_{max} = maximum detectable radiation wave length in m

photovoltaic detectors

$$V_c = V_0 \ln (1 + I_R)$$

where

V_c = open-circuit cell voltage
V_0 = material constant
I_R = light intensity

I_{sc} varies linearly with light intensity

photodiode detectors

phototransistors

photoemissive detectors

photomultiplier tube
gains of 10^5 to 10^7

pyrometry

total radiation
$E \alpha T^4$

E in J/s per unit area or w/m^2
T temperature in K

wide band for noncontact temperature measurement

optical sources

LASER = light amplification by stimulated emission radiation