Hochschule
Bonn-Rhein-Sieg
University of Applied Sciences

b-it Bonn-Aachen International Center for Information Technology

Institute for AI and Autonomous Systems

Calculus

Speaker Sven Ludwig

Authors Divin, Santosh, Michal, Musharraf, Kishan, Sven

Course Calculus Refresher

Program Autonomous Systems (MSc)

2024-09-18

Introduction

What is Calculus?

A branch of mathematics which deals with "*study of continuous change*" of functions or sequences

Introduction

What is Calculus?

A branch of mathematics which deals with "*study of continuous change*" of functions or sequences

Major Branches

- Differential Calculus
- Integral Calculus

Introduction

What is Calculus?

A branch of mathematics which deals with "*study of continuous change*" of functions or sequences

Major Branches

- Differential Calculus
- Integral Calculus

General Applications

- Optimisation
- To Solve Differential Equations
- Function Approximations
- Length, Area, Volume, Center of Mass, Moment of Inertia Calculations

Applications in Robotics

- **Kinematics**
  - Modelling mobile robot (eg: differential drive) and environment
  - Transform between different spaces

Applications in Robotics

- Kinematics
  - Modelling mobile robot (eg: differential drive) and environment
  - Transform between different spaces
- Dynamics
  - Euler-Lagrange equations

Applications in Robotics

- **Kinematics**
  - Modelling mobile robot (eg: differential drive) and environment
  - Transform between different spaces
- **Dynamics**
  - Euler-Lagrange equations
- **Machine Learning**
  - Optimisation

Calculus in Deep Learning

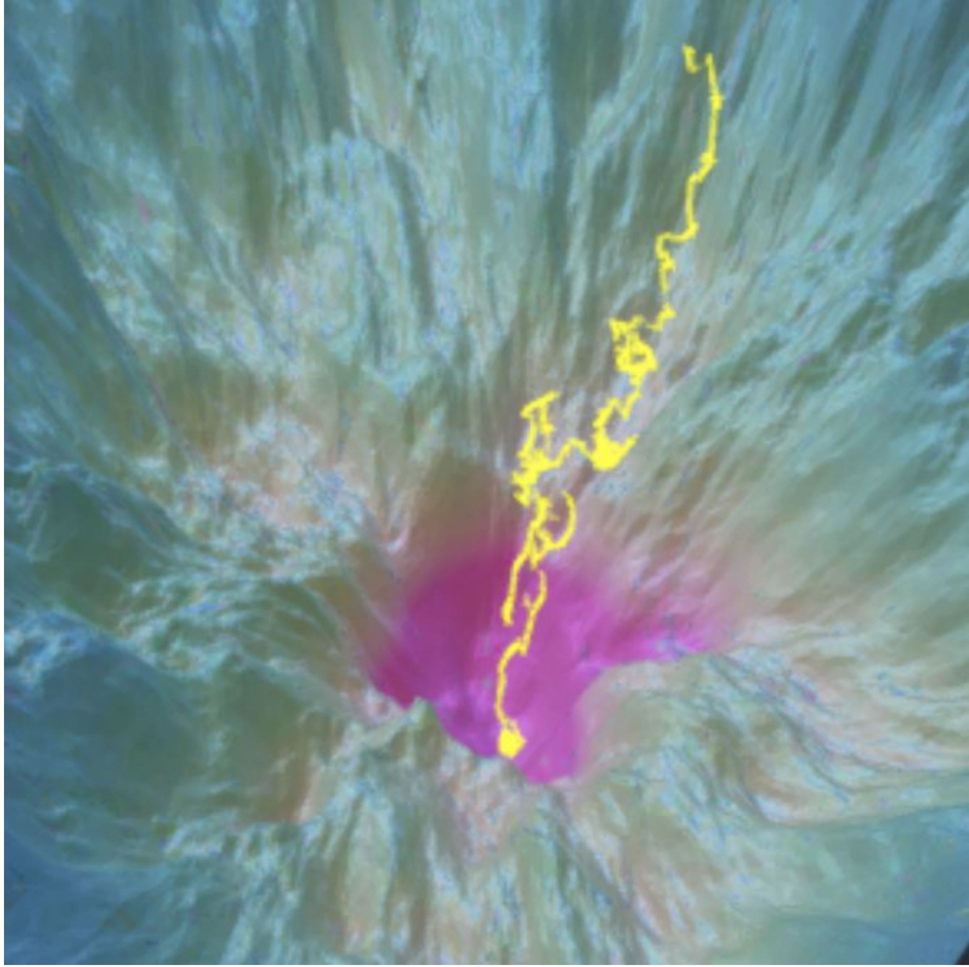
Prominent application of computing the gradient, i.e. first derivatives, when minimizing loss function

- Loss landscape is all loss values over all combinations of parameter values.

- Computing gradient, i.e. partial first derivatives, requires some form of differentiation of the loss function.

- Gradient indicates how to update the model parameters θ to minimize loss.

- With *stochastic* gradient descent, model parameters are updated iteratively with respect to subsets (batches) of the training dataset.



Javier Ideami (2020-2024). Online Loss Landscapes Explorer https://losslandscape.com/explorer accessed 2024-07-05

Calculating first derivatives in batches on the GPU with PyTorch

```python
# given x values
x = torch.tensor([-3., -2., -1., 0., 1., 2., 3.], dtype=torch.float32, device=device, requires_grad=True)

# f(x) = x^2
two_dot_zero = torch.tensor(2.0, dtype=torch.float32, device=device)
y = x.pow(two_dot_zero)

# f'(x) via autograd
ones = torch.ones_like(y, dtype=torch.float32, device=device)
y.backward(ones)
gradients = x.grad

print(f"x     = {x}")
print(f"f(x)  = {y}")
print(f"f'(x) = {gradients}")
```

```
CUDA is available. Using GPU.
x     = tensor([-3., -2., -1.,  0.,  1.,  2.,  3.], device='cuda:0',
       requires_grad=True)
f(x)  = tensor([9., 4., 1., 0., 1., 4., 9.], device='cuda:0', grad_fn=<PowBackward1>)
f'(x) = tensor([-6., -4., -2.,  0.,  2.,  4.,  6.], device='cuda:0')
```

Adam Paszke et al. (2019). PyTorch: An Imperative Style, High-Performance Deep Learning Library https://pytorch.org/

Calculating first derivatives in batches on the GPU with JAX

```
print(jax.devices())

def f(x):
    return x**2.0

x = jnp.array([-3., -2., -1., 0., 1., 2., 3.])
f_prime = jax.grad(f)
gradients = []
for i in range(0, len(x)):
  y_prime = f_prime(x[i])
  gradients.append(float(y_prime))

print(f"x     = {x.tolist()}")
print(f"f(x)  = {f(x).tolist()}")
print(f"f'(x) = {gradients}")
```

```
[cuda(id=0)]
x     = [-3.0, -2.0, -1.0, 0.0, 1.0, 2.0, 3.0]
f(x)  = [9.0, 4.0, 1.0, 0.0, 1.0, 4.0, 9.0]
f'(x) = [-6.0, -4.0, -2.0, 0.0, 2.0, 4.0, 6.0]
```

James Bradbury et al. (2018). JAX: Composable transformations of Python+NumPy programs https://jax.readthedocs.io

Applications in Robotics

- **Kinematics**
  - Modelling mobile robot (eg: differential drive) and environment
  - Transform between different spaces
- **Dynamics**
  - Euler-Lagrange equations
- **Machine Learning**
  - Optimisation
- **Modelling Robot Control Systems**
  - PID controllers

Applications in Robotics

- **Kinematics**
  - Modelling mobile robot (eg: differential drive) and environment
  - Transform between different spaces
- **Dynamics**
  - Euler-Lagrange equations
- **Machine Learning**
  - Optimisation
- **Modelling Robot Control Systems**
  - PID controllers
- **Computer Vision**
  - Changes in color and intensities (Eg: MRI scan)

Applications in Robotics

- Kinematics
  - Modelling mobile robot (eg: differential drive) and environment
  - Transform between different spaces
- Dynamics
  - Euler-Lagrange equations
- Machine Learning
  - Optimisation
- Modelling Robot Control Systems
  - PID controllers
- Computer Vision
  - Changes in color and intensities (Eg: MRI scan)
- Signal Processing
  - Data analysis (Eg: stock market, speech signals, sensor data)
  - Fourier transformations

Applications in Robotics

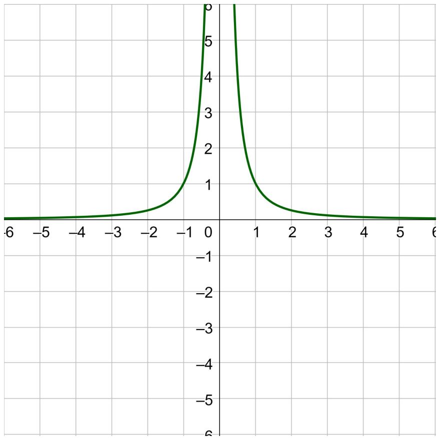
- Kinematics
  - Modelling mobile robot (eg: differential drive) and environment
  - Transform between different spaces
- Dynamics
  - Euler-Lagrange equations
- Machine Learning
  - Optimisation
- Modelling Robot Control Systems
  - PID controllers
- Computer Vision
  - Changes in color and intensities (Eg: MRI scan)
- Signal Processing
  - Data analysis (Eg: stock market, speech signals, sensor data)
  - Fourier transformations
- Logic
  - Situational calculus

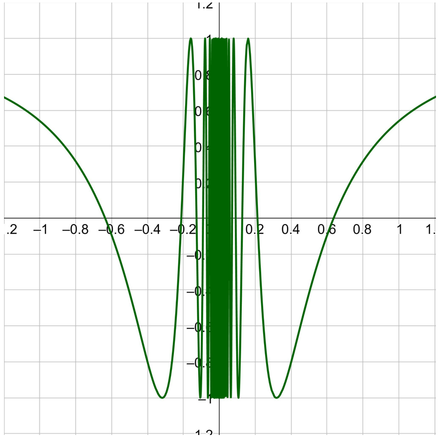
Limits and Continuity

Functions vs Relations

- Functions
  Eg: $y = x^2+4x$

- Relations
  Eg: $x^2+y^2 = 25$

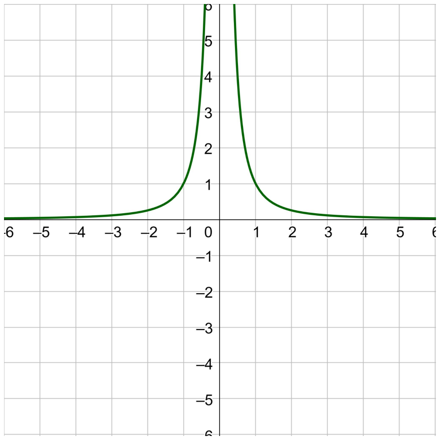


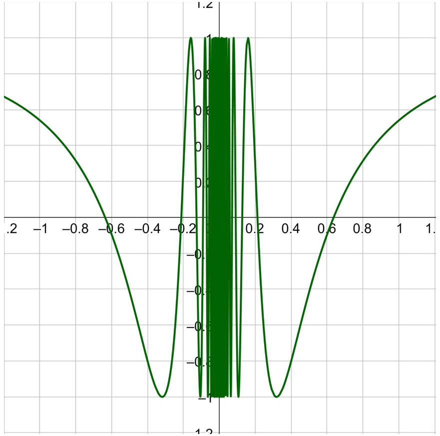


Limits and Continuity

Functions vs Relations

- Functions have one unique output for a given input
  Eg: $y = x^2+4x$
- Relations can have more than one output
  Eg: $x^2+y^2 = 25$





Limits and Continuity

Functions vs Relations

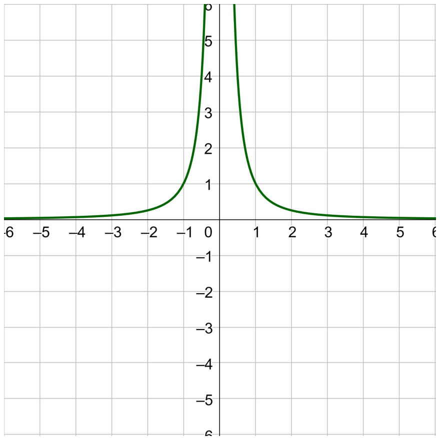
- Functions have one unique output for a given input
  Eg: $y = x^2+4x$
- Relations can have more than one output
  Eg: $x^2+y^2 = 25$

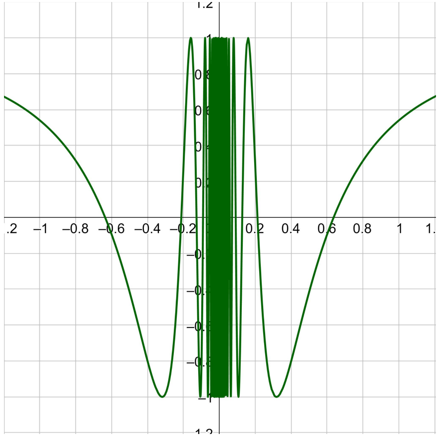
Limit

- Unique value of a function when its input approaches a particular number from both sides

  $\lim_{x \to c} f(x) = L$

  Eg: $f(x) = \frac{x^2-1}{x-1}$.





Limits and Continuity

Functions vs Relations

- Functions have one unique output for a given input
  Eg: $y = x^2+4x$
- Relations can have more than one output
  Eg: $x^2+y^2 = 25$

Limit

- Unique value of a function when its input approaches a particular number from both sides

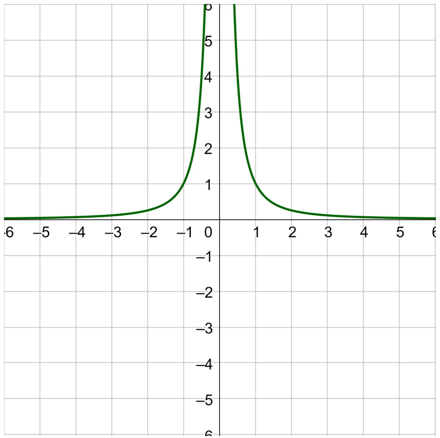
$$\lim_{x\to c} f(x) = L$$

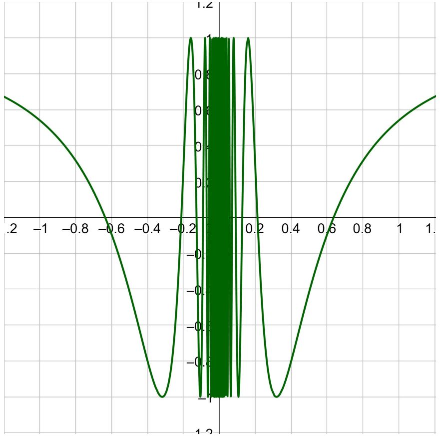
Eg: $f(x) = \frac{x^2-1}{x-1}$.

Continuity

- For all values of input to a function, it must satisfy,

$$\lim_{x\to c} f(x) = f(c)$$





Characteristics of Data

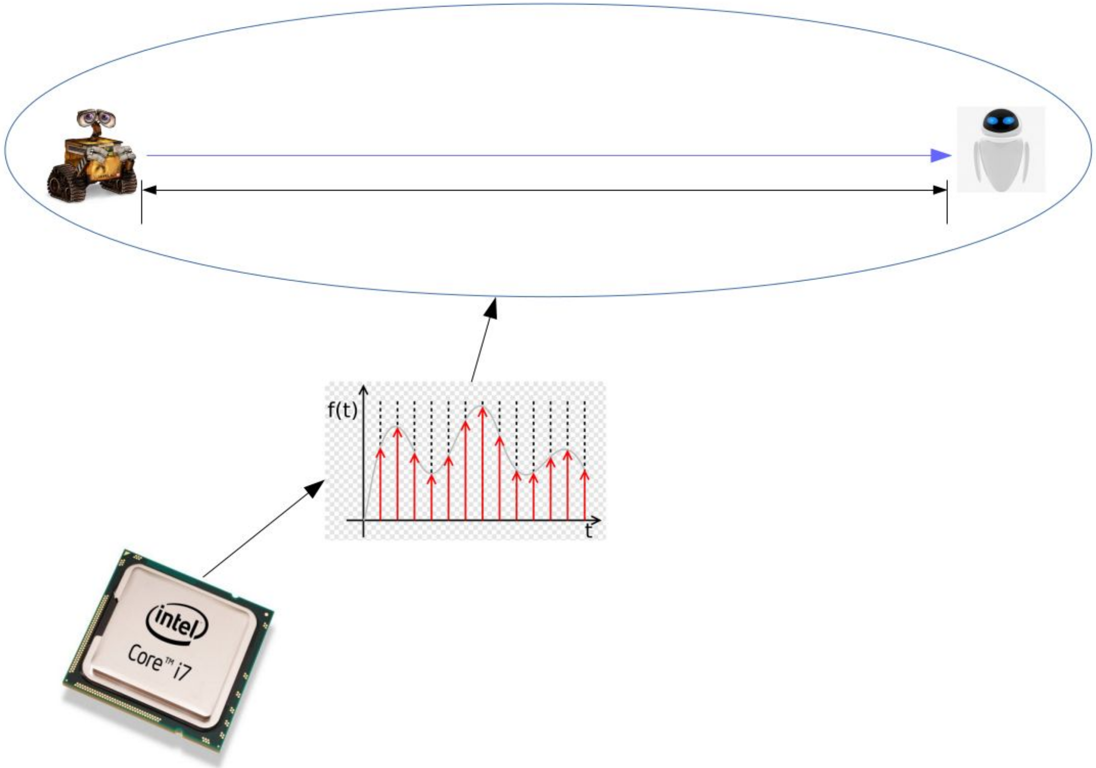
Analog

- continuous

Digital

- discrete

Differential Calculus

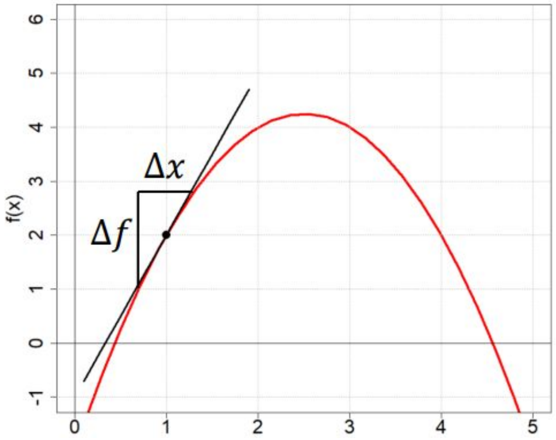


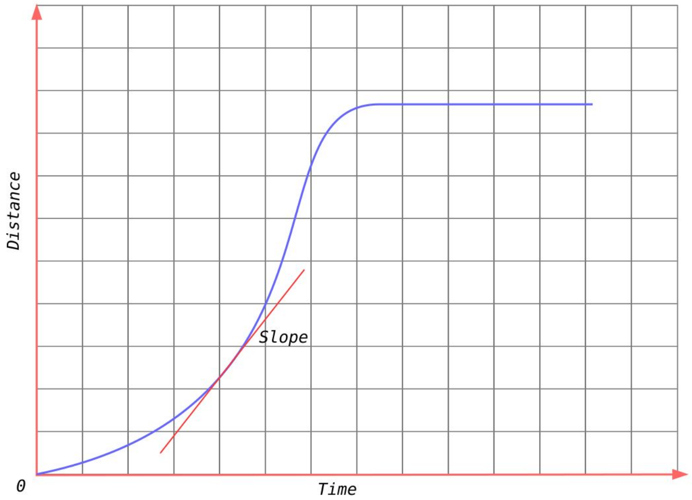
Differential Calculus

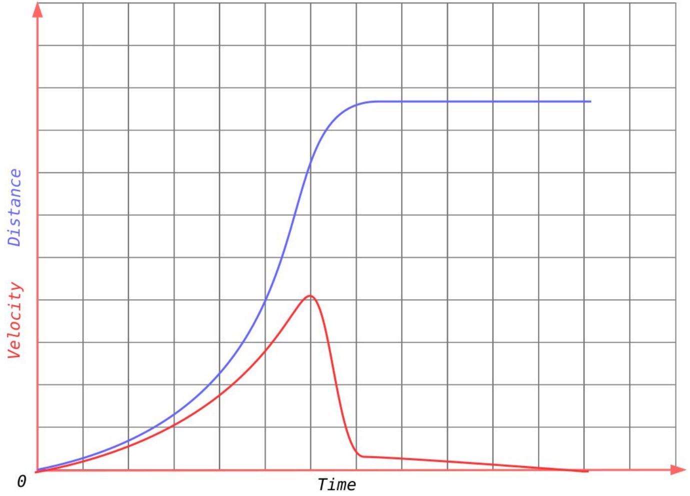
Derivative of a Function

- Rate of change
- In a 2D system, derivatives = slope = tan(θ)

Leibniz notation: $y = x^2; \quad \frac{dy}{dx} = 2x$







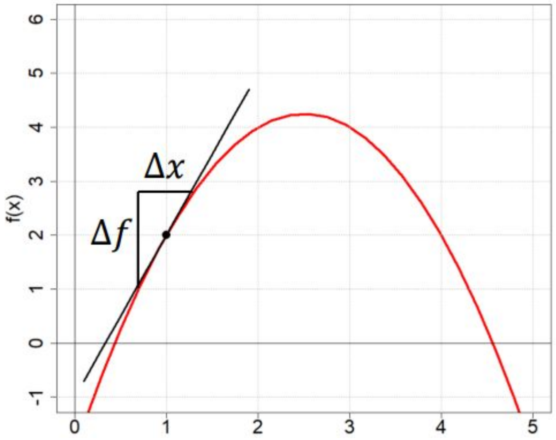
Differential Calculus

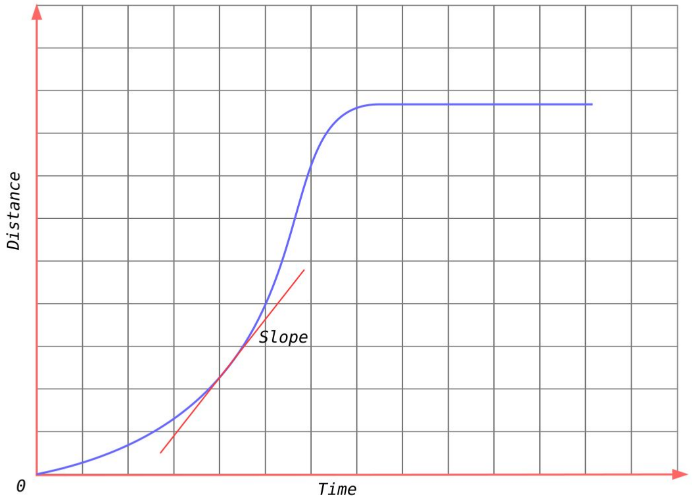
Derivative of a Function

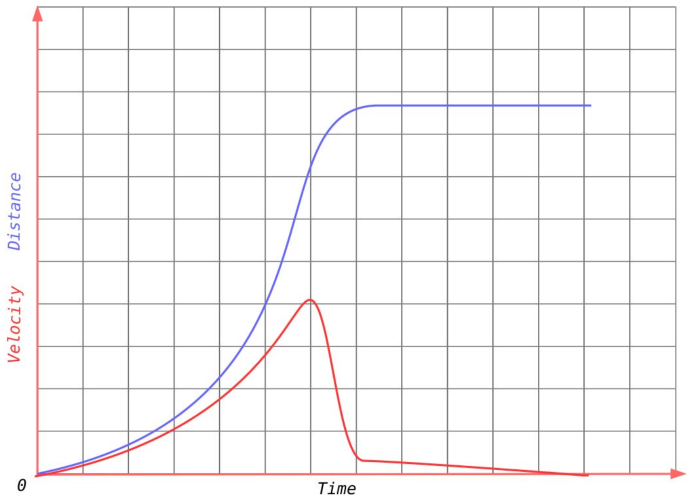
- Rate of change
- In a 2D system, derivatives = slope = tan(θ)
- *Another view:*

  The line itself is the best linear approximation

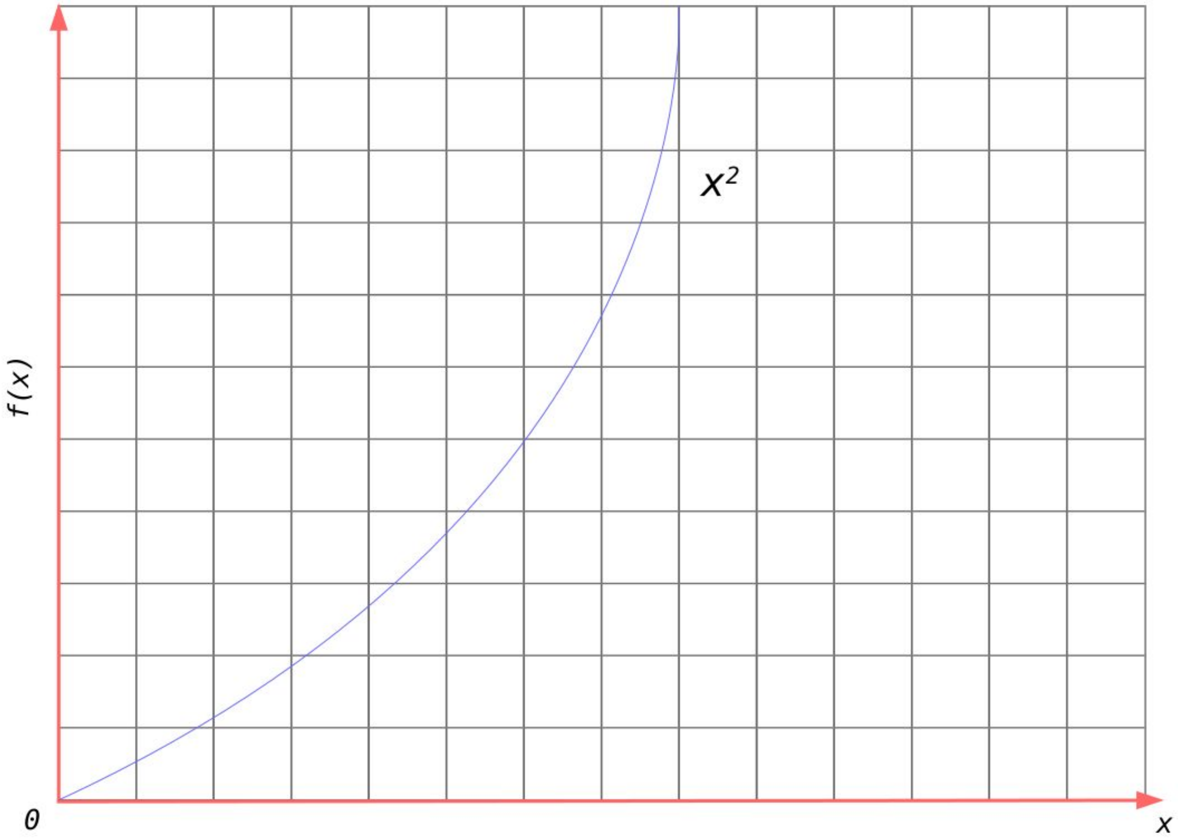
Leibniz notation: $y = x^2; \quad \frac{dy}{dx} = 2x$





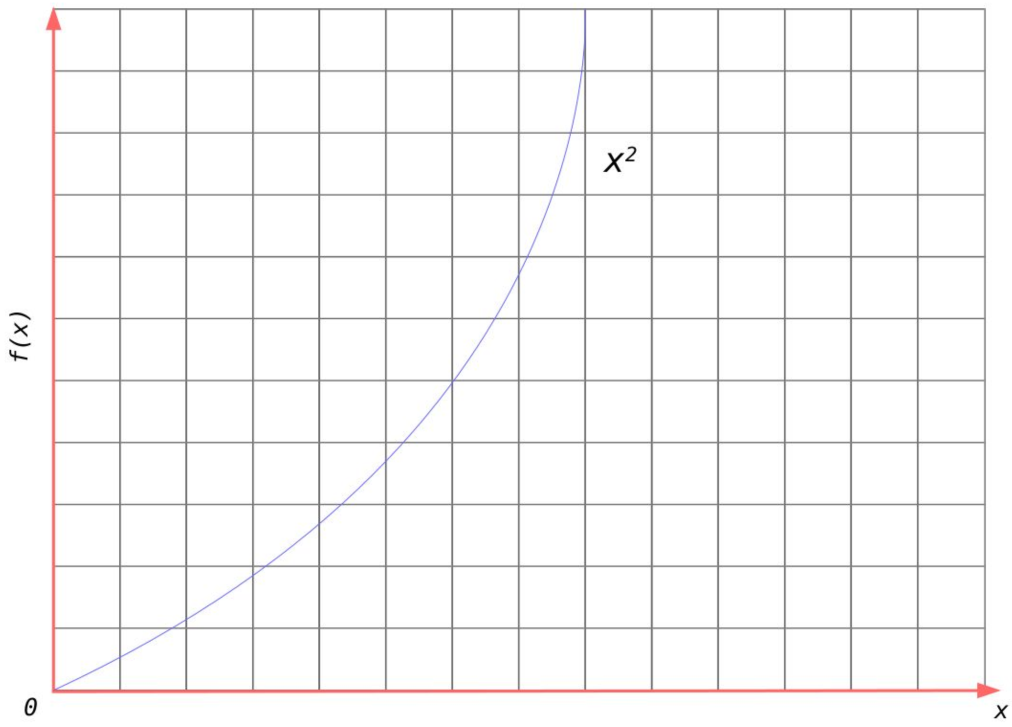


Differential Calculus



Differential Calculus

How to handle discrete functions?



Differential Calculus

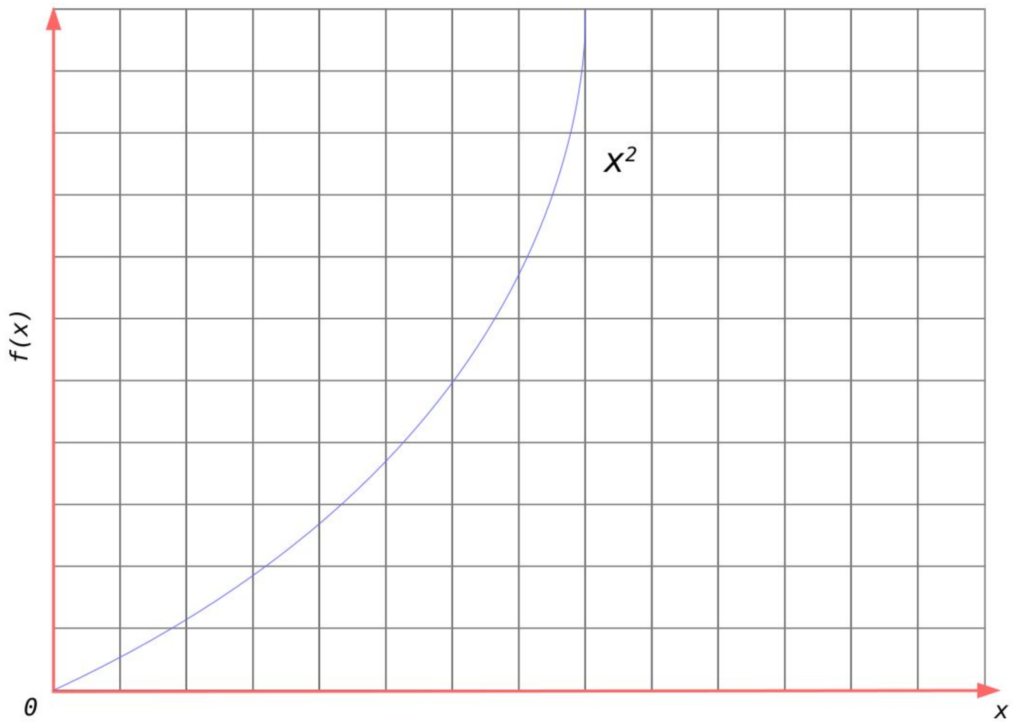
How to handle discrete functions?

Approximate!

$$f'(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

h – small, positive, fixed epsilon



Differential Calculus - Rules (single variable)

Addition Rule ⟶ $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

Product Rule ⟶ $\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)$

Power Rule ⟶ $\frac{d}{dx}[x^n] = n\cdot x^{n-1}$

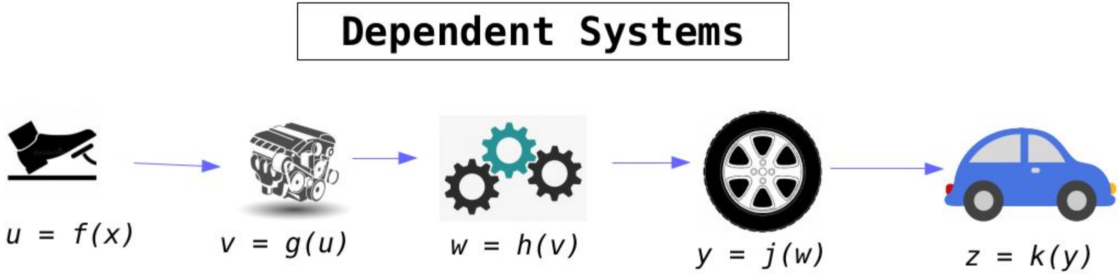
Quotient Rule ⟶ $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)}{(g(x))^2}$

Derivative of Constant ⟶ $\frac{d}{dx}[c] = 0$

Chain Rule ⟶ $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Other Rules: For logarithmic and exponential functions

Differential Calculus - Chain Rule



$$\frac{dz}{dx} = \frac{dz}{dy} * \frac{dy}{dw} * \frac{dw}{dv} * \frac{dv}{du} * \frac{du}{dx}$$

Differential Calculus - Multivariable Functions

How to handle derivatives of multivariable functions?

Partial Derivatives: Taking derivative of each variable and holding others as constants

- Notations: $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

Differential Calculus - Multivariable Functions

How to handle derivatives of multivariable functions?

Partial Derivatives: Taking derivative of each variable and holding others as constants

- Notations: $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

  $$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

- Application: *Jacobians*

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

Differential Calculus - Multivariable Functions

How to handle derivatives of multivariable functions?

Partial Derivatives: Taking derivative of each variable and holding others as constants

- Notations: $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

  $$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

- Application: *Jacobians*

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

Cross Partial Derivatives

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Differential Calculus - Multivariable Functions

Total Derivatives: if there is a function f(x,y), then the total derivative is represented as sum f partial derivatives of each variable times the derivative of that variable

Differential Calculus - Multivariable Functions

Total Derivatives: if there is a function f(x,y), then the total derivative is represented as sum f partial derivatives of each variable times the derivative of that variable

In this example, assuming function ‘f’ is not directly dependent on ‘t’ and ‘t’ is independent variable, the total derivative can be represented as,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Differential Calculus - Multivariable Functions

Total Derivatives: if there is a function f(x,y), then the total derivative is represented as sum f partial derivatives of each variable times the derivative of that variable

In this example, assuming function 'f' is not directly dependent on 't' and 't' is independent variable, the total derivative can be represented as,

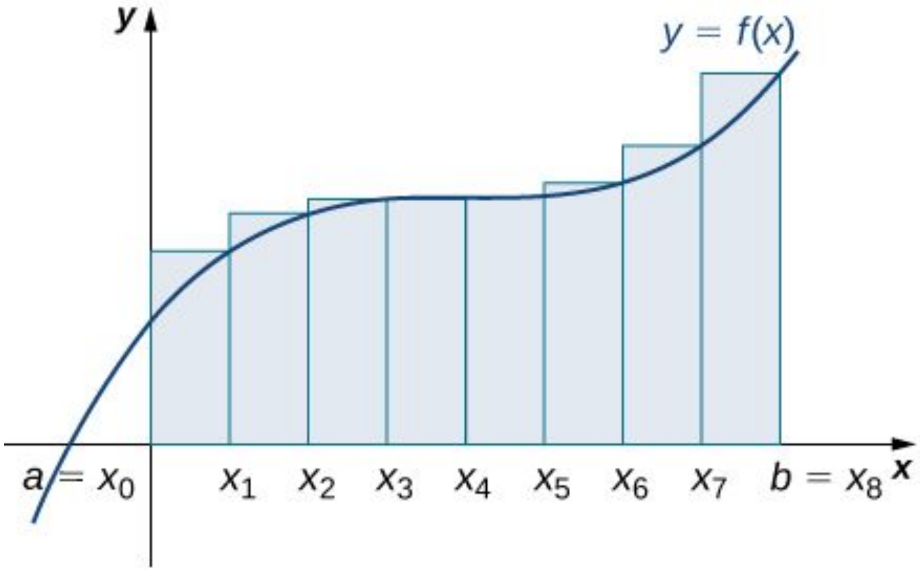
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Taylor Series: Approximations of functions. At x=0, this series is also termed as Maclaurin series.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Integral Calculus

Integration: it is analogous to summation. It is also termed as antiderivative

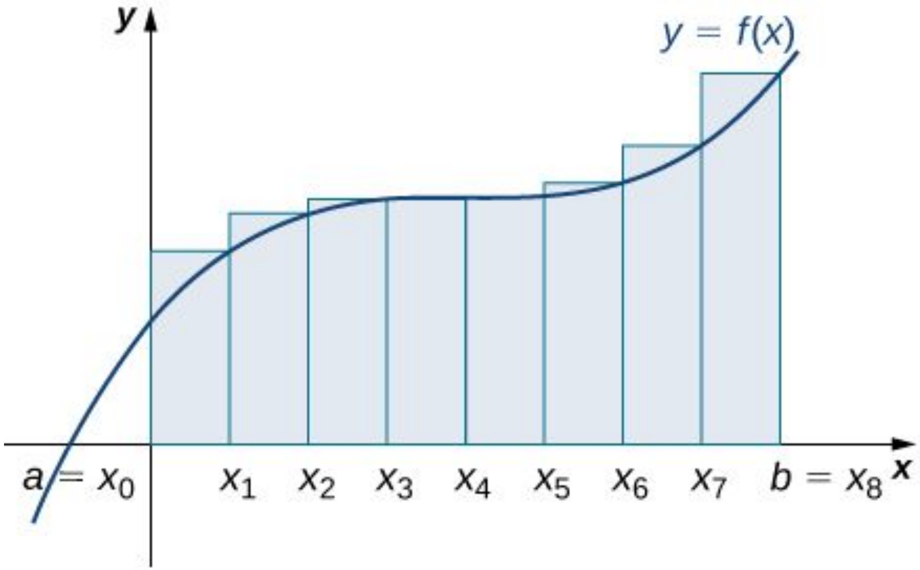


Integral Calculus

Integration: it is analogous to summation. It is also termed as antiderivative

Types:

- *Indefinite integrals:* bounds are undefined
- *Definite integrals:* bounds are defined



Integral Calculus

Integration: it is analogous to summation. It is also termed as antiderivative

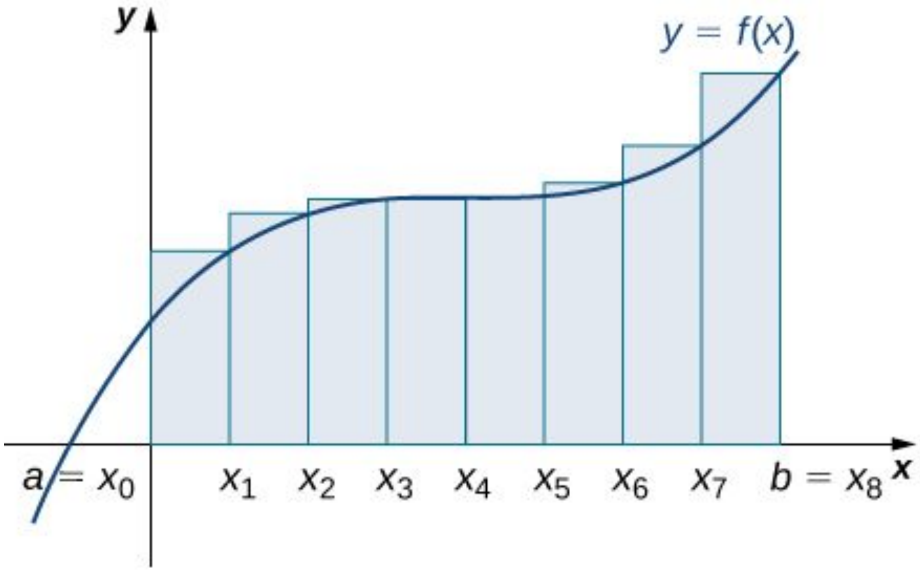
Types:
- *Indefinite integrals:* bounds are undefined
- *Definite integrals:* bounds are defined

Example:

$y_1 = x^2 + 20$
$y_2 = x^2 + 100$

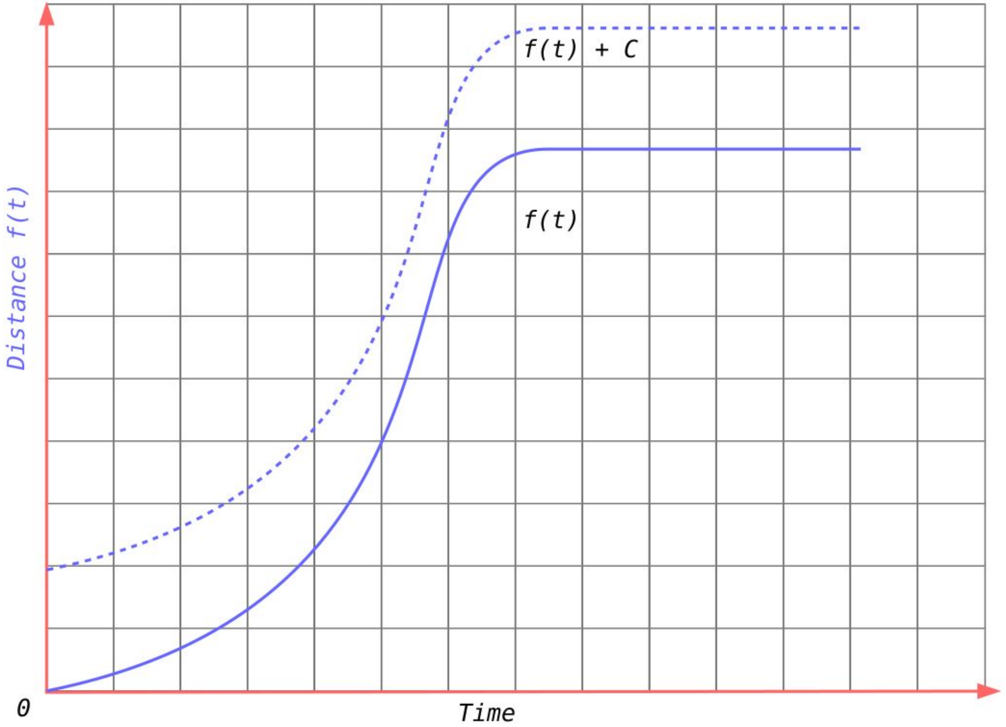
$y'_1 = y'_2 = 2x$

Thus,
$\int 2x \, dx = x^2 + c$

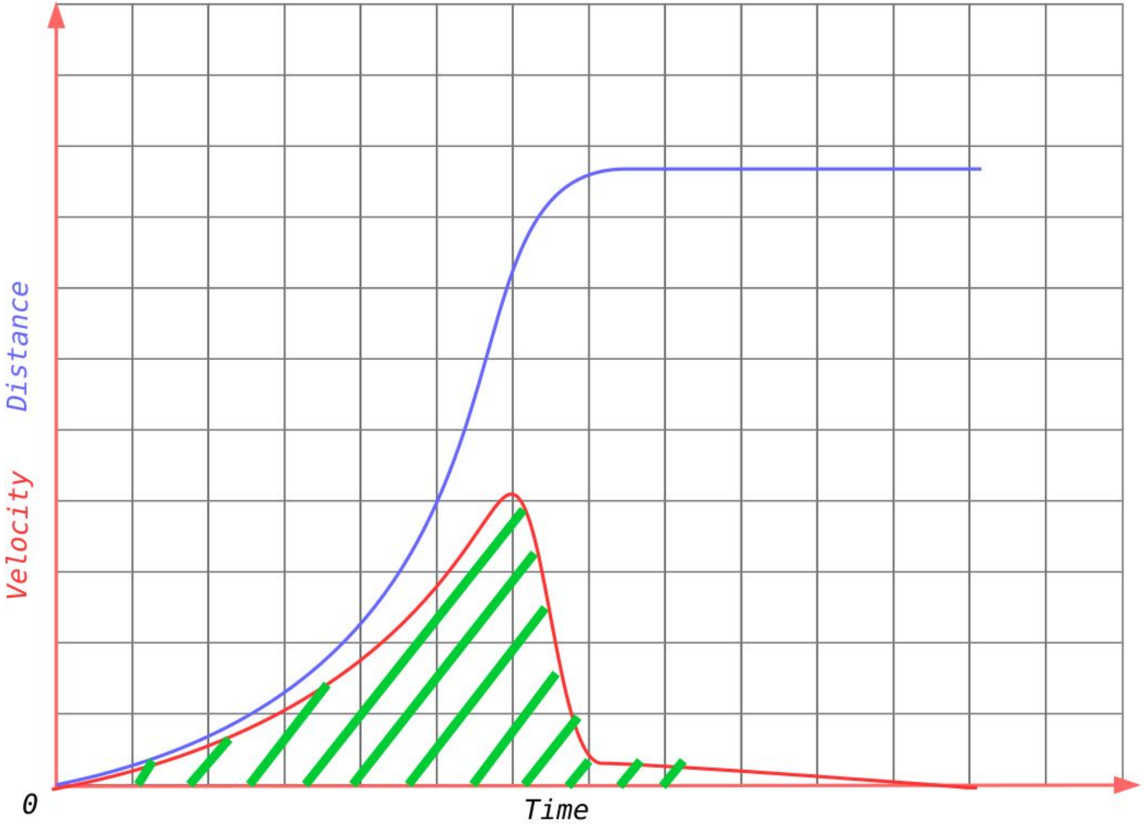


Integral Calculus

Bounds



Integral Calculus



Integral Calculus

What about discrete functions?

Integral Calculus

What about discrete functions?

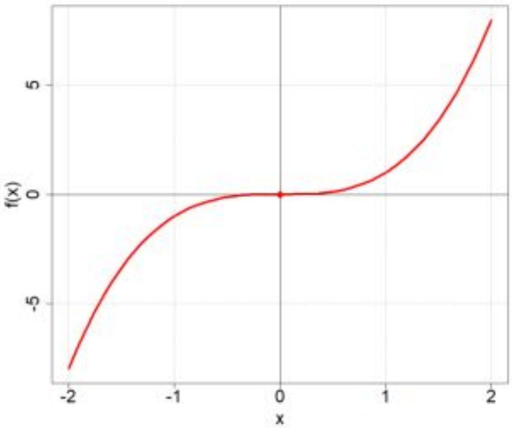
Approximate!

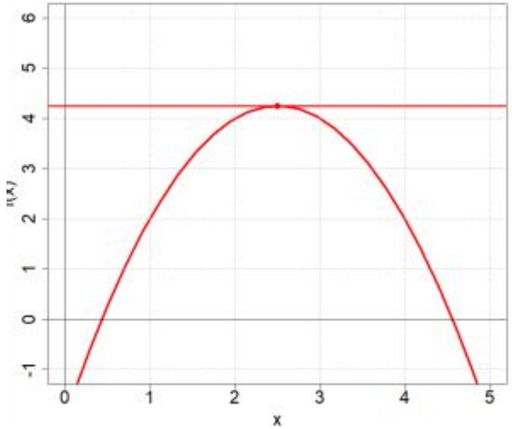
$$\int_a^b f(x)dx \approx \frac{h}{2}\sum_{k=0}^{n-1}(s_{k+1} + s_k)$$

h – small, positive, fixed epsilon

Optimisation

Inflection Point vs Critical Point





References

1. https://www.math.utoronto.ca/vohuan/mat13717.html
2. https://sites.nicholas.duke.edu/onboarding/files/2015/05/Calculus-Review-Presentation.pdf
3. https://www.sfu.ca/~vjungic//Precalculus/Precalculus.pdf
4. https://www.andrews.edu/~rwright/Precalculus-RLW/Text/12.html
5. https://en.wikipedia.org/wiki/Total_derivative
6. https://www.youtube.com/watch?v=wCZ1VEmVjVo
7. https://losslandscape.com/explorer
8. https://pytorch.org/
9. https://jax.readthedocs.io/en/latest/quickstart.html
10. https://www.youtube.com/watch?v=WUvTyaaNkzM

Thank You!