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Linear Algebra

A (re-)introduction

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Linear Algebra

What is it?

- Deals with linear equations of the form:

$$a_1x_1 + ... + a_nx_n = b$$

- Represented as systems of equations, matrices and vectors
- **Python Coding for Linear Algebra** in a separate session

Linear Algebra

Why is it important?

- A powerful way of representing and solving problems in:
  - physics and engineering
  - computer vision and graphics
  - robotics
  - economics
  - and more
- Fundamental in understanding:
  - geometry
  - optimization
  - machine learning
  - and more

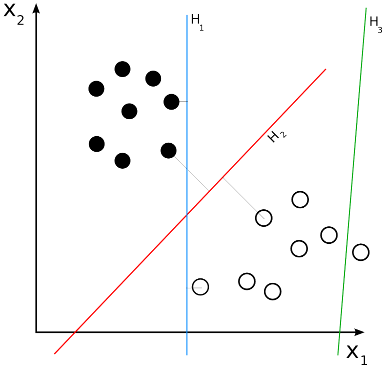

Figure 1: Linear Classifiers - Image from Wikipedia, by Cyc; Public Domain

In Machine Learning

Example in Single-Layer Perceptron

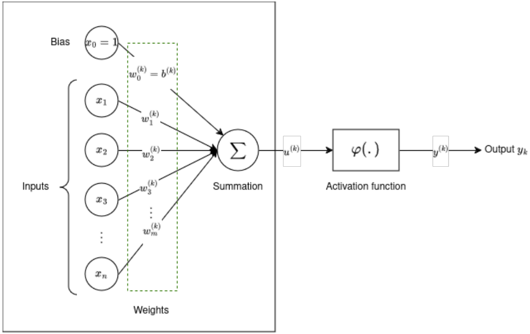

Figure 2: Single-Layer Perceptron - Image by Harley Lara

The highlighted part is of the form: $x_1 w_1 + ... + x_n w_n + x_0$

In Computer Graphics

Example from Blender

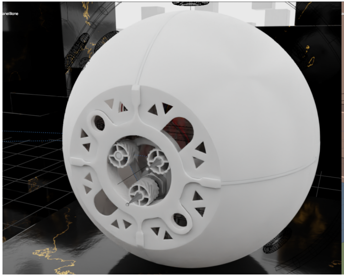

Figure 3: Rendered view

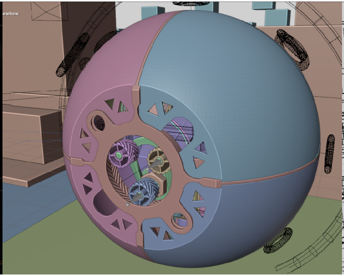

Figure 4: Solid view

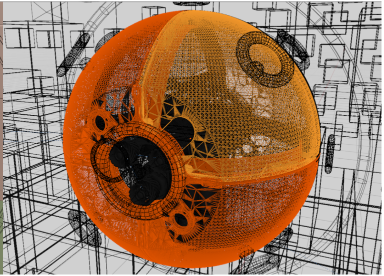

Figure 5: Geometry / wireframe

It's all lines (mostly)

Vectors

- Quantities with magnitude and direction
- Represent positions, velocities, accelerations in space
- Consider vector P from $(0, 0)$ to $(5, 4)$
  - $||P|| = 6.4031$ - distance from origin
  - $\angle P = 38.659°$ - angle from origin
- How was this calculated?

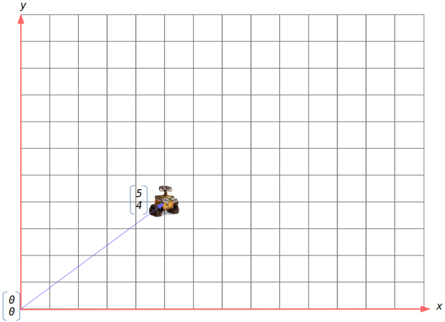

Figure 6: Position vector - Image from a previous session by Divin and Santosh

Vector Norms

- Represent the magnitude of a vector
- Mapping from vector space to non-negative real numbers
  - L1 norm:
    $||x||_1 = \sum_{i=1}^{n} |x_i|$
  - L2 norm (Euclidean distance):
    $||x||_2 = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$
  - p-norm:
    $||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$
- Use `numpy.linalg.norm()`

Vector Angles

- Represent direction of vector
- Given by $\tan^{-1}(\frac{y}{x})$
- Also given by $\tan^{-1}(\frac{\text{opposite}}{\text{adjacent}})$
- Use `numpy.arctan2()` and `numpy.arctan()`
- `numpy.arctan2()` considers quadrants
- Use `numpy.rad2deg()` or `numpy.deg2rad()` for conversions

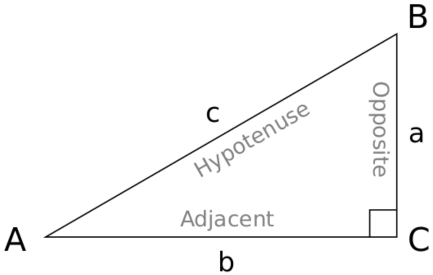

Figure 7: Triangle sides for $\angle A$ From Wikipedia; By TheOtherJesse - Own work, Public Domain

Matrices

- Multi-dimensional arrays of elements
- Dimensions given by number of rows, columns, depth, and so on
- Can be added, subtracted, multiplied (according to rules)
- Elegant way to represent
  - vectors, rotations, translations (and composite transformations)
  - systems of equations
  - masks, kernels, images, and so on
- `numpy.array()`, `numpy.zeros()`, `numpy.ones()`, `numpy.identity()`, `numpy.eye()` useful for matrix construction

Matrix Operations

Notable rules

- Addition (**translation**) possible with matrices of same dimensions
  e.g. $(m \times n)$ and $(m \times n)$, or (n) and (n)
- Scalar multiplication with a scalar
- Matrix multiplication possible with arrays of equal adjacent dimension
  e.g. $(m \times n)$ and $(n \times o)$ giving $(m \times o)$
- Important to note when performing operations, debugging code

Systems of Linear Equations

and their representations

- Two or more equations of the same set of variables, e.g.:

$$2x + 3y = 7$$

$$x - y = 1$$

- Various methods exist to solve such systems:
  - substitution
  - elimination
  - graphical methods
  - matrix methods

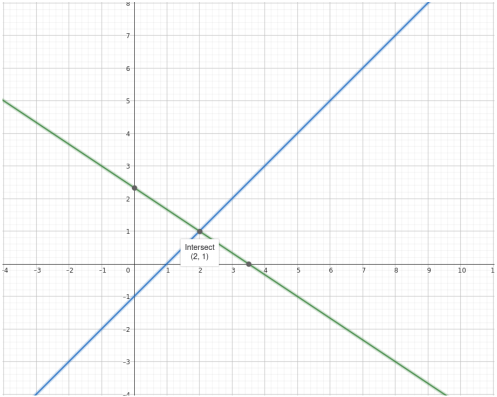

Figure 8: Graphical representation plotted in GeoGebra

Systems of Linear Equations

Matrix Representations

- Represented in matrix form as:

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

- or:

$$\begin{bmatrix} 2 & 3 & -7 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Can then be solved with
  - Gaussian elimination
  - Least-squares (`numpy.linalg.lstsq()`)
  - Singular Value Decomposition (`numpy.linalg.svd()`) (this may seem magical)
- More on these methods in **Mathematics for Robotics and Control**

Basis Vectors and Vector Spaces

What are they?

- Set of linearly independent vectors B that span a vector space V
- **Unique** linear combination of basis vectors B can represent any vector in space V
- Minimal number of vectors, but maximal span
- B can be ordered
- Number of basis vectors – **dimension**
- e.g. 2D Cartesian space in $\mathbf{R}^2$ spanned by $[1, 0]$ and $[0, 1]$

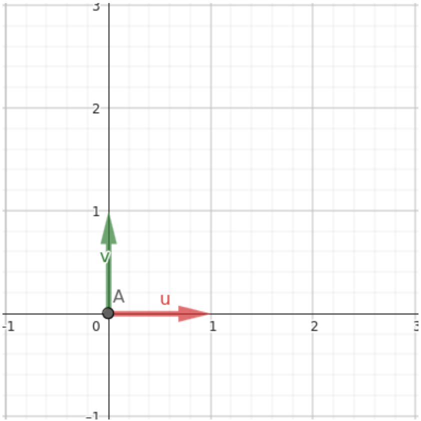

Figure 9: 2D Cartesian Basis Vectors, plotted in GeoGebra

Basis Vectors and Vector Spaces

How are they used?

- Unit vectors of desired basis directions

  e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

- Compose as column vectors –

  we get $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

- Dot product to project vectors into this space
- Useful in **Principal Component Analysis** (PCA)

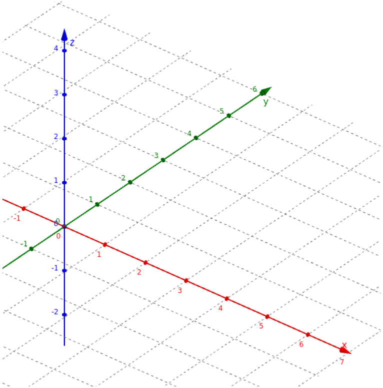

Figure 10: 3D Cartesian Space, shown in GeoGebra

Basis Vectors

Prof. Plöger's notes

- "Any set of vectors which fulfills:
  - number == N (dimension of domain), and
  - are linearly independent"
- Consider $\mathbf{R}^2$ basis B_{my} with vectors $b_1 = [1, 1]$ and $b_2 = [-1, 1]$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -0.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

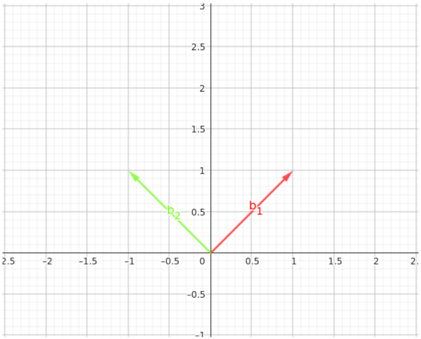

Figure 11: An alternative basis for $\mathbf{R}^2$– Prof. Paul G. Plöger – plotted in GeoGebra

Basis Vectors

Prof. Plöger's notes

- Compared to regular unit vectors, vectors in B_{my}:
  - are **not** unit length
  - are **not** axis aligned
- However, still perpendicular
- Need not be the case for a general basis B

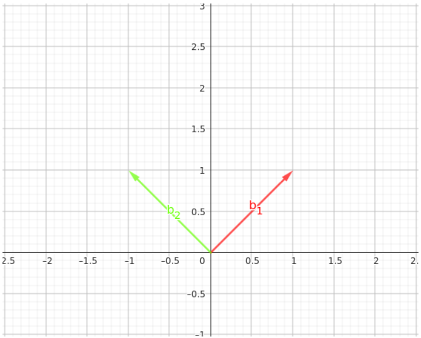

Figure 12: An alternative basis for $\mathbf{R}^2$– Prof. Paul G. Plöger – plotted in GeoGebra

Change of Basis

What is it?

- Expresses vector coordinates in one basis relative to another basis
- In matrix notation, written as:

$$\mathbf{x}_1 = A\mathbf{x}_2$$

  where
  - $\mathbf{x}_1$ and $\mathbf{x}_2$ are vector coordinates in each basis
  - A is the **change of basis matrix**
- A projection into a vector space
- Projection into spaces useful in **Principal Component Analysis** (PCA) and **dimensionality reduction**

Linear Transformations

Definitions and Properties

- Transforms that **preserve** vector addition, scalar multiplication
  - rotation
  - scaling
  - reflection
- Defined as a function $T : V \to W$, where V, W are vector spaces with $\mathbf{u}, \mathbf{v} \in V$ such that:
  1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  2. $T(k\mathbf{v}) = kT(\mathbf{v})$
- **Matrices** can represent arbitrary linear transformations
- Transform vectors with a **transformation matrix**
- Translation is **not** a linear transform
  - Does not preserve above properties
  - Shifts vector / space without changing orientation, shape

Rotations

As Linear Transforms

- Represented by **rotation matrices**
- Consider **counterclockwise** rotation by θ in 2D

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

- Can be represented in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

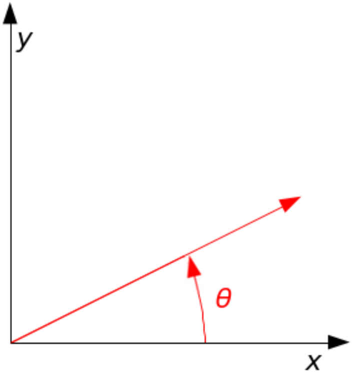

Figure 13: Vector rotation – From Wikipedia; By MathsPoetry - Own work, CC BY-SA 3.0

Rotation

As change in basis

- Rotation using rotation matrix R also **change in basis**
  - Sufficiently many column vectors (since square and $R^{-1} = R^T$)
  - Linearly independent column vectors (since $\det(R) = 1 \neq 0$)
- Written as

$$\mathbf{x}' = R\mathbf{x}$$

  - R is a **change of basis** matrix
- More on rotations in **Mathematics for Robotics and Control**
  - Axis angle and Euler angle representations
  - 3D composite rotations and rotation matrices

Scaling

As a Linear Transform

- Changes length or magnitude of vector
- Uniform 2D scaling:

$$k\mathbf{x} = k\mathbf{I}\mathbf{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \mathbf{x}$$

- 2D scaling along an axis:

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

Homogeneous Transforms

Affine Transforms

- Represent rotations, translations, scaling, and combinations
- Square matrix of $N + 1$
  - N columns represent rotation, last column represents translation

$$T = \begin{bmatrix} R & & t \\ & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}$$

- Compose transformations by multiplication

$$T = T_a \cdot T_b$$

- Inverse transform represented by T^{-1}
- Important to note transformation **frames**, order of composition
  - More on transforms and frames in **Mathematics for Robotics and Control**

Eigenvalues and Eigenvectors

What are they?

- Vectors $\mathbf{v}$ with unchanged direction under a linear transform T – **eigenvectors**
- Only scaled by constant factor λ under linear transform T – **eigenvalues**

$$T(\mathbf{v}) = \lambda \mathbf{v}$$

- For a transformation matrix A

$$A\mathbf{v} = \lambda \mathbf{v}$$

- Compute by solving **characteristic equation** of A

$$\det(A - \lambda I) = 0$$

Eigenvalues and Eigenvectors

Visual Example

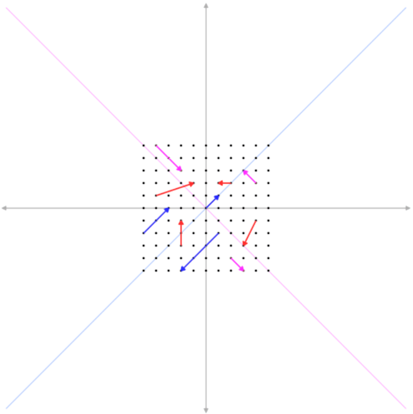

Figure 14: Vectors on a grid – From Wikipedia; By Lucas Vieira - Own work, Public Domain

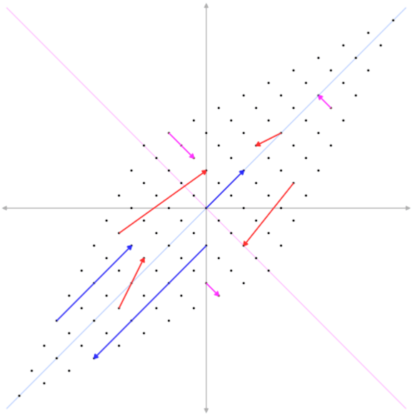

Figure 15: Scaling diagonally – note vectors parallel to drawn diagonals – From Wikipedia; By Lucas Vieira - Own work, Public Domain

Eigenvalues and Eigenvectors

From Visual Example

- Shown was the transformation of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Eigenvectors and eigenvalues of this transform are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, 1 \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, 3$$

- Vectors along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are scaled by 3; vectors along $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are scaled by 1
- Help understand the basic effects of a transformation, among others

Eigenvalues and Eigenvectors

How to find them

- Use `numpy.linalg.eig()` to get eigenvalues and eigenvectors
- Can use `numpy.linalg.eigh()` for symmetric matrices – sorted output
- **Singular Value Decomposition** (SVD) returns a decomposition of data matrix A such that (for real valued matrices):

$$A = U\Sigma V^T$$

  where
  - Columns of U are eigenvectors of AA^T; left-singular vectors of A
  - Elements of Σ represent eigenvalues of A^TA or AA^T; squared singular values of A
  - Rows of V^T are eigenvectors of A^TA; right-singular vectors of A
- A^TA or AA^T is symmetric, positive semi-definite (non-negative eigenvalues, orthogonal eigenvectors)
- Use `numpy.linalg.svd()` for computing SVD

Singular Value Decomposition

and its uses

- Singular vectors capture directions where data in A is spread or aligned
- Closely related to **principal components** – directions in which data points vary the most – capture maximum variance
- SVD useful for
  - **Principal Component Analysis** (PCA)
  - **dimensionality reduction**
  - **data compression** and **reconstruction**
  - **pseudoinverse** (A^+)
  - solution of homogeneous linear equations
  - least squares solution
  - und so weiter…
- More on SVD and its applications in **Mathematics for Robotics and Control**

Singular Value Decomposition

Principal Component Analysis

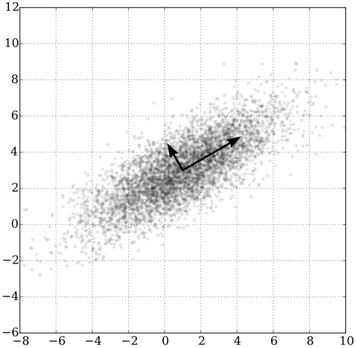

Figure 16: PCA of a multivariate Gaussian distribution – note principal vectors along directions of maximum variance – From Wikipedia; By Nicoguaro - Own work, CC BY 4.0