(**Note that the handwritten solutions for problems 2,3,4 are numbered 3,4,5 (from last year)**)

1) Starting with

$$B_\lambda = \frac{2hc^2\lambda^{-5}}{\exp\left(\frac{hc}{k_b \lambda T}\right) - 1} \tag{1}$$

derive the equivalent relationship for frequency $\hat{\nu}$:

$$B_{\tilde{\nu}}(T) = \frac{2h\tilde{\nu}^3}{c^2}\frac{1}{e^{\frac{h\tilde{\nu}}{kT}} - 1} \tag{2}$$

and show that it has the correct unit of $\mathrm{W\,m^{-2}\,sr^{-1}s}$

solution: From the notes we know that

$$\int_{\lambda_1}^{\lambda_2} B_\lambda d\lambda = \int_{\hat{\nu}_2}^{\hat{\nu}_1} B_{\hat{\nu}} d\hat{\nu} = -\int_{\hat{\nu}_1}^{\hat{\nu}_2} B_{\hat{\nu}} d\hat{\nu} \tag{3}$$

where $\lambda = c\hat{\nu}^{-1}$.

By the rules for substitution of variables we have:

$$B_\lambda \frac{d\lambda}{d\hat{\nu}} d\hat{\nu} = B_{\hat{\nu}} d\hat{\nu} \tag{4}$$

where

$$\frac{d\lambda}{d\hat{\nu}} = -\frac{c}{\hat{\nu}^2} = -c\hat{\nu}^{-2} \tag{5}$$

So

$$B_{\hat{\nu}} = -B_\lambda \frac{d\lambda}{d\hat{\nu}} \tag{6}$$

with the limits of integration taken from large $\hat{\nu}_1$ (small λ_1) to smaller $\hat{\nu}_2$ (larger λ_2).

Replace λ by the equivalent $c\hat{\nu}^{-1}$ in (1):

$$\frac{2hc^2c^{-5}\hat{\nu}^5}{\exp\left(\frac{h\hat{\nu}}{k_b}\right) - 1} \tag{7}$$

multiply (8) by (5)

$$-B_{\hat{\nu}} = -c\hat{\nu}^{-2}\frac{2hc^2c^{-5}\hat{\nu}^5}{\exp\left(\frac{h\hat{\nu}}{k_b}\right) - 1} = -\frac{2hc^{-2}\hat{\nu}^3}{\exp\left(\frac{h\hat{\nu}}{k_b}\right) - 1} \tag{8}$$

Units: $(\mathrm{Js})(\mathrm{ms^{-1}})^{-2}(\mathrm{s^{-1}})^3 = \mathrm{Jm^{-2}s^{-1}s} = \mathrm{Wm^{-2}s} = \mathrm{W\,m^{-2}\,sr^{-1}/s^{-1}}$

3) $\int B_\lambda d\lambda = \int_0^\infty \frac{2hc^2}{\lambda^5 \exp\left(\frac{hc}{k_b \lambda T}\right) - 1} d\lambda$

Only λ varies, T is constant.

Take the hint and write

$$x = \frac{hc}{k_b \lambda T}, \quad dx = \frac{-hc}{k_b \lambda^2 T} d\lambda$$

so $\quad dx = -x \frac{d\lambda}{\lambda}$ and $\frac{1}{\lambda} = \frac{x k_b T}{hc}$

$$d\lambda = \frac{-hc}{k_b T} x^{-2} dx$$

Use this to rewrite the integration in terms of x: for the limits note that as $\lambda \to \infty$, $x \to 0$.

$$\int_0^\infty B_\lambda d\lambda = -\int_\infty^0 \frac{2hc^2 \left(\frac{hc}{k_b x T}\right)^{-5} \left(\frac{hc}{k_b T} x^{-2}\right) dx}{\exp(x) - 1}$$

Rearrange, flipping the limits of integration

$$\int_0^\infty B_\lambda d\lambda = 2hc^2 \left(\frac{hc}{k_b T}\right)^{-5} \left(\frac{hc}{k_b T}\right) \int_0^\infty \frac{x^5 x^{-2}}{\exp(x) - 1} dx$$

$$= 2hc^2 \left(\frac{hc}{k_b}\right)^{-4} T^4 \int_0^\infty \frac{x^3}{\exp(x) - 1} \propto T^4$$

Also: $\int_0^\infty \frac{x^3}{\exp(x) - 1} dx = \frac{\pi^4}{15}$

So $$\int_0^\infty B_\lambda d\lambda = \frac{2hc^2 k_b^4}{h^4 c^4} \frac{\pi^4}{15} T^4$$

$$= \frac{2\pi^4 k_b^4}{15 h^3 c^2} T^4 = \frac{5.67\times10^{-8}}{\pi} T^4$$

as expected.

PS1 p4 and PS1 p5

4) $$B_\lambda = \frac{2hc^2\lambda^{-5}}{\exp\left(\frac{hc}{k_b\lambda T}\right)-1}$$

If $x \ll 1$, $\exp(x) \sim 1+x$

So $$B_\lambda \sim \frac{2hc^2\lambda^{-5}}{1+\frac{hc}{k_b\lambda T}-1} = 2k_b\lambda T c\lambda^{-5}$$

$$= \frac{2ck_bT}{\lambda^4}$$

with $\lambda = 0.01$ m $= 10000\ \mu m$

and $T = 300$ K, $\frac{hc}{k_b\lambda T} \sim 0.0048$

$\ll 1$

Spreadsheet says: exact 2.427×10^{-10}

$W m^{-2} \mu m^{-1} sr^{-1}$

Rayleigh-Jeans $=$

2.4834×10^{-10}

$W m^{-2} \mu m^{-1} sr^{-1}$

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5) Take the hint to disregard the -1 term:

$$B_\lambda = \frac{2hc^2}{\lambda^5 \exp\left(\frac{hc}{k_b\lambda T}\right)} = 2hc^2\lambda^{-5}\exp\left(\frac{-hc}{k_b\lambda T}\right)$$

As the text says, we want to set $\frac{dB_\lambda}{d\lambda}=0$ and solve for λ. Use the chain rule:

$$\frac{dB_\lambda}{d\lambda} = \frac{2hc^2}{\lambda^6\exp\left(\frac{hc}{\lambda k_bT}\right)}(-5) + 2hc^2\lambda^{-5}\left(\frac{hc}{k_b\lambda^2T}\right)\exp(\ldots)$$

5 cont. p5 in p5 and p6

set this to zero:

$$\frac{dB_\lambda}{d\lambda} = \frac{2hc^2}{\lambda^6 \exp\left(\frac{hc}{k_b \lambda T}\right)} \left(\frac{hc}{k_b \lambda T} - 5\right) = 0$$

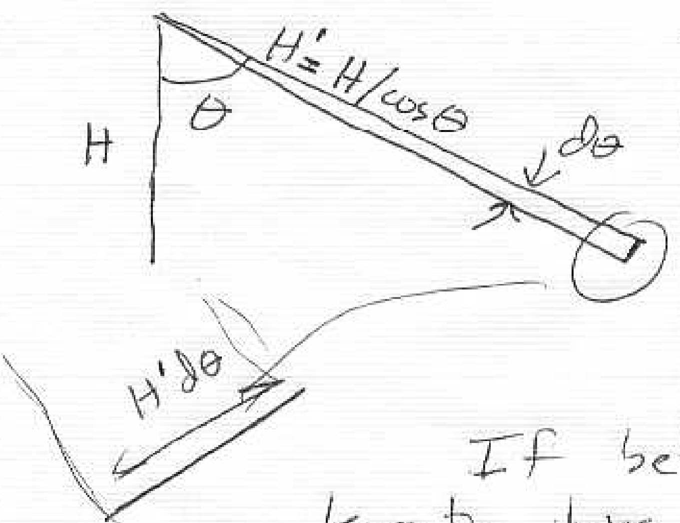
so $\frac{hc}{k_b \lambda_{max} T} - 5 = 0$ and

$$\lambda_{max} = \frac{hc}{k_b 5T} = \frac{2.877 \times 10^{-3}}{T} \text{ (meters)}$$

6) i) $\delta\omega = \frac{\text{area}}{r^2} = \frac{1 km^2}{(830 km)^2} = 1.45 \times 10^{-6}$ sr

ii) Changing the "look angle" doesn't change $\delta\omega$, because it is a property of the telescope.

iii) If the pixel is square, then as θ increases, the distance from ground to satellite increases



This affects both the width and the ~~height~~ length

If before, the length was $H d\theta = 1$ km