Read for Friday, Day 5

1. Coverage: W&H 113-116, Stull 2: 34-36
2. Develop the definitions of solid angle, field of view, flux, intensity
3. Questions we want to answer:
   - How does the flux (in $\mathrm{Watts/m^2}$) reaching a satellite instrument depend on the orbital height and instrument field of view?
   - Related question: Why doesn't the percieved brightness of a piece of white paper change as you move away from it?

Wavenumber and wavelength

First a question from W&H 4.1:

$$\nu = 1/\lambda \tag{4.1}$$

where λ is the wavelength (traditionally measured in cm).

- Question: Why have two ways of talking about the same thing?

Wavelength gives us a feeling for what size a particles will interact with a particular wave, but wave number has two advantages:

1. Photon energy is proportional to ν:

$$\text{Photon Energy (Joules)} = ch\nu \tag{1}$$

where c is the speed of light and h is Plank's constant 6.34×10^{-34} Js

Wavenumber continued

2. The wavenumber appears in the equation for the electric or magnetic fields:

$$H = H_0 \sin \frac{2\pi}{\lambda}(x - ct) \tag{2}$$

More accurately, suppose we have a 2-dimensional plane wave:

$$\vec{H} = H_0 \sin\left[2\pi(\vec{\nu} \cdot \vec{x} - ct)\right] = H_0 \sin\left[2\pi(\nu_x x + \nu_y y - ct)\right]$$

where $\vec{\nu} = \vec{i}\nu_x + \vec{j}\nu_y$ defines the direction of propagation for the phase of the wave and $|\nu|^2 = \nu_x^2 + \nu_y^2$

3. Bottom line: expect to move back and forth between wavelength (usually in microns) and wavenumber (usually in cm^{-1}). e.g., if $\lambda = 15\ \mu\mathrm{m}$, $\nu = 1/0.0015 = 667\ \mathrm{cm}^{-1}$.

Monochromatic flux

Suppose you have 100 Watts flowing across a flat surface with an area of 0.5 m^2. The *total flux density or irradiance* is given by:

$$E = \frac{power}{area} = \frac{P}{A} = \frac{100}{0.5} = 200 \ \mathrm{W\,m^{-2}} \tag{3}$$

Unfortunately, Wallace and Hobbs use F for the irradiance, Stull uses E, which is the internationally accepted symbol according to Wikipedia

But what if that flux density was being transmitted in a 2 $\mu\mathrm{m}$ wavelength range between $10 < \lambda < 12 \ \mu\mathrm{m}$? Then we write the *monochromatic* flux density as:

$$E_\lambda = \frac{E}{\Delta\lambda} = \frac{200}{2} = 100 \ \mathrm{W\,m^{-2}} \ \mu\mathrm{m}^{-1} \tag{4}$$

(pronounced *Watts per square meter per micron wavelength interval*)

Monochromatic flux, cont.

But we could also say that the power was coming from the *wavenumber* range $1/12 < \nu < 1/10 \;\; \mu\mathrm{m}^{-1}$ with $\Delta\nu = 1/10 - 1/12 = 0.1 - 0.08333 = 0.01666 \;\; \mu\mathrm{m}^{-1}$, or more commonly, $\Delta\nu = 1/0.0010 - 1/0.0012 = 166.7 \; cm^{-1} =$ so that the monochromatic flux density is:

$$E_\nu = \frac{F}{\Delta\nu} = \frac{200}{166.7} = 1.20 \;\; \mathrm{W\,m^{-2}/cm^{-1}} \tag{5}$$

(pronounced *Watts per square meter per inverse cm wavenumber interval*)

Relating E_ν to E_λ

- The power coming through a surface can't depend on whether you are working with ν or λ, that is:
- $E(\mathrm{W\,m^{-2}}) = E_\lambda \Delta\lambda = E_\nu \Delta\nu$
- $E_\lambda = E_\nu \frac{\Delta\nu}{\Delta\lambda} = E_\nu \frac{d\nu}{d\lambda}$ (in the limit of $\Delta\lambda \rightarrow d\lambda$)
- But since $\nu = 1/\lambda$:

  $$\frac{d\nu}{d\lambda} = -\frac{1}{\lambda^2} d\lambda$$

  put this together to get a version of W&H equation 4.4:

  $$E_\nu = -\lambda^2 F_\lambda \tag{4.4}$$

  (note that the minus sign doesn't produce negative flux because $\Delta\lambda$ and $\Delta\nu$ will have different signs, i.e. if $\lambda_2 > \lambda_1$ then $\nu_2 < \nu_1$, so that E_ν will be a positive number).

Taylor series

Here's another way to get

$$F_\nu = \lambda^2 F_\lambda \tag{4.4}$$

using Taylor series (see the wikipedia page for review). Remember that for any differentiable function $f(x)$ we can write:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{2 \cdot 3}(x - x_0)^3 + \dots \tag{6}$$

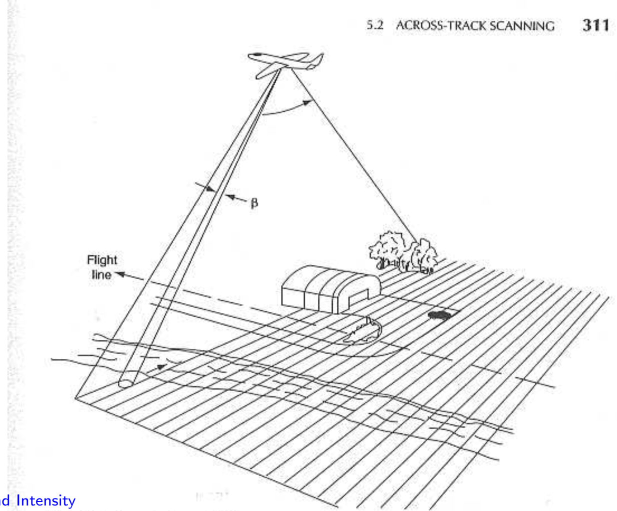
or keeping just the first order term and rearranging:

$$f(x) - f(x_0) = \Delta f = f'(x_0)\Delta x \tag{7}$$

If we let $\Delta\nu = \Delta f$, $\Delta x = \Delta\lambda$ and $f'(x_0) = \frac{d\nu}{d\lambda}$ then we get back to W&H 4.4.

Flux vs. Intensity

When we calculate the flux, we count every photon crossing the surface, no matter what angle it is coming in at. This doesn't work for instruments (like satellite sensors or our eyeballs) that take in photons only for a particular set of angles (the "field of view").



Flux vs. Intensity: coordinates

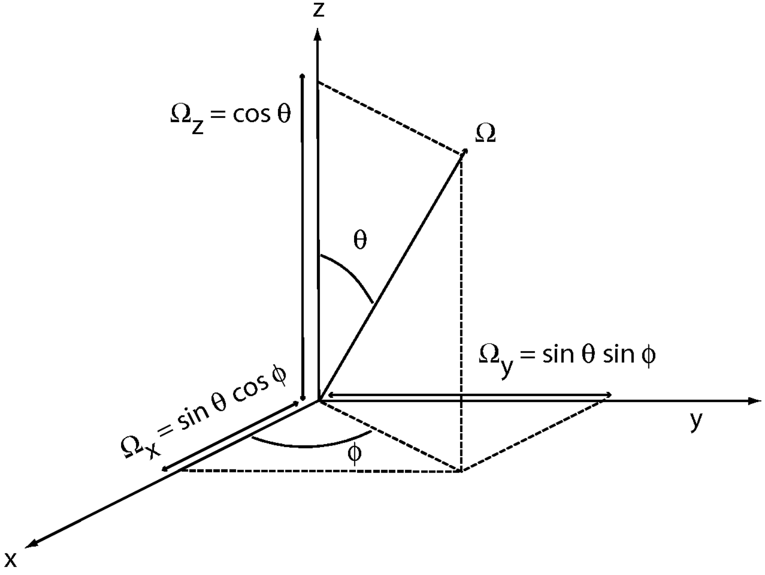


Fig. 2.3: The relationship between Cartesian and spherical coordinates.

Flux vs. Intensity: solid angle

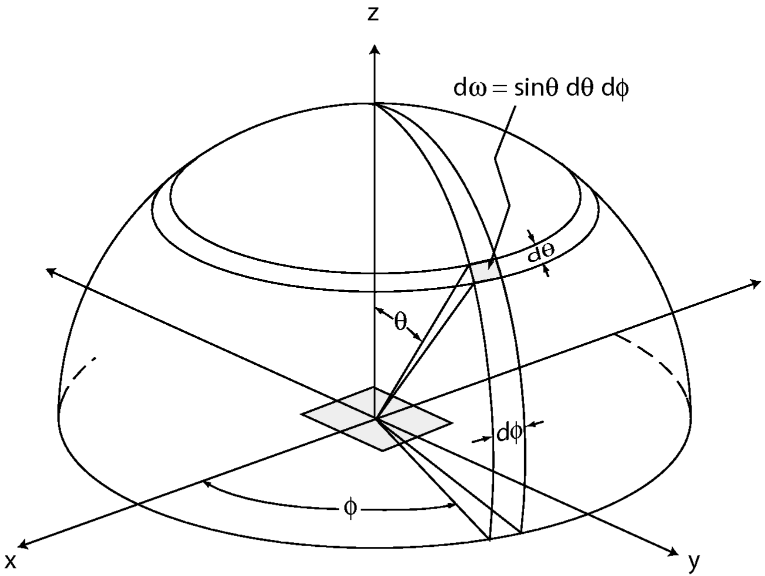


Fig. 2.4: The relationship between solid angle and polar coordinates.

Solid angle, continued

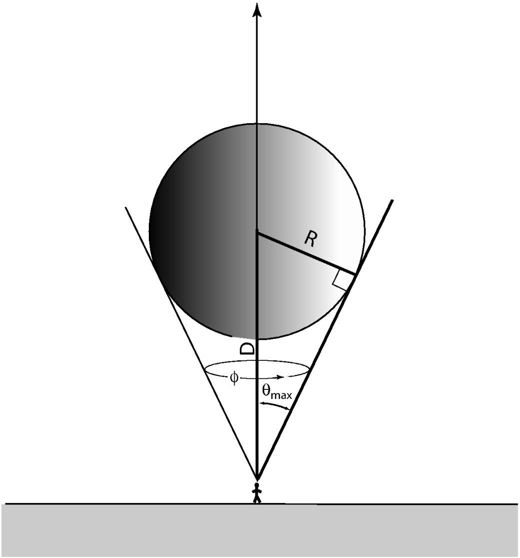


Fig. 2.5: Geometric framework for computing the solid angle subtended by a sphere of radius R whose center is a distance D from the observer.

Solid angle, continued

Since $d\omega = \sin\theta d\theta d\phi$, integrate this to get:

$$\omega = \int_0^{\phi_1} \int_0^{\theta_1} \sin\theta d\theta d\phi$$

for the case the $D = 5$, $R = 2$, $\theta = \sin^{-1}(R/D) = \sin^{-1}(0.4) =$ 0.41 radians $= 23$ degrees

$$\begin{aligned} \omega = \int_0^{2\pi} \int_0^{0.4} \sin\theta d\theta d\phi &= 2\pi(-(\cos(0.4) - \cos(0))) \\ &= 2\pi(1 - 0.92) = 0.5 \ \text{steradian} \end{aligned}$$

Intensity

So use the solid angle to define the power per unit area falling into a particular field of view

$$I_\lambda(\,\mathrm{W\,m^{-2}}\ \mu\mathrm{m^{-1}\,sr^{-1}}) = \frac{\text{Energy (Joules)}}{\text{time } (\textit{seconds}) \times \text{area } (m^2) \times \text{channel width } (\,\mu\mathrm{m}) \times \text{fov } \ (\mathrm{sr^{-1}})}$$

- Show that while the flux decreases as $1/r^2$, the intensity *remains constant*

Perpendicular beam

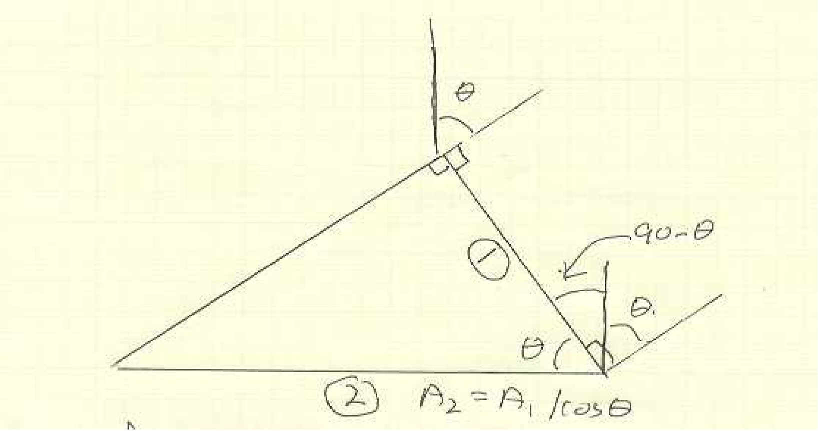
- Consider a very narrow bean of radiation, with tiny solid angle $\Delta\omega$, so that all photons are coming from the same direction (the sun satisfies this approximation). We know from the definition of flux density (or flux) (3) and intensity (previous page) that this beam is making a small contribution to the total flux:

$$dF_\lambda = I_\lambda d\omega \tag{8}$$

Slanted surfaces

- But what about directions that aren't perpendicular to the surface? We need to account for this:

increasing the area as in this figure:



Slant surface, cont.

If there is no absorption as the photons travel from side 1 to side 2, then the power is constant, and the irradiance dF_2 arriving at the ground (side 2) from this direction is given by:

$$dE_2 = \frac{\text{power}}{A_2} = \cos\theta \, \frac{\text{power}}{A_1} = \cos\theta \, dE_1 = \cos\theta \, I_1 d\omega \qquad (9)$$

where I've used (8) and taken the limit of infinitesimal $d\omega$. Integrating the contributions over every field of view gives

$$E_\lambda = \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \, I_\lambda \sin\theta d\theta d\phi \qquad (10)$$

- Look at change of variables $\mu = \cos\theta$, $d\mu = -\sin\theta \, d\theta$, and show that $E_\lambda = \int_0^{2\pi} \int_0^1 \mu I_\lambda d\mu d\phi$
- Show that for isotropic radiation $E_\lambda = \pi I_\lambda$ (W&H Exercise 4.3).

flux and intensity

- Show that radiance is conserved with distance for a fixed area (or a fixed field of view over a uniform surface), while the irradiance decreases as $1/r^2$. Why don't white surfaces get dimmer as we move away from them? Because intensity is independent of distance. (show this holds for a telephoto lens and and eyeball)
- W&H exercise 4.2 (slightly different starting point)
  1. *Looking outward:* The sun's power ouput at its surface is $P = 3.8679 \times 10^{26}$ W
     The sun-earth distance is $R_{se} = 1.5 \times 10^{11}$ m
     So the flux density at earth orbit is $P/(4\pi R_{se}^2)$=1368 $\mathrm{W\,m^{-2}}$
     The solar radius is $R_{sun} = 7 \times 10^8$ m
     Solid angle of the sun seen from earth is about $\frac{disk\ area}{R_{se}^2} = \frac{\pi R_{sun}^2}{R_{se}^2}$
     $= 6.84 \times 10^{-5}$ sr
     Intensity from sun is $\frac{1368\ \mathrm{W\,m^{-2}}}{6.84 \times 10^{-5}\ sr} = 2 \times 10^7\ \mathrm{W\,m^{-2}sr^{-1}}$

Summary

1. We need both wavelength λ for our intuition and wavenumber $\nu = 1/\lambda$ for wave vector and energy calculations
2. Flux density F ($\mathrm{W\,m^{-2}}$) is the energy passing through a flat surface (which includes a spherical surface, when zoomed in infinitely close)
3. But no instrument can count photons from every direction –they all have a "field of view"
4. Intensity, which is the flux density per unit field of view, measures the flux coming from a set of directions. It's the variable of choice for remote sensing, while flux density is the variable of choice for heating/cooling calculations
5. Read for Wendesday: W&H 116 to 120, answer 3 short multiple choice problems on Connect by Wednesday 8 am.