MATH:ODE

order → highest derivative

degree → power of highest derivative

→ problem solving flow ←

I $\boxed{1\text{'st O}}$

1. No addition of $f(x)$ & $f(y)$ ⟹ seperable

→ turn to $\frac{M(x)}{dx} = \frac{N(y)}{dy}$ and integrate.

2. eqn of form: $y' = f(ax+by+c)$ ⟹ seperable

→ put $ax+by+c$ then $\xrightarrow{goto}$ #1

II

3. every term's power sum is equal ⟹ homog.

→ set $v = \frac{y}{x}$ then $\xrightarrow{goto}$ #1

4. eqn of form $y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$ ⟹ homog.

→ find intersection, move origin there to eliminate constants then $\xrightarrow{goto}$ #1

III

5. eqn of form $\overbrace{M(x,y)dx}^{masz} + \overbrace{N(x,y)dy}^{nie} = 0$

- case '$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$' : $\xrightarrow{goto}$ #6
- '$\frac{M_y - N_x}{M} = f(y)$' : $\xrightarrow{goto}$ #7
- '$\frac{M_y - N_x}{N} = f(x)$' : $\xrightarrow{goto}$ #8
- none : $\xrightarrow{goto}$ #9

6. ⇒ Exact eqn of the form $Mdx + Ndy = 0$

→ Result : $\int M dx + \int N dy = C$
(during addition, repeated terms are taken once only)

7. ⇒ Reducible to Exact (form: $\frac{M_y - N_x}{M} = f(y)$)

¿put negative? yes

→ $\mu(\boxed{y}) = e^{-\int f(y) dy}$, then multiply original eqn by $\mu(y)$ then $\xrightarrow{\text{go to}}$ # 6

8. ⇒ Reducible to Exact (form: $\frac{M_y - N_x}{N} = f(x)$)

¿put negative? no

→ $\mu(x) = e^{\int f(x) dx}$, then multiply original eqn by $\mu(x)$ then $\xrightarrow{\text{go to}}$ # 6

9. ⇒ Linear first order O.D.E (in disguise)

- turn $Mdx + Ndy = 0$ into $\frac{dy}{dx} + P(x)y = q(x)$ then $\xrightarrow{\text{go to}}$ # 10
- if not possible:
  - can be turned into $y' + P(x)y = q(x)y^n$? $\xrightarrow{\text{go to}}$ # 11
  - (to be continued)

10. eqn of form: $y' + p(x)y = q(x) \Rightarrow$ Linear 1st O

$\rightarrow$ calculate $\mu(x) = e^{\int p(x)\,dx}$, General sol. will be: $y_{g.s} = \frac{1}{\mu}\left[\int \mu q(x)\,dx + c\right]$ (one $\frac{1}{\mu}$ outside, μ inside)

11. eqn of form: $y' + p(x)y = q(x)y^n \Rightarrow$ Bernoulli's

$\rightarrow$ put $z = y^{1-n} \Rightarrow z' = (1-n)y^{-n} \cdot y'$
then $\xrightarrow{\text{goto}}$ #10

$\boxed{\text{2'nd O}}$

12. eqn of form: $y'' = f(x, y, y') \Rightarrow$ Reducible to 1st

- case 'y' is absent: put $y' = z(x)$, then $y'' = z'(x) \xrightarrow{\text{goto}}$ #1
- case 'x' is absent: put $y' = z(y)$, then $y'' = \frac{dz}{dx} = \frac{dz \cdot dy}{dy \cdot dx} = z \cdot \frac{dz}{dy} \xrightarrow{\text{goto}}$ #1
- both are there: $\xrightarrow{\text{goto}}$ #13

$\boxed{n^{th}\ \text{O}}$

$y^{(3)} \Rightarrow y'''$

$\boxed{\text{orthagonal trajectories}}$ $\rightarrow$ $\boxed{\text{goto} \rightarrow \#18}$

13. eqn of form: $a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(x)$
$\Rightarrow$ Linear D.E with const. coeffecients, operator $D = \frac{d}{dx}$

$\rightarrow$ Find y_{cf} [general solution of $(a_0 D^n + a_1 D^{n-1} + \dots + a_n)y = 0$], write H.E (Homog. Eqn) $\xrightarrow{\text{goto}}$ #14

$\rightarrow$ Find y_{pi} [particular Integral]

- Operator method (preferred) $\xrightarrow{\text{goto}}$ #15
- Variation of parameters $\xrightarrow{\text{goto}}$ #16

$\rightarrow y_{g.s} = y_{c.f} + y_{p.i}$

14• Homg. eqn. of form $L(D)y=(a_0D^n+a_1D^{n-1}+\dots+a_n)y=f(x)$

→ **H.E**: state the homog. eqn.

- **A.E** (Auxillary Eqn): replace every $\underline{D}$ with m and solve eqn. for number $\underline{n}$ of roots

- **C.F**
  - roots are real and distinct:
    $\rightarrow y_{cf}=\sum_{i=1}^{n} c_i e^{m_i x} = c_1e^{m_1x}+c_2e^{m_2x}+\dots+c_ne^{m_nx}$
  - roots are equal:
    $\rightarrow y_{cf}=\sum_{i=0}^{n-1} c_{(i+1)}x^i \cdot e^{mx}=(c_1x^0+c_2x^1+\dots+c_nx^{n-1})e^{mx}$
  - roots are complex of form: $\underline{m=a+ib}$
    $\rightarrow y_{c.f}=\sum_{j=1}^{n} c_j e^{ibx}\cdot e^{ax}$ → using euler's identity, "$e^{\pi i}+1=0$"
    → $y_{cf}=e^{ax}(A_1\cos bx + A_2\sin bx)$
    $:\{A_1,A_2\}\subset\mathbb{R}$
  - combination of above: (m count y_{cf})
    → multiply them : $Y_{cf}=\prod_{j=1}^{m}(y_{c.f})_j$

γ 3 ∝ k̄ ṽ

↑ ↑ ↑ ↑ ↑

σ ∂ ξ ∢ Ã

> e.g: for "$m^4+6m^2+9=0$"
> roots are : $m_1=m_3=3i$, $m_2=m_4=-3i$
> thus y_{cf} will be:
>
> $$y_{cf}=\overbrace{(c_1'\cos[3x]+c_2'\sin[3x])}^{\text{complex}}\overbrace{(c_3'+c_4'x)}^{\text{equal}[m=0]}$$

- For a homogeneous eqn, G.S = C.F
  - if original eqn is homog.: $y_{g.s}=y_{c.f}$
  - if non-homog. $y_{g.s}=y_{cf}+y_{pi}$

$\xrightarrow{\text{goto}}$ #13

15. For $L(D)y = g(x) \Rightarrow y_{pi} = \frac{1}{L(D)} g(x)$
$\because \frac{1}{D} \equiv \int$, non-homog. part $[g(x)]$ is of form:

- $ae^{mx} \rightarrow$ put $D = m$
- $\cos mx$ or $\sin mx \rightarrow$ put $D^2 = -m^2$
- $\cosh mx$ or $\sinh mx \rightarrow$ put $D^2 = m^2$
- polynomial in $x \rightarrow$ turn to form $\frac{1}{1 \pm L(D)}$

$\rightarrow$ Rewrite using following expansions till lowest order of $\underline{D}$ in the term is bigger than highest power of x ← in the polynomial

negative odd terms

- $(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 - \dots$
- $(1-D)^{-1} = 1 + D + D^2 + D^3 + D^4 + \dots$

- $e^{mx} \cdot g(x) \rightarrow D = (D+m)$ and move e^{mx} out

e.g: $\frac{1}{D^2+2D} e^{2x} \cdot x$

$\Rightarrow e^{2x} \cdot \left(\frac{1}{[D+2]^2 + 2[D+2]}\right) \cdot x$

- if combined of sum of above
$\rightarrow$ seperate to parts

e.g $\frac{1}{D+1}(e^x \cdot x + 2x) = \frac{1}{D+1} e^x \cdot x + \frac{1}{D+1} \cdot 2x$

- if constant $\rightarrow$ use any of above (1^{st} preferred)
- if result is $\frac{1}{0} g(x) \rightarrow$ use 5th case

16. After calculating $y_{c.f} = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$:

→ $y_{pi} = c_1(x)\, y_1 + c_2(x)\, y_2 + \dots + c_n(x)\, y_n$

> Solve following system:

$$\left\| \begin{array}{l} c_1' y_1 + c_2' y_2 + \dots + c_n' y_n = 0 \rightarrow \boxed{1} \\ c_1' y_1' + c_2' y_2' + \dots + c_n' y_n' = 0 \rightarrow \boxed{2} \\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ c_1' y_1^{(n-1)} + c_2' y_2^{(n-1)} + \dots + c_n' y_n^{(n-1)} = g(x) \rightarrow \boxed{n} \end{array} \right.$$

i.e: we solve $\underline{n}$ eqn's ($g(x)$ → non-homog. part)

> Integrate all c' w.r.t x

> substitute in 1st step