INDUCTION EXERCISES

1 A palindrome is a string that can be read in the same way both from left-to-right and from right-to-left. Consider the following inductive definition of the related concept pal:

1. Each character of the alphabet is a pal.
2. If α is a pal so is the result of concatenating any character before and after α.
3. Nothing is a pal except if obtained from 1. and 2.

Prove by induction that every pal is a palindrome. Is the converse true? If so, prove it. Otherwise, correct the definition so that it becomes true.

2 Definition of tree:

1. A simple node is a tree.
2. If T_1, T_2, ..., T_k are trees, the structure that results from taking a new node N and connecting with an edge each of the trees T_1, T_2, ..., T_k to N is a tree.
3. Nothing else is a tree other than that obtained from 1. and 2.

Prove that a tree has a number of nodes N larger than the number of edge E by one unit.

3 Prove that the sum of the first n perfect cubes is a perfect square. [Note: you need to proof a stronger affirmation (Inventor's Paradox).]

4 Prove that, for every natural number $1+3+5+ ...+(2n+1) = (n+1)^2$.

5 Prove that, for every natural number $n\geq 2$, $(1-1/2)(1-1/3)\dots(1-1/n) = 1/n$.

6 Prove that, for every natural number $n \geq 1$, 2 is a factor of n^2+n.

7 An ATM has only 20€ and 50€ bills. Prove by induction that this machine can supply any amount that is a multiple of 10€, equal or larger than 40€.

Define the structure over which you'll make your proof. Show the different steps of the proof.

8 A binary tree can be recursively defined in the following manner: a binary tree is constituted by one node and 0, 1 or 2 (sub-)trees connected by edges to that node. A full node is a node that has two sub-trees. A leaf is a node that has no sub-trees. We want to prove, using mathematical induction, that, for every binary tree, the number of leafs equals the number of full nodes plus one.

State the structure over which you'll be performing the induction, explaining its inductive definition.