# ASP FINAL PROJECT

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# abstract

In this course, I went through a lot of adaptive filter. Also, I have learned about array data model and adaptive beamforming.Recently, communications with moving sources have become a significant research topic.If we use these adaptive filter properly, We are possible to recover the desired signal.

This project formulates a moving source and interference propagating to a uniform linear array with N elements and the inter-element spacing d =$\frac{\lambda}{2}$. The input x(t) of the array is

$$\mathbf{x}(t) = \mathbf{a}(\theta_s(t))\mathbf{s}(t) + \mathbf{a}(\theta_i(t))\mathbf{i}(t) + \mathbf{n}(t)$$

with time index t= 1,2,...,L.The steering vector is defined as

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 \\ e^{j\pi sin\theta} \\ e^{j2\pi sin\theta} \\ ... \\ e^{j(N-1)\pi sin\theta} \end{bmatrix}$$

The source signal is $\mathbf{s}$(t), the interference signal is $\mathbf{i}$(t), the noise vector is $\mathbf{n}$(t), the source DOA is $\theta_s$(t), and the interference DOA is $\theta_i$(t). The goal of the project is to estimate $\theta_s$(t), $\theta_i$(t) and design a beamformer to estimate the source signal.


# 1 Details of the following beamformers

Consider a uniform array with N attenas and the inter-element spacing d =$\frac{\lambda}{2}$. The received signal is

$$\mathbf{x}(t) = \mathbf{a}(\theta_s(t))\mathbf{s}(t) + \mathbf{n}(t)$$

with steering vector

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 \\ e^{j\pi sin\theta} \\ e^{j2\pi sin\theta} \\ ... \\ e^{j(N-1)\pi sin\theta} \end{bmatrix}$$

### 1.1 Beamformer with uniform weights

The weight vector of uniform weight beamformer is

$$\mathbf{w} = \frac{1}{N} \begin{bmatrix} 1 \\ 1 \\ 1 \\ ... \\ 1 \end{bmatrix}$$

Now, we can derive the output as below

$$\mathbf{y}(t) = \mathbf{w}^H \mathbf{a}(\theta)\mathbf{s}(t) + \mathbf{w}^H \mathbf{n}(t) = \mathbf{B}_\theta(\theta)\mathbf{s}(t) + \mathbf{w}^H \mathbf{n}(t)$$

where $\mathbf{B}_\theta(\theta)$ is the beampattern of this beamformer,and it can be shown as below

$$\mathbf{B}_\theta(\theta) = \mathbf{w}^H \mathbf{a}(\theta) = \frac{1}{N} \begin{bmatrix} 1 \\ 1 \\ 1 \\ ... \\ 1 \end{bmatrix}^H \begin{bmatrix} 1 \\ e^{j\pi sin\theta} \\ e^{j2\pi sin\theta} \\ ... \\ e^{j(N-1)\pi sin\theta} \end{bmatrix} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j\pi sin\theta n}$$

$$= e^{j\frac{N-1}{2}\pi sin\theta} * \frac{1}{N} * \frac{sin(\frac{N}{2}\pi sin\theta)}{sin(\frac{1}{2}\pi sin\theta)} \ , \ where - 90^o < \theta < 90^o$$

Obviously, $\mathbf{B}_\theta(\theta)$ has the maximum 1 at $\theta = 0$.Also we can derive SNR as below

$$SNR = |B_\theta(\theta)|^2 \frac{N\sigma_1^2}{\sigma_n^2}$$

$$\text{SNR}_{max} = |B_0(0)|^2 \tfrac{N\sigma_1^2}{\sigma_n^2} = \tfrac{N\sigma_1^2}{\sigma_n^2}$$

### 1.2 Beamformer with array steering

If we are given DOA $\theta_s$.The weight vector of array steering beamformer is

$$\mathbf{w} = \frac{1}{N}\begin{bmatrix} 1 \\ e^{j\pi sin\theta_s} \\ e^{j2\pi sin\theta_s} \\ ... \\ e^{j(N-1)\pi sin\theta_s} \end{bmatrix}$$

Now, we can derive the output as below

$$\mathbf{y}(t) = \mathbf{w}^H\mathbf{a}(\theta)\mathbf{s}(t) + \mathbf{w}^H\mathbf{n}(t) = \mathbf{B}_\theta(\theta)\mathbf{s}(t) + \mathbf{w}^H\mathbf{n}(t)$$

where $\mathbf{B}_\theta(\theta)$ is the beampattern of this beamformer,and it can be shown as below

$$\mathbf{B}_\theta(\theta) = \frac{1}{N}\mathbf{w}^H\mathbf{a}(\theta) = \frac{1}{N}\begin{bmatrix} 1 \\ e^{j\pi sin\theta_s} \\ e^{j2\pi sin\theta_s} \\ ... \\ e^{j(N-1)\pi sin\theta_s} \end{bmatrix}^H \begin{bmatrix} 1 \\ e^{j\pi sin\theta} \\ e^{j2\pi sin\theta} \\ ... \\ e^{j(N-1)\pi sin\theta} \end{bmatrix} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\pi sin\theta n}e^{-j\pi sin\theta_s n}$$

$$= e^{j\frac{N-1}{2}\pi(sin\theta - sin\theta_s)} * \frac{1}{N} * \frac{sin[\frac{N}{2}\pi(sin\theta - sin\theta_s)]}{sin[\frac{1}{2}\pi(sin\theta - sin\theta_s)]}\ ,\ where - 90^o < \theta < 90^o$$

Obviously, $\mathbf{B}_\theta(\theta)$ has the maximum 1 at $\theta = \theta_s$.

### 1.3 MVDR Beamformer

The weight vector of for the minimum variance distortionless response (MVDR) beamformer is the solution of the following optimization problem

$$\mathbf{w}_{MVDR} = \underset{x}{\operatorname{argmin}} E[|y(t)|^2] = \mathbf{w}^H \mathbf{R} \mathbf{w}$$

$$\text{subject to } \mathbf{w}^H \mathbf{a}(\theta)_s = 1$$

where $\mathbf{R} = E[\mathbf{x}\mathbf{x}^H]$,
we can solve this problem by Lagrange multiplier

$$\mathcal{L} = \mathbf{w}^H \mathbf{R} \mathbf{w} - \frac{1}{2}\lambda[\mathbf{w}^H \mathbf{a}(\theta_s) - 1] - \frac{1}{2}\lambda^*[\mathbf{w}^T \mathbf{a}^*(\theta_s) - 1]$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}^*} = \mathbf{R}\mathbf{w} - \frac{1}{2}\lambda \mathbf{a}\theta_s = 0$$

then we can obtain the solution as below

$$\mathbf{w}_{MVDR} = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta_s)}{\mathbf{a}^H(\theta_s)R^{-1}\mathbf{a}^H(\theta_s)}$$

By this weight,we can supress noise optimally in this beamformer

### 1.4 LCMV Beamformer

Although the MVDR beamformer can suppress noise optimally, the performance decreases in interfered environment. So we develop another beamformer which is called LCMV.It is different from MVDR by adding more constraint to the optimization problem.

$$\mathbf{w}_{LCMV} = \underset{x}{\operatorname{argmin}} E[|y(t)|^2] = \mathbf{w}^H \mathbf{R} \mathbf{w}$$

$$\text{subject to } \mathbf{C}^H \mathbf{w} = \mathbf{g}$$

where $\mathbf{R} = E[\mathbf{x}\mathbf{x}^H]$,
we can solve this problem by Lagrange multiplier

$$\mathcal{L} = \mathbf{w}^H \mathbf{R} \mathbf{w} - \frac{1}{2}\lambda[\mathbf{C}^H \mathbf{w} - \mathbf{g}] - \frac{1}{2}\lambda^*[\mathbf{C}^T \mathbf{w}^* - \mathbf{g}^*]$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}^*} = \mathbf{R}\mathbf{w} - \frac{1}{2}\mathbf{C}\lambda = 0$$

then we can obtain the solution as below

$$\mathbf{w}_{MVDR} = \mathbf{R}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C})^{-1}\mathbf{g}$$

By this weight,we can supress noise and interference optimally in this beamformer.

# 2 DOA Denosing

### 2.1 Details of denoising method

In the denoising method , I propose the denoising method based on singular value decomposition(SVD) combined with Akaike information criterion (AIC) which is called AIC-SVD algorithm.In SVD theorem we can extract the source signal by divides the two kinds singular values

$$\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}^T = (\mathbf{U}_S\mathbf{U}_n)\begin{bmatrix} \Sigma_s & 0 \\ 0 & \Sigma_n \end{bmatrix}(\mathbf{V}_S\mathbf{V}_n)^T$$

where $\mathbf{H}$ is Hankel matrix of input signal s(t), t=1,2,...,N

$$\mathbf{H}_{m\times n} = \begin{bmatrix} s(1) & s(1) & ... & s(n) \\ s(2) & s(3) & ... & s(n+1) \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ s(m) & s(m+1) & ... & s(N) \end{bmatrix}$$

where m=N-n+1 and 1<n<N,**H** can be shown as below before extraction

$$\mathbf{H} = \mathbf{U}_s\ \Sigma_s\mathbf{V}_s^T + \mathbf{U}_n\Sigma_n\mathbf{V}_n^T$$

After evaluating breaking point of singular values ,we can extract the source signal $\mathbf{H}_s^{'}$ from received signals

$$\mathbf{H}_s^{'} = \mathbf{U}_s \ \Sigma_s \mathbf{V}_s^T$$

However, we cannot always choose the breaking point for the singular values. Therefore, we propose a method to address the issues of selecting the matrix structure and determining the order of the singular values. Based on the energy characteristics of the singular values, the optimal structure of the Hankel matrix is determined to serve as the trajectory matrix of the signals. During the process of Singular Value Decomposition (SVD), the effective singular values are accurately selected using the improved Akaike Information Criterion (AIC). After eliminating the noise components, the remaining singular components are used to reconstruct an approximate matrix. Finally, the averaging method is applied to obtain the denoised time-series signal.

The AIC is an estimated measure of the fitting goodness of statistical models , and is currently used in the estimation of the source number. The decision functions of the AIC are as follows :

$$AIC(d) = -2N(n-d)log_{10}(L_d) + 2d(2n-d)$$

and

$$L_d = \frac{\prod_{i=d+1}^{n} \lambda_i^{\frac{1}{n-d}}}{\frac{1}{n-d}\sum_{i=d+1}^{n} \lambda_i}$$

where $\lambda_i = \sigma_i^2$ denotes the eigenvalues of the unitary matrices, $L_d$ is the maximum likelihood estimation of the eigenvalues, and d = 1, 2, ···, n − 1 denotes the number of sources.When the sum of the two terms is minimum, the best estimate of the effective order is obtained by balancing both the terms as

$$k = \underset{d}{\operatorname{argmin}} AIC(d)$$

The AIC can achieve variable noise separation, which is beneficial for reasonable noise reduction and feature extraction.Apart from a white noise of uniform power, the actual vibration signals are also mixed with an uneven colored noise. To smooth the interference components in the background of the colored noise,the eigenvalues are modified by the diagonal loading technique as follows

$$\mu_i = \sigma_i^2 + \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

Substituting the modified eigenvalues into the maximum likelihood estimation of the signals, the improved AIC function becomes as expressed like this

$$AIC(d) = -2N(n-d)log_{10}(\frac{\prod_{i=d+1}^{n} \mu_i^{\frac{1}{n-d}}}{\frac{1}{n-d}\sum_{i=d+1}^{n} \mu_i}) + 2d(2n-d)$$

Therefore, the adaptive determination of the singular components can be achieved by minimizing the AIC objective function for the signals containing the colored noise. Also we can obtain estimated source signal from Reconstrct matrix $\hat{A}$ by averaging method expressed in Equation

$$\hat{x}(i) = \frac{1}{h-l+1} \sum_{j=l}^{h} \hat{A}(i-j+1, j),$$

where $i = 1, 2, \ldots, N$, $l = \max(1, i-n+1)$, and $h = \min(n, i)$.

### 2.2 Summarize the steps of my denoising method

**Algorithm 1** AIC-SVD

**Input:**

received Signal , $x(t)$;

**Output:**

estimated source signal , $s(t)$;

1: construct Hankel matrix H
2: Apply SVD to matrix **H**
3: Obtain singular value $\sigma$
4: X(t) has colored noise $\rightarrow \mu_i = \sigma_i^2$
5: X(t) has white noise $\rightarrow \mu_i = \sigma_i^2 + \sqrt{\sum_{i=1}^{n} \sigma_i^2}$
6: Determine minimum index K
7: Reconstrrct matrix $\hat{A} = \sum_{i=1}^{K} u_i \sigma_i v_i^T$
8: Using averaging method to obtain estimated source signal s(t)

### 2.3 advantages of my method

In my method,I achieveda powerful denoising purpose compared to original received signal. Simulation analysis and are undertaken to demonstrate the effectiveness of the proposed AIC—SVD, and the following conclusions can be drawn:

(1) In the signal processing of SVD based on Hankel matrix, the energy of the singular values is maximum when the matrix structure is a square or an approximate square. Currently, the feature components provide the largest degree of distinction, which is convenient for the order determination of the effective singular values.

(2) The method of order determination based on the AIC possesses high accuracy and robustness.

Last but not the least,we can observe the effectiveness of our method between orginal signals as following figures in topic **3**.

# 3 Plot of the estimated DOA

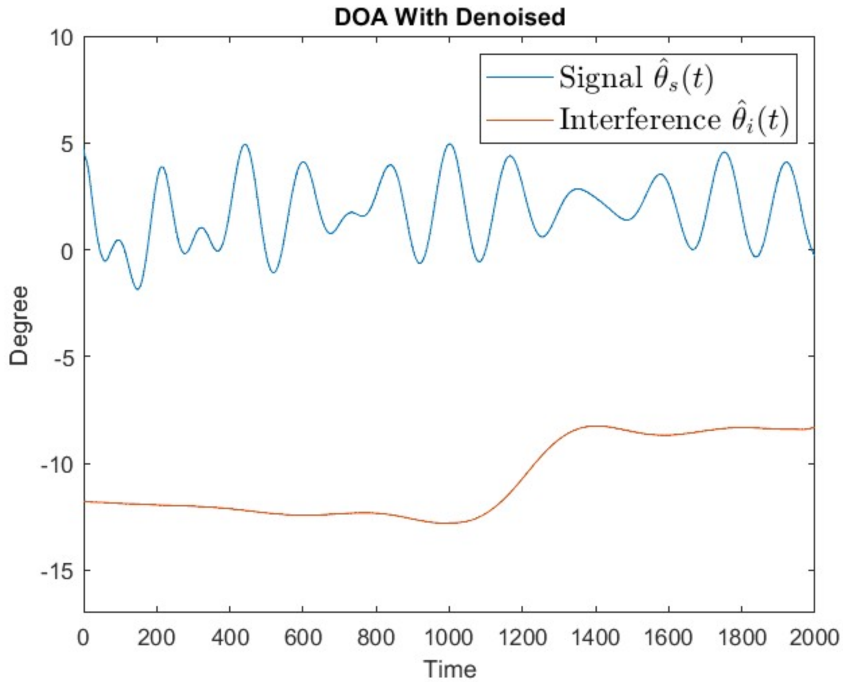



Figure 1: Denoised DOA

# 4 Beamformer Design

## 4.0 Results of the previous beamformers

Before discussing the proposed beamformer design, I have simulated the results of all beamformers in section 1, which are shown in figure 2. We can first take a look at these beamformers and discuss their pros and cons.

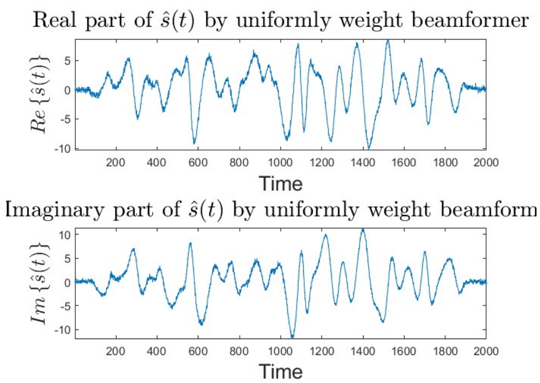
- **Uniform Beamforming with uniform weighting**: This method can guaranteed the SNR is maximum in zero degree and attenuate the power of noise by a factor of $N$, but it cannot fully filter out the interference signal (i.e., $\mathbf{i}(t)$ steered by $\theta_i(t)$ in this project).

- **Uniform Beamforming with Array Steering**: This method can improve the SNR in a certain direction and attenuate the power of noise by a factor of $N$, but it cannot fully filter out the interference signal (i.e., $\mathbf{i}(t)$ steered by $\theta_i(t)$ in this project).

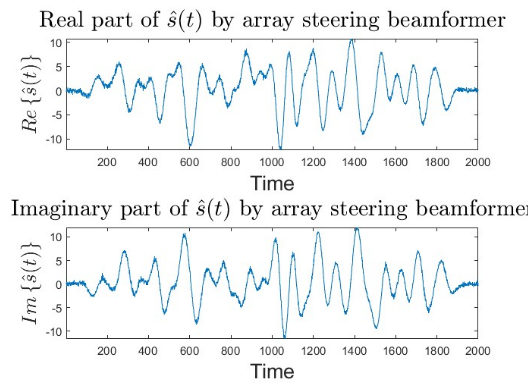
- **MVDR (Minimum Variance Distortionless Response)**:

This method can extract the original signal without distortion, but it cannot filter out $\mathbf{i}(t)$, and the attenuation of noise power is not guaranteed.

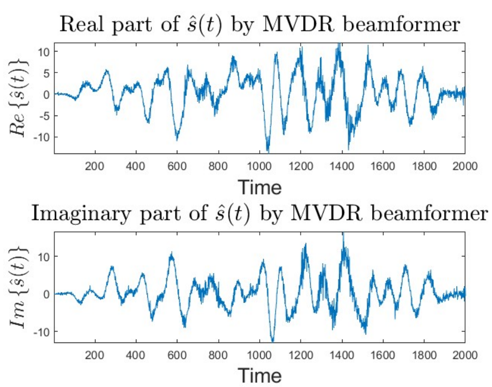
- **LCMV (Linearly Constrained Minimum Variance)**: This method can be considered a more general version of **MVDR**. It can strongly filter out $\mathbf{n}(t)$, but the attenuation of noise power is also not guaranteed.



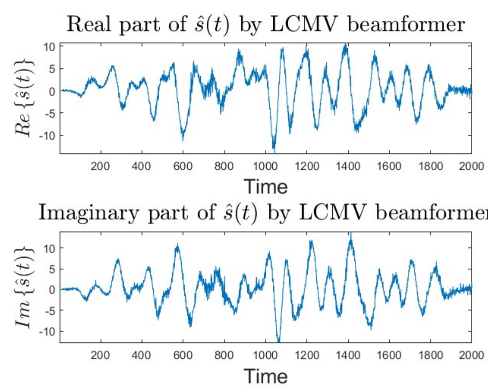

(a) Uniformly weighted beamformer




(b) Array steering beamformer




(c) MVDR beamformer




(d) LCMV beamformer

Figure 2: The results of the beamformers in section 1

### 4.1 details of my beamformer

In my method, I considered the characteristics of LCMV, which is robust to interference. However, LCMV can be attenuated by noise power. Therefore, I use the LMS algorithm to estimate suboptimal weights that efficiently suppress noise power (PSL).In order to derivean iterative suboptimal solution of weight vector $\mathbf{w}$, the proposed beamformer is designed using the following optimization problem in real valued form

$$J(\mathbf{w}) = \frac{1}{2}\mathbf{w}^H\mathbf{R}_{xx} + \mathbf{Real}\left\{\lambda^H(\mathbf{C}^H\mathbf{w} - \mathbf{f})\right\}$$

where $\mathbf{Real}\{\cdot\}$is an operator of taking real part and $\lambda \in \mathbf{C}^{D\times 1}$ is a Lagrange multipliers vector. Calculating the conjugate gradient of $\mathbf{J}(\mathbf{w})$ with respect to $\mathbf{w}$,we have

$$\nabla_w J(\mathbf{w}) = \mathbf{R}_{xx}\mathbf{w} + \frac{\partial \mathbf{Real}\left\{\lambda^H(\mathbf{C}^H\mathbf{w} - \mathbf{f})\right\}}{\partial \mathbf{w}^*} = \mathbf{R}_{xx}\mathbf{w} - \mathbf{C}\lambda$$

The following updating formulation is used to compute the weight vector $\mathbf{w}$(n+1)

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu\left[-\nabla_w J(\mathbf{w})\right] = \mathbf{w}(n) - \mu\left[\mathbf{R}_{xx}\mathbf{w} - \mathbf{C}\lambda\right]$$

where $\lambda$(n) varies with index n.We constrain that the weight vector $\mathbf{w}$(n+1) must satisfy the original constraint condition of LCMV in each iteration

$$\mathbf{C}^H\mathbf{w}(n+1) = \mathbf{f} = \mathbf{C}^H\mathbf{w}(n) - \mu\mathbf{C}^H\left[\mathbf{R}_{xx}\mathbf{w} - \mathbf{C}\lambda\right]$$

Simplifying above equation further yields

$$\mathbf{w}(n+1) = \left[\mathbf{I} - \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H\lambda\right]\left[\mathbf{I} - \mu\mathbf{R}_{xx}\right]\mathbf{w}_n + \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{f}$$

Denoting

$$\mathbf{P} = \left[\mathbf{I} - \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H\lambda\right]$$

and

$$\mathbf{G} = \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{f}$$

we can rewrite the iterative process of weight vector $\mathbf{w}$(n+1) as the following form

$$\mathbf{w}(n+1) = \mathbf{P}\left[\mathbf{I} - \mu\mathbf{R}_{xx}\right]\mathbf{w}(n) + \mathbf{G}$$

From SMI in chapter 12 p.44, we can rewrite an available approximation $\mathbf{R}_{xx}$ in the following form

$$\mathbf{R}_{xx} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}(k)\mathbf{x}(k)^H$$

However, one can imagine that the error will be large due to the poor approximation of approximated covariance matrix.So I introduced a forgetting factor $\beta$ to estimate covariance matrix in the kth and (k−1)th observation time to avoid this situation

$$\mathbf{R}_{xx}(k) = \beta\mathbf{R}_{xx}(k-1) + (1-\beta)\mathbf{x}(k)\mathbf{x}(k)^H$$

(12)

the boundary of $\mu$ can be found in LMS,in here I set $\mu$ to a nonfixed dumber due to change of time

$$\mu = \frac{2}{\lambda_{max}} - 0.1$$

(13)

Based on the recursive estimate of covariance matrix abovementiened, we obtain algorithm as follows

$$\mathbf{w}(k+1) = \mathbf{P}\left[\mathbf{I} - \mu\mathbf{R}_{xx}(k)\right]\mathbf{w}(k) + \mathbf{G}$$

(14)

### 4.2 Summarize the steps of my beamformer

**Algorithm 2** RAIS-LCMV

**Input:**

initial covariance matrix , $\mathbf{R}_{xx}(0) = \mathbf{I}_N$;

initial weight , $\mathbf{w}(0) = 0_{N\times 1}$;

number of iteration , $K$=99;

updated factor , $\mu$=1;

forgetting factor , $\beta$=0.998;

**Output:**

estimated weight vector , $\mathbf{w}(t)$;

1. calculate $\mathbf{P}$

2.calculate $\mathbf{G}$

**for** $k = 1; k < K; k++$ **do**

3. Recursive estimation of covariance matrix using (12)

4. update $\mu$ using (13)

5. Update weight vector using (14)

**end for**

## 4.3 Adavantages of my beamformer

In this method, we derive a low-complexity RAIS-LCMV algirithm using conjugate gradient (CG) optimization method. Our proposed algorithms can, in certain applications, substantially improve the robustness to low SNR, JNRs, and small number of iteration which means less computation time.Also ,it can reject the interference too.However, in this project we don't know the noise power and jamming power.So we can not estimate the comparison of SNR between proposed method and traditional LCMV method.But we can see strong noise reduction propose in section.5 compared to LCMV in figure 2.

## 4.4 weight comparison

ALL weight has been stored in weight.mat.Dear TA and teacher can check out that all of these are different.

# 5 Plot of the proposed beamformer

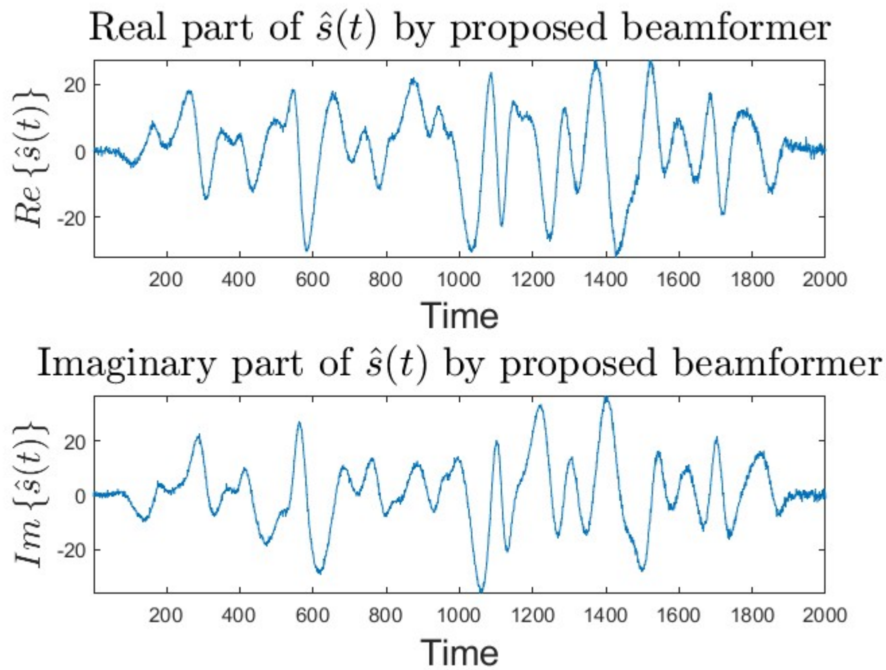



Figure 3: proposed beamformer

## References

[1] Yin, X.; Xu, Y.; Sheng, X.; Shen, Y. Signal Denoising Method Using AIC—SVD and Its Application to Micro-Vibration in Reaction Wheels. Sensors 2019, 19, 5032.

[2] GUO, X., CHU, L., LI, B. Robust adaptive LCMV beamformer based on an iterative suboptimal solution. Radioengineering, 2015, vol. 24, no. 2, p. 572—582. DOI: 10.13164/re.2015.0572