QF620-Stochastic Modelling finance

Group project report

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1 Analytical Option Formulae

1.1 Objective

In this part of project, derive and implement the following model,
- Black-Scholes model
- Bachelier model
- Black 76 model
- Displaced-diffusion model

and use the model to value the following options in Python.
- Vanilla call/put
- Digital cash-or-nothing call/put
- Digital asset-or-nothing call/put

1.2 Model definitions and assumptions

The Black-Scholes Merton model prices a contract where the premium is valued at issuance or on purchase, with closing transactions being subject to the option exercise either at maturity or before. Hence this model applies a discounted present value price. The statistical analysis applies a continuously compounded return with the dispersal being described by a lognormal distribution.[1]

Louis Bachelier's model prices an option contract over a futures contract with all payments at maturity, making a stated discount unnecessary. The statistical analysis applies an arithmetic return method with normally distributed dispersal measure.(Thomson, 2016)

The Louis Bachelier and the Black-Scholes Merton models apply asset returns or asset prices in distinctively different manners relating to assumptions on the underlying asset price path and contractual form.

Black-76, is an adjustment of his earlier Black-Scholes options pricing model. Unlike the earlier model, the revised model is useful for valuing options on futures. Black's Model is used in the application of capped variable rate loans, and is also applied to price a variety of derivatives. Future prices are log-normally distributed and that the expected change in futures price is zero. One of the key differences between his 1976 model and the Black-Scholes model(which assumes a known risk free interest rate,options that can only be exercised at maturity, no commissions and that volatility is held constant),is that his revised model uses forward prices to model the value of a futures option at maturity versus the spot prices Black-Scholes used. It also assumes that volatility is dependent on time, rather than being constant.

The displaced-diffusion model is based on the Black-Scholes formula,which combine the BS lognormal and Bachlier normal together, with several desirable features. First, it contains the Black-Scholes formula,as a special case, a = b = 0. Second, it derives the stochastic process driving the stock from a more fundamental analysis of the characteristics of the firm. Like the compound option model (Geske), it brings in debt but it goes deeper into the structure of the firm by decomposing its assets. Since its assets consist of a portfolio of a risky and riskless asset, the volatility of the value of the firm will not be a constant as in the compound option case, but rather it will be stochastic.[2]

Here we can interpret Black-Scholes (call) option formula as Vanilla Call = Asset Digital Call - K $\times$ Cash Digital Call,which is also satisfied to all the four models. And besides the small difference in Φ between these 4 models ,the main difference is as following ,for example, Bachelier cash or nothing model have no discount factor D(0,T).

Model	**Cash-or-nothing**	**Asset-or-nothing**
Black-Sholes	D(0,T)	$S_0\Phi$
Bachelier	-	$S_0(1-\sigma\sqrt{T})\Phi$
D-Diffusion	D(0,T)	$F_0 e^{-rT}\Phi$

[1] Ian A.Thomson."option pricing model:comparing Louis Bachelier with Black-Sholes Merton" The Journal of Finance Vol.38,No.1 (Mar.,1983),pp.213-217.

[2] Robert Geske."The Valuation of Compound Options."Journal of Financial Economics 7 (Mar 1979), PP.63-81.

2 Model Calibration

In this part of project, two models are calibrated and match the option prices:

- Displaced-diffusion model
- SABR model (fix beta = 0.8)

2.1 Background

To get the data of at-the-money directly, let K = 850 to be the parameters in the model calibration as in the google call and google put both have 850 as strike price. And the mid price of best bid and offer price is set as the option price for at-the-money.

2.2 Procedure

Step 1: Analyse the option data and calculate the implied volatility

From the google put data, use the strike price < 850 as the left scatter data. From the google call data, use the strike price > 850 as the right scatter data. (Due to call and put option are out-of-money when the strike price is greater or less than the S0 price. And option with out-of-money has much more liquidity to be priced reasonably than in-the-money, thus it is chosen to plot the scatter for precision purpose.) After obtaining the call and put strike price and option price, the implied volatility is calculated by backstepping the BS call and BS put, and use for future scatter plotting.

Figure 1:

```
gcall=gcall[gcall['strike']>S]
gput=gput[gput['strike']<S]
df=pd.DataFrame(columns=['strike','impliedvol'])
for i in gcall.index:
    t=pd.DataFrame([gcall.loc[i,'strike'],impvcall(S, gcall.loc[i,'strike'], r,
(gcall.loc[i,'best_bid']+gcall.loc[i,'best_offer'])/2, T)[0]]).T
    t.columns=['strike','impliedvol']
    df=df.append(t)
for i in gput.index:
    t=pd.DataFrame([gput.loc[i,'strike'],impvput(S, gput.loc[i,'strike'], r,
(gput.loc[i,'best_bid']+gput.loc[i,'best_offer'])/2, T)[0]]).T
    t.columns=['strike','impliedvol']
    df=df.append(t)
impvol=impliedCallVolatility(S, 850, r, 102.5, T)
```

To be noted is that the implied volatility = 0.2570 is going to be used in the whole project.

Step 2: Obtain the parameters of SABR and Displaced-Diffusion model

Get the parameter of SABR, by OLS method, a best fitted implied volatility is simulated. Which generates three important parameters α,ρ,υ

Figure 2:

```
initialGuess = [0.02, 0.2, 0.1]
res = least_squares(lambda x: sabrcalibration(x,
                                              df['strike'].values,
```

Figure 3:

```
                                              df['impliedvol'].values,
                                              S,
                                              T),
                    initialGuess)
alpha = res.x[0]
beta = 0.8
rho = res.x[1]
nu = res.x[2]
print('alpha: %.4f   rho: %.4f   nu:%.4f'%(alpha,rho,nu))
```

output:α:0.9909,ρ:-0.2886,υ:0.3529

Similarly, by using OLS method to minimize the sum of squares of residuals. The beta of Displaced-Diffusion could be obtained. What to be noted is the implied volatility that used are at-the-money in order to make the plot passing the point of at-the-money in the final chart.

Figure 4:

```
initialGuess = [0.02, 0.2, 0.1]
res = least_squares(lambda x: sabrcalibration(x,
                                              df['strike'].values,
                                              df['impliedvol'].values,
                                              S,
                                              T),
                    initialGuess)
alpha = res.x[0]
beta = 0.8
rho = res.x[1]
nu = res.x[2]
print('alpha: %.4f , rho: %.4f , nu:%.4f'%(alpha,rho,nu))
```

output: betaDD: 0.3651

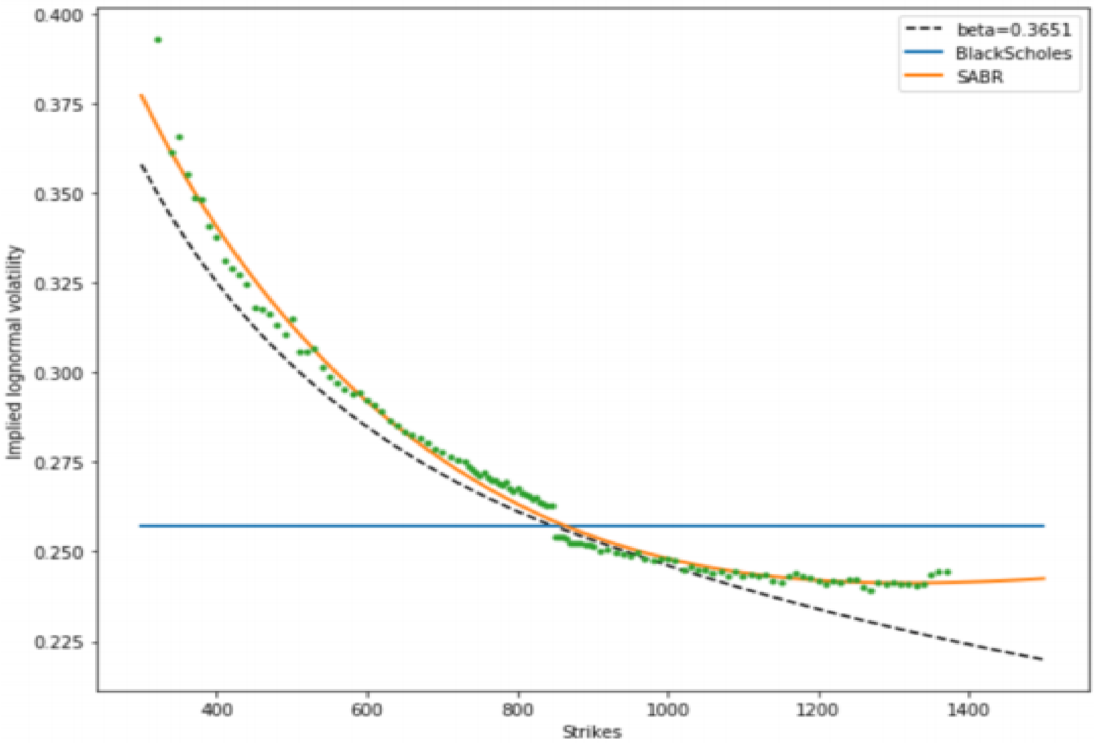
Step 3: Calibration of SABR, Displaced-Diffusion model and plotting

From the parameters obtained in step 1 and step 2, a scatter with K as independent variable, implied volatility as the dependent variable is plotted. To be noted in the plotting method, the implied volatility of at-the-money is taken into the BS call and put model to get the price. And take the two price into the Displaced-Diffusion function separately to get the corresponding sigma to plot the curve. For instance, the left side curve is displaced-diffusion put, using the displaced-diffusion sigma to calculate the displaced-diffusion put price and take in into BS put to calculate every single implied volatility of K as the result. Right side is similar. And negative correlation between strike price and volatility are expected here, when strike price increase, the volatility decrease, vice versa.

By comparing the scatter plotting, SABR model is well fitted including the smile skew; and the Displaced-Diffusion with beta = 0.3651 is not fitted well especially when the strike price at two side, the difference is obvious.

The implied volatility of BS is simply the sigma of BS that shows up as horizontal line.

Figure 5:



Step 4: Analysis of changing parameters of model

i. Beta of Displaced-diffusion

Based on the $dF_t = \sigma\,[\beta F_t + (1-\beta)F_0]\,dW_t^*$, with the increase of beta, the lognormal part become more significant, Displaced-Diffusion model is more likely to have a flat plot trend.

Figure 6:

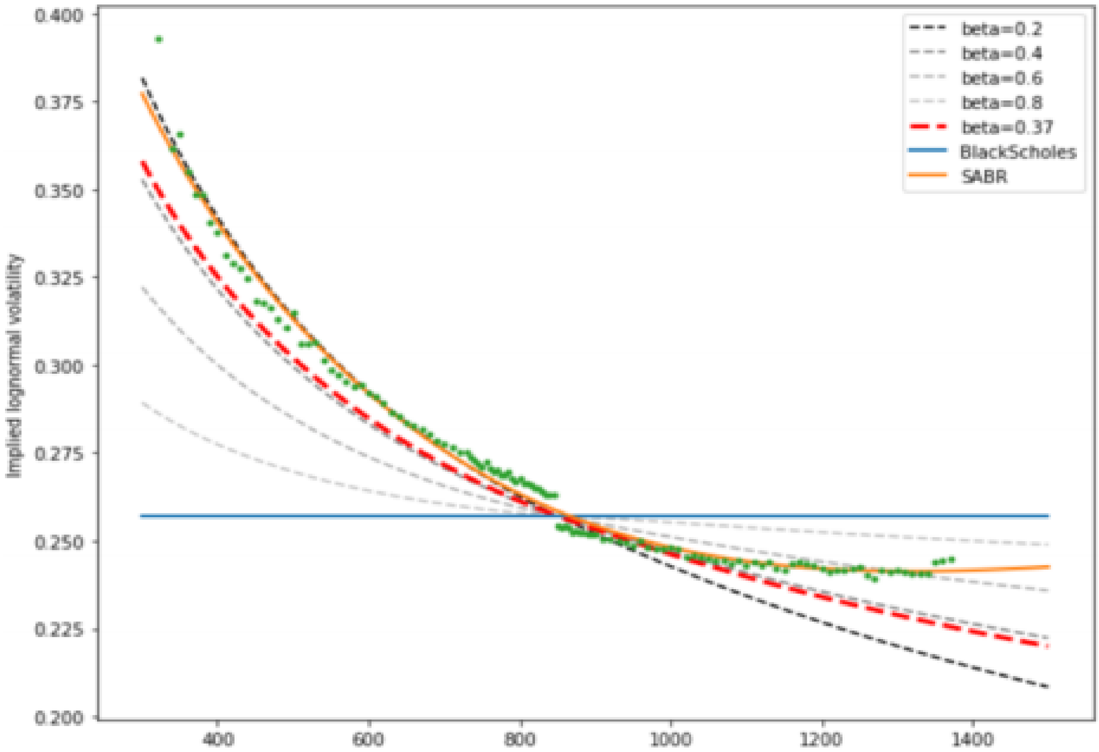
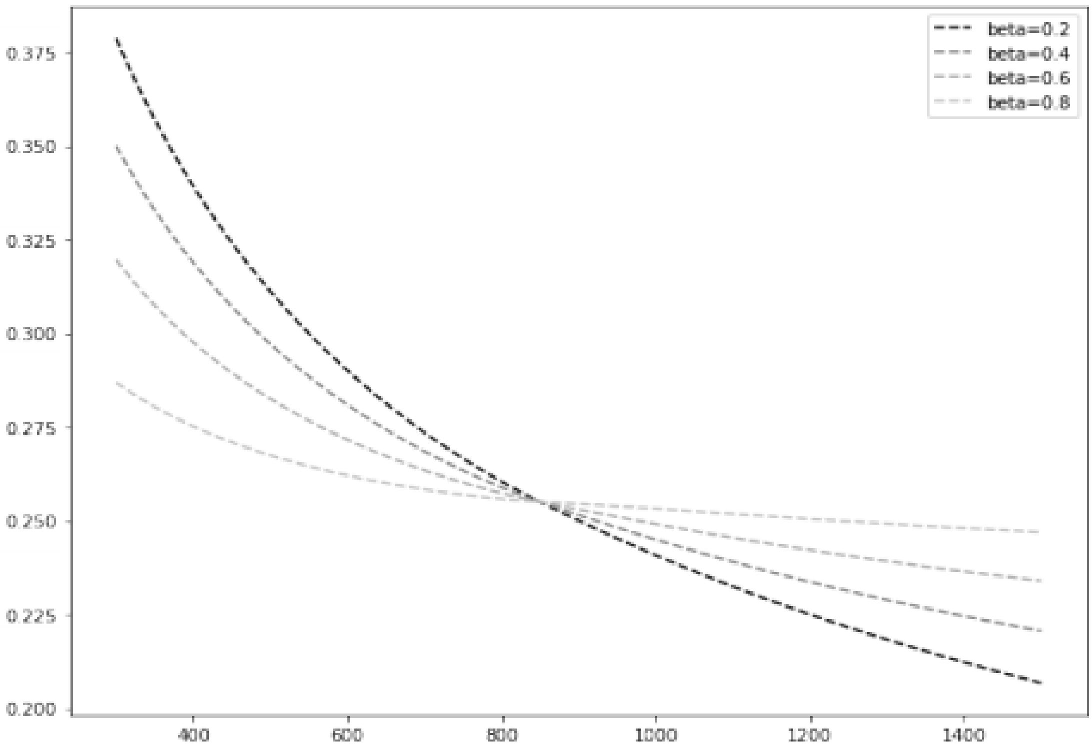


Figure 7:

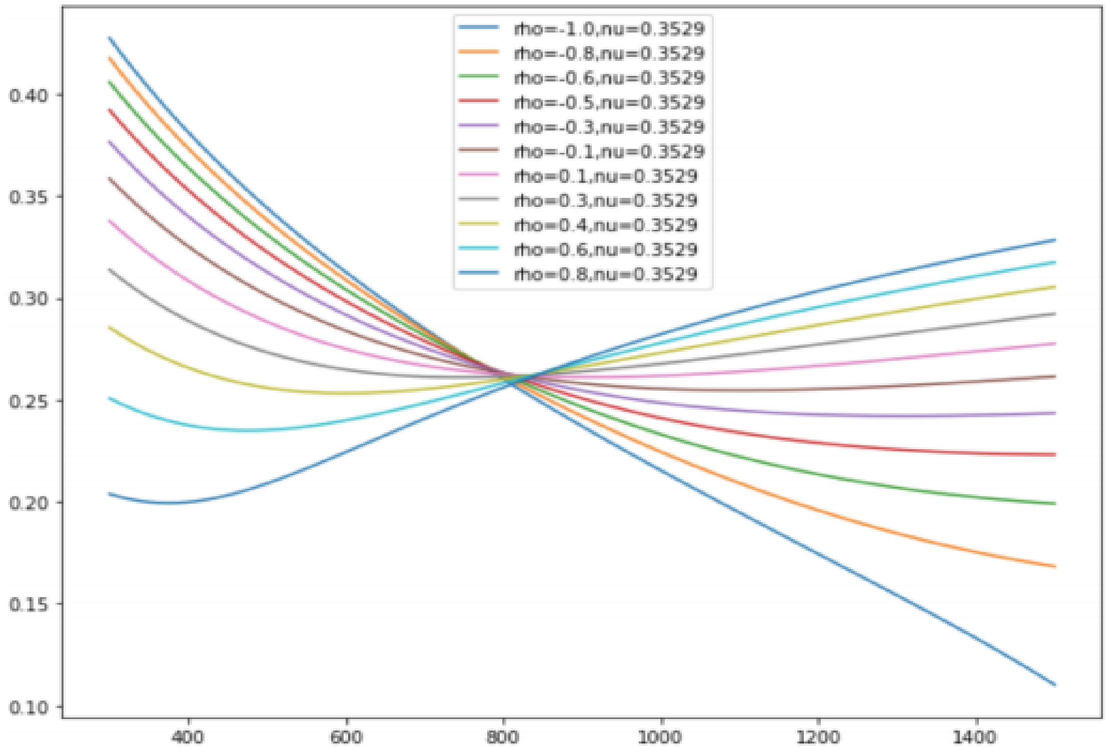


ii. ρ of SABR

ρ basically represents the correlation between price and volatility. When υ remains constant, with the change of ρ, the SABR model plotting will rotate near the centre of at-the-money. In particular, with the increase of ρ, the SABR plotting will rotate anti-clockwise.

A negative correlation results in high volatility when the stock price drops, which increases the prices of out-of-money puts and decreases the prices of out-of-money calls relative to the Black-Scholes model price.

Figure 8:



iii. υ of SABR

In SABR model, υ as the volatility of volatility, when υ becomes greater the curve of volatility become more curved.Increasing the volatility of volatility has the effect of increasing the kurtosis of return which is increases the price of out-of-money call and put options.

With the υ changes in the same direction,the greater the absolute value of υ is, the greater the convexity of SABR is, and the overall trend is inward.

Figure 9:

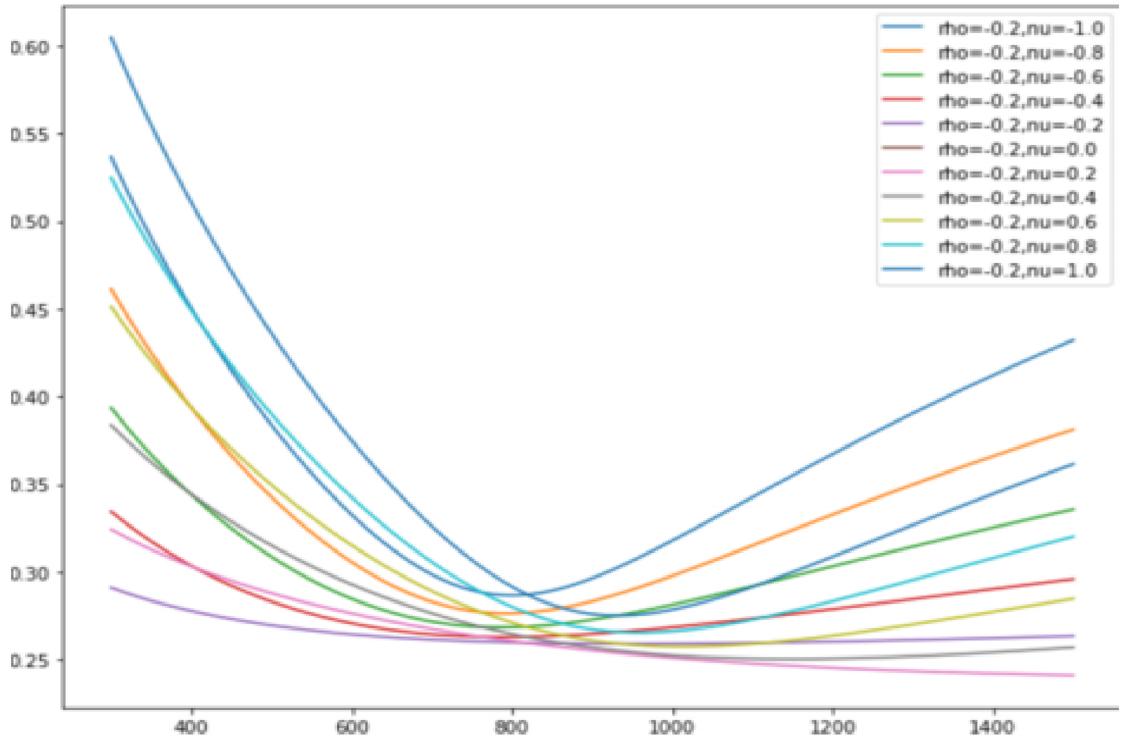


Figure 10:

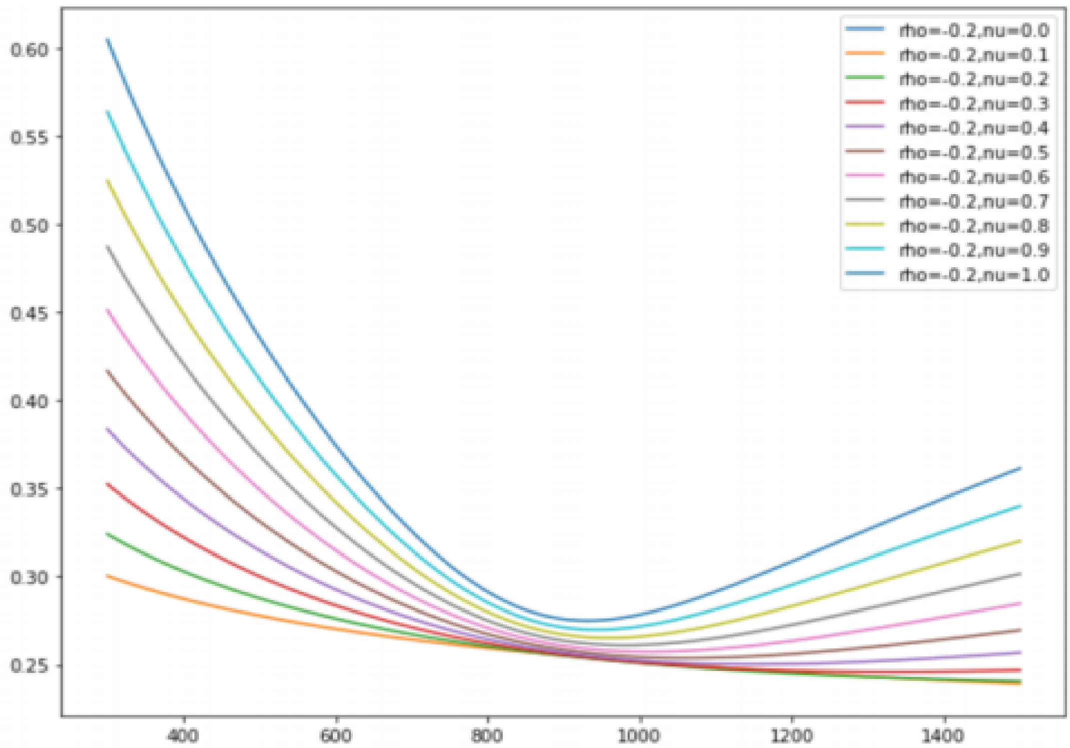
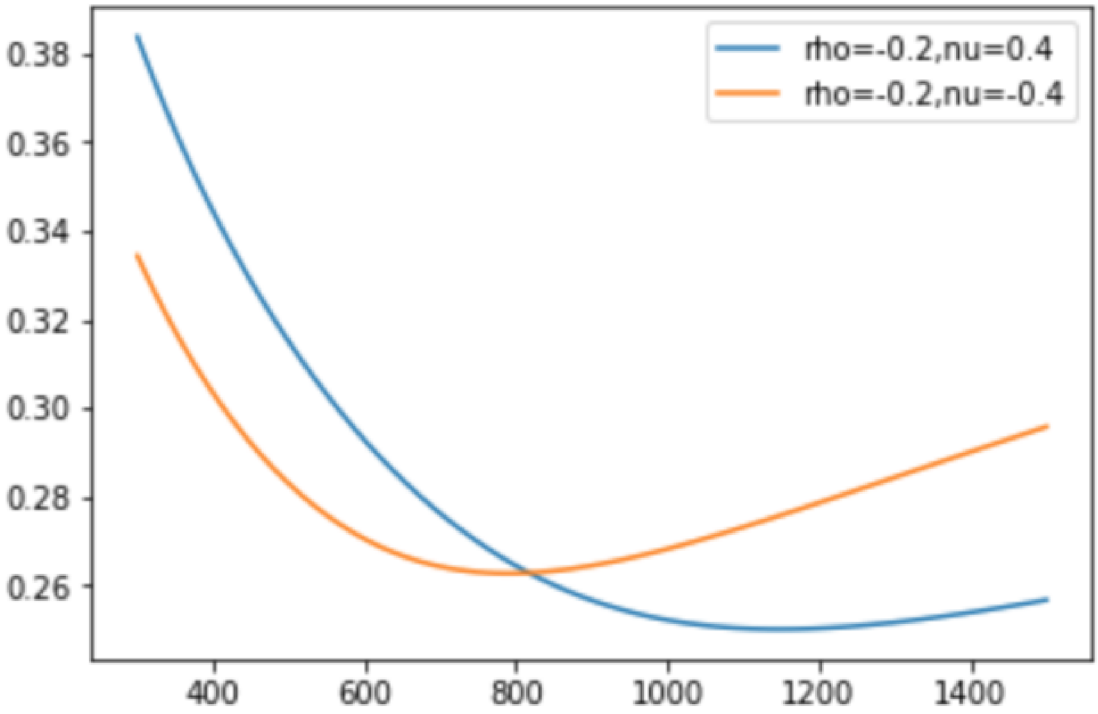


Figure 11:



For instance, when there is a change of same absolute value of stock price, the rate of drop in volatility will become larger, which results that with the drop of strike price, the absolute tangent value become larger, the volatility change faster.

The Displaced-Diffusion Model can balance the degree of log normal and normal, so that different pricing models can be obtained by changing beta to fit the market better.

The reason we can fit the market volatility perfectly through SABR is that the breaker in the market is actually use the SABR model to quote the price so this is it.

3 Static Replication

On 30-Aug-2013, an European derivatives expiring on 17-Jan-2015, whose payoff function only depends on S_t and time, can be determined in following way.

Based on Part 2, the inital price S_0 is 846.9\$, ATM $Sigma$ is 0.257 in both Black-Scholes and Bachelier contexts, time T is 1.38 year and interest rate R is intepolated as 0.405%.Input these variables into the two models will yield the corresponding prices for the derivatives.

1. The payoff using Black Scholes model is as follow

$$
\begin{aligned}
V_0 &= e^{-rt}E^{Q^*}\left[S_t^3 \times 10^{-8} + 0.5 \times \log\left(S_T\right) + 10\right] \\
&= e^{-rt}E^{Q^*}\left[S_0^3 e^{3\left(r-\frac{1}{2}\sigma^2\right)t+3\sigma W_t} \times 10^{-8} + 0.5 \times \left[\log\left(S_0\right) + \left(r-\frac{1}{2}\sigma^2\right)t + \sigma W_t\right] + 10\right] \\
&= e^{-rt} \times (10^{-8} \times S_0^3 e^{3(r-\frac{1}{2}\sigma^2)t} \times E[e^{3\sigma W_t}] + 0.5 \times (\log(S_0) + \left(r-\frac{1}{2}\sigma^2\right)t) + 10) \\
&= e^{-rt} \times (10^{-8} \times S_0^3 e^{3(r-\frac{1}{2}\sigma^2)t} \times e^{\frac{9\sigma^2 t}{2}} + 0.5 \times (\log(S_0) + \left(r-\frac{1}{2}\sigma^2\right)t) + 10) \\
&= e^{-rt} \times (10^{-8} \times S_0^3 e^{3(r+\sigma^2)t} + 0.5 \times (\log(S_0) + \left(r-\frac{1}{2}\sigma^2\right)t) + 10) \\
&= 21.36
\end{aligned}
$$

2. The payoff using Bachelier model is as follow:

$$dS_t = \sigma S_0 dW_t, \text{which means } S_t = S_0 + \sigma S_0 W_t$$

$$\begin{aligned}
V_0 =& E[(S_0 + \sigma S_0 W_t)^3 \times 10^{-8} + 0.5 \times \log(S_0 + \sigma S_0 W_t) + 10] \\
=& E[{S_0}^3 + \sigma^3 {S_0}^3 {W_t}^3 + 3{S_0}^2 \sigma S_0 W_t + 3S_0 \sigma^2 {S_0}^2 {W_t}^2] \times 10^{-8} \\
&+ 0.5 \times E[\log(S_0 + \sigma S_0 W_t)] + 10 \\
=& ({S_0}^3 + 3{S_0}^3 \sigma^2 t) \times 10^{-8} + 0.5 \log(S_0) + \frac{1}{2\sqrt{2\pi}} \int_{x^*}^{\infty} \log(1 + \sigma\sqrt{t}X) e^{-\frac{x^2}{2}} dx + 10 \\
=& 21.61
\end{aligned}$$

$$Where\ x^* = -\frac{1}{\sigma\sqrt{t}}$$

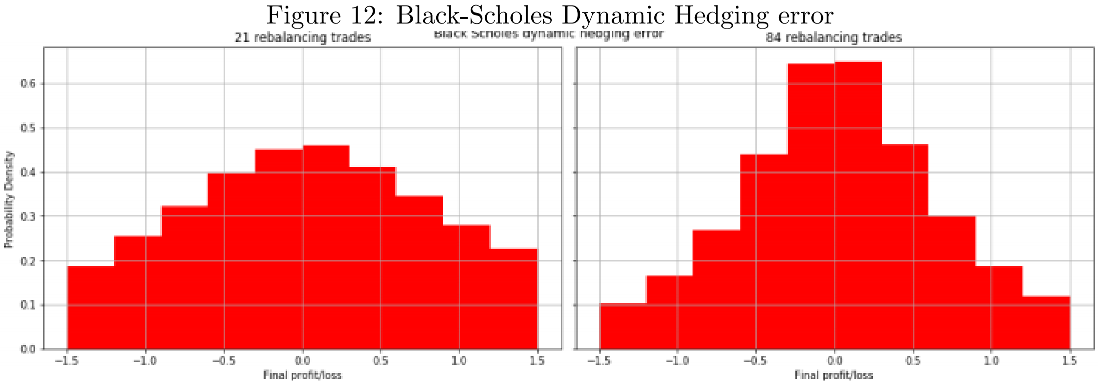
This function could then be solved by python package scipy.integrate.

In boths frameworks, the at the money σ for options have been used. Since when the european contracts expire at the money, in which the stock price equals the strike, any option holders could exercise the options to buy or sell the stocks at the prevailing stock prices. This has the holding/disposal effect that equates option holders with stock holders.Any transactions made will not bias towards stocks or options and any investors are indifferent between stocks and options.Thus the option volatility at the money could best mimic the volatility of stocks.

The prices of the derivatives derived from both frameworks are similar with the notice that Black Scholes model's price is slightly lower than that from Bachelier model. This could be explained by the fact that Black Scholes model assumes expected future price will be today's price grows at risk free rate while Bachelier assmues expected future price will just the price today, thus no discount component.

4 Dynamic Hedging

Figure 12: Black-Scholes Dynamic Hedging error



The Black Scholes dynamic hedging strategy's hedging errors after 21/84 smulations have been presented in figure 12. The Hedging error is defined as "Value of Black-Scholes hedge at T - Final option payoff". The stock price evolves within the Black-Scholes framework. It operates lognoramlly and drifts in a risk free manner. It has a Brownian motion component with a fixed known volatility. If the options are not hedged, option writers will be exposed to the future payoff risks. Thus option writers should follow Black-Scholes hedging strategy, rehedge continously during the time T,deliver the option payoff and unwind the hedge. If the hedger had followed the exact Black-Scholes hedging strategy, the final hedging error will be zero as Black-Scholes formula is expected to provide the fair value of the option. However, as in reality the hedges are conducted in discrete, so the hedging error will occur.

The dynamic hedging stragey for an option is:

$$C_t = \phi_t S_t - \psi_t Bt$$

.

where

$$\phi_t = \frac{\partial C}{\partial S} = \Phi\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

and

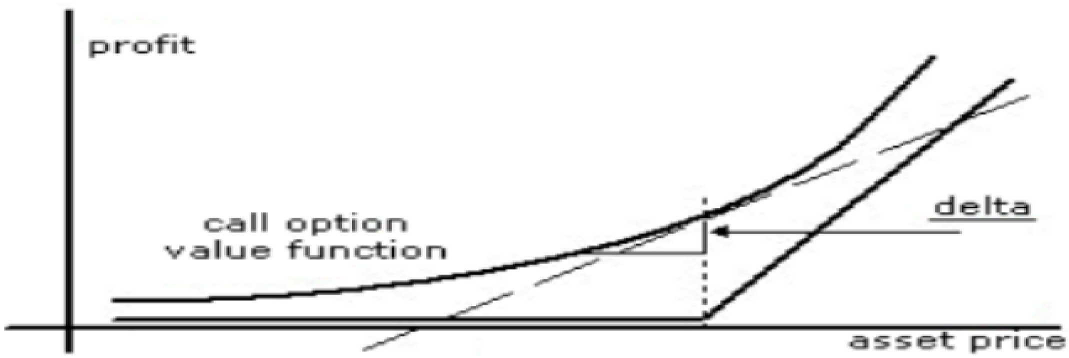
$$\psi_t B_t = -Ke^{-r(T-t)}\Phi\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

The hedger hold Δ amount of stocks and short reasonable amounts of bonds to replicate the Black-Scholes prcing formula and hedge the option risks. Based on Black-Scholes PDE $\frac{\partial V}{\partial S} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$, the Black-Scholes hedged portfolio is risk free and its value at time T will just be the call premium grows at risk free rate untill time T.However, it is only a static replication instead of dynamic hedging if no rebalance made after price changes.

At any given time, the portfolio:$V_t = \phi_t S_t - \psi_t Bt = C_t$,but in next period: $V_{t+1} = \phi_t S_{t+1} - \psi_t Bt + 1 \neq C_{t+1}$. The stock and bond price evolve with time but the amounts of stocks and bonds will still be the same amounts in t if no rebalance made, which will cause failure in replicating the Black Scholes pricing formula and contribute to hedging error. In this report, the hedging are conducted from a cash flow perspective that the inital call premium received will be kepted in the cash balance and grows at risk free rate. At the beginning, hedger replicates the call by shorting bonds and buying Δ amounts of stocks. Every rebalancing period, the assets evolve and thus result in profit and loss in the cash balance. The hedging then is conducted by adjusting the portfolio through buying/selling right amounts of stocks and bonds to replicate the Black-Scholes option pricing formula at each time and the costs of which will be deducted from cash balance. The hedger then unwind the hedges at time T and final hedging error will be the total cash balance(which grows at risk free rate during the priod) at time T minus final option payoffs.

As can be seen in the graph, both hedging error distributions resemble normal distribution and no bias in either distribution. The means for 21 and 84 simulations are 0.02 and 0.017 respectively, which essential are zero. The standard deviations for both are 1.52 and 0.79 respectively, the more hedges made, the smaller hedging error will be and the hedging error will more likely to cluster around 0. It can be concluded that Black Scholes dynamic hedging strategy could hedge risk reasonably well even in discrete time and is expected to completely hedge away option riks if hedging frequency goes to infinity in this context.

Figure 13: Error of delta



It should be noted that this context has a short time interval and very small stock price deviations. The Black Scholes dynamic hedging which relies on delta hedging in discrete time could result in more hedging error if price moves too quickly and too far away. As can be seen in figure 13, delta hedging works reasonable well as option value has litte deviation from its tangent line-delta. However, when price becomes more volatile, the gaps are expected to be bigger, so more hedging error will be experienced.In this case, the hedging should be made more frequently to match the stock price changes,or consider Gamma hedging,in order to reduce the hedging error.