QF 605 Fixed-Income Securities Group Project

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Part 1 Bootstrapping Swap Curves

1.1 Bootstrap the OIS discount factor

Introduction of OIS discount factor

Overnight index swap (OIS) discounting has been implementing for derivative valuation because the bankruptcy of banks in 2008 financial crisis indicated that the interbank lending rates (LIBOR) were not truly risk-free as previously thought. Collateralized derivatives with standard CSA agreements mitigate counterparty risk, thus OIS discount factor is the rational choice for valuation.

Bootstrapping procedure and result

Given a bunch of the overnight index swap rates paid annually and varied from tenor, the OIS discount factor is implied in the swap rate formulated as $S = \frac{D(0,T_0)-D(0,T_n)}{\sum_{i=1}^{n}\Delta_{i-1}D(0,T_i)}$, so we can extract the discount factor in sequence from these swap rates. Since tenor of the swaps in the dataset has gaps in between, the gaps somewhat cause difficulty in computing the discount factor of next year directly from the previous years. Therefore, linear interpolation on discount factors is applied as a way to obtain the estimated values. Specifically, for the period of years without raw data to calculate the discount factor, we linear interpolate between the discount factor calculated from the latest step and the one waiting for solve to get a series of discount factors.

In detail, based on present value of fixed leg equals to the present value of floating leg, we can get the OIS discount factor $D(0,T_n)$ by solving the below equation every step from scratch. Note that because the floating leg and discount rate are both based on OIS rate, the present value of floating leg can be simplified as $1-D(0,T_n)$.

$$PV_{fixed}-PV_{float}=0 \quad and \quad OIS\times\sum_{i=1}^{n}\Delta_{i-1}D(0,T_i)-(1-D(0,T_n))=0$$

The OIS discount curve is shown as following.

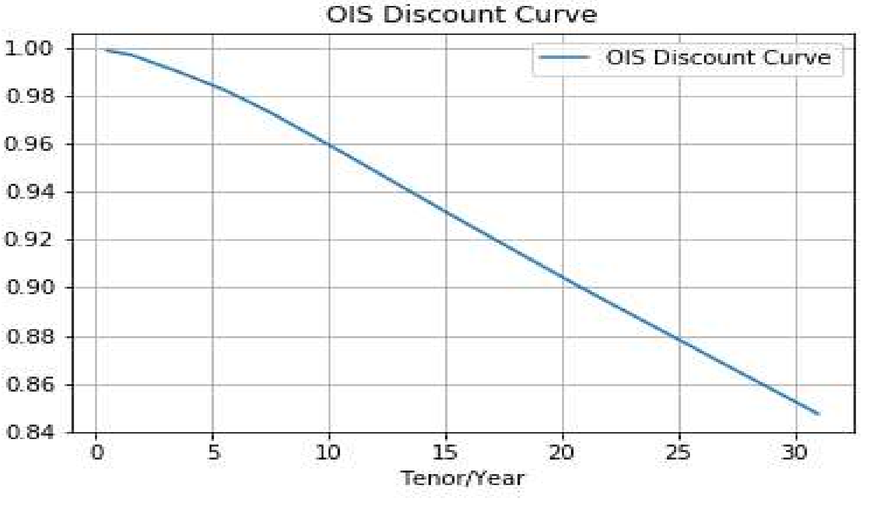

Figure 1: OIS Discount Curve

1.2 Bootstrap the LIBOR discount factor

Introduction of LIBOR discount factor

Before the 2008 financial crisis, LIBOR, the short-term borrowing rate of highly-rated financial institutions, is regarded as a proxy for the risk-free rate. The use of LIBOR to value derivatives brings convenience of valuation because the reference interest rate of traded derivatives is the same as the discount rate, but was called into question by the credit crisis. For non-collateralized derivatives, LIBOR rates are still used for valuation. LIBOR curve can be bootstrapped from a interest rate swap where LIBOR is exchanged for a fixed rate in that the swap rate can be viewed as an average of the forward LIBOR curve in the time-value-of-money sense.

Bootstrapping procedure and result

Considering that the swap market is collateralized, OIS rate is the appropriate rate for discounting. On the account of the semi-annual cash flow frequency of the interest rate swap, we should first continue to interpolate the OIS discount rate already got from the overnight index swap to have the complete OIS discount rates with the same time step.

Just like the procedure of bootstrapping OIS discount factor, we can get the LIBOR discount factor by utilizing the feature of the interest rate swap that the fixed leg and floating leg share the same present value to construct the equation as follows.

$$PV_{fixed} - PV_{float} = 0$$
$$IRS \times \sum_{i=1}^{T} D_o(0, t_i) \times \Delta_{i-1} - \sum_{i=1}^{T} D_o(0, t_i) \times L(t_{i-1}, t_i) \times \Delta_{i-1} = 0$$

However, the key difference between bootstrapping these two discount factors from interest derivatives lies in the aspect of the inconsistency of benchmark rate used by floating leg and discounting. Therefore, before employing the above equation, the forward LIBOR rates are required to be inferred from LIBOR discount factor in the first place. More importantly, linear interpolation was conducted on the LIBOR discount factor between the one just calculated from previous step and the one needed to be solved, in such a way that all these forward LIBOR rates are expressed into the linear combination included the unknown discount factor, in order to get a sequence of forward LIBOR matches the cash flow frequency of the interest rate swap.

$$L_{(i-1,i)} = \frac{D(0, t_{i-1}) - D(0, t_i)}{\Delta_{i-1} \times D(0, t_i)}$$

The following plot is the bootstrapping result of LIBOR discount factor.

Figure 2: Libor Discount Curve

1.3 Calculate the following forward swap rates

The forward par swaps rates have been calculated as below. Based on the pricing principle that the total future discount cash flow of both fixed leg and floating leg should be the same, the forward swap rates shown in the below formula can be interpreted as the weighted average OIS discount factor of forward LIBOR rates.

$$F_{(E,T)} = \frac{\sum_{i=E+1}^{E+T} D_o(0,t_i) \times L(t_{i-1},t_i) \times \Delta_{i-1}}{\sum_{i=E+1}^{E+T} D_o(0,t_i) \times \Delta_{i-1}}$$

The following list is summary of calculation result of forward swap rates.

Forward Swap (Expire*Tenor)	Rate
1y*1y	0.032007
1y*2y	0.033259
1y*3y	0.034011
1y*5y	0.035255
1y*10y	0.038428
5y*1y	0.039274
5y*2y	0.040075
5y*3y	0.040072
5y*5y	0.041093
5y*10y	0.043634
10y*1y	0.042189
10y*2y	0.043116
10y*3y	0.044097
10y*5y	0.046249
10y*10y	0.053458

Table 1: Calculation result of forward swap rates

Part 2 Swaption Calibration

2.1 Calibration the Displaced-Diffusion Model and SABR model

Companies enter in an interest rate swap contract to lock in a profit by receiving a fixed rate payment in exchange of paying a floating rate, or convert its fixed rate liability to an agreed floating rate liability. They choose to enter a swapiton contract to manage interest rate risk arising from their core business. Swaption is the option on interest rate swap. It gives the owner the right to enter in an IRS at a predetermined fixed rate in the future.

To price Swaptions, we can use Displaced-Diffusion Model or SABR model.

The swaption price under the displaced-diffusion model is:

$$V_{n,N}(0) = P_{n+1,N}(0) Black76\left(\frac{S_{n,N}(0)}{\beta}, K + \frac{1-\beta}{\beta} S_{n,N}(0), \sigma\beta, T\right)$$

To calibrate with Displaced-Diffusion Model, we need first calibrate the σ by using a random beta (in our model, we choose $\beta = 0.5$), as β actually has no effect on calculating sigma. Then we move on using the calibrated σ to calibrate β.

The SABR model attempts to capture the volatility smile structure by modelling a single forward rate with stochastic volatility dynamics. From previous study, we already know the close-form algebraic formulas for the implied volatility as functions of today's forward price and the strike.

In our project, the β is pre-determined as 0.9. The calibration of the SABR model also requires minimization procedure for each expire and tenor pairs. That is to say, the calibration of the whole swaption matrix will generally produce $m*n*3$ parameters where m and n are numbers of different expires and tenors respectively.

The minimization problem to be solved by SABR model calibration is matching the market implied volatility with the SABR volatility evaluated for a given strike set and current forward rate for each expiry and each tenor. Empirically, the model parameter Θ satisfies:

$$\Theta = argmin \sum_{i=0}^{N} [\sigma - \sigma_{SABR}(F, \sigma_{ATM}, T, \alpha, \beta, \rho, \nu)]^2$$

Calibrated Displaced-Diffusion Model Parameters

Sigma

Expire\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.22464432	0.28646066	0.29697582	0.26014689	0.24424255
5Y	0.26945341	0.29418178	0.29561965	0.26307564	0.24281025
10Y	0.27823008	0.28506327	0.28616868	0.26149323	0.23921982

Beta

Expire\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	-1.4836485	-1.2307152	-0.9902123	-0.6799511	-0.1390707
5Y	-1.1866368	-0.719661	-0.5449013	-0.2672982	0.03091165
10Y	-0.6798581	-0.5153394	-0.4151315	-0.3211231	-0.1755765

Figure 3: Displaced-Diffusion Model Calibration Parameters

Calibrated SABR Model Parameters

Alpha

Expire\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.1461853	0.18923562	0.20270663	0.18494815	0.17774561
5Y	0.16394598	0.19897449	0.21086318	0.19212527	0.17638789
10Y	0.17358313	0.1911166	0.20189134	0.195694	0.1777596

Rho

Expire\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	-0.6093267	-0.5248881	-0.4943506	-0.4498734	-0.3457903
5Y	-0.5658088	-0.540923	-0.5488723	-0.516649	-0.4393099
10Y	-0.5298166	-0.5301994	-0.5348937	-0.5438715	-0.4994586

Nu

Expire\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	1.93765629	1.62938274	1.38471526	0.99940018	0.68558572
5Y	1.306989	1.05279096	0.92952492	0.66580131	0.50768922
10Y	0.98859841	0.91090995	0.85432933	0.71314926	0.58936707

Figure 4: SABR Model Calibration Parameters

Our calibrated parameters are shown in Figure 3 and Figure 4.

To examine more closely the goodness of fit of two models, i.e. the volatility smile, we plot the volatility smile for all different expires and tenors, where market volatility is dot, SABR is dash-dot line, Displaced-Diffusion is solid line.

From Figure 5 we can see that SABR model fits the swapiton implied volatility fairly well. However it seems that SABR fitted volatility smile does not quite match the market implied volatility for longer maturity, especially at lower strike. The potential explanation is that in addition to the fact that long expiry swaptions are generally less liquid, SABR model also suffers from negative density function issue at low strike. It is also known that swaption data and corresponding forward rates are not updated at the same frequency. All those facts would cause the inconsistency among the model input thus bad results might be produced.

The Displaced-Diffusion model however has worse fitting than SABR model. The calibrated beta are mostly negative and the implied volatility shows it only fits well when close to the ATM strikes. This is because Displaced-Diffusion model can only fit to implied volatility skew, which means if the implied volatility surface also exhibit 'smile' characteristics, there will be mismatch.

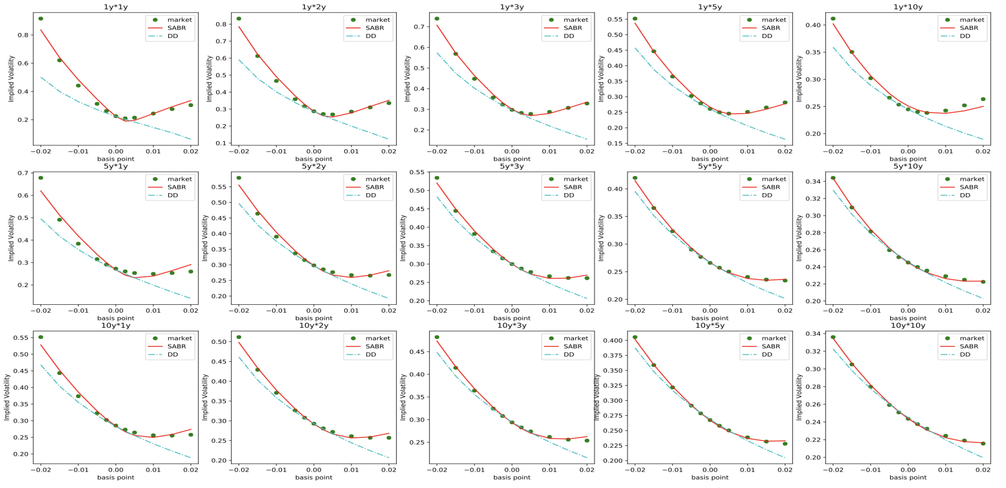

Figure 5: Implied Volatilities of Market, SABR, DD

2.2 Swaption Pricing

We can use our calibrated Displaced-Diffusion and SABR parameters to interpolate the 10y tenor $\alpha, \rho, \nu, \sigma, \beta$ between the 1y and 5y expiry to price the $2yx10y$ payer swaption and between 5y and 10y expires to price the $8yx10y$ receiver swaption. The swapiton prices are shown in Figure 6. We can see the Displaced-Diffusion model and SABR model have similar results.

swaption\K	1%	2%	3%	4%	5%	6%	7%	8%
DD payer 2Y*10Y	0.288345	0.195388	0.112723	0.051117	0.016642	0.003552	0.000457	3.26E-05
DD receiver 8Y*10Y	0.021985	0.036837	0.058935	0.090072	0.131636	0.184279	0.247707	0.320704
SABR payer 2Y*10Y	0.289206	0.197774	0.115164	0.052599	0.02058	0.009174	0.005005	0.003164
SABR receiver 8Y*10Y	0.027601	0.049525	0.07255	0.100756	0.14095	0.200174	0.275942	0.360493

Figure 6: Swaption Prices using Calibrated DD and SABR model

Part 3 Convexity Correction

3.1 Valuing CMS legs

Constant Maturity Swap contract pays a swap rate rather than Libor rate on its floating leg. A CMS paying the swap rate $S_{n,N}(T)$ at time $T = T_n$ can be expressed as $V0 = D(0,T)E[S_{n,N}(T)]$. The exact value of this contract can be obtained through convexity correction. In this case, static replication has been applied to achieve the model independent convexity correction.

CMS rates can be static replicated as:

$$E[S_n,N(T)] = g(F) + D(0,T)\int_0^F h(K)V^{rec}(K)dK + \int_F^\infty h(K)V^{pay}(K)dK$$

where $V^{rec}(K)$ and $V^{pay}(K)$ are the values of liquid receiver and payer swaptions. Those swaptions' values can be just be valued by Black 76 market model, with its volatility backed by SABR models.

A CMS leg is a collection of CMS rates paid over a period. For the leg of receiving CMS10y semi-annually over 5 years, its PV is just

$$PV = D(0,6m)\times 0.5\times E^T[S_{6m,10y6m}(6m)] + D(0,1y)\times 0.5\times E^T[S_{1y,11y}(1y)] + ...$$

$$+D(0,4y6m)\times 0.5\times E^T[S_{4y6m,14y6m}(4y6m)] + D(0,5y)\times 0.5\times E^T[S_{5y,15y}(5y)]$$

All the individual CMS rates can just be obtained through static replication mentioned above. For the leg of receiving CMS2y quarterly over the next 10 years, the valuation method is exact, despite the fact that it is quarterly payment, which requires adjustment on discount factors. Its PV is

$$PV = D(0,3m)\times 0.25\times E^T[S_{3m,2y3m}(3m)] + D(0,6m)\times 0.25\times E^T[S_{6my,2y6y}(6m)] + ...$$

$$+D(0,9y9m)\times 0.25\times E^T[S_{9y9m,11y9m}(9y9m)] + D(0,10y)\times 0.25\times E^T[S_{10y,12y}(10y)]$$

It should be noted that the payment frequency won't affect settlement period [Ti,Ti+1] for the underlying swaptions, which remain at semi-annual.

The calculated PV for 5 years CMS10y semi-annual leg is 0.2087; the PV for 10 year CMS2y quarterly leg is 0.2054. It can be noticed that although the former only receives 10 payments, compare with 40 payments from the latter, the former's PV is larger than the latter. It can be explained as in a typical upward sloping yield cruve, CMS10y has higher rates than CMS2y.Moreover, the CMS10y leg has been discounted less, which may lead to its higher PV.

3.2 CMS Rates and Forward Swap Rates Comparison

	Fwd Swap rates	**CMS rates**
1y*1y	0.032007	0.032095
1y*2y	0.033259	0.03345
1y*3y	0.034011	0.034273
1y*5y	0.035255	0.035552
1y*10y	0.038428	0.038958
5y*1y	0.039274	0.039863
5y*2y	0.040075	0.041055
5y*3y	0.040072	0.041364
5y*5y	0.041093	0.042726
5y*10y	0.043634	0.046508
10y*1y	0.042189	0.0432
10y*2y	0.043116	0.044822
10y*3y	0.044097	0.046503
10y*5y	0.046249	0.049693
10y*10y	0.053458	0.060522

Table 2: forward swap rates VS CMS rates

Expiry \Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.884459	1.912444	2.621561	2.966981	5.3
5Y	5.88575	9.801444	12.921389	16.32754	28.743548
10Y	10.109095	17.06528	24.055759	34.4418	70.649174

Table 3: Convexity Correction for CMS rates in basis points

The CMS rates are calculated through static replication, using liquidly traded receiver and payer swaption, valued by Black 76 models. The inputted volatility was obtained through SABR model, since the SABR model calibrated to market volatilities well, whose strikes range from F-200bps to F+200bps, it is good practice to cap the lower bound to F-200bps minus 1% and the upper bound to F+200bps plus 1% for integration in static replication, in order to keep consistency and generate meaningful values within the boundaries of the models.

As indicated in Table 2, both Swap rates and CMS rates are monotonically increasing with increased expiry and tenor, which is expected in a upward sloping yield curve framework. Convexity correction is required to obtain correct values for CMS rates by avoiding directly calculation in a wrong measure. Convexity corrections are the differences between CMS rates and forward swap rates. As demonstrated in the table 3, the longer the expiry, the longer the tenor, the more convexity corrections should be made. In each expiry year, the longer the tenor, the more convexity corrections in basis points; for shorter expiry year, the magnitude of increasing column by column is less than those CMS rates with longer expiry year, which is less liquidly traded. While long tenor CMS rates always require more corrections, very extreme differences can be noticed for 70.65 basis points correction for $10Y \times 10Y$ CMS rate and only 0.88 basis points correction for $1Y \times 1Y$.

Part 4 Decompounded Options

4.1 Static replication for $\text{CMS10y}^{1/p} - 0.04^{1/q}$

Using quotient rule, the first and second order derivatives of $h(K)$ are given by:

$$g(k) = k^{0.25} - 0.04^{0.5} \quad g'(k) = 0.25 \cdot K^{-0.75} \quad g''(k) = -0.1875k^{-1.75}$$

$$h(K) = \frac{g(K)}{\text{IRR}(K)} = \frac{k^{0.25} - 0.04^{0.5}}{IRR(k)}$$

$$h'(K) = \frac{\text{IRR}(K)g'(K) - g(K)\text{IRR}'(K)}{\text{IRR}(K)^2} = \frac{\text{IRR}(K) \cdot 0.25 \cdot K^{-0.75} - \left(K^{0.25} - 0.04^{0.5}\right)\text{IRR}'(K)}{\text{IRR}(K)^2}$$

$$\begin{aligned} h''(K) &= \frac{\text{IRR}(K)g''(K) - \text{IRR}''(K)g(K) - 2 \cdot \text{IRR}'(K)g'(K)}{\text{IRR}(K)^2} + \frac{2 \cdot \text{IRR}'(K)^2 g(K)}{\text{IRR}(K)^3} \\ &= \frac{\text{IRR}(k) \cdot \left(-0.1875k^{-1.75}\right) - \text{IRR}''(k)\left(k^{0.25} - 0.04^{0.5}\right) - 2 \cdot \text{IRR}'(k) \cdot 0.25 \cdot k^{-0.75}}{\text{IRR}(k)^2} \\ &\quad + \frac{2 \cdot \text{IRR}'(k)^2 \cdot \left(k^{0.25} - 0.04^{0.5}\right)}{IRR(k)^3} \end{aligned}$$

For CMS static replication,

$$V_0 = D(0,T)g(F) + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK$$

Generally,use receiver swaption value to replication for interval [0,F] since it is liquid and payer swaption value for interval [F, ∞] since payer swaption is liquid within this interval. In this case, as Black 76 model with calibrated SABR sigma has been used to generate values for payer/receiver swaptions, the upper bound in integration has been capped at forward plus highest strike plus 1%, so that the values outputted by the models are sensible. The value is **0.227304** for this option.

4.2 Static replication for $\left(\text{CMS10y}^{1/p} - 0.04^{1/q}\right)^+$

Since the payoff function is to take the maximum among $\text{CMS10y}^{1/p} - 0.04^{1/q}$ and 0, this option becomes a product similar to CMS caplet. The boundary where payoff is 0 is when CMS rate is 0.0016. Therefore we could use just payer swaption for replication and integral from 0.0016 to ∞.

$$\begin{aligned} V_0 &= \int_b^\infty h(k)\frac{\partial^2 V^{pay}(k)}{\partial k^2}dk \\ &= \left[h(k)\frac{\partial V^{pay(k)}}{\partial k}\right]_b^\infty - \int_b^\infty h'(k)\frac{\partial V^{(pay)(k)}}{\partial k}dk \\ &= -h(b)\frac{\partial V^{pay}(b)}{\partial k} + h'(b)V^{pay}(b) + \int_b^\infty h''(k)V^{pay}(k)dk \\ \because h(b) &= \frac{g(b)}{IRR(b)} = \frac{0.0016^{0.25} - 0.04^{0.5}}{IRR(b)} = 0 \\ &= h'(b)V^{pay}(b) + \int_b^\infty h''(k)V^{pay}(k)dk \end{aligned}$$

The value of this option is **0.241939**, which is a little bit higher than the first option, because the payoff is always positive, in contrast with possible negative payoffs from the previous option.