# Application of learning algorithms to nonlinear filtering and MCMC

PhD Dissertation defense

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COGNITION & CONTROL IN COMPLEX SYSTEMS

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Committee members : Prof. José Principe, Prof. Kamran Mohseni, Prof. James Hobert

Thanks to the National Science Foundation and Army Research Office
Thanks to friends & family

# List of Publications

## Feedback Particle Filter:

- A. Radhakrishnan, A. M. Devraj, S. P. Meyn, **"Learning Techniques for Feedback Particle Filter Design"** *IEEE Conference on Decision and Control, Dec 2016.*
- A. Radhakrishnan, and S. P. Meyn, **"Feedback Particle Filter Design Using a Differential-Loss Reproducing Kernel Hilbert Space"** *American Control Conference, June 2018.*
- A. Radhakrishnan, and S. P. Meyn, **"Gain Function Tracking in the Feedback Particle Filter"** *American Control Conference, July 2019.*

## Markov chain Monte Carlo methods:

- N. Brosse, A. Durmus, S. P. Meyn, E. Moulines, and A. Radhakrishnan, **"Diffusion Approximation and Control Variates for MCMC"** Submitted to *Annals of Applied Probability in July 2019.*

## Code on Github:

- FPF package in Julia by S. Surace - **https://github.com/simsurace/FeedbackParticleFilters.jl**
- Matlab and Python code - **https://github.com/a4anandr/FPF-code**

# Outline

# PhD Proposal - Recap

## Work till then

- Presented the feedback particle filter (FPF) - an approximation of the nonlinear filter that requires computing a gain function by solving a version of *Poisson's equation*.
- Proposed a differential LSTD-learning algorithm to approximate the gradient of the solution. [R et al. 16]

# PhD Proposal - Recap

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- Presented the feedback particle filter (FPF) - an approximation of the nonlinear filter that requires computing a gain function by solving a version of *Poisson's equation*.
- Proposed a differential LSTD-learning algorithm to approximate the gradient of the solution. [R et al. 16]
- Presented numerical examples for gain approximation and state estimation for scalar systems.

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## Tasks promised and accomplished

- Extend the algorithm to a multidimensional setting

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- Impact of gain approximation error on filtering performance

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- Impact of gain approximation error on filtering performance - X [Taghvaei et al. 18]

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- Extend the algorithm to a multidimensional setting ✓
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[R et al. 18]

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### New directions

- Simplified differential LSTD algorithm for Langevin diffusion
- Application of kernel methods to gain approximation [R et al. 18]
- Improvements for online gain estimation [R et al. 19]
- Explored the application to MCMC algorithms [Brosse et al. 19]

## Poisson's Equation

$$0 = \tilde{c} + \mathcal{D}h$$

$$h(x) = \mathsf{E}\Big[\int_0^{\tau} \tilde{c}(X(t))\,dt\Big]$$

with $X(0) = x$

## Optimal FPF Gain

$$\mathsf{K} = \nabla h$$

## Optimal Control

$$\phi_{n+1}(x) = \operatorname*{argmin}_u \{c(x,u) + \mathcal{D}_u h_n(x)\}$$

## Optimal MCMC CV

$$\begin{aligned}\gamma^2_{\mathrm{CLT}} &= \langle 2\nabla h, \tilde{c}\rangle \\ \gamma^2_{\theta} &= 2\langle h^{\theta}, \tilde{c}^{\theta}\rangle \\ &= 2\|\nabla h^{\theta}\|^2\end{aligned}$$

# Poisson's Equation

# Poisson's Equation

- Second order partial differential equation with applications in various fields.
- General form in stochastic systems

$$\mathcal{D}h = -f$$

$\mathcal{D}$ - differential operator
$f$ - forcing function, usually centered

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$\mathcal{D}$ - differential operator
$f$ - forcing function, usually centered
$h$ - solution to Poisson's equation

# Poisson's Equation

## Stochastic optimal control

Example: In average cost optimal control problems,

$$c(x,u) + P_u h(x) = h(x) + \eta$$

| | |
|---|---|
| $c(x,u)$ | - Cost function associated with state $x$ and action $u$. |
| $P_u$ | - Transition kernel of the controlled Markov chain. |
| $\eta$ | - Average cost. |
| $h$ | - Relative value function |

Poisson's equation is the average-cost dynamic programming equation.

# Poisson's Equation

## Langevin Diffusion

Langevin diffusion is given by the SDE,

$$\mathrm{d}\Phi_t = \underbrace{-\nabla U(\Phi_t)\,\mathrm{d}t}_{\text{Drift term}} + \underbrace{\sqrt{2}\,\mathrm{d}W_t}_{\text{Diffusion term}}, \qquad \Phi \in \mathbb{R}^d$$

$U \in C^1$ is called the *potential function*.

$\boldsymbol{W} = \{W_t : t \geq 0\}$ is a standard Brownian motion on $\mathbb{R}^d$.

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- May be regarded as a $d$-dimensional gradient flow with "noise".
- Diffusion is reversible, with unique *invariant density* $\rho = e^{-U+\Lambda}$, where $\Lambda$ is a normalizing constant.

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## Langevin Diffusion

Differential generator:

$$\mathcal{D}f := \lim_{t \to 0} \frac{\mathsf{E}[f(\Phi_t)|\Phi_0 = x] - f(x)}{t}$$

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where $\nabla$ is the gradient and $\Delta$ is the Laplacian.

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Let $c\colon \mathbb{R}^d \to \mathbb{R}$ be a function of interest, and

$$\eta = \int c(x)\rho(x)dx = \langle c, 1 \rangle_{L^2}.$$

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Function $h \in C^2$ solves Poisson's equation with forcing function $c$ if

$$\mathcal{D}h := -\tilde{c}, \qquad \tilde{c} = c - \eta.$$

$$h := \int_0^\infty \mathsf{E}[\tilde{c}(\Phi_t)]\mathrm{d}t$$

# Poisson's Equation

## Existence of a solution

- A solution exists under weak assumptions on $U$ and $c$ [Glynn & Meyn 96, Kontoyiannis et al. 12].
- Representations for the gradient of $h$ and bounds are obtained in [Laugesen et al. 15, Devraj et al. 18].
- A smooth solution $h \in C^2$ exists under stronger conditions in [Pardoux et al. 01], subject to growth conditions on $c$ similar to [Glynn & Meyn 96].

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Goal: For a given function class $\mathcal{H}$, find the minimizer of

$$g^* := \arg\min_{g \in \mathcal{H}} \|h - g\|^2_{L^2} \tag{$\star$}$$

Such minimum norm optimization problems can be solved using *TD learning* [Tsitsikilis 99].

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Such minimum norm optimization problems can be solved using *TD learning* [Tsitsikilis 99].

Challenge - No algorithm exists to solve $(\star)$ if the process does not regenerate (for diffusions, dim $> 1$ ruled out).

# LSTD Learning

## Discounted cost case

Discounted-cost value function:

$$h^\gamma(x) := \int_0^\infty e^{-\gamma t} \mathsf{E}_x[c(\Phi_t)]\mathrm{d}t, \qquad \gamma > 0 : \text{ discount factor}$$

Discounted-cost optimality equation:

$$\gamma h^\gamma = c + \mathcal{D} h^\gamma$$

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# LSTD Learning

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LSTD goal: $g^* := \arg\min_{g \in \mathcal{H}} \|h^\gamma - g\|_{L^2}^2$

For a linear parameterization

$$g = h^\theta := \sum_{i=1}^{\ell} \theta_i \psi_i$$

$$\theta^* = M^{-1} b$$

$$M_{ij} = \langle \psi_i, \psi_j \rangle_{L^2}, \quad b_i = \langle \psi_i, h^\gamma \rangle_{L^2}$$

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$$= \langle \psi_i,\, R_\gamma c \rangle_{L^2}$$

Resolvent kernel: $R_\gamma c\,(x) := \int_0^\infty \mathsf{E}_x\Big[e^{-\gamma t} c(\Phi_t)\Big] \mathrm{d}t$

$$R_\gamma c = (I\gamma - \mathcal{D})^{-1} c$$

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$$= \langle R_\gamma^\dagger \psi_i,\, c \rangle_{L^2}$$

Using an adjoint operation and applying the stationarity of $\Phi$.

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$$\text{Eligibility vector: } \varphi(t) := \int_0^\infty e^{-\gamma r} \psi(\Phi_{t-r}) \mathrm{d}r$$

$$R_\gamma^\dagger \psi_i(x) = \mathsf{E}[\varphi_i(t) | \Phi_t = x]$$

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# LSTD Learning

## Discounted cost case

ODE formulation of the LSTD algorithm:

$$
\begin{aligned}
\frac{d}{dt} M(t) &= \psi(\Phi_t)\psi^{\intercal}(\Phi_t) \\
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\theta(t) &:= M(t)^{-1} b(t)
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For average-cost ($\gamma = 0$) LSTD requires the existence of a regenerating state.

# Differential LSTD Learning ($\nabla$-LSTD)

## Poisson's equation

Idea: Approximate the gradient of $h$ directly [R et al. 16, Devraj et al. 16]:

$$g^* := \arg\min_{g \in \mathcal{H}} \|\nabla h - \nabla g\|_{L^2}^2$$

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Need to choose a function class $\mathcal{H}$ for $g$ (or $\nabla g$)

- A finitely parameterized family of functions.
- A reproducing kernel Hilbert space (RKHS).

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- A reproducing kernel Hilbert space (RKHS).
  Choice of basis is not an easy task
  $\Longrightarrow$ RKHS framework is far easier to implement.

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$$\nabla\text{-LSTD goal: } g^* := \operatorname*{arg\,min}_{g \in \mathcal{H}} \|\nabla h - \nabla g\|_{L^2}^2$$

Challenge: the function $h$ is not known,

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$\nabla$-LSTD: For a linear parameterization

$$g = h^\theta := \sum_{i=1}^{\ell} \theta_i \psi_i \implies \nabla g = \sum_{i=1}^{\ell} \theta_i \nabla \psi_i$$

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Gen.resolvent kernel: $R_{U''} \nabla c\,(x) := \int_0^\infty \mathsf{E}_x \Big[ \exp\Big( - \int_0^t U''(\Phi_s)\; ds \Big) \nabla c(\Phi_t) \Big] dt$

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$\nabla$-LSTD-L: For Langevin diffusion, if $f, g \in L^2(\rho)$

$$\langle \nabla f, \nabla g \rangle_{L^2} = -\langle f, \mathcal{D} g \rangle_{L^2} = -\langle \mathcal{D} f, g \rangle_{L^2}.$$

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Applying this and Poisson's equation $\mathcal{D}h = -\tilde{c}$:

$$\begin{aligned} \|\nabla h - \nabla g\|^2_{L^2} &= \|\nabla h\|^2_{L^2} + \|\nabla g\|^2_{L^2} - 2\langle \nabla h, \nabla g \rangle_{L^2} \\ &= \|\nabla h\|^2_{L^2} + \|\nabla g\|^2_{L^2} + 2\langle \mathcal{D}h, g \rangle_{L^2} \end{aligned}$$

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# Differential LSTD Learning for Langevin ($\nabla$-LSTD-L)

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$$M_{ij} = \langle \nabla\psi_i, \nabla\psi_j \rangle_{L^2} \approx \frac{1}{t}\int_0^t \nabla\psi(\Phi_s)\,\nabla\psi^{\mathsf{T}}(\Phi_s)\,\mathrm{d}s$$

$$b_i = \langle \nabla\psi_i, \nabla h\rangle_{L^2} = \langle \psi_i, \tilde{c}\rangle_{L^2} \approx \frac{1}{t}\int_0^t \psi_i(\Phi_s)\,\tilde{c}(\Phi_s)\,\mathrm{d}s$$

# Differential LSTD Learning on RKHS ($\nabla$-LSTD-RKHS)

## Basics of RKHS

Choose a kernel function $K(x, y)$ that is

- *Symmetric*: $K(x, y) = K(y, x)$ for any $x, y \in \mathbb{R}^d$
- *Positive definite*: For any $\{x^i\} \subset \mathbb{R}^d$, matrix $\{M_{ij} := K(x^i, x^j)\}$ is positive definite.
- *Smooth*: $K \in C^2$

$K$ defines a unique RKHS $\mathcal{H}$ [Moore-Aronszajn theorem].

Inner product: If $g_\alpha = \sum_i \alpha_i K(x^i, \cdot\,)$ and $g_\beta = \sum_j \beta_j K(y^j, \cdot\,)$,

$$\langle g_\alpha, g_\beta \rangle_{\mathcal{H}} := \sum_{i,j} \alpha_i \beta_j K(x^i, y^j)$$

Reproducing property: $g_\alpha(x) = \langle g_\alpha, K(x, \cdot\,) \rangle_{\mathcal{H}}\,, \quad x \in \mathbb{R}^d.$

# Differential LSTD learning on RKHS

## Empirical risk minimization (ERM)

Recall ∇-LSTD-L goal:

$$g^* = \arg\min_{g \in \mathcal{H}} \Big\{ \|\nabla g\|_{L^2}^2 - 2\langle \tilde{c}, g \rangle_{L^2} \Big\}$$

Approximation via empirical risk minimization (ERM):

$$\arg\min_{g \in \mathcal{H}} \underbrace{\frac{1}{N} \sum_{i=1}^{N} \Big[ \|\nabla g(x^i)\|^2 - 2\tilde{c}_N(x^i) g(x^i) \Big]}_{\text{Empirical risk}} + \underbrace{\lambda \|g\|_{\mathcal{H}}^2}_{\text{Regularization}}$$

where $\tilde{c}_N(x) = c(x) - \frac{1}{N}\sum_{i=1}^{N} c(x^i)\,, \quad x \in \mathbb{R}^d$

# Differential LSTD Learning on RKHS ($\nabla$-LSTD-RKHS)

## $\nabla$-LSTD-RKHS-Opt

Empirical risk minimization (ERM):

$$\arg\min_{g\in\mathcal{H}} \frac{1}{N}\sum_{i=1}^{N}\Big[\|\nabla g(x^i)\|^2 - 2\tilde{c}_N(x^i)g(x^i)\Big] + \lambda\|g\|_{\mathcal{H}}^2$$

Classical representer theorem [Wahba 70] is a remarkable result for ERMs in RKHS.

# Differential LSTD Learning on RKHS ($\nabla$-LSTD-RKHS)

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Classical representer theorem [Wahba 70] is a remarkable result for ERMs in RKHS.

Not applicable to our loss function due to gradient term.

# Differential LSTD Learning on RKHS ($\nabla$-LSTD-RKHS)

## $\nabla$-LSTD-RKHS-Opt

Empirical risk minimization (ERM):

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Extended Representer Theorem [Zhou 08]

If loss function $L(x, \cdot, \cdot)$ is convex on $\mathbb{R}^{d+1}$ for each $x \in \mathbb{R}^d$, then the optimizer $g^*$ over $g \in \mathcal{H}$ exists:

$$g^*(\cdot) = \sum_{i=1}^{N}\Big[\beta_i^{0*}K(x^i, \cdot) + \sum_{k=1}^{d}\beta_i^{k*}\frac{\partial}{\partial x_k}K(x^i, \cdot)\Big]$$

# Differential LSTD Learning on RKHS ($\nabla$-LSTD-RKHS)

## $\nabla$-LSTD-RKHS-N

Drawback : Complexity grows linearly with $d$, since $\beta^* \in \mathbb{R}^{(d+1)\times N}$

Simplified solution : By considering the finite-dimensional function class - $\mathcal{H}_N :=$ span$\{K_{x^i} : 1 \leq i \leq N\}$

$$g^*(y) = \sum_{j=1}^{N} \beta_i^* K(x^i, y)$$

$$\beta^* = M^{-1} b$$

Surprising empirical observation : Simplified solution does as good as the optimal solution for $d \leq 5$.

# Applications to Nonlinear Filtering

## Feedback Particle Filter

Goal: To obtain estimates of the state of a stochastic dynamical system based on noisy partial observations.

# Applications to Nonlinear Filtering

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# Applications to Nonlinear Filtering

## Feedback Particle Filter

Goal: To obtain estimates of the state of a stochastic dynamical system based on noisy partial observations.

Kalman filter is optimal for a linear Gaussian system.

For nonlinear systems, conditional distribution fails to be Gaussian, cannot be captured by a finite set of parameters.

Particle filters are popular Monte-Carlo approximations of the nonlinear filter.

# Applications to Nonlinear Filtering

## Feedback Particle Filter

Problem:

Signal: $$\mathrm{d}X_t = a(X_t)\mathrm{d}t + \mathrm{d}B_t, \quad X_0 \sim \rho_0^*,$$
Observation: $$\mathrm{d}Z_t = c(X_t)\mathrm{d}t + \mathrm{d}W_t,$$

- $X_t \in \mathbb{R}^d$ is the state at time $t$.
- $\{Z_t : t \geq 0\}$ is the observation process.
- $a(.)$,$c(.)$ are $C^1$ functions.
- $\{B_t\}$,$\{W_t\}$ are mutually independent Wiener processes.

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- $a(.)$,$c(.)$ are $C^1$ functions.
- $\{B_t\}$,$\{W_t\}$ are mutually independent Wiener processes.
- $\rho_t^* := P(X_t | Z_s : s \leq t)$ is the posterior distribution.

# Applications to Nonlinear Filtering

## Feedback Particle Filter

Feedback particle filter (FPF) [Yang et al. 13] is motivated by techniques from mean-field optimal control.

# Applications to Nonlinear Filtering

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$N$ *particles* are propagated in the form of a controlled system.

$$\mathrm{d}X_t^i = \underbrace{a(X_t^i)dt + \mathrm{d}B_t^i}_{\text{Propagation}} + \underbrace{\mathrm{d}U_t^i}_{\text{Update}}, \quad i = 1 \text{ to } N$$

- $X_t^i \in \mathbb{R}$ is the state of the $i^{th}$ particle at time $t$
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Approximation of $\rho_t^*$:

$$\rho_t^* \approx \rho_t^{(N)}(A) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\{X_t^i \in A\}, \quad A \subset \mathbb{R}.$$

# Applications to Nonlinear Filtering

## Feedback Particle Filter

$$\mathsf{d}U_t^i = \mathsf{K}_t(X_t^i) \circ \overbrace{(\mathsf{d}Z_t - \tfrac{1}{2}[c(X_t^i) + \hat{c}_t]\mathsf{d}t)}^{\mathsf{d}I_t^i}$$

$I_t^i$ : Innovations process.

$\mathsf{K}_t$ : FPF gain, similar in nature to the Kalman gain.

# Applications to Nonlinear Filtering

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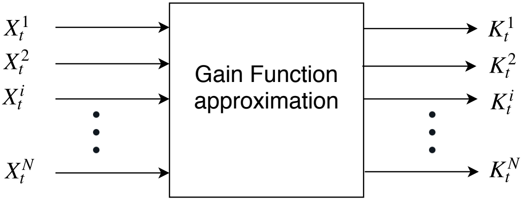
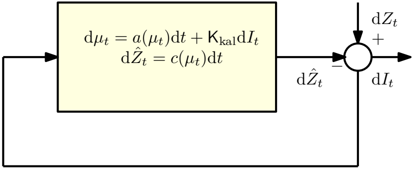



Finite-$N$ implementation

# Applications to Nonlinear Filtering

## Feedback Particle Filter

KF: $$\mathsf{d}\mu_t = a(\mu_t)\mathsf{d}t + \underbrace{\mathsf{K}_{\mathsf{kal}}(\mathsf{d}Z_t - c(\mu_t)\mathsf{d}t)}_{\text{update}}$$




Kalman filter

# Applications to Nonlinear Filtering

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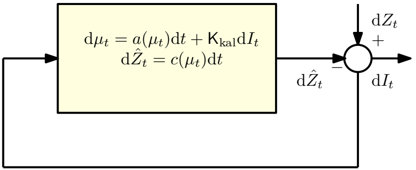
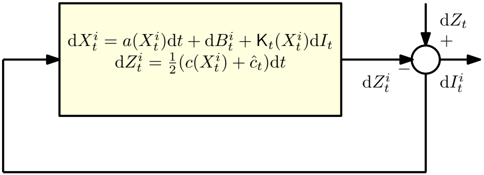
FPF: $$\mathsf{d}X^i_t = a(X^i_t)\mathsf{d}t + \mathsf{d}B^i_t + \underbrace{\mathsf{K}_t(X^i_t) \circ (\mathsf{d}Z_t - \tfrac{1}{2}[c(X^i_t) + \hat{c}_t]\mathsf{d}t)}_{\text{update}}$$




Kalman filter




Feedback particle filter (FPF)

# Applications to Nonlinear Filtering

## FPF gain function

Representation: $\mathsf{K}_t = \nabla h$

$h$ solves Poisson's equation: $\mathcal{D}h = -\nabla U \cdot \nabla h + \Delta h = -\tilde{c}$.

# Applications to Nonlinear Filtering

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Approximations to K can be obtained by

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Can be solved using $\nabla$-LSTD learning

# Applications to Nonlinear Filtering

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FPF implementation requires online gain estimation for each $t$.

- $\nabla$-LSTD-RKHS with optimal mean
- $\nabla$-LSTD-RKHS with memory

# Applications to Nonlinear filtering

## $\nabla$-LSTD-RKHS-OM

Constant gain approximation for K is the minimizer obtained over all deterministic vectors:

$$\widehat{\mathsf{K}}^* := \arg\min_{\widehat{\mathsf{K}} \in \mathbb{R}^d} \|\mathsf{K} - \widehat{\mathsf{K}}\|_{L^2}^2$$

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Solution is evidently the mean, $\widehat{\mathsf{K}}^* = \mathsf{E}[\mathsf{K}]$.

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Empirical approximation:

$$\widehat{\mathsf{K}}_k^* \approx \frac{1}{N} \sum_{i=1}^{N} [c(x^i) - \hat{c}] x_k^i$$

# Applications to Nonlinear filtering

## ∇-LSTD-RKHS-OM

Redefine the approximation to K as,

$$\nabla g = \widehat{\mathsf{K}}^* + \nabla \tilde{g}$$

Modified ERM with constaints is:

$$\tilde{g}^* := \arg\min_{\tilde{g} \in \mathcal{H}} \; \|\nabla h - \widehat{\mathsf{K}}^* - \nabla \tilde{g}\|_{L_2}^2$$
$$\text{s.t.} \quad \langle \partial_{x_k} \tilde{g}, 1 \rangle_{L_2} = 0, \quad 1 \leq k \leq d$$

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Solution obtained by finding a saddle point for the Lagrangian

$$L(\tilde{g}, \mu) := \|\nabla h - \widehat{\mathsf{K}}^* - \nabla \tilde{g}\|_{L_2}^2 + \langle \mu, \nabla \tilde{g} \rangle_{L_2}$$

where $\mu \in \mathbb{R}^d$ are the Lagrange multipliers.

# Applications to Nonlinear filtering

## $\nabla$-LSTD-RKHS-OM

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$$\text{s.t.} \quad \langle \partial_{x_k} \tilde{g}, 1 \rangle_{L_2} = 0, \quad 1 \le k \le d$$

Using $\mathcal{H}_N := \text{span}\{K_{x^j} : 1 \le j \le N\}$, $\beta$ and $\mu$ can be obtained by solving $N + d$ linear equations

$$\mathsf{K} = \widehat{\mathsf{K}}^* + \nabla \tilde{g}^*$$

# Applications to Nonlinear filtering

## $\nabla$-LSTD-RKHS-memory

Gain updates are done at $t = n\delta$, where $\delta$ is the inter-sampling time.
Continuity: $\mathsf{K}_n = \mathsf{K}_{t_n} \approx \mathsf{K}_{t_{n-1}}$ if $\delta \approx 0$.

# Applications to Nonlinear filtering

## ∇-LSTD-RKHS-memory

Gain updates are done at $t = n\delta$, where $\delta$ is the inter-sampling time.
Continuity: $\mathsf{K}_n = \mathsf{K}_{t_n} \approx \mathsf{K}_{t_{n-1}}$ if $\delta \approx 0$.
Adding a regularization term to the loss function:

$$g_n^* := \arg\min_{g \in \mathcal{H}} \frac{1}{N} \sum_{j=1}^{N} L_n(x_n^j, g, \nabla g) + \lambda \|g\|_{\mathcal{H}}^2$$

$$L_n(x, g, \nabla g) := \|\nabla g(x)\|^2 - 2\tilde{c}_N(x) g(x) + \underbrace{\lambda_{mem} \|\nabla g(x) - \nabla g_{n-1}(x)\|^2}_{\text{continuity penalty}}$$

$$\beta_n^* = M^{-1} b$$

# Application to nonlinear filtering

## Numerical example - Gain approximation for a fixed $t$

Example: For a fixed $t$, $\rho_t$ a Gaussian mixture
$c(x) \equiv x$, $d = 1$
$T = 10^4, 10^5, 10^6$ with $\delta = 0.01$, $N = 1000$

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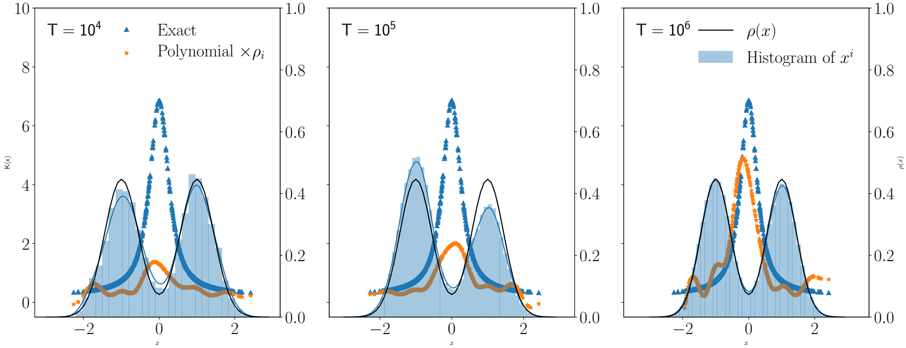


$\nabla$-LSTD with $\psi_i = x^i \rho_1(x)$, $\psi_{i+1} = x^i \rho_2(x)$ with $1 \leq i \leq 5$.

# Application to nonlinear filtering

## Numerical example - Gain approximation for a fixed $t$

Example: For a fixed $t$, $\rho_t$ a Gaussian mixture
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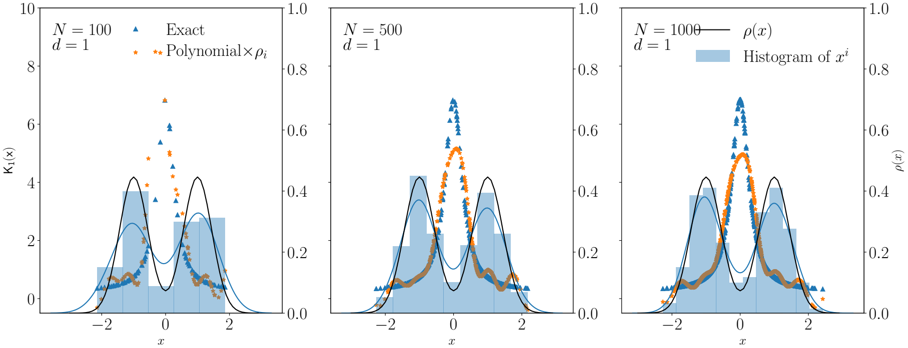
$N = 100, 500, 1000$




$\nabla$-LSTD-L with $\psi_i = x^i \rho_1(x)$, $\psi_{i+1} = x^i \rho_2(x)$ with $1 \le i \le 5$.

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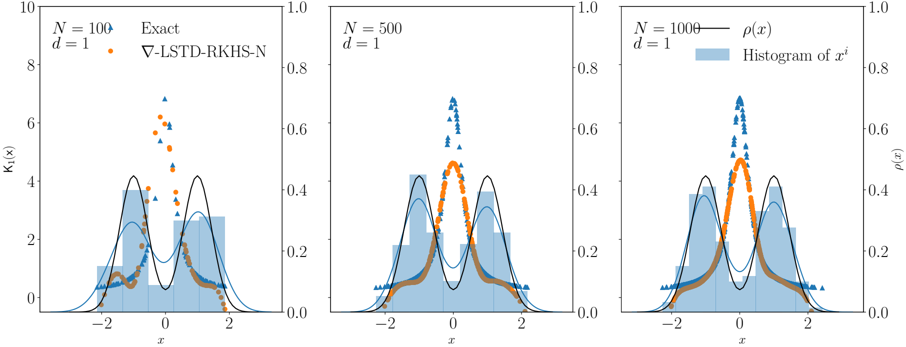


$\nabla$-LSTD-RKHS-N with Gaussian kernel, $\varepsilon = 0.1$ and $\lambda = 10^{-2}$ (best)

# Application to nonlinear filtering

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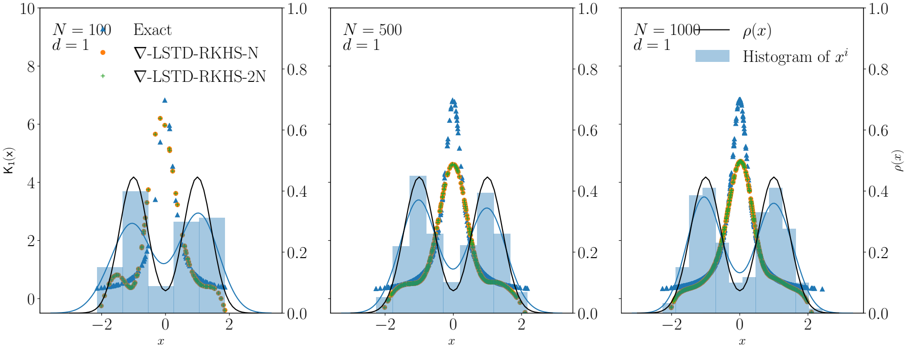


$\nabla$-LSTD-RKHS-2N with Gaussian kernel, $\varepsilon = 0.1$ and $\lambda = 10^{-2}$ (best)

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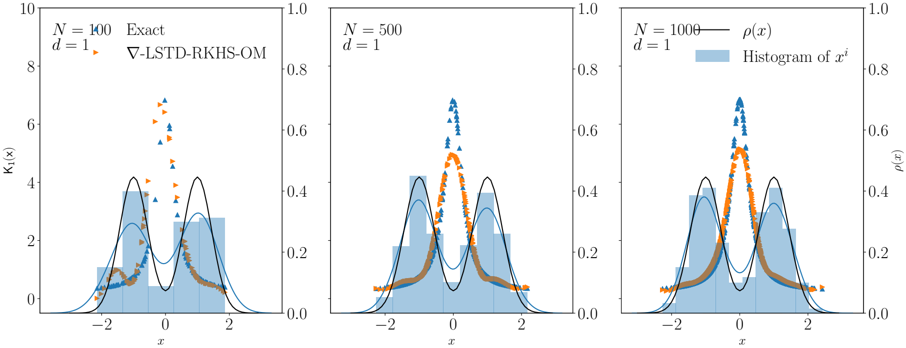


$\nabla$-LSTD-RKHS-OM with Gaussian kernel, $\varepsilon = 0.1$ and $\lambda = 10^{-2}$ (best)

# Application to nonlinear filtering

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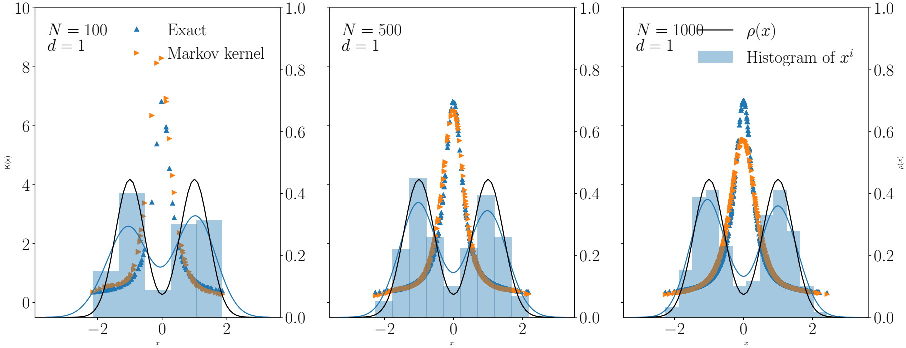


Markov kernel approximation [Taghvaei et al. 18] with $\epsilon = 0.1$ (best)

# Application to nonlinear filtering

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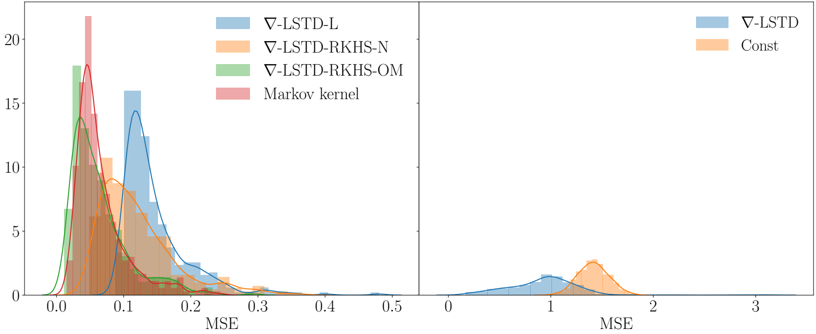
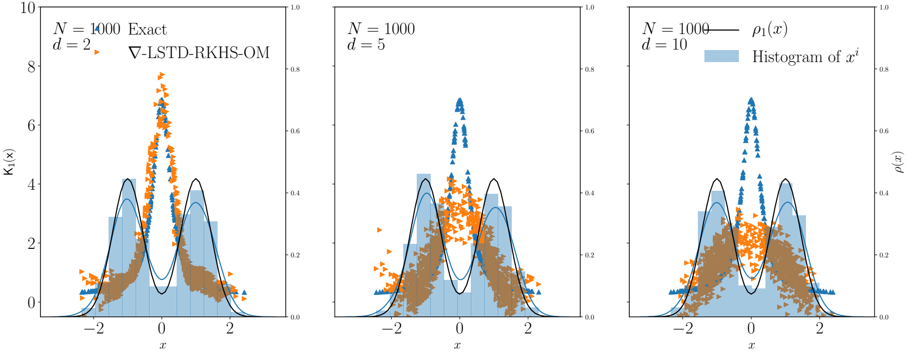



Histogram of MSEs obtained from 1000 independent trials

# Application to nonlinear filtering

## Numerical example - Gain approximation for a fixed $t$

Example: $\rho(x) = \prod_{k=1}^{d} \rho_k(x_k)$, each $\rho_k$ a Gaussian mixture
$c(x) = C^T x$, where $C = \mathbb{I}_d$
$d = 2, 5, 10, \quad N = 1000$




∇-LSTD-RKHS-OM

# Application to nonlinear filtering

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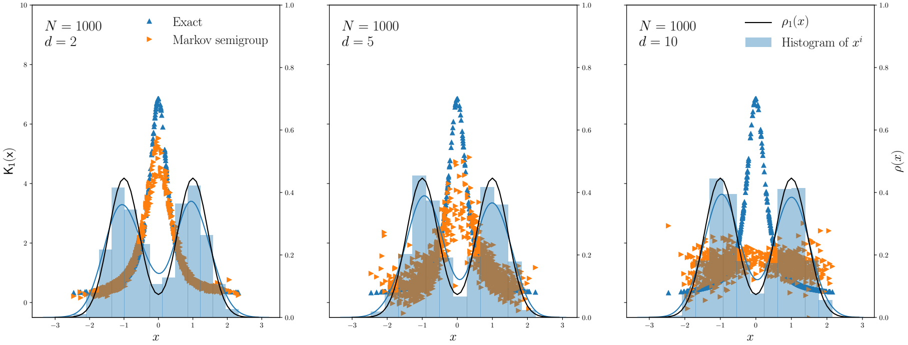



Markov kernel approximation

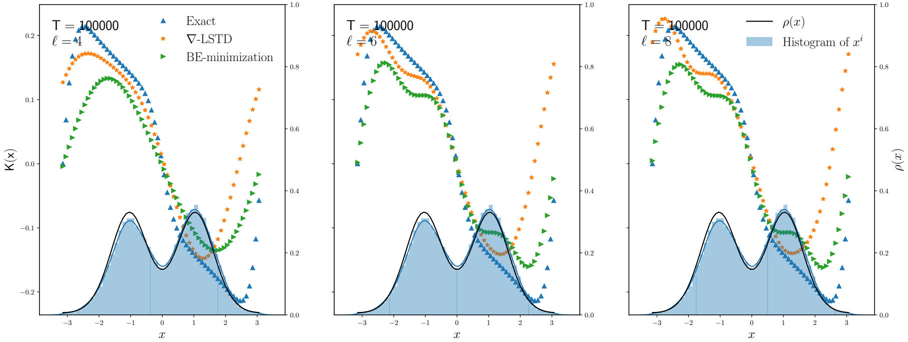
# Application to nonlinear filtering

## Numerical example - Gain for a nonlinear oscillator model

Example: $\rho(x)$ is a mixture of von Mises densities on a circle

$$\mathrm{d}\vartheta = \omega \mathrm{d}t + \sigma_B \mathrm{d}B_t \mod 2\pi,$$

$$\mathrm{d}Z_t = c(\vartheta)\mathrm{d}t + \sigma_W \mathrm{d}W_t, \quad c(\vartheta) = \frac{1}{2}[1+\cos(\vartheta)]$$

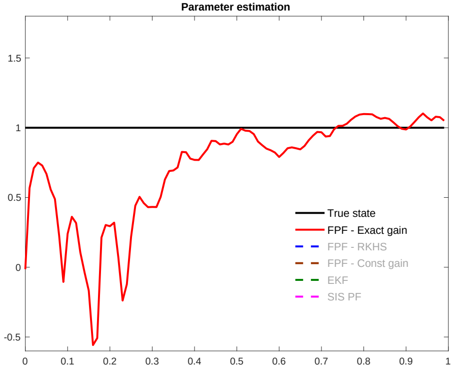



∇-LSTD-L with $\psi_i = \sin(ix), \psi_{i+1} = \cos(ix)$ with $1 \le i \le \ell/2$, $\ell = 4, 6, 8$.

# Applications to Nonlinear filtering

## Numerical example - Parameter estimation

Example: Parameter estimation with bimodal prior
Observations: parameter plus additive noise with $\sigma_W = 1$.



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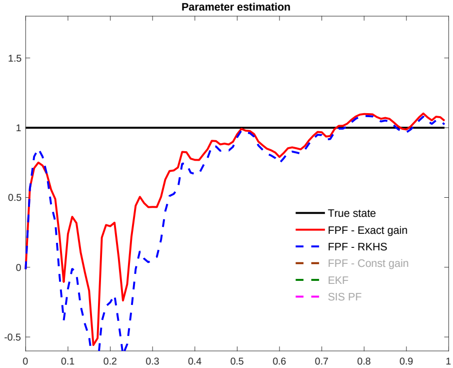

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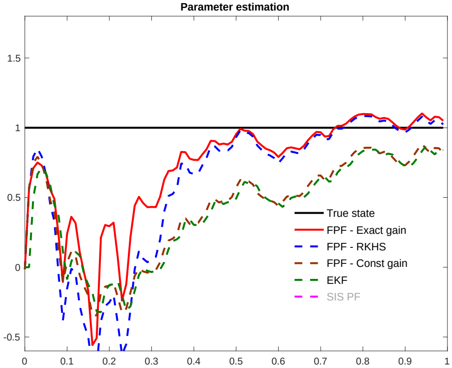

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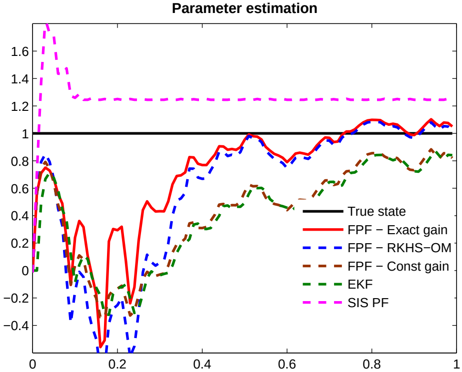
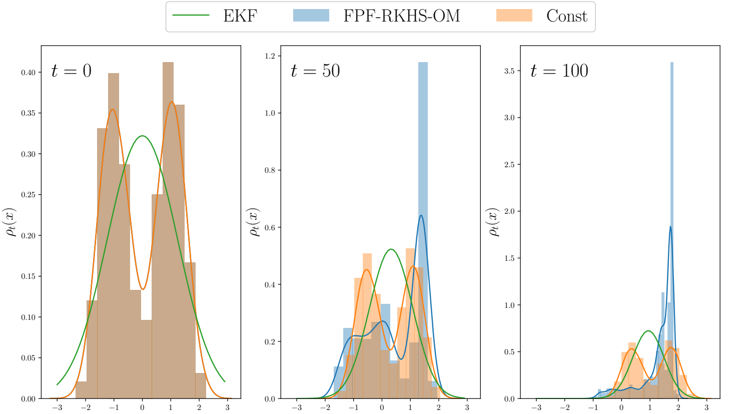


# Applications to Nonlinear filtering

## Numerical example - Parameter estimation

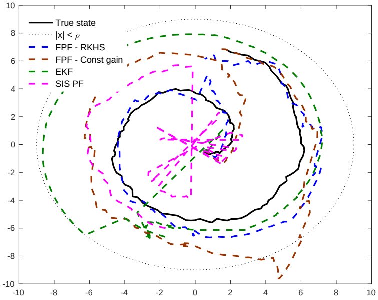
Example: Parameter estimation with bimodal prior
Observations: parameter plus additive noise with $\sigma_W = 1$.



# Applications to Nonlinear filtering

## Numerical example - Nonlinear $2d$ ship dynamics model

Example: Nonlinear ship dynamics model in $2d$.
Observations: $c(x) = \arctan(x_1/x_2)$ with std. deviation $\approx 18°$.




| Filter | $\Sigma_1$ | Lost track |
|---|---|---|
| RKHS-OM | 0.90 | 4 |
| RKHS-mem. | 0.91 | 7 |
| Const. | 1.30 | 14 |
| SIR PF | 3.14 | 57 |
| EKF | 6.52 | 93 |

# Applications to MCMC

## Introduction to MCMC

In many applications, we need to compute

$$\eta = \int c(x)\rho(x)\,\mathrm{d}x$$

- $c\colon \mathbb{R}^\ell \to \mathbb{R}$ is a measurable function.
- $\rho$ is a target probability density in $\mathbb{R}^\ell$.

Markov-Chain Monte Carlo (MCMC) methods provide numerical algorithms to obtain estimates:

$$\eta_t = \frac{1}{t}\int_0^t c(\mathbf{\Phi}(s))\,ds$$

$\mathbf{\Phi}$ is a Markov process with steady state distribution $\rho$.

# Applications to MCMC

## Asymptotic Variance

Estimates $\eta_t$ obey a Central Limit Theorem,

$$\sqrt{t}(\eta_t - \eta) \xrightarrow{d} N(0, \gamma^2)$$

Representation in terms of $h$ [see eg: Glynn & Meyn 96]:

$$\gamma^2 = 2\langle h, \, \tilde{c} \rangle$$

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$$\begin{aligned} \gamma^2 &= 2\langle h,\, \tilde{c}\rangle \\ &= 2\|\nabla h\|^2_{L^2} \end{aligned}$$

(For Langevin diffusion)

# Applications to MCMC

## Control Variates

Goal: To minimize asymptotic variance.

Idea: Modify the estimator using control variates [Henderson 97, CTCN[1]]

$$c_g = c + \underbrace{\mathcal{D}g}_{\text{Control variate}}\,, \quad \textit{where} \quad g \in \mathcal{H}$$

$$\eta_t^g = \frac{1}{t}\int_0^t c_g(\Phi_s)\,ds$$

For any $g \in C^2 \cap L^2(\rho)$, $\langle \mathcal{D}g,\, 1\rangle_{L^2} = 0$.

[1] Control Techniques for Complex Networks, S.Meyn

# Applications to MCMC

## Optimal control variates

$$\mathcal{D}(h - g) = -c_g + \eta$$

$h - g$ is the solution to Poisson's equation with $\tilde{c}_g$ as the forcing function.

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Can be minimized using $\nabla$-LSTD algorithms.

# Applications to MCMC

## Numerical Examples - Variance v Asymptotic variance

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$$\sigma^2 = \langle \tilde{c}, \tilde{c} \rangle_{L^2} = R(0)$$ - Variance

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But does minimizing $\sigma^2 \implies$ minimizing $\gamma^2$ ?

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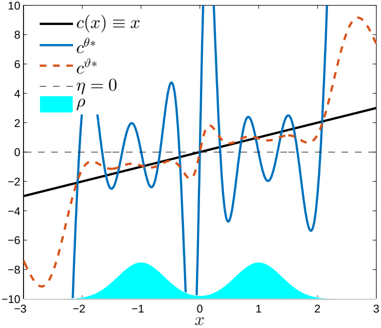
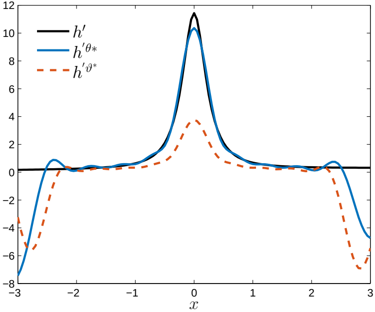
# Applications to MCMC

## Numerical Examples - Variance vs Asymptotic variance

**Example:** Unadjusted Langevin algorithm (ULA)

$c(x) \equiv x$



# Applications to MCMC

## Numerical Examples - Variance vs Asymptotic variance

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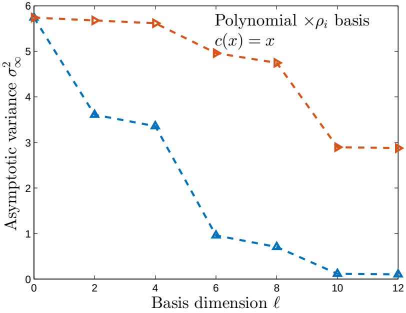
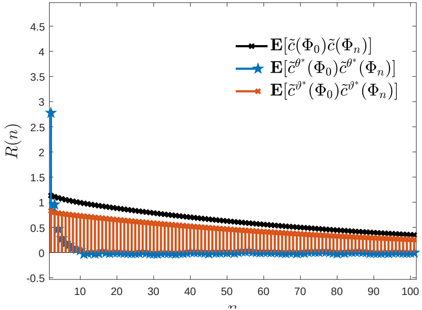


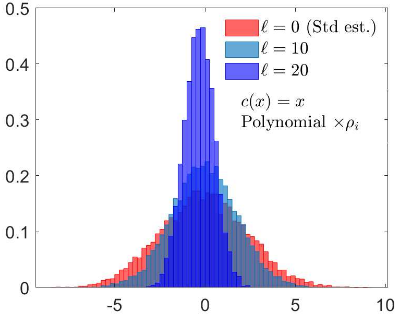



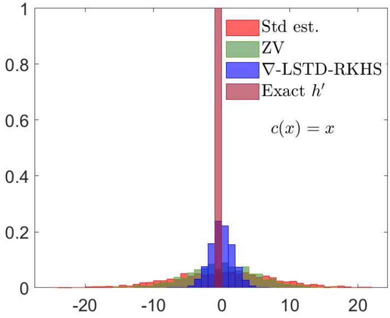
# Applications to MCMC

## Numerical Examples - ULA

Example: Unadjusted Langevin algorithm (ULA)

$c(x) \equiv x,\ \rho \sim 0.5\mathcal{N}(-1, 0.4472) + 0.5\mathcal{N}(1, 0.4472)$

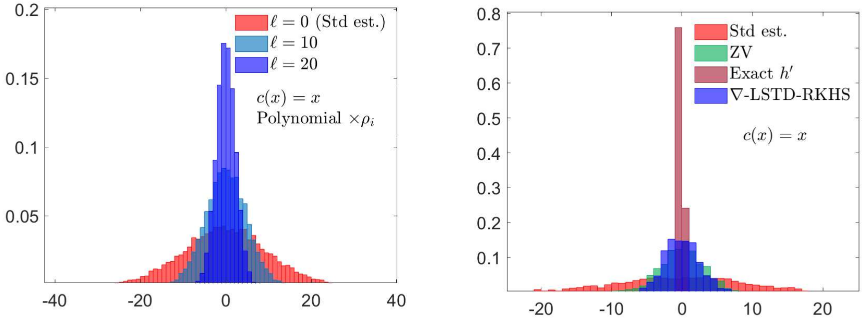


# Applications to MCMC

## Numerical Examples - RWM

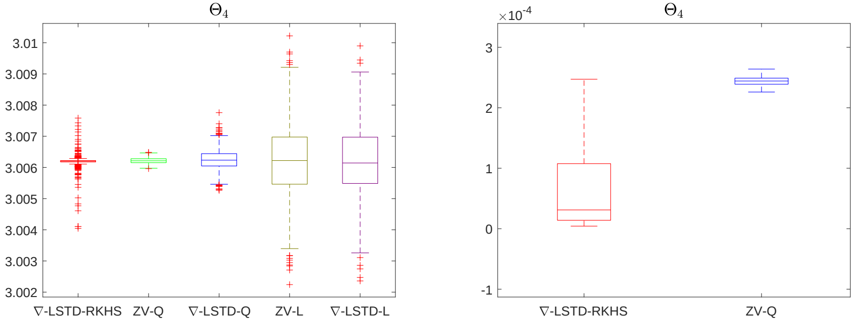
Example: Random walk Metropolis (RWM)
$c(x) \equiv x$, $\rho \sim 0.5\mathcal{N}(-1, 0.4472) + 0.5\mathcal{N}(1, 0.4472)$
Theoretical justification based on [Brosse et al. 19]



# Applications to MCMC

## Numerical Examples - Logistic Regression with RWM sampling

Example: Logistic Regression for Swiss bank notes
RWM sampling




Box plots of estimates of $\Theta_4$.

# Conclusions

- Differential LSTD learning algorithms to approximate solution to Poisson's equation for the Langevin diffusion.
    - RKHS based approaches solve the basis selection problem and enable easy extensions to higher dimensions.
    - Optimal mean gain is a useful design tool and makes the algorithm more robust to $\varepsilon$ and $\lambda$.
- Two interesting applications
    - Gain function approximation in feedback particle filter.
    - Asymptotic variance reduction in MCMC algorithms.
- Recent research extended to include reversible Markov chains.

# Future Work

Analysis:

- Error analysis of the $\nabla$-LSTD-RKHS method, which could lead to proper choices of hyper parameters.
- A more thorough comparison of the various gain approximation algorithms.
- Nagging question: Why reduced complexity solution is as good as the optimal?

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- $\nabla$-LSTD-RKHS algorithm with a differential regularizer. This is limited by the scope of representer theorem.
- An algorithm based on semiparametric representer theorem, that allows the use of additional parameterized functions.

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Applications:

- Real time filtering problem - Potential application to battery SOC estimation is being explored currently.
- Application of FPF to control in the context of POMDPs.

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# Thank You!

Questions?