積分

問題 1

$$\int_{-1}^{1} \frac{x^2}{1+e^x}\,dx$$

解法 1. ここで，$t=-x$ とおくと，[1]

$$\frac{dt}{dx} = -1$$

であるから，

$$\begin{aligned}
I &= \int_{-1}^{1} \frac{x^2}{1+e^x}\,dx \\
&= \int_{1}^{-1} \frac{t^2}{1+e^{-t}} \cdot (-1)dt \\
&= \int_{-1}^{1} \frac{t^2 e^t}{e^t+1}\,dt = \int_{-1}^{1} \frac{x^2 e^x}{1+e^x}\,dx \\
2I &= \int_{-1}^{1} \frac{x^2}{1+e^x}\,dx + \int_{-1}^{1} \frac{x^2 e^x}{1+e^x}\,dx \\
&= \int_{-1}^{1} \frac{(1+e^x)\,x^2}{1+e^x}\,dx \\
&= \int_{-1}^{1} x^2\,dx \\
&= 2\int_{0}^{1} x^2\,dx \\
&= \frac{2}{3}
\end{aligned}$$

以上より，

$$\int_{-1}^{1} \frac{x^2}{1+e^x}\,dx = \frac{1}{3}$$

[1] King Property 参照

問題 2

$f(x) = \tan(x)\,(0 \leqq x \leqq \frac{\pi}{4})$ の時において

$$\int_{0}^{1} f^{-1}(x)\,dx$$

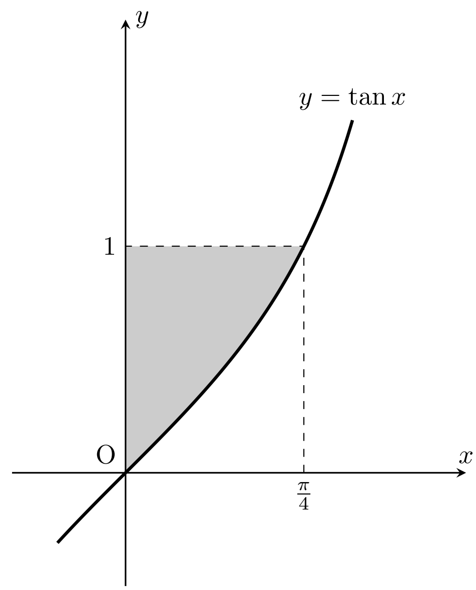

図 1: $y=\tan x$

解法 2. 図 1 より，

$$\begin{aligned}
\int_{0}^{1} f^{-1}(x)\,dx &= 1\cdot\frac{\pi}{4} - \int_{0}^{\frac{\pi}{4}} \tan x\,dx \\
&= \frac{\pi}{4} - \Big[-\log|\cos x|\Big]_{0}^{\frac{\pi}{4}} \\
&= \frac{\pi}{4} - \frac{1}{2}\log 2
\end{aligned}$$

問題 3

$$\int_0^{\frac{\pi}{3}} \frac{1}{\sin x \cos x}\, dx$$

解法 3.

$$\begin{aligned}(\log|\tan x|)' &= \frac{1}{\tan x}\cdot(\tan x)' \\ &= \frac{1}{\sin x \cos x}\end{aligned}$$

したがって，

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \frac{1}{\sin x \cos x}\, dx &= \Big[\log|\tan x|\Big]_0^{\frac{\pi}{3}} \\ &= \log\sqrt{3} \\ &= \frac{1}{2}\log 3\end{aligned}$$

別解 1.

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \frac{1}{\sin x \cos x}\, dx &= \int_0^{\frac{\pi}{3}} \left(\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}\right) dx \\ &= \Big[\log(\sin x) - \log(\cos x)\Big]_0^{\frac{\pi}{3}} \\ &= \Big[\log|\tan x|\Big]_0^{\frac{\pi}{3}} \\ &= \log\sqrt{3} \\ &= \frac{1}{2}\log 3\end{aligned}$$

問題 4

$$\int \frac{1}{\cos^3 x}\, dx$$

解法 4.

$$\begin{aligned}I &= \int \frac{1}{\cos^3 x}\, dx \\ &= \int \frac{1}{\cos^2 x}\cdot\frac{1}{\cos x}\, dx \\ &= \int (\tan x)'\cdot\frac{1}{\cos x}\, dx \\ &= \tan x\cdot\frac{1}{\cos x} - \int \tan x\cdot\frac{\sin x}{\cos^2 x}\, dx \\ &= \frac{\sin x}{\cos^2 x} - \int \frac{\sin^2 x}{\cos^3 x}\, dx \\ &= \frac{\sin x}{\cos^2 x} - \int \frac{1-\cos^2 x}{\cos^3 x}\, dx \\ &= \frac{\sin x}{\cos^2 x} - I + \int \frac{1}{\cos x}\, dx \\ 2I &= \frac{\sin x}{\cos^2 x} + \int \frac{\cos x}{1-\sin^2 x}\, dx\end{aligned}$$

ここで，$t = \sin x$ とおくと，

$$\frac{dt}{dx} = \cos x$$

であるから，

$$\begin{aligned}I &= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\int \frac{1}{1-t^2}\, dt \\ &= \frac{\sin x}{2\cos^2 x} + \frac{1}{4}\int \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt \\ &= \frac{\sin x}{2\cos^2 x} + \frac{1}{4}\left\{\log|1+t| - \log|1-t|\right\} + C \\ &= \frac{\sin x}{2\cos^2 x} + \frac{1}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right| + C\end{aligned}$$

問題 5

$$\int_0^{2\pi} \sqrt{2\left(1-\cos\theta\right)}\,d\theta$$

サイクロイドの長さ

解法 5.

$$\begin{aligned} & \cos 2\theta = 1 - 2\sin^2\theta \\ \iff \quad & 1 - \cos 2\theta = 2\sin^2\theta \end{aligned}$$

であるから,

$$\begin{aligned} \int_0^{2\pi} \sqrt{2\left(1-\cos\theta\right)}\,d\theta &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\,d\theta \\ &= 2\int_0^{2\pi} \left|\sin\frac{\theta}{2}\right| d\theta \\ &= 2\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} \\ &= 8 \end{aligned}$$

問題 6

$$\int \frac{1}{1+e^{-x}}\,dx$$

シグモイド関数

解法 6.

$$\begin{aligned} \int \frac{1}{1+e^{-x}}\,dx &= \int_0^1 \frac{e^x}{e^x+1}\,dx \\ &= \int \frac{\left(e^x+1\right)'}{e^x+1}\,dx \\ &= \log\left(e^x+1\right) + C \end{aligned}$$

別解 1. $t = 1 + e^{-x}$ とおくと,

$$\begin{aligned} \frac{dt}{dx} &= -e^{-x} \\ &= 1 - t \end{aligned}$$

であるから,

$$\begin{aligned} \int \frac{1}{1+e^{-x}}\,dx &= \int \frac{1}{t} \cdot \frac{1}{1-t}\,dt \\ &= \int \left(\frac{1}{t} + \frac{1}{1-t}\right) dx \\ &= \log t - \log\left(1-t\right) + C \\ &= \log\left(\frac{1+e^{-x}}{e^{-x}}\right) + C \\ &= \log\left(e^x+1\right) + C \end{aligned}$$

問題 7

$$\int_0^\pi \frac{x\sin x}{8+\sin^2 x}\,dx$$

2018 年横浜国立大学第 1 問

解法 7. $x=\pi-\theta$ とおくと,[2]

$$\frac{d\theta}{dx}=-1$$

であるから,

$$\begin{aligned}
I &= \int_0^\pi \frac{x\sin x}{8+\sin^2 x}\,dx \\
&= \int_\pi^0 \frac{(\pi-\theta)\sin(\pi-\theta)}{8+\sin^2(\pi-\theta)}\cdot(-1)d\theta \\
&= \int_0^\pi \frac{(\pi-\theta)\sin\theta}{8+\sin^2 t}\,dt = \int_0^\pi \frac{(\pi-x)\sin x}{8+\sin^2 x}\,dx \\
2I &= \int_0^\pi \frac{x\sin x}{8+\sin^2 x}\,dx + \int_0^\pi \frac{(\pi-x)\sin x}{8+\sin^2 x}\,dx \\
&= \pi\int_0^\pi \frac{\sin x}{8+\sin^2 x}\,dx \\
I &= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{8+\sin^2 x}\,dx \\
&= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{9-\cos^2 x}\,dx
\end{aligned}$$

ここで, $t=\cos x$ とおくと,

$$\frac{dt}{dx}=-\sin x$$

であるから,

$$\begin{aligned}
I &= \frac{\pi}{2}\int_1^{-1}\frac{1}{9-t^2}\cdot(-1)dt \\
&= \pi\int_0^1 \frac{1}{(3-t)(3+t)}\,dt \\
&= \pi\int_0^1 \frac{1}{6}\left(\frac{1}{3-t}+\frac{1}{3+t}\right)dt \\
&= \frac{\pi}{6}\Big[\log|3+t|-\log|3-t|\Big]_0^1 \\
&= \frac{\pi}{6}\log 2
\end{aligned}$$

[2] King Property 参照

問題 8

$$\int_0^{\pi} \frac{x\sin^3 x}{4-\cos^2 x}\,dx$$

2005 年名古屋大学第 4 問

解法 8. $x = \pi - \theta$ とおくと,[3]

$$\frac{dx}{d\theta} = -1$$

であるから,

$$\begin{aligned} I &= \int_0^{\pi} \frac{x\sin^3 x}{4-\cos^2 x}\,dx \\ &= \int_{\pi}^{0} \frac{(\pi-\theta)\sin^3(\pi-\theta)}{4-\cos^2(\pi-\theta)}\cdot(-1)d\theta \\ &= \int_0^{\pi} \frac{(\pi-\theta)\sin^3\theta}{4-\cos^2\theta}\,d\theta = \int_0^{\pi} \frac{(\pi-x)\sin^3 x}{4-\cos^2 x}\,dx \\ &= \pi\int_0^{\pi} \frac{\sin^3 x}{4-\cos^2 x}\,dx - I \end{aligned}$$

以上から,

$$\begin{aligned} I &= \frac{\pi}{2}\int_0^{\pi} \frac{\sin^3 x}{4-\cos^2 x}\,dx \\ &= \frac{\pi}{2}\int_0^{\pi} \frac{1-\cos^2 x}{4-\cos^2 x}\cdot \sin x dx \end{aligned}$$

ここで, $t = \cos x$ とおくと,

$$\frac{dt}{dx} = -\sin x$$

であるから,

$$\begin{aligned} I &= \frac{\pi}{2}\int_1^{-1} \frac{1-t^2}{4-t^2}\cdot(-1)dt \\ &= \frac{\pi}{2}\int_{-1}^{1} \left(\frac{3}{t^2-4}+1\right)dt \\ &= \pi\int_0^{1} \left\{\frac{3}{(t-2)(t+2)}+1\right\}dt \\ &= \pi\int_0^{1} \left\{\frac{3}{4}\left(\frac{1}{t-2}-\frac{1}{t+2}\right)+1\right\}dt \\ &= \pi\left[\frac{3}{4}\Big(\log|t-2|-|t+2|\Big)+t\right]_0^1 \\ &= \left(1-\frac{3}{4}\log 3\right)\pi \end{aligned}$$

[3] King Property 参照

問題 9

$$\int \frac{1}{\sin x+\cos x+1}\,dx$$

解法 9. $t=\tan\dfrac{x}{2}$ とおいた時に,

$$\begin{aligned}
\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\
&= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \\
&= \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} \\
&= \frac{2t}{t^2+1} \\
\cos x &= \cos^2\frac{x}{2}-\sin^2\frac{x}{2} \\
&= \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \\
&= \frac{1-\tan^2\frac{x}{2}}{\tan^2\frac{x}{2}+1} \\
&= \frac{1-t^2}{t^2+1}
\end{aligned}$$

となる．また,

$$\begin{aligned}
\frac{dt}{dx} &= \frac{1}{2\cos^2\frac{x}{2}} \\
&= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) \\
&= \frac{t^2+1}{2}
\end{aligned}$$

となるので,

$$\begin{aligned}
&\int \frac{1}{\sin x+\cos x+1}\,dx \\
&= \int \frac{1}{\frac{2t}{t^2+1}+\frac{1-t^2}{t^2+1}+1}\cdot\frac{2}{t^2+1}dt \\
&= \int \frac{2}{2t+2}\,dt \\
&= \int \frac{1}{t+1}\,dt \\
&= \log|\tan t+1|+C \\
&= \log\left|\tan\frac{x}{2}+1\right|+C
\end{aligned}$$

別解 1.

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{1}{\sin x+\cos x+1}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}+2\cos^2\frac{x}{2}}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{\tan\frac{x}{2}+1}\cdot\frac{1}{2\cos^2\frac{x}{2}}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{\tan\frac{x}{2}+1}\cdot\left(\tan\frac{x}{2}\right)'dx \\
&= \log\left|\tan\frac{x}{2}+1\right|+C
\end{aligned}$$

問題 10

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx$$

解法 10. $x=\dfrac{\pi}{2}-t$ とおくと，[4]

$$\frac{dx}{dt}=-1$$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx \\ &= \int_{\frac{\pi}{2}}^{0} \frac{\sin\left(\frac{\pi}{2}-t\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)}\cdot(-1)dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx \end{aligned}$$

であるので，

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx+\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sin x+\cos x}\,dx \\ &= \int_0^{\frac{\pi}{2}} dx \\ &= \frac{\pi}{2} \end{aligned}$$

したがって，

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx=\frac{\pi}{4}$$

[4] King Property 参照

別解 1.

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx \\ =\ & \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)-(\cos x-\sin x)}{\sin x+\cos x}\,dx \\ =\ & \frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{1-\frac{(\sin x+\cos x)'}{\sin x+\cos x}\right\}dx \\ =\ & \frac{1}{2}\Big[x-\log|\sin x+\cos x|\Big]_0^{\frac{\pi}{2}} \\ =\ & \frac{\pi}{4} \end{aligned}$$

別解 2.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}\,dx$$

ここで，$t=x+\dfrac{\pi}{4}$ とおくと，

$$\frac{dt}{dx}=1$$

であるので，

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx &= \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{\sin\left(t-\frac{\pi}{4}\right)}{\sqrt{2}\sin t}\,dt \\ &= \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{\frac{1}{\sqrt{2}}\sin t-\frac{1}{\sqrt{2}}\cos t}{\sqrt{2}\sin t}\,dt \\ &= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \left\{1-\frac{(\sin t)'}{\sin t}\right\}dt \\ &= \frac{1}{2}\Big[t-\log|\sin t|\Big]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \\ &= \frac{\pi}{4} \end{aligned}$$

問題 11

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$$

解法 11. $t=\sqrt{\dfrac{1-x}{1+x}}$ とおくと，

$$\begin{aligned}
t^2 &= \frac{1-x}{1+x}\\
(1+x)\,t^2 &= (1-x)\\
\left(t^2+1\right)x &= 1-t^2\\
x &= \frac{1-t^2}{1+t^2}\\
\frac{dx}{dt} &= \frac{-2t\left(1+t^2\right)-2t\left(1-t^2\right)}{\left(1+t^2\right)^2}\\
&= -\frac{4t}{\left(1+t^2\right)^2}
\end{aligned}$$

となるので，

$$\begin{aligned}
\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx &= \int_1^0 t\cdot\left\{-\frac{4t}{\left(1+t^2\right)^2}\right\}dt\\
&= 4\int_0^1 \frac{t^2}{\left(t^2+1\right)^2}\,dt
\end{aligned}$$

ここで，$t=\tan\theta$ とおくと，

$$\frac{dt}{d\theta}=\frac{1}{\cos^2\theta}$$

であるから，

$$\begin{aligned}
\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx &= 4\int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\left(\tan^2\theta+1\right)^2}\cdot\frac{1}{\cos^2\theta}\,d\theta\\
&= 4\int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta\\
&= 2\int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\,d\theta\\
&= \Big[2\theta-\sin 2\theta\Big]_0^{\frac{\pi}{4}}\\
&= \frac{\pi}{2}-1
\end{aligned}$$

別解 1.

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx$$

ここで，$x=\sin\theta$ とおくと，

$$\frac{dx}{d\theta}=\cos\theta$$

となるので，

$$\begin{aligned}
\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx &= \int_0^{\frac{\pi}{2}} \frac{1-\sin\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta\\
&= \int_0^{\frac{\pi}{2}} \frac{1-\sin\theta}{|\cos\theta|}\cdot\cos\theta\,d\theta\\
&= \int_0^{\frac{\pi}{2}} (1-\sin\theta)\,d\theta\\
&= \Big[\theta-\cos\theta\Big]_0^{\frac{\pi}{2}}\\
&= \frac{\pi}{2}-1
\end{aligned}$$

問題 12

$$\int \frac{1}{\sqrt{x^2+1}}\,dx$$

解法 12. $x = \dfrac{e^t - e^{-t}}{2}$ とおくと，[5]

$$\frac{dx}{dt} = \frac{e^t + e^{-t}}{2}$$

であるから，

$$\begin{aligned}
&\int \frac{1}{\sqrt{x^2+1}}\,dx \\
=\ &\int \frac{1}{\sqrt{\dfrac{e^{2t}-2+e^{-2t}}{4}+1}} \cdot \frac{e^t+e^{-t}}{2}\,dt \\
=\ &\int \frac{1}{\sqrt{\dfrac{e^{2t}+2+e^{-2t}}{4}}} \cdot \frac{e^t+e^{-t}}{2}\,dt \\
=\ &\int \frac{1}{\dfrac{e^t+e^{-t}}{2}} \cdot \frac{e^t+e^{-t}}{2}\,dt \\
=\ &t + C
\end{aligned}$$

ここで，

$$\begin{aligned}
x &= \frac{e^t - e^{-t}}{2} \\
e^{2t} - 2xe^t - 1 &= 0 \\
e^t &= x \pm \sqrt{x^2+1}
\end{aligned}$$

となるが，$e^t \geqq 0$ であるので，

$$\begin{aligned}
e^t &= x + \sqrt{x^2+1} \\
t &= \log\left(x + \sqrt{x^2+1}\right)
\end{aligned}$$

以上より，

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \log\left(x+\sqrt{x^2+1}\right) + C$$

[5]双曲線関数参照

別解 1. $t = x + \sqrt{x^2+1}$ とおくと，

$$\begin{aligned}
\frac{dt}{dx} &= 1 + \frac{2x}{2\sqrt{x^2+1}} \\
&= \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}} \\
&= \frac{t}{\sqrt{x^2+1}}
\end{aligned}$$

となるので，

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+1}}\,dx &= \int \frac{1}{\sqrt{x^2+1}} \cdot \frac{\sqrt{x^2+1}}{t}\,dt \\
&= \int \frac{1}{t}\,dt \\
&= \log t + C \\
&= \log\left(x+\sqrt{x^2+1}\right) + C \quad (1)
\end{aligned}$$

問題 13

$$\int \sqrt{x^2+1}\,dx$$

二次関数の長さ

解法 13. $x = \dfrac{e^t - e^{-t}}{2}$ とおくと，[6]

$$\frac{dx}{dt} = \frac{e^t + e^{-t}}{2}$$

となるので，

$$\begin{aligned}
&\int \sqrt{x^2+1}\,dx \\
=\ & \int \sqrt{\frac{e^{2t} - 2 + e^{-2t}}{4} + 1} \cdot \frac{e^t + e^{-t}}{2}\,dt \\
=\ & \int \sqrt{\frac{e^{2t} + 2 + e^{-2t}}{4}} \cdot \frac{e^t + e^{-t}}{2}\,dt \\
=\ & \frac{1}{4}\int \left(e^t + e^{-t}\right)^2 dt \\
=\ & \frac{1}{4}\int \left(e^{2t} + 2 + e^{-2t}\right) dt \\
=\ & \frac{1}{4}\left(\frac{1}{2}e^{2t} + 2t - \frac{1}{2}e^{-2t}\right) + C \\
=\ & \frac{1}{8}\left(e^{2t} - e^{-2t}\right) + \frac{1}{2}t + C \\
=\ & \frac{1}{8}\left(e^t - e^{-t}\right)\left(e^t + e^{-t}\right) + \frac{1}{2}t + C \\
=\ & \frac{1}{8} \cdot 2x \cdot \left(e^t + e^{-t}\right) + \frac{1}{2}t + C
\end{aligned}$$

となる．ここで，

$$\begin{aligned}
x &= \frac{e^t - e^{-t}}{2} \\
x^2 + 1 &= \frac{e^{2t} - 2 + e^{-2t}}{4} + 1 \\
&= \frac{e^{2t} + 2 + e^{-2t}}{4} \\
2\sqrt{x^2+1} &= e^t + e^{-t}
\end{aligned}$$

であり，

$$\begin{aligned}
x &= \frac{e^t - e^{-t}}{2} \\
e^{2t} - 2xe^t - 1 &= 0 \\
e^t &= x \pm \sqrt{x^2+1}
\end{aligned}$$

となるが，$e^t \geqq 0$ であるので，

$$\begin{aligned}
e^t &= x + \sqrt{x^2+1} \\
t &= \log x + \sqrt{x^2+1}
\end{aligned}$$

以上から，

$$\begin{aligned}
&\int \sqrt{x^2+1}\,dx \\
=\ & \frac{1}{2}\left\{x\sqrt{x^2+1} + \log\left(x + \sqrt{x^2+1}\right)\right\} + C
\end{aligned}$$

別解 1.

$$\begin{aligned}
I &= \int \sqrt{x^2+1}\,dx \\
&= \int (x)'\sqrt{x^2+1}\,dx \\
&= x\sqrt{x^2+1} - \int x \cdot \frac{2x}{2\sqrt{x^2+1}}\,dx + C' \\
&= x\sqrt{x^2+1} - \int \frac{x^2}{\sqrt{x^2+1}}\,dx + C' \\
&= x\sqrt{x^2+1} - \int \frac{\left(x^2+1\right) - 1}{\sqrt{x^2+1}}\,dx + C' \\
&= x\sqrt{x^2+1} - I + \int \frac{1}{\sqrt{x^2+1}}\,dx + C' \\
2I &= x\sqrt{x^2+1} + \int \frac{1}{\sqrt{x^2+1}}\,dx + C''
\end{aligned}$$

したがって，式 (1) より，

$$\begin{aligned}
I &= \int \sqrt{x^2+1}\,dx \\
&= \frac{1}{2}\left\{x\sqrt{x^2+1} + \log\left(x + \sqrt{x^2+1}\right)\right\} + C
\end{aligned}$$

[6] 双曲線関数参照