Section 6 – Midterm Review

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1 Marginal Value of Another "Unit"

1.1 Lagrangian case

In the Lagrangian case, we have a budget constraint. Therefore, we may be interested in the marginal value of another unit of budget.

$$\mathcal{L} = U(x,y) + \lambda(B - wx - vy) \tag{1}$$

To find the marginal value of another unit of budget, we can take the derivative of the Lagrangian **??** (which measures value) with respect to the budget B,

$$\frac{\partial \mathcal{L}}{\partial B} = \lambda. \tag{2}$$

Therefore λ is the marginal value of another unit of budget (*aka.* the marginal utility of money) in the Lagrangian case.

1.2 Hamiltonian Case

In the Hamiltonian case, we are interested in a stock that grows following $\dot{s}$. Our current value Hamiltonian is

$$\begin{aligned} H_t &= W(s_t, h(s_t)) + \lambda_t \dot{s}_t && (3) \\ &= \textit{dividends} + \textit{capital gains} && (4) \end{aligned}$$

Recognize that λ_t is the marginal value of another unit of stock in the Hamiltonian case.

In problem set three, you were asked what the marginal value of a mammoth is in the first time period. You should have recognized that this was asking for the shadow price in time period one, *i.e.,* the marginal value of another unit of stock (which is the definition of a price). The first optimality condition gave you a solution for the shadow price as a function of harvest, which is what we expected you to use to calculate the shadow price.

To solve for an explicit equation for the shadow price λ refer to section **??** on the Euler equation.

2 Implicit Functions review

Implicit versus explicit $U(x,y)$ versus $x^{\beta}y^{1-\beta}$
$U(C)$ versus $\frac{C^{1-\eta}}{1-\eta}$
$G(s)$ versus $sr(1-\frac{s}{K})$

Q: If I was a company and produced widgets using produced capital natural capital, how would you write an implicit function to represent production?
A: $F(N,P)$ where N is natural capital and P is produced capital.

3 Logistic Growth and Graphs

$$\dot{x} = \frac{\partial x}{\partial t} = rx(1-\frac{x}{K})$$

Where will $\dot{x}=0$? There are 2 roots, so there are 2 spots it will equal zero. It's already factored, so set each part equal to zero.

$$1 - x/K = 0 \implies x_{ss} = K$$
$$rx = 0 \implies x_{ss} = 0$$

When $x > 0$ but $x < K$ we know the growth rate is positive. When $x > K$, the growth rate is negative. Therefore we know when $x_{ss} = K$ is a stable point, b/c if x is perturbed, it will return to K.

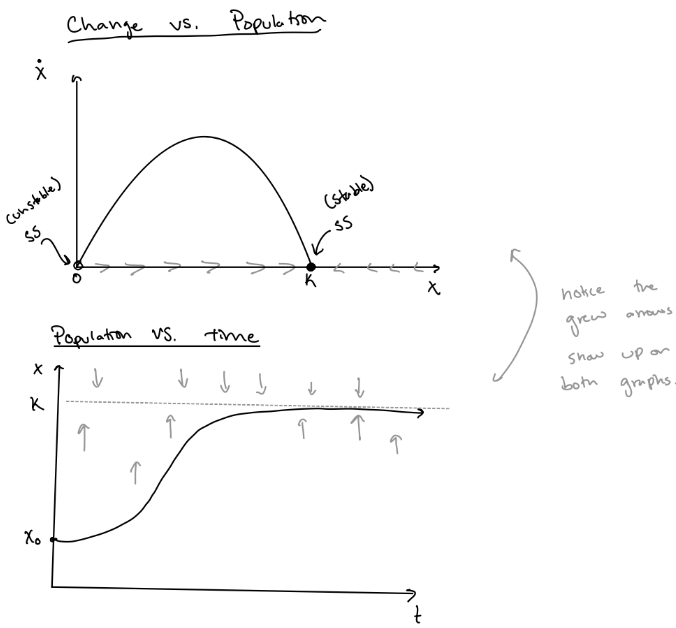

Figure 1: "grew" should say grey

4 Logistic Growth with Harvest

$$\dot{x} = rx(1 - \frac{x}{K}) - h$$

Where will $\dot{x} = 0$? When $h = rx(1 - \frac{x}{K})$. Consider the case where our harvest program is $h = \frac{x}{4}$. Could then set $rx(1 - \frac{x}{K}) = \frac{x}{4}$ and solve for x_{ss}.

When $h < rx(1 - \frac{x}{K})$, the growth rate $\dot{x}$ is positive. When $h > rx(1 - \frac{x}{K})$, the growth rate $\dot{x}$ is negative.

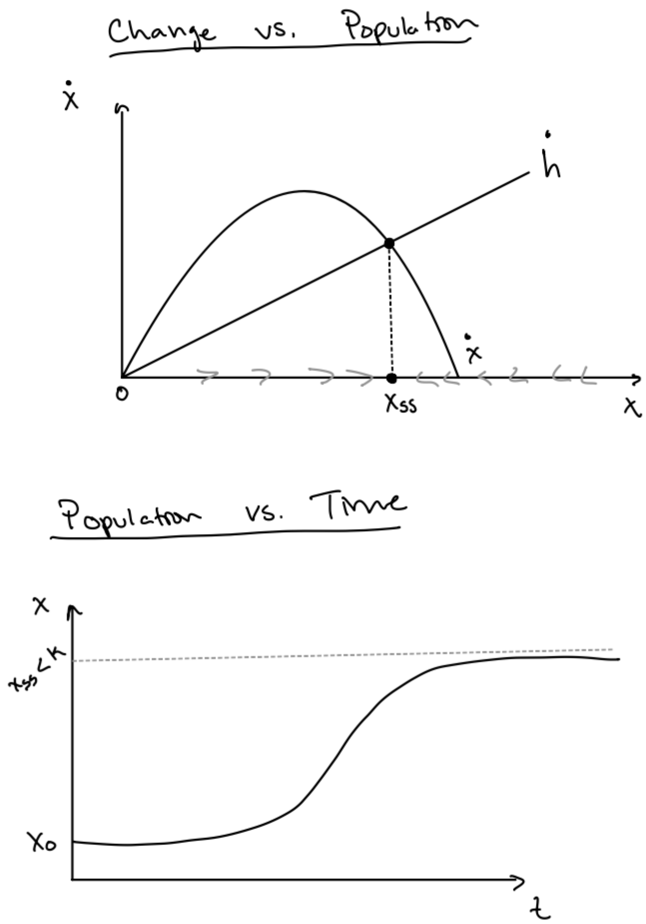

Figure 2: The steady-state level of population is less than its carrying capacity when there is a harvest

Note: I would focus on reviewing logistic growth, then be conceptually familiar with a few other types of ecological growth. Review the week two class notes on ecological models.

5 Euler Equation to get Shadow Price

5.1 Easy Version

Start with the Current Value Hamiltonian

$$\delta V = H = W(s,h) + \lambda \dot{s}. \tag{5}$$

Solve for the value, V,

$$V = \frac{W(s,h) + \lambda \dot{s}}{\delta}.$$

Recall the shadow price is defined as the marginal value of an additional unit of stock. Find $\frac{\partial V}{\partial s}$ and solve for λ to get the Euler equation which gives us the shadow price,

$$\lambda = \frac{\partial V}{\partial s} = \frac{W_s + \dot{\lambda}}{\delta - \dot{s}_s}. \tag{6}$$

?? is the Euler equation and tells us what our shadow price λ is equal to.

5.2 Not so hard but slightly longer version

We can derive the current value Hamiltonian from the intertemporal welfare function and then get the Euler equation.

The present value of a stock where the "present" time period is 0 can be found using our intertemporal welfare function,

$$V(s(t)) = \int_0^{\infty} W(s(\tau), x(s(\tau)) \exp(-\delta \tau) d\tau. \tag{7}$$

For small changes in time, we can take the derivative of the LHS and RHS. This exercise will eventually reveal the CVH in equation **??**.

Left Hand Side of **??** (uses the chain rule, also recall marginal value = capital price (i.e., $\frac{\partial V}{\partial s} = \lambda(s)$) and that the derivative with respect to time uses dot notation (i.e., $\frac{\partial s}{\partial t} = \dot{s}$)):

$$\frac{\partial V(s(t))}{\partial t} = \frac{\partial V}{\partial s}\frac{\partial s}{\partial t} = \lambda(s)\dot{s} \tag{8}$$

$$= \textit{capital gains} \tag{9}$$

Right Hand Side of **??** (Leibniz rule for the derivative of integral, don't stress if you don't follow this calculus step):

$$\frac{\partial \int_t^{\infty} W(s(\tau), x(s(\tau)) \exp(-\delta(\tau - t)) d\tau}{\partial t} = \delta V(s(t)) - W(s, x(s)) \tag{10}$$

Setting **??** = **??** and doing a tiny bit of algebra, we get

$$\delta V(s(t)) = W(s, x(s)) + \lambda(s)\dot{s} \tag{11}$$

which is the same as our current value Hamiltonian from equation **??**!

You can then follow the same steps in section **??** to get the Euler Equation for the capital/shadow price.

6 Calculating Wealth

Not doing p*q except if you're finding a lower bound.

7 Review from in Class

7.1 Present vs Current Value Hamiltonian

PVH:
Discrete value: $H_p = W(S,h)\beta^t + \lambda(G(s) - h)$
Continuous value: $H_p = W(S,h)e^{-\delta t} + \lambda(G(s) - h)$
The units of a PVH will always be in 2023 dollars (or whatever time period zero is). *The units* of λ are in present-year dollars.

CVH:
Continuous time: $H_c = W(S,h) + \lambda e^{\delta t}(G(S) - h)$
which can be rewritten as $H_c = W(S,h) + \mu_t(G(S) - h)$.
The unit of the CVH will be in year t dollars. *The units* of μ are in current-year dollars.

Going between the two:
$\mu_t = \lambda e^{\delta t}$
$e^{\delta t}$*PVH = CVH
The only reason we use the CVH more often is that the CVH has easier algebra.

Recall that the shadow price comes from the Lagrangian

$$\mathcal{L} = U(x,y) + \lambda(B - vx - wy) \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial B} = \lambda \tag{13}$$

Note that how a stock grows can also be a constraint:

$$\mathcal{L} = \sum_t U(S_t, h_t)\beta^t + \beta\lambda_{t+1}(S_t + G(S_t, h_t) - S_{t+1}) \tag{14}$$

$$\tag{15}$$

7.2 Steady States

Consider the mammoth problem:

$$M_{t+1} = M_t + G(M_t) - h_t \tag{16}$$

$$\frac{\Delta M}{\Delta t} = M_{t+1} - M_t = G(M_t) - h_t \tag{17}$$

In steady state, $\frac{\Delta M}{\Delta t} = 0$, because the population isn't changing anymore.
$\frac{\Delta M}{\Delta t} = 0$ & **??** $\implies h_{10} = G(h_{10})$.
You could then solve for h_{10} to get the harvest you can take from period 10 until forever.
The get the annuity value of this harvest level you can find

$$\textit{annuity value} = \frac{U(h_{10})\beta^{10}}{\delta}$$

7.3 Changes in wealth and changes in welfare

$$V(S_t) = \int U(S(\tau))e^{\delta(\tau - t)}\partial\tau \tag{18}$$

$$s.t.\ \dot{S} \tag{19}$$

$$\frac{\partial V(S_t)}{\partial t} = U_S - \delta V \tag{20}$$

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial S}\frac{\partial S}{\partial t} \tag{21}$$

$$\delta V = U_s + \frac{\partial V}{\partial S}\dot{S} \tag{22}$$

Notice that **??** is a current value Hamiltonian. **??** and **??** imply **??**. This is how we derive the CVH.

$$W = \sum pS \tag{23}$$

$$\frac{\partial W}{\partial t} = \sum p\dot{S} \tag{24}$$

7.4 Discounting

Ramsey Rule: Start with constant elasticity of utility

$$U = \frac{C^{1-\eta} - 1}{1 - \eta}$$

$$\dot{S} = F(s) - C$$

$$\int_0^\infty U(C)e^{-\delta t}$$

$$r = \delta + \eta F'(S)$$

There are two ways to discount, δ and η. δ is how we discount between generations. η is how we discount the growth of wealth. $F'(S)$ is a growth rate. Typically we think future generations are going to be wealthier. Ramsey argues that δ should be set equal to zero, but that η should be non-zero because we can borrow from the rich of tomorrow in order to benefit the poor of today.

A high discount rate would be that we need to borrow from tomorrow to redistribute wealth to today. A low discount rate would be that we favor future generations.

7.5 "Combine mathematical expressions"

If you have two equations equal to the same thing (say 0), then "combining mathematical expressions" means setting those equations equal to one another.