Calculus Review

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1 Common Derivative Rules

Consider a function, $f(x)$, that depends on the variable x. Derivatives tell us the slope of the function at a specific point. The slope is the rate of change.

Notation: There are many ways to denote that you are taking an derivative. They are all (essentially) equivalent. I will switch between notations to keep the math neat.

$$\frac{d}{dx}f(x) = f'(x) = f_x(x) = \Delta f(x)$$

Constant rule: Let c be a constant.

$$\frac{d}{dx}cx = c$$

Power rule:

$$\frac{d}{dx}x^n = nx^{n-1}$$

Product rule:

$$\frac{d}{dx}(f(x) * g(x)) = f'(x) * g(x) + f(x) * g'(x)$$

Quotient rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x) * f'(x) - f(x) * g'(x)}{g(x)^2}$$

A rhyme that helps me remember the quotient rule: "Low d High minus High d Low all over the square of what's below".

Chain Rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x)$$

Log Rules:

$$\frac{d}{dx}\log(x) = \frac{1}{x}$$

I am not sure if you'll need the following log rules for this course but wanted to include them here in case!

$$\log(x * y) = \log(x) + \log(y)$$

$$\log(\frac{x}{y}) = \log(x) - \log(y)$$

In this course, unless it's stated otherwise, you can assume all logs are "base e", i.e.,

$$\log_e(x) = \log(x) = \ln(x).$$

Negative Exponent Rule: Derivatives of functions with negative exponents follow the same power rule.

$$f(x) = \frac{1}{x^2} = x^{-2}$$

You could find the derivative of $f(x)$ using the quotient rule (because it is a fraction) or the power rule.

Quotient rule:

$$f'(x) = \frac{x^2 * 0 - 1 * 2x}{x^4} = \frac{-2x}{x^4} = \frac{-2}{x^3} = -2x^{-3}$$

Power rule:

$$f'(x) = -2x^{-3}$$

With negative exponents, I find the power rule much easier.

Partial Derivatives: An function can depend on more than one variable, i.e., $U(x, y)$. You can take the derivative of that function with respect to a single variable and treat the other as a constant.

Example:

$$U(x, y) = x^{1/3} * y^{2/3}$$
$$U_x(x, y) = \frac{1}{3}x^{-2/3} * y^{2/3}$$
$$U_y(x, y) = \frac{2}{3}x^{1/3}y^{-1/3}$$

1.1 Examples

Example 1, power rule:
x is a variable.

$$f(x) = 3 + x^2 + 4x^3$$
$$f'(x) = 2x + 12x^2$$

Example 2, logistic growth:
r and K are constants. x is a variable.

$$g(x) = rx(1 - \frac{x}{K})$$
$$g(x) = rx - \frac{rx^2}{K}$$
$$g'(x) = r - \frac{2rx}{K}$$

Example 3, profit function:
a, b and c are constants. x and w are variables.

$$\pi(x, w) = aw - \frac{b}{2}w^2 - c\frac{w}{x}$$
$$\frac{d\pi(x, w)}{dx} = cwx^{-2}$$
$$\frac{d\pi(x, w)}{dw} = a - bw - \frac{c}{x}$$

2 Finding Max & Min of Functions / Unconstrained Optimization

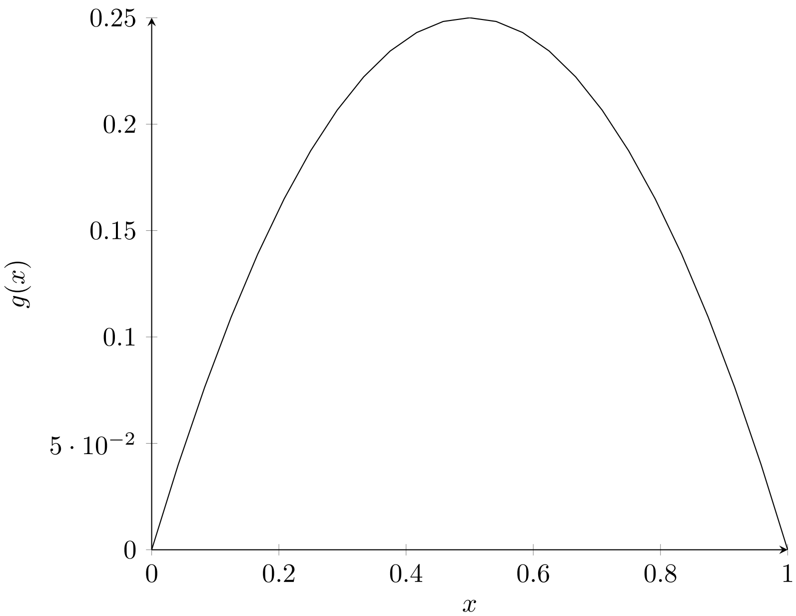
We use derivatives to find the maximum or minimum of a function. Recall that derivatives tell us the slope of a function at a specific point.

At a maximum or minimum, the slope of the function will be zero. Therefore, to find the maximum or minimum of a function, we can:

1. Find the derivative of the function, $f'(x)$.
2. Set the derivative of the function equal to zero, $f'(x) = 0$.
3. Solve for x. This value of x maximizes the function $f(x)$.
4. Plug the x into $f(x)$ to find the maximum or minimum value of $f(x)$.

2.1 Example One

Consider $g(x) = x - x^2$



Immediately we can see that the maximum of $g(x)$ is 0.25 and occurs at $x = 0.5$. However, let's solve for it following the steps above.

1. $g'(x) = 1 - 2x$
2. $1 - 2x = 0$.
3. $x = \frac{1}{2}$
4. $g(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2}^2 = 1/2 - 1/4 = 1/4$.

2.2 Example two

Consider the manager of a fishery. She knows that the fish population grows at the rate

$$g(x) = 0.5x(1 - x/100),$$

where x is the population of fish. She wants to find the population level that will lead to the maximum growth rate. How do you solve this problem?

First, rewrite $g(x)$

$$g(x) = \frac{1}{2}x - \frac{x^2}{200}$$

1) Find derivative:

$$g'(x) = 1/2 - x/100$$

2) Set equal to zero:

$$1/2 - x/100 = 0$$

3) Solve for x:

$$x = 50$$

4) Plug back into $g(x)$:

$$g(50) = 1/2 * 50 - 50^2/200 = 25 - 12.5 = 12.5$$