

Fundamentals of Convexity

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These notes are subject to change and may contain errors.

1 Preliminaries

- Notation: \mathbb{R} for reals, \mathbb{R}_+ for non-negative reals, and \mathbb{R}_{++} for positive reals. $[n] = \{1, \dots, n\}$ for integer intervals. $e_1, \dots, e_n \in \mathbb{R}^n$ for the standard basis. $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ for standard Euclidean inner product, and $\|x\|_2^2 = \langle x, x \rangle = \sum_{i=1}^n x_i^2$ for the Euclidean norm.
- Complexity Theory - big-O and poly time
- Topology - open vs closed

Definition 1. $S \subseteq \mathbb{R}^n$ is open if for every point $x \in S$, there is an $\varepsilon > 0$ such that the open ball $B^\circ(x, \varepsilon) := \{y \mid \|y - x\|_2 < \varepsilon\} \subseteq S$.

S is closed if the complement \mathbb{R}^n / S is closed; equivalently, if S contains the limit point of every convergent sequence in S : $\{x_i \in S\} \implies \lim_i x_i \in S$.

S is compact if it is closed and bounded.

- Differentiability and Taylor approximation

Definition 2. f is differentiable at x in the interior of the domain $\text{dom}(f)^\circ$ if all partial derivatives exist:

$$\lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t},$$

and further the above limit is a linear function of v .

It is k -times differentiable if all k -th order partial derivatives exist, and k -times continuously differentiable if furthermore the k -th derivative is continuous in a neighborhood of x .

Definition 3. If f is differentiable at x , and an inner product $\langle \cdot, \cdot \rangle$ is given, then the gradient $\nabla f(x)$ is uniquely defined by

$$\forall v \in \mathbb{R}^n : \quad \langle \nabla f(x), v \rangle = D_v f(x).$$

For the standard inner product, this induces the more familiar definition

$$(\nabla f(x))_i = \partial_{x_i} f(x).$$

Definition 4. Similary, if f is twice-differentiable at x , the Hessian $\nabla^2 f(x)$ is uniquely defined by

$$\forall u, v \in \mathbb{R}^n : \langle u, \nabla^2 f(x)v \rangle = D_u D_v f(x).$$

For the standard inner product, this induces the more familiar definition

$$(\nabla^2 f(x))_{ij} = \partial_{x_i} \partial_{x_j} f(x).$$

- Linear and quadratic functions

Definition 5. An affine function $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ is of the form

$$\ell(x) = \langle a, x \rangle + b$$

for $a \in \mathbb{R}^n, b \in \mathbb{R}$.

A quadratic function $q : \mathbb{R}^n \rightarrow \mathbb{R}$ is of the form

$$q(x) := \langle x, Ax \rangle + \langle b, x \rangle + c$$

where $A \in \mathbb{R}^{n \times n}$ (symmetric matrix without loss of generality), $b \in \mathbb{R}^n, c \in \mathbb{R}$.

Definition 6. For function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the linear and quadratic approximation at x , for once- and twice-differentiable functions, respectively, are

$$\ell_x(y) := f(x) + \langle \nabla f(x), y - x \rangle;$$

$$q_x(y) := f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \langle (y - x), \nabla^2 f(x)(y - x) \rangle.$$

Remark 7. By e.g. intermediate value theorem, the remainder $f - \ell_x, f - q_x$ are small in the neighborhood of x if f is appropriately differentiable at x .

2 Introduction

2.1 Convex Sets

Definition 8. A set $C \subseteq \mathbb{R}^d$ is convex if for all $x, y \in C$ and $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda)y \in C.$$

Definition 9. For subset $S \subseteq \mathbb{R}^n$ we can define the span, affine, and convex hull in terms of linear combinations as

$$\begin{aligned} \text{span}(S) &:= \left\{ \sum_i^N a_i x_i \mid x_i \in S \right\}; \\ \text{aff}(S) &:= \left\{ \sum_i^N a_i x_i \mid x_i \in S, \sum_i^N a_i = 1 \right\}; \\ \text{conv}(S) &:= \left\{ \sum_i^N a_i x_i \mid x_i \in S, a \geq 0, \sum_i^N a_i = 1 \right\}. \end{aligned}$$

Note $\text{conv}(S) \subseteq \text{aff}(S) \subseteq \text{span}(S)$. Try to visualize these sets for small examples.

Fact 10. *The following operations preserve convexity of sets*

- *Scalar multiplication:* $K \rightarrow cK$;
- *Addition:* $K_1 + K_2$
- *Intersection:* $\cap_i K_i$
- *Affine transform:* $K \rightarrow AK + b$

2.2 Convex Functions

Definition 11. *A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in \text{dom}(f)$ and $\lambda \in [0, 1]$.

Here $\text{dom}(f)$ is the domain of f , where it is well-defined. E.g. \mathbb{R}_+ for \sqrt{x} or \mathbb{R}_{++} for $\log(x)$. Function convexity depends on the domain, e.g. $\sqrt{|x|}$ is convex for $\text{dom} = \mathbb{R}_+$ but not for $\text{dom} = \mathbb{R}$.

Fact 12. *The following operations preserve convexity of functions*

- *Non-negative scalar multiplication:* $f \rightarrow cf$;
- *Addition:* $f_1 + f_2$
- *Point-wise supremum:* $\sup_i f_i$
- *Restriction:* $t \rightarrow f((1 - t)x + ty)$ (or more generally a line or subspace).
- *Affine transform:* $x \rightarrow f(Ax + b)$
- *Perspective:* $(x, t) \rightarrow tf(x/t)$ for $t > 0$.

Theorem 13. *If f is differentiable, then f is convex iff*

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \quad \text{for all } x, y.$$

Proposition 14. *For convex f , x is a local minimum iff it is a global minimum.*

Proof: Clearly a global minimum is also a local minimum. For the converse, let $\varepsilon > 0$ such that $f(y) \geq f(x)$ for all $\|y - x\|_2 \leq \varepsilon$. Assume for contradiction x is not the global minimum, so $f(x_*) < f(x)$. Then letting $x_\delta := (1 - \delta)x + \delta x_*$, by convexity we have

$$f(x_\delta) \leq (1 - \delta)f(x) + \delta f(x_*) < f(x).$$

But for small enough δ this contradicts local minimality of x .

Definition 15. *For $f : \mathbb{R}^n \rightarrow \mathbb{R}$, let the epigraph be*

$$\text{epi}(f) := \{(x, t) \mid f(x) \leq t\}$$

i.e. the 'upwards closure' of the graph of f in \mathbb{R}^{n+1} .

Theorem 16. $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a (closed) convex function iff the epigraph $\text{epi}(f) \subseteq \mathbb{R}^{n+1}$ is a (closed) convex set.

Proof: First consider f convex, and we want to show $\text{epi}(f)$ is a convex set. Let $(x, t), (y, s) \in \text{epi}(f)$, which by definition means

$$f(x) \leq t, \quad f(y) \leq s.$$

Now consider convex combination $z := (1 - \lambda)x + \lambda y$ for $\lambda \in [0, 1]$. Then by convexity

$$f(z) \leq (1 - \lambda)f(x) + \lambda f(y) \leq (1 - \lambda)t + \lambda s,$$

where the first step was by convexity of f , and in the second step we used that $(x, t), (y, s) \in \text{epi}(f)$. Rewriting this, we have shown

$$(z, (1 - \lambda)t + \lambda s) = (1 - \lambda)(x, t) + \lambda(y, s) \in \text{epi}(f),$$

which shows $\text{epi}(f)$ is a convex set.

For the converse, assume $\text{epi}(f)$ is closed and convex. Then for $(x, f(x)), (y, f(y)) \in \text{epi}(f)$ we have $(z, t) \in \text{epi}(f)$ for $z := (1 - \lambda)x + \lambda y, t := (1 - \lambda)f(x) + \lambda f(y)$, which implies

$$f((1 - \lambda)x + \lambda y) = f(z) \leq t = (1 - \lambda)f(x) + \lambda f(y),$$

verifying convexity of f .

Lemma 17. For convex $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and any $t \in \mathbb{R}$, the sub-level set

$$L_t(f) := \{x \in \mathbb{R}^n \mid f(x) \leq t\}$$

is a convex set. Further, if $f(x) = t$ and f is differentiable at x , then the gradient $\nabla f(x)$ gives a supporting hyperplane for the sub-level set L_t at x .

Proof: The sub-level set is the projection of the intersection of the epigraph and a halfspace

$$\text{epi}(f) \cap \{(z, s) \mid s \leq t\},$$

and intersection and projection both preserve convexity, so L_t is convex.

3 Separation Theorems

3.1 Separating Hyperplane Theorem

Theorem 18. For closed convex K and $p \in \mathbb{R}^n$ with $p \notin K$, there is a separating hyperplane $w \in \mathbb{R}^n$ satisfying

$$\max_{x \in K} \langle w, x \rangle < b < \langle w, p \rangle.$$

Proof: The proof plan is simple: let $x_* := \arg \inf_{x \in K} \|x - p\|_2^2$, then we claim the direction $w := x_* - p$ gives a separating hyperplane.

To show this is well-defined, we first reduce to the case when K is compact. Take any $x_0 \in K$ and consider

$$L := K \cap \{y \in K \mid \|y - p\|_2^2 \leq \|x_0 - p\|_2^2\},$$

which is convex since K is convex and the second set is the sub-level set of convex function $y \rightarrow \|y - p\|_2^2$. Therefore we assume K is compact for the remainder of the proof and note $x_* = \arg \inf_{x \in K} \|x - p\|_2^2$ is attained by the extreme value theorem. We can in fact show that the optimizer x_* is unique, as $\|\cdot\|_2^2$ is strictly convex, but we will not need this for the proof.

Now we claim that $x_* = \arg \min_{x \in K} \|x - p\|_2^2$ iff $\forall x \in K : \langle x - x_*, x_* - p \rangle \geq 0$. Indeed, we can rewrite the function difference

$$\|x - p\|_2^2 - \|x_* - p\|_2^2 = 2\langle x - x_*, x_* - p \rangle + \|x - x_*\|_2^2,$$

as can be verified directly. Now if the inner product term is ≥ 0 for all x , then since the norm term is always ≥ 0 , we get $\|x - p\|_2^2 \geq \|x_* - p\|_2^2 \forall x \in K$. Conversely, if there exists $x \in K : \langle x - x_*, x_* - p \rangle < 0$, then consider $x_\varepsilon := (1 - \varepsilon)x_* + \varepsilon x$, which is in K by convexity, and note that

$$\|x_\varepsilon - p\|_2^2 - \|x_* - p\|_2^2 = 2\langle x_\varepsilon - x_*, x_* - p \rangle + \|x_\varepsilon - x_*\|_2^2 = \varepsilon\langle x - x_*, x_* - p \rangle + \varepsilon^2\|x - x_*\|_2^2 < 0,$$

where we used $x_\varepsilon - x_* = \varepsilon(x - x_*)$, and the last step follows as the ε term is negative and therefore dominates the quadratic term for small enough $\varepsilon > 0$.

Finally, we verify that $w := x_* - p$ gives a separating hyperplane:

$$\langle p - x_*, w \rangle = -\|p - x_*\|_2^2 < 0, \quad \forall x \in K : \langle x - x_*, w \rangle \geq 0$$

which by rearranging gives

$$\implies \langle p, w \rangle < \langle x_*, w \rangle \leq \inf_{x \in K} \langle x, w \rangle.$$

We also get a version for two sets.

Theorem 19. *For closed convex $C, D \subseteq \mathbb{R}^n$, if they are disjoint $C \cap D = \emptyset$ and one of C or D is bounded, then there is a separating hyperplane $(w, b) \in \mathbb{R}^{n+1}$ satisfying*

$$\sup_{x \in C} \langle w, x \rangle < b < \inf_{x \in D} \langle w, x \rangle.$$

Proof: Consider $K = C - D$ and $p = 0$: there is a hyperplane separating C, D iff there is a hyperplane separating 0 from $C - D$. We need the condition that one of C or D is bounded in order for K to be closed.

Remark 20. *Both the assumptions (closed and bounded) are necessary for strict separation: consider open convex intervals $(0, 1), (1, 2) \subseteq \mathbb{R}$, which are disjoint but cannot be strictly separated; and consider $\{(x, y) \mid y \leq 0\}$ and the epigraph of the function $1/x$, both in \mathbb{R}^2 , which are disjoint closed convex sets that ‘meet at ∞ ’ and so cannot be strictly separated.*

If we only require separation $\sup_{x \in C} \langle w, x \rangle \leq \inf_{x \in D} \langle w, x \rangle$, then some of these technical conditions can be dropped, see [BV Section 2.5] for further discussion.

3.2 Supporting Hyperplanes

Definition 21. *A supporting halfspace H of $K \subseteq \mathbb{R}^n$ at $x \in \partial K$ satisfies: (1) $K \subseteq H$; (2) $x \in \partial H$.*

Corollary 22. *For every closed convex K and $x \in \partial K$, there is a supporting hyperplane $H \supseteq K$ such that $x \in \partial H$.*

Proof: See Theorem 3.1.12 in [Nesterov] and page 51 of [Boyd, Vanderberghe].

Exercise 1 (Exercise 2.27 in BV). *$K \subseteq \mathbb{R}^n$ closed with non-empty interior, then K is convex iff it has a supporting hyperplane at every point of its boundary.*

4 Strong Duality

Theorem 23. *Closed convex $K \subseteq \mathbb{R}^n$ is the intersection of all containing halfspaces*

$$K \equiv \cap_{\text{halfspace } H \supseteq K} H.$$

Proof: One direction is clear: for $x \in K, H \supseteq K \implies x \in H$, so $K \subseteq \cap_{H \supseteq K} H$. For the converse, consider $z \notin K$, then the separating hyperplane theorem 18 gives $w \in \mathbb{R}^n$ such that

$$\langle w, z \rangle > b > \sup_{x \in K} \langle w, x \rangle.$$

So we can consider halfspace $H := \{x \in \mathbb{R}^n \mid \langle w, x \rangle \leq b\}$, which contains K and does not contain z . Therefore $z \notin \cap_{H \supseteq K} H$.

Theorem 24. *Closed convex $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the supremum of all affine minorants*

$$f(x) = \sup_{f \geq h} h(x).$$

The proof reduces f to its epigraph and then applies strong duality for convex sets. It is valuable to consider what exactly the supporting hyperplanes of $\text{epi}(f)$ correspond to.

5 First Order Definition of Convexity

Recall that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $x \in \text{dom}(f)^\circ$ if the limit

$$\lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

exists, and further is a linear function of v . If we have an inner product, then the gradient is defined as

$$\forall v : \quad \langle \nabla f(x), v \rangle = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Proposition 25. *Let f be differentiable. Then f is convex iff*

$$\forall x, y : \quad f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle.$$

We can apply this theory even in the non-differentiable setting.

Definition 26. *For $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $x \in \text{dom}(f)$, the set of subgradients $\partial f(x)$ are the elements $g \in \mathbb{R}^n$ satisfying*

$$\forall y \in \mathbb{R}^n : \quad f(y) \geq f(x) + \langle g, y - x \rangle.$$

Note that the set of subgradients is convex: since it is defined by the above inequalities, $g, h \in \partial f(x) \implies (1 - \lambda)g + \lambda h \in \partial f(x)$.

Proposition 27. *If f is convex, then for every $x \in \text{dom}(f)^\circ$, there is a subgradient $\partial f(x) \neq \emptyset$.*

Conversely, if $\partial f(x) \neq \emptyset$ for every $x \in \text{dom}(f)$, then f is convex.