Convex Optimization: Problem Set 1

January 21, 2026

Some problems borrowed from Optimization course of Daniel Dadush.

1. Recall the definition of convex hull:

$$\text{conv}(S) := \left\{ \sum_{i=1}^{N} \lambda_i \mathbf{x}_i \mid \mathbf{x}_i \in S, \lambda \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

   1. Prove that $\text{conv}(S) = \cap_{K \supseteq S} K$ where the intersection is over all closed convex K containing S. Therefore $\text{conv}(S)$ is the smallest convex set containing S.
   2. Prove Jensen's inequality: for convex $f : \mathbb{R}^n \to \mathbb{R}$ and input $\mathbf{x} = \sum_{i=1}^{N} \lambda_i \mathbf{x}_i$ where $\lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1$

$$f(\mathbf{x}) \leq \sum_{i=1}^{N} \lambda_i f(\mathbf{x}_i).$$

   3. Show that if $\mathcal{C} \subseteq \mathbb{R}^n$ is a compact convex set and $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function then the **supremum** of f over $\mathcal{C}$ is attained at an extreme point of $\mathcal{C}$.

2. Describe the set of boundary points for the following norm balls, i.e. points not in the interior. For each boundary point $\mathbf{x}$ give the set of supporting hyperplanes at $\mathbf{x}$.

   1. $B_1^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_1 := \sum_{i=1}^{n} |x_i| \leq 1\}$
   2. $B_2^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2} \leq 1\}$
   3. $B_\infty^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty := \max_{i \in [n]} |x_i| \leq 1\}$.

3. For convex closed $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \text{dom}(f)$:

   - Show for any subgradient $g \in \partial f(x)$, $(g, -1)$ gives a *supporting hyperplane* at $(x, f(x))$ for the epigraph

$$\text{epi}(f) := \{(x, t) \mid f(x) \leq t\}.$$

   - Show that $g \in \partial f(x)$ gives a *supporting hyperplane* at x for the sub-level set

$$L := \{y \in \mathbb{R}^n \mid f(y) \leq f(x)\}.$$

4. Prove Exercise 2.27 in [BV]: Let $K \subseteq \mathbb{R}^n$ be closed, bounded, with non-empty interior, such that there exists a supporting hyperplane of K at every point of the boundary $x \in \partial K$. Show this implies K is convex.

5. Affine and quadratic functions are the most basic convex functions. We will prove some properties about them:

   - Show $h : \mathbb{R}^n \to \mathbb{R}$ is convex *and* concave (i.e. h is convex and $-h$ is convex) iff h is an affine function, i.e. $h(x) = \langle a, x\rangle + b$ for $a \in \mathbb{R}^n, b \in \mathbb{R}$.
   - Let $\tilde{q}(x) := \langle x, Qx\rangle + \langle a, x\rangle + b$ for symmetric matrix $Q \in \mathbb{R}^{n\times n}, a \in \mathbb{R}^n, b \in \mathbb{R}$. Show $\tilde{q}$ is convex iff $q(x) := \langle x, Qx\rangle$ is convex. (Hint: use first part)
   - Show $q(x) = \langle x, Qx\rangle$ is convex iff

     $$\forall v \in \mathbb{R}^n : \quad \langle v, Qv\rangle \geq 0,$$

     and similarly show it is strictly convex iff the inequality is strict.
     (These conditions are known as positive-semi-definiteness and positive-definiteness, and are denoted $Q \succeq 0, Q \succ 0$.)
   - Find the optimizer and optimum value of strictly convex quadratic

     $$\tilde{q}(x) := \langle x, Qx\rangle + \langle a, x\rangle + b.$$

6. An Ellipsoid is an affine image of the Euclidean ball

   $$\mathcal{E} = c + AB_2^n \quad \text{where} \quad B_2^n := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\},$$

   for some $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ invertible.

   - Let $q(x) := \langle x, Qx\rangle + \langle d, x\rangle + e$ be *strictly convex*. Show any sub-level set

     $$L_t := \{x \in \mathbb{R}^n \mid q(x) \leq t\}$$

     is either empty or an Ellipsoid (i.e. find c, A such that $L_t = c + AB_2$).
   - Conversely, given Ellipsoid $\mathcal{E} = c + AB_2$ as above, find convex quadratic $q(x) := \langle x, Qx\rangle + \langle d, x\rangle + e$ such that

     $$\mathcal{E} = \{x \in \mathbb{R}^n \mid q(x) \leq 1\}.$$

7. Recall the GLS oracle model for convex sets. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex closed function with $\mathrm{dom}(f) = \mathbb{R}^n$. We investigate natural function oracles in terms of the epigraph:

   - Show MEM(epi(f)) can be implemented using an EVALuation oracle for f; show EVAL(f) can be approximately implemented using MEM(epi), i.e. for input $x \in \mathbb{R}^n$ compute $t \in \mathbb{R}$ such that $|t - f(x)| \leq \varepsilon$.

- Show SEP(epi(f)) can be implemented using an GRADient (or subgradient) oracle for f; show GRAD(f) can be approximately implemented using SEP(epi(f)), i.e. for input $x \in \mathbb{R}^n$ compute $g \in \mathbb{R}^n$ such that

$$\forall y \in \mathbb{R}^n : f(y) \geq f(x) - \varepsilon + \langle g, y - x \rangle.$$

- Relate OPT(epi(f)) and EVAL and GRAD for the Fenchel dual defined as

$$f^*(w) := \sup_{x \in \mathbb{R}^n} \langle w, x \rangle - f(x).$$

- Relate MEM, SEP for epi(f^*) with VAL, VIOL for epi(f). Note the relation does not need to be exact.