Convex Optimization: Problem Set 1

February 9, 2026

Some problems borrowed from Optimization course of Daniel Dadush.

1. Recall the definition of convex hull:

$$\mathrm{conv}(S) := \left\{ \sum_{i=1}^{N} \lambda_i \mathbf{x}_i \mid \mathbf{x}_i \in S, \lambda \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

  1. Prove that for finite $S \subseteq \mathbb{R}^n, \mathrm{conv}(S) = \cap_{K \supseteq S} K$ where the intersection is over all closed convex K containing S. Therefore $\mathrm{conv}(S)$ is the smallest convex set containing S.

  2. Prove Jensen's inequality: for convex $f : \mathbb{R}^n \to \mathbb{R}$ and input $\mathbf{x} = \sum_{i=1}^{N} \lambda_i \mathbf{x}_i$ where $\lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1$

$$f(\mathbf{x}) \leq \sum_{i=1}^{N} \lambda_i f(\mathbf{x}_i).$$

  3. Show that if $\mathcal{C} \subseteq \mathbb{R}^n$ is a compact convex set and $f : \mathbb{R}^n \to \mathbb{R}$ is a convex continuous function then the **supremum** of f over $\mathcal{C}$ is attained at an extreme point of $\mathcal{C}$. (Here $x \in \mathcal{C}$ is an extreme point if it cannot be written as a convex combination of two other points $y, z \in \mathcal{C}$ such that $y \neq x, z \neq x$.)

2. Describe the set of boundary points for the following norm balls, i.e. points not in the interior. For each boundary point $\mathbf{x}$ give the set of supporting hyperplanes at $\mathbf{x}$.

  1. $B_1^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_1 := \sum_{i=1}^{n} |x_i| \leq 1\}$
  2. $B_2^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2} \leq 1\}$
  3. $B_\infty^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty := \max_{i \in [n]} |x_i| \leq 1\}$.

3. For convex closed $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathrm{dom}(f)$:

  - Show for any subgradient $g \in \partial f(x)$, $(g, -1)$ gives a *supporting hyperplane* at $(x, f(x))$ for the epigraph

$$\mathrm{epi}(f) := \{(x, t) \mid f(x) \leq t\}.$$

- Show that $g \in \partial f(x)$ gives a *supporting hyperplane* at x for the sub-level set

$$L := \{y \in \mathbb{R}^n \mid f(y) \le f(x)\}.$$

4. Prove Exercise 2.27 in [BV]: Let $K \subseteq \mathbb{R}^n$ be closed, bounded, with non-empty interior, such that there exists a supporting hyperplane of K at every point of the boundary $x \in \partial K$. Show this implies K is convex.

5. Affine and quadratic functions are the most basic convex functions. We will prove some properties about them:

- Show $h : \mathbb{R}^n \to \mathbb{R}$ is convex *and* concave (i.e. h is convex and $-h$ is convex) iff h is an affine function, i.e. $h(x) = \langle a, x \rangle + b$ for $a \in \mathbb{R}^n, b \in \mathbb{R}$.

- Let $\tilde{q}(x) := \langle x, Qx \rangle + \langle a, x \rangle + b$ for symmetric matrix $Q \in \mathbb{R}^{n \times n}, a \in \mathbb{R}^n, b \in \mathbb{R}$. Show $\tilde{q}$ is convex iff $q(x) := \langle x, Qx \rangle$ is convex. (Hint: use first part)

- Show $q(x) = \langle x, Qx \rangle$ is convex iff

$$\forall v \in \mathbb{R}^n : \quad \langle v, Qv \rangle \ge 0,$$

and similarly show it is strictly convex iff the inequality is strict.
(These conditions are known as positive-semi-definiteness and positive-definiteness, and are denoted $Q \succeq 0, Q \succ 0$.)

- Find the optimizer and optimum value of strictly convex quadratic

$$\tilde{q}(x) := \langle x, Qx \rangle + \langle a, x \rangle + b.$$

6. An Ellipsoid is an affine image of the Euclidean ball

$$\mathcal{E} = c + AB_2^n \quad \text{where} \quad B_2^n := \{x \in \mathbb{R}^n \mid \|x\|_2 \le 1\},$$

for some $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ invertible.

- Let $q(x) := \langle x, Qx \rangle + \langle d, x \rangle + e$ be *strictly convex*. Show any sub-level set

$$L_t := \{x \in \mathbb{R}^n \mid q(x) \le t\}$$

is either empty or an Ellipsoid (i.e. find c, A such that $L_t = c + AB_2$).

- Conversely, given Ellipsoid $\mathcal{E} = c + AB_2$ as above, find convex quadratic $q(x) := \langle x, Qx \rangle + \langle d, x \rangle + e$ such that

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid q(x) \le 1\}.$$

7. Recall the GLS oracle model for convex sets. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex closed function with $\operatorname{dom}(f) = \mathbb{R}^n$. We investigate natural function oracles in terms of the epigraph:

- Show MEM(epi(f)) can be implemented using an EVALuation oracle for f; show EVAL(f) can be approximately implemented using MEM(epi(f)), i.e. for input $x \in \mathbb{R}^n$ compute $t \in \mathbb{R}$ such that $|t - f(x)| \leq \varepsilon$.

- Show SEP(epi(f)) can be implemented using EVALuation and GRADient (or subgradient) oracle for f; show GRAD(f) can be approximately implemented using SEP(epi(f)), i.e. for input $x \in \mathbb{R}^n$ compute $g \in \mathbb{R}^n$ such that

$$\forall y \in \mathbb{R}^n : f(y) \geq f(x) - \varepsilon + \langle g, y - x \rangle.$$