CPSC 536C: Algorithms for Convex Optimization
University of British Columbia

Lecture X: Cutting Plane Methods

Akshay Ram
January 5, 2025

These notes are subject to change and may contain errors.

1 GLS Oracle Model

1.1 Definitions

Definition 1 (Oracle Model). *The following are algorithmic problems for closed convex set $K \subseteq \mathbb{R}^n$:*

MEM bership($x \in \mathbb{R}^n, K \subseteq \mathbb{R}^n$):
Output: YES if $x \in K$; NO otherwise.

SEP aration($x \in \mathbb{R}^n, K \subseteq \mathbb{R}^n$):
Output: YES if $x \in K$;
otherwise output separating w such that $\langle w, x\rangle > \sup_{z\in K}\langle w, z\rangle$;

OPT imization($w \in \mathbb{R}^n, K \subseteq \mathbb{R}^n$)
Output: $h_K(w) = \sup_{x\in K}\langle w, x\rangle$
and optimizer $x_* \in K$ such that $h_K(w) = \langle w, x_*\rangle$;

VAL idity($w \in \mathbb{R}^n, b \in \mathbb{R}, K \subseteq \mathbb{R}^n$):
Output: YES if $\langle w, x\rangle \le b$ for all $x \in K$; NO otherwise.

VIOL ation($w \in \mathbb{R}^n, b \in \mathbb{R}, K \subseteq \mathbb{R}^n$):
Output: YES if $\langle w, x\rangle \le b$ for all $x \in K$;
otherwise output $y \in K$ such that $\langle w, y\rangle > b$.

We can also consider *approximate* versions of all these oracles. For the formal definitions, see [GLS 2.1]. Many of the algorithms we study in this class can be thought of as reductions between these algorithmic problems.

1.2 Relations between Oracles

There are also relations between these oracles for polar sets.

Proposition 2. *In the following, we assume $B(0, \varepsilon) \in int(K)$ and $B(0, \varepsilon) \in int(K^\circ)$.*

- *OPTimization oracle for K can be (approximately) implemented using VIOLation or VALidity oracle for K;*
- *MEMbership oracle for K is equivalent to VALidity oracle for K°; and vice-versa MEM(K°) is equivalent to VAL(K);*

- *SEParation oracle for K is equivalent to VIOLation oracle for K°; and vice-versa SEP(K°) is equivalent to VIOL(K);*

Proof:

- In the following we use VAL if we only need the optimum value, and VIOL if the optimizer is also required. Using the VAL/VIOL oracles we can check if $\exists x \in K : \langle w, x\rangle \geq b$. Therefore, to compute $h_K(w) = \sup_{x\in K}\langle w, x\rangle$ approximately, we perform binary search over b.

- We first implement MEM(K°) using VAL(K). Given input w we want to check if $w \in K^\circ$ iff $\sup_{x\in K}\langle w, x\rangle \leq 1$. For this we return the output of VAL($w, 1, K$).

  Conversely, we want to implement VAL(K) using MEM(K°). So given an input (w, b) we want to check if $\forall x \in K : \langle w, x\rangle \leq b$. First we use that $B(0, \varepsilon) \subseteq K$ so if $b \leq 0$ we output NO. Otherwise note that $v \in K^\circ$ iff $\sup_{x\in K}\langle v, x\rangle \leq 1$. So we can return the output of MEM($w/b, K^\circ$).

- We first implement SEP(K°) using VIOL(K). Given input w we want to check if $w \in K^\circ$, and if not to output a separating hyperplane. Note $w \in K^\circ$ iff $\sup_{x\in K}\langle w, x\rangle \leq 1$, so we query VIOL($w, 1, K$); if the output is YES, we return YES; if the output is NO then the oracle returns $x \in K$ such that $\langle w, x\rangle > 1 \geq \sup_{v\in K^\circ}\langle v, x\rangle$, where the last step is by definition of K°, so we can output x as our separating hyperplane for K°.

  Conversely, we want to implement VIOL(K) using SEP(K°). So given an input (w, b) we want to check if $\sup_{x\in K}\langle w, x\rangle \leq b$, and if not to output $x \in K$ such that $\langle w, x\rangle > b$. First we use that $B(0, \varepsilon) \subseteq K$ so if $b \leq 0$ we output $0 \in K$ as $\langle w, 0\rangle = 0 > b$. Otherwise we query SEP($w/b, K^\circ$); if the output is YES, we return YES; if the output is NO then the oracle returns separating hyperplane $x \in \mathbb{R}^n$ such that $\langle w, x\rangle > \sup_{v\in K^\circ}\langle v, x\rangle$, so we can output $\frac{\varepsilon x}{\|x\|_2} \in B(0, \varepsilon) \subseteq K$ as our violating point.

1.3 Function Oracles: TODO

What is the relation to function oracles? In particular, recall that we prove duality of convex functions by reducing to duality of the epigraph, which is a convex set.

Definition 3 (Fenchel dual)**.** *For convex $f : \mathbb{R}^n \to \mathbb{R}$, the Fenchel dual is*

$$f^*(w) := \sup_y \langle w, y\rangle - f(y).$$

Lemma 4. *For differentiable closed convex f, the supporting hyperplane for the epigraph of f at $(x, f(x))$ is given by slope $(\nabla f(x), -1)$ and value $f^*(\nabla f(x)) = \sup_y\langle \nabla f(x), y\rangle - f(y)$.*

Similarly, the supporting hyperplane for the sub-level set $L_{f(x)} := \{y \in \mathbb{R}^n \mid f(y) \leq f(x)\}$ is given by slope $\nabla f(x)$ and value $\langle \nabla f(x), x\rangle$.

Both these statements can be generalized to non-differentiable f using subgradients.

Proposition 5. *In the following, we assume $f : \mathbb{R}^n \to \mathbb{R}$ is a closed convex function.*

- *EVALuation oracle for f can be implemented using MEMbership oracle for epi(f);*

- *GRADient oracle for f can be implemented using SEParation oracle for epi(f);*

- *OPTimization oracle for epi(f) is equivalent to EVALuation for the Fenchel dual $f^*(w) := \sup_y\langle w, y\rangle - f(y)$.*

2 Center-of-Gravity Method

Theorem 6 (Grunbaum's Theorem)**.** *For convex $K \subseteq \mathbb{R}^n$, let $c := \mathbb{E}_{x \in K} x$ be the center-of-gravity of K. Then for any hyperplane $H \ni c$, the two halfspaces H_+, H_- satisfy*

$$\min\{vol(K \cap H_+), vol(K \cap H_-)\} \geq vol(K)/e.$$

Theorem 7. *Given $K \subseteq B(0, R) \subseteq \mathbb{R}^n$ via separation oracle, the center-of-gravity method requires $O(n \log(R/\varepsilon))$ iterations to either*

1. *find $x \in K$;*
2. *certify K does not contain a ball of radius ε*

3 Lower Bound

Theorem 8. *Given $K \subseteq B(0, R) \subseteq \mathbb{R}^n$ via separation oracle, any first-order method requires $\Omega(n \log(R/\varepsilon))$ oracle calls to either*

1. *find $x \in K$;*
2. *certify K does not contain a ball of radius ε.*

4 Ellipsoid Method [Khachiyan 1979]

Theorem 9. *Given $K \subseteq B(0, R) \subseteq \mathbb{R}^n$ via separation oracle, the Ellipsoid method requires $O(n^2 \log(R/\varepsilon))$ iterations to either*

1. *find $x \in K$;*
2. *certify K does not contain a ball of radius ε.*

Lemma 10 (Lemma 2.3 in [Bubeck)**.** *] Let $\mathcal{E} := \{x \in \mathbb{R}^n \mid \langle (x-c), Q^{-1}(x-c)\rangle \leq 1\}$ be an Ellipsoid with $Q \succ 0$; and let $H := \{x \in \mathbb{R}^n \mid \langle w, x\rangle \leq \langle w, c\rangle\}$ be a halfspace through the center of $\mathcal{E}$. Then there exists ellipsoid $\overline{\mathcal{E}}$ such that*

$$\overline{\mathcal{E}} \supset \mathcal{E} \cap H$$

such that $vol(\overline{\mathcal{E}}) \leq vol(\mathcal{E}) \exp(-1/2n)$.

Further, if $n \geq 2$ then $\overline{\mathcal{E}} := \{x \in \mathbb{R}^n \mid \langle (x - \overline{c}), \overline{Q}^{-1}(x - \overline{c})\rangle \leq 1\}$ where

$$\overline{c} := c - \frac{1}{n+1} \frac{Qw}{\sqrt{\langle w, Qw\rangle}}, \qquad \overline{Q} := \frac{n^2}{n^2 - 1}\left(Q - \frac{2}{n+1} \frac{(Qw)(Qw)^*}{\langle w, Qw\rangle}\right).$$