

# Convex Optimization: Problem Set 1

January 27, 2026

Some problems borrowed from Optimization course of Daniel Dadush.

1. Recall the definition of convex hull:

$$\text{conv}(S) := \left\{ \sum_{i=1}^N \lambda_i \mathbf{x}_i \mid \mathbf{x}_i \in S, \lambda \geq 0, \sum_{i=1}^N \lambda_i = 1 \right\}.$$

1. Prove that for compact  $S$ ,  $\text{conv}(S) = \cap_{K \supseteq S} K$  where the intersection is over all closed convex  $K$  containing  $S$ . Therefore  $\text{conv}(S)$  is the smallest convex set containing  $S$ .
2. Prove Jensen's inequality: for convex  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and input  $\mathbf{x} = \sum_{i=1}^N \lambda_i \mathbf{x}_i$  where  $\lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1$

$$f(\mathbf{x}) \leq \sum_{i=1}^N \lambda_i f(\mathbf{x}_i).$$

3. Show that if  $\mathcal{C} \subseteq \mathbb{R}^n$  is a compact convex set and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function then the **supremum** of  $f$  over  $\mathcal{C}$  is attained at an extreme point of  $\mathcal{C}$ .
2. Describe the set of boundary points for the following norm balls, i.e. points not in the interior. For each boundary point  $\mathbf{x}$  give the set of supporting hyperplanes at  $\mathbf{x}$ .

1.  $B_1^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_1 := \sum_{i=1}^n |x_i| \leq 1\}$
2.  $B_2^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^n x_i^2} \leq 1\}$
3.  $B_\infty^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty := \max_{i \in [n]} |x_i| \leq 1\}.$

3. For convex closed  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x \in \text{dom}(f)$ :

- Show for any subgradient  $g \in \partial f(x)$ ,  $(g, -1)$  gives a *supporting hyperplane* at  $(x, f(x))$  for the epigraph

$$\text{epi}(f) := \{(x, t) \mid f(x) \leq t\}.$$

- Show that  $g \in \partial f(x)$  gives a *supporting hyperplane* at  $x$  for the sub-level set

$$L := \{y \in \mathbb{R}^n \mid f(y) \leq f(x)\}.$$

4. Prove Exercise 2.27 in [BV]: Let  $K \subseteq \mathbb{R}^n$  be closed, bounded, with non-empty interior, such that there exists a supporting hyperplane of  $K$  at every point of the boundary  $x \in \partial K$ . Show this implies  $K$  is convex.
5. Affine and quadratic functions are the most basic convex functions. We will prove some properties about them:

- Show  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex and concave (i.e.  $h$  is convex and  $-h$  is convex) iff  $h$  is an affine function, i.e.  $h(x) = \langle a, x \rangle + b$  for  $a \in \mathbb{R}^n, b \in \mathbb{R}$ .
- Let  $\tilde{q}(x) := \langle x, Qx \rangle + \langle a, x \rangle + b$  for symmetric matrix  $Q \in \mathbb{R}^{n \times n}, a \in \mathbb{R}^n, b \in \mathbb{R}$ . Show  $\tilde{q}$  is convex iff  $q(x) := \langle x, Qx \rangle$  is convex. (Hint: use first part)
- Show  $q(x) = \langle x, Qx \rangle$  is convex iff

$$\forall v \in \mathbb{R}^n : \langle v, Qv \rangle \geq 0,$$

and similarly show it is strictly convex iff the inequality is strict.

(These conditions are known as positive-semi-definiteness and positive-definiteness, and are denoted  $Q \succeq 0, Q \succ 0$ .)

- Find the optimizer and optimum value of strictly convex quadratic

$$\tilde{q}(x) := \langle x, Qx \rangle + \langle a, x \rangle + b.$$

6. An Ellipsoid is an affine image of the Euclidean ball

$$\mathcal{E} = c + AB_2^n \quad \text{where} \quad B_2^n := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\},$$

for some  $c \in \mathbb{R}^n$  and  $A \in \mathbb{R}^{n \times n}$  invertible.

- Let  $q(x) := \langle x, Qx \rangle + \langle d, x \rangle + e$  be *strictly convex*. Show any sub-level set

$$L_t := \{x \in \mathbb{R}^n \mid q(x) \leq t\}$$

is either empty or an Ellipsoid (i.e. find  $c, A$  such that  $L_t = c + AB_2$ ).

- Conversely, given Ellipsoid  $\mathcal{E} = c + AB_2$  as above, find convex quadratic  $q(x) := \langle x, Qx \rangle + \langle d, x \rangle + e$  such that

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid q(x) \leq 1\}.$$

7. Recall the GLS oracle model for convex sets. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex closed function with  $\text{dom}(f) = \mathbb{R}^n$ . We investigate natural function oracles in terms of the epigraph:

- Show  $\text{MEM}(\text{epi}(f))$  can be implemented using an EVALuation oracle for  $f$ ; show  $\text{EVAL}(f)$  can be approximately implemented using  $\text{MEM}(\text{epi}(f))$ , i.e. for input  $x \in \mathbb{R}^n$  compute  $t \in \mathbb{R}$  such that  $|t - f(x)| \leq \varepsilon$ .
- Show  $\text{SEP}(\text{epi}(f))$  can be implemented using an GRADient (or subgradient) oracle for  $f$ ; show  $\text{GRAD}(f)$  can be approximately implemented using  $\text{SEP}(\text{epi}(f))$ , i.e. for input  $x \in \mathbb{R}^n$  compute  $g \in \mathbb{R}^n$  such that

$$\forall y \in \mathbb{R}^n : f(y) \geq f(x) - \varepsilon + \langle g, y - x \rangle.$$