Lab 03: Spring-mass Oscillations

Written by:
Andrew Zhang
Maya L. A. Rehem

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Abstract

We performed experiments of recording the oscillations of two vertical spring-mass systems: undamped and damped. The undamped spring-mass experiment had a period of $T = 0.72 \pm 1 \times 10^{-3}$ and from this was able to calculate the angular frequency $\Omega_0 = 8.7 \pm 0.01\, rad\, s^{-1}$ and spring constant $k = 15 \pm 0.008\, rad\, s^{-1}\, kg$. Similarly, for the damped mass we found $T = 0.76 \pm 0.001\, sec$ and an angular frequency of $\Omega_0 = 8.3 \pm 0.01\, rad\, s^{-1}$. The decay constant we took was $\gamma = 0.027$. We modeled the undamped mass with both the forward and symplectic Euler formulas. We found that the symplectic method did the best at modeling the experimental value as it was able to take into account energy conservation. We then proceeded to model the damped mass with the symplectic simulation and also found that it was able to display similar mechanical energy decay trends.

1 Introduction

A spring mass system will follow Hooke's law: $F_{spring} =- ky$. Where $y$is the displacement from the position of equilibrium and $k$is the spring constant. (**Q1**) Using this and Newton's 2nd law we find

$$
\begin{aligned}
F_{spring} = ma = -ky \\
ma + ky = 0 \\
m\frac{d^2y}{dt^2} + ky = 0 \\
\frac{d^2y}{dt^2} + \frac{k}{m}y = 0
\end{aligned} \tag{1}
$$

Assuming we have no dampening force. For a vertical spring mass system, where , we have that

$$\Omega_0 = \sqrt{\frac{k}{m}} \tag{2}$$

And thus we have that

$$\frac{dy}{dt} = v \qquad\qquad \frac{dv}{dt} = -\Omega_0^2 y$$

(**Q2**) Since both the displacement and velocity, as functions of time, are continuous we can say that for small [delta t] we have that

$$y(t + \Delta t) = y(t) + \Delta t v(t) \qquad\qquad v(t + \Delta t) = v(t) + \Delta t a(t)$$

Thus, letting $y_i = y(t_i)$and $y_{i+1} = y(t_i + \Delta t)$ and similar relationships for v we have that, using equation (1) and the forward Euler's method,

$$
\begin{aligned}
y_{i+1} = y(t_{i+1}) &= y(t_i + \Delta t) \\
&= y(t_i) + \Delta t v(t_i) \\
&= y_i + \Delta t v_i
\end{aligned}
$$

$$
\begin{aligned}
v_{i+1} = v(t_{i+1}) &= v(t_i + \Delta t) \\
&= v(t_i) + \Delta t a(t_i) \\
&= v(t_i) + \Delta t \frac{dv}{dt}(t_i) \\
&= v(t_i) - \Delta t \Omega_0^2 y(t_i) \\
&= v_i - \Delta t \Omega_0^2 y_i
\end{aligned}
$$

The equations above will be the base of the forward Euler simulation for the undamped mass-spring experiment.

However, we may also view this system not simply kinematically, but also through the lance of energy. For a spring-mass system that is not damped we expect that the mechanical energy of the system will be conserved. In other words,

$$\begin{aligned} E_{tot} &= K(\dot{y}) + U(y) \\ &= \frac{1}{2}mv^2 + \frac{1}{2}ky^2 \end{aligned}$$

With this in mind we will also use the Symplectic Euler method to simulate the spring-mass system that remains stable in regards to energy. The equations for such method are as follows

$$\begin{aligned} y[i+1] &= y[i] + \Delta t\, v[i] \\ v[i+1] &= v[i] - \Delta t\, \frac{k}{m} y[i+1] \end{aligned}$$

We know however that in the real world we cannot conduct an experiment that will perfectly conserve energy. In the case of our spring-mass, there exist a damping for that acts in the opposite direction of the velocity and is proportional on the density of the medium the mass is traveling through, the cross-section of the mass perpendicular to the flow, and the velocity.

However this force is also very unique to the system at hand via the Reynolds number. In Laminar flow the Reynolds number is less than 2300 and the drag is directly proportional to velocity. However, in turbulent mediums the number can be more than 4000 and the drag is then proportional to the square of the velocity.

(**Q9**) We expect that the Reynold number of our set up to be in the "small" range (i.e. < 2300) as we intentionally kept the amplitude of the set up small so that the movement would be controlled and this would ideally constrain turbulence that would otherwise significantly change our Reynold number.

2 Materials and methods

2.1 Materials

- Mass/bob ($0.2 \pm 1 \times 10^{-4}\, kg$)
- Digital scale (uncertainty: $1 \times 10^{-4}\, kg$)
- Spring
- Motion sensor (uncertainty: $\pm\ 0.239\, cm$)
- Data acquisition device
- Computer with LabView application
- Dampening disk
    - Bob + disk mass: $0.2011 \pm 1 \times 10^{-4}\, kg$

2.2 Methods

2.2.1 Undamped spring-mass (mechanical energy conservation)

1. Measure the bob of the mass using a digital scale.
2. Place the mass attached to the string 20 cm above the motion sensor.

3. With the application set up to measure position, gently pull the bob down approximately 4 cm and release.
4. Begin collecting data for about 10 seconds and then stop the program.
5. On the analyze panel place 5 oscillations (measured from peak to peak) between the yellow cursors. The time elapsed between the cursors divided by the number of oscillations was taken to be the period.

2.2.2 Damped spring-mass

1. We attached the dampen disk to the bottom of the bob and then remeasured the mass.
2. Repeat the method from 2.2.1 with the disk and bob, but in step 4 collect data for 2 minutes instead of 10 seconds.

3 Data and analysis

3.1 Undamped mass-spring

The mass of the bob was found to be $0.2 \pm 1 \times 10^{-4}\, kg$. The uncertainty of the mass was taken to be the smallest increment the digital scale went down to. The following is data from the undamped mass-spring set up. The measurement errors of displacement of the mass was taken to be the uncertainty given by the measuring apparatus ($\pm\ 0.239\, cm$). We found the period to be $T = 0.72\ \pm 1 \times 10^{-3}$. The uncertainty of the period was similarly taken to be the smallest increment of the value the program displayed. Thus we can calculate the value of the angular frequency to be

$$\begin{aligned}\Omega_0 &= \frac{2\pi}{T} \\ &= \frac{2\pi\, rad}{0.72 \pm 1 \times 10^{-3}\, sec} \\ &= 8.7 \pm 0.01\, \frac{rad}{sec}\end{aligned}$$

(**Q3**) We may rearrange equation (2) in order to find the spring constant as follows

$$\begin{aligned}\sqrt{\frac{k}{m}} &= \Omega_0 \\ \frac{k}{m} &= \Omega_0^2 \\ k &= \Omega_0^2 m \\ &= \left(8.7 \pm 0.01\, \frac{rad}{sec}\right)^2 (0.2 \pm 1 \times 10^{-4}\, kg) \\ &= 15 \pm .008\, \frac{rad}{sec} kg\end{aligned}$$

We used both forward and symplectic Euler equations in order to model and simulate the oscillatory motion of the mass-spring system. Figure 1 below displays the data and Euler modeled simulations plotted as displacement and velocity as functions of time. The measured data is indeed plotted with error bars, however the error is so small relative to the scale of the vertical axis that they are not easily visible on the plots below.

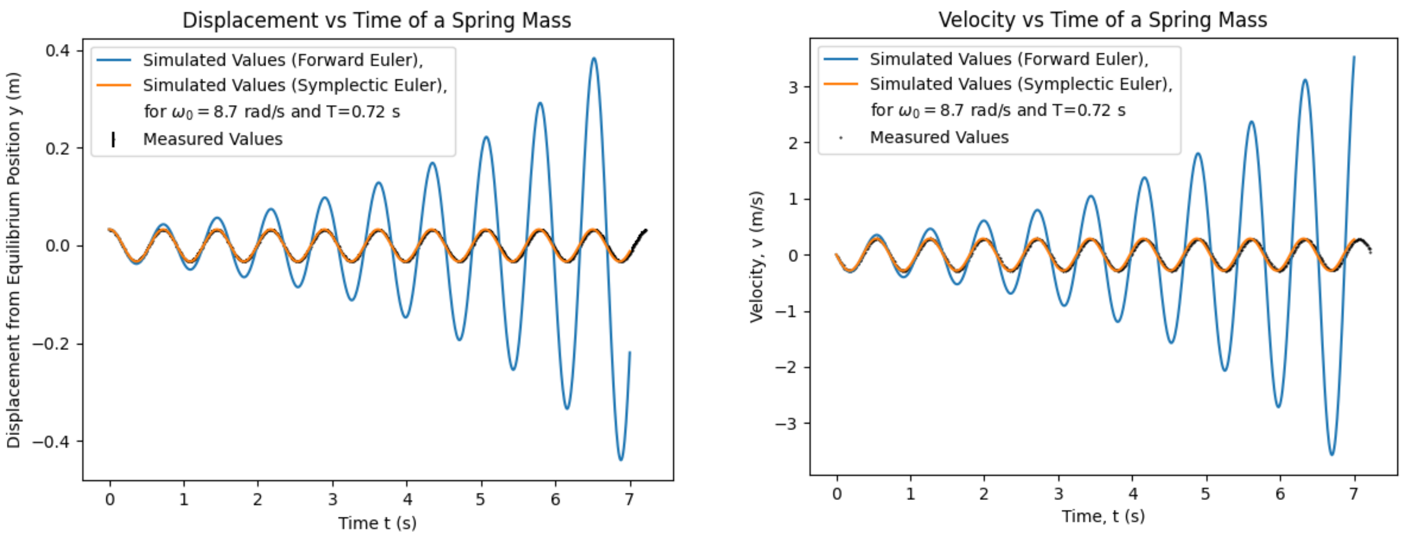

Figure 1: (left) displacement of the undamped mass as a function of time with the forward and symplectic Euler models. (right) velocity of undamped mass as a function of time with forward and symplectic models.

(Q4) From Figure 1 above, we can see that the Forward Euler method (blue graph) does a reasonable job modeling the data early in the experiment. However, as more time went on we see that the forward model amplitude begins to grow while the data's amplitude appears to stay relatively constant. This simulation would suggest that the mass-spring system is gaining velocity and traveling a farther displacement for each cycle. This is something we know we shouldn't physically expect from a system that obeys energy conservation. Anything past about 1 second and it is clear that the forward Euler does a poor job properly simulating the physical motion of the measured values and this deviation from the measured data becomes more stark as time elapses.

Figure 2 also provides us with the phase plots of our system I.e. the velocity vs. position of the measured data and both simulations. Figure 3 displays this discrepancy between energy conservation between the forward simulation and both the symplectic simulation and data. From these plots it's clear that the forward Euler simulation will quite literally "spiral" out of control while the symplectic simulation remains a closed ellipse. (**Q6**&**8**) The reason why the symplectic simulation and the data appear to be an ellipse is because of energy conservation. In this system where mechanical energy is being approximately conserved the velocity at any given position must be that same velocity when it returns to that position. Else we have a system with less kinetic energy, but the same potential energy and thus the mechanical energy is not conserved. Thus the phase plot manifests as a closed ellipse.

From figure 3, we see that the actual data's energy appears constant with time, supporting that this system indeed approximately conserves mechanical energy. (**Q8**) The symplectic simulation's energy similarly remains constant with time as one would expect as the model should account for the mechanical energy conservation. However it is the forward Euler simulation that displays the most obvious difference. (**Q5**) We see that the energy of a system simulated with the forward method gains energy exponentially with time which explains the unexpected behaviour displayed in figure 1 and 2. Thus, the symplectic Euler simulation, not surprisingly, appears to do a significantly better job at modeling the physical data.

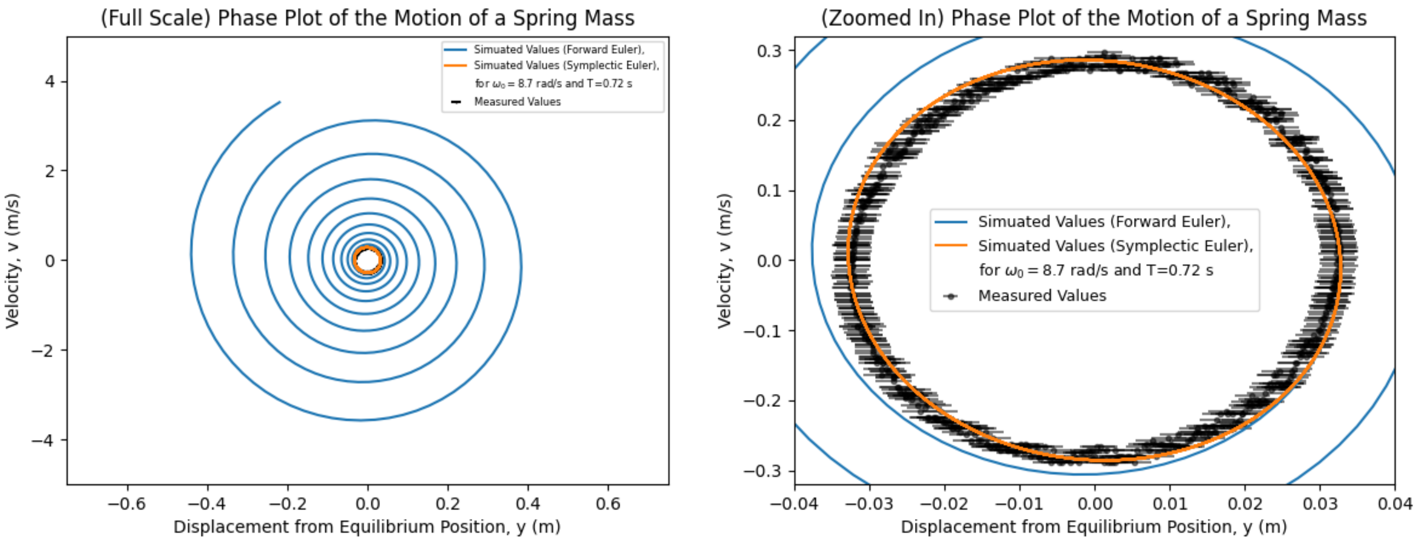

Figure 2: (left) Velocity versus displacement of the undamped mass-spring with forward and symplectic models. (right) left plot zoomed in about 15x

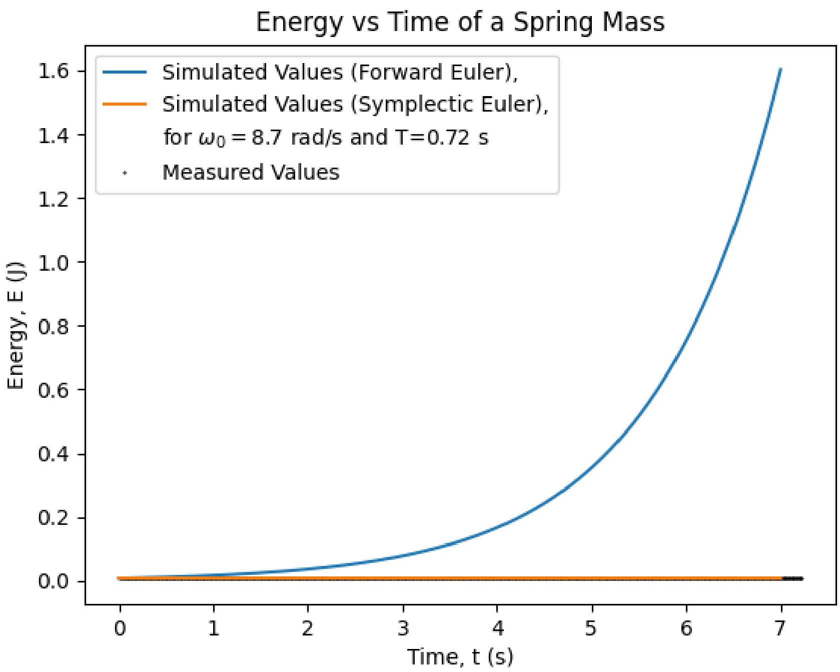

Figure 3: Energy as a function of time for the measured data, forward and symplectic Euler.

3.2 Damped mass-spring

The next following analysis is on a damped mass-spring system. We found that the new period of oscillation was $T = 0.76 \pm 0.001\ sec$, where the uncertainty was taken as it was taken in 3.1. Thus, using the same previous calculation methods we find that $\Omega_0 = 8.3 \pm 0.01\ rad\ s^{-1}$. Assuming an exponential decay of the oscillation envelope, we estimated that the decay constant was $\gamma = 0.027$.

Below is the the measured data plotted along with the symplectic Euler simulation and the amplitude envelope from our estimated value of γ. Again, the uncertainty/error of the displacement is small on the graph and thus are not easily visible.

As well, we have plotted the mechanical energy of the system as a function of time. Figure 5 also includes the energy from the symplectic Euler simulation along with the decay envelope.

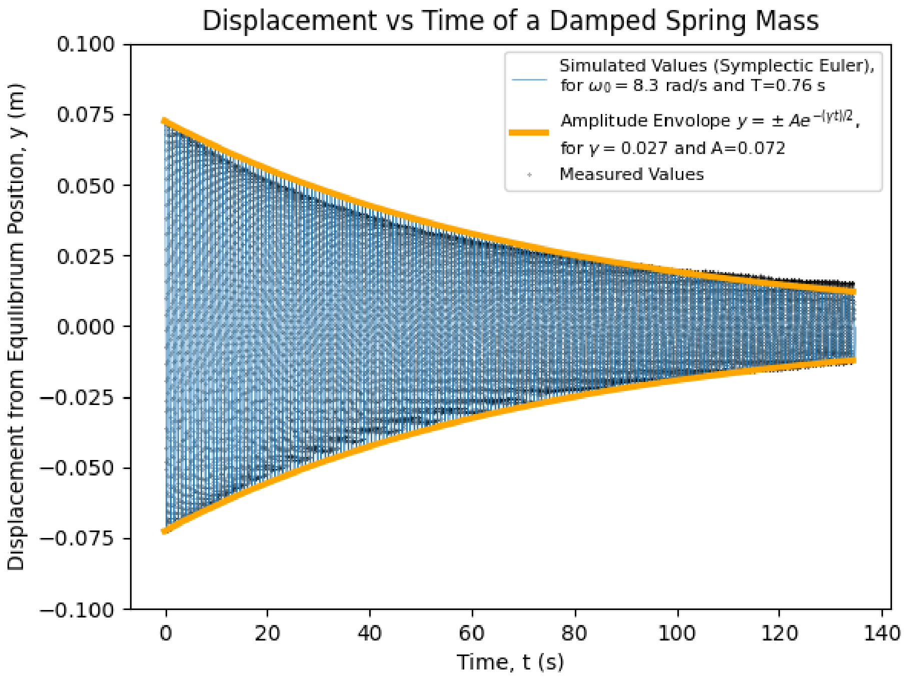

Figure 4: displacement measured data for the entire duration of the experiment plotted with the symplectic Euler simulation (blue) along with the approximated decay envelope (orange) as a function of time.

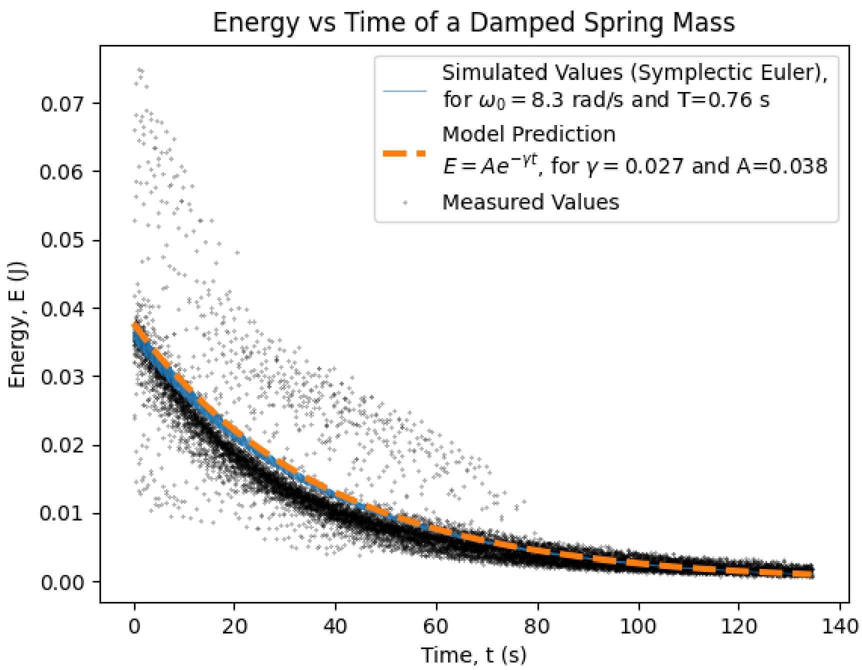

Figure 5: The mechanical energy of the measured data plotted along with the mechanical energy of the Symplectic Euler simulation along with the decay envelope as a function of time for the entire duration of the experiment.

(**Q10**) Figure 5 suggests that in a system with a damped spring-mass set up mechanical energy is not conserved. We can see that the total mechanical energy of the system decreases with time quickly in the beginning of the experiment and closer to the end the energy begins to plateau with time. We also see that the simulation and decay envelope also display similar decay trends.

(**Q11**) From looking at figure 4, we see the experimental data shows a slightly faster drop in amplitude in comparison to our simulation. However, the displacement data later in the experiment (approximately after 70 seconds) begins to match the simulation again. A similar discrepancy can be seen in figure 5, where the experimental data appears to have lost energy more rapidly in the beginning of the data being recorded before flatting and coming to match the simulation (more discussion in section 4).

4 Discussion and conclusion

We found that the undamped spring-mass experiment had a period of $T = 0.72 \pm 1 \times 10^{-3}$ and from this was able to calculate the angular frequency $\Omega_0 = 8.7 \pm 0.01\, rad\, s^{-1}$ and spring constant $k = 15 \pm 0.008\, rad\, s^{-1}\, kg$.

We attempted to simulate the oscillating behaviour with both the Forward Euler procedure and the Symplectic Euler procedure. We found that the Symplectic Euler procedure performed the best compared to the forward procedure at modeling the physical behaviour. This is because the Symplectic procedure took into account the conservation of mechanical energy while the forward procedure clearly displayed 'unstable' behaviour as the energy of the simulated system would have grown exponentially.

We then proceeded to conduct the same experiment on a damped mass. Only the symplectic simulation was used to model this data. As the damped mass-spring is being acted upon by an additional drag force that works against the velocity we see the mass begin to decelerate. Thus the average kinetic energy of the system decreases with time and along with that so does the total mechanical energy of the system as the mechanical energy is being converted to thermal. We found a period of $T = 0.76 \pm 0.001\, sec$ and an angular frequency of $\Omega_0 = 8.3 \pm 0.01\, rad\, s^{-1}$. The decay constant we took was $\gamma = 0.027$.

The symplectic simulation was able to display a similar decay trend in amplitude of displacement and thus also decay in energy over time. We notice however that the experimental data appears to initially decay at a more rapid rate than our simulated system before converging again later when the energy loss begins to flatten with respect to time. One possible reason for this minor discrepancy is that the symplectic system assumes lamor flow and thus a small Reynold number that dictates the drag force.

However, the fluid that our mass moves through (the air of the lab room) is an inherently chaotic environment. Thus there was possible minor turbulence that would have possibly increased the Reynold number. We note that our amplitude for the damped mass was larger than that of the undamped mass--possibly contributing to a less laminar flow. In which case we would expect that the drag is no longer directionally proportional to the velocity, but instead would have a greater contribution to the drag and thus dissipate the energy/slow the mass down slightly faster in the beginning.