Lecture 2: two-qubit system

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In the last chapter, we have learned about one qubit system. Here, we shall learn about a two-qubit system. It turns out two-qubit system has much more fun. It will take three weeks for us to cover these materials. Guess how many qubits we can cover by the end of the course?

Two-qubit state, mixed state, Bloch "ball", positive-operator valued measure (POVM); Super operator, purification of mixed state and POVM; Bell's inequality, CHSH/CH/Eberhard inequality; Experiment development and loopholes, entanglement; Quantum dense coding, teleportation (experiment development); Using teleportation for operation (Gottesman-Chuang'99), remote state preparation; Quantum dense coding, teleportation (experiment development); Using teleportation for operation (Gottesman-Chuang'99), remote state preparation

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I.   REVIEW: ONE-QUBIT SYSTEM

1. state: ray.

2. normalize state: vector.

3. representation way: $|u\rangle = \cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}$. (Bloch Sphere[1])

4. density matrix: $\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$.[2]

5. $\rho^+ = \rho, Tr(\rho) = 1$.

6. $\forall|\phi\rangle, \langle\phi|\rho|\phi\rangle \geq 0$.[3]

7. For pure qubit, Assume $\rho = |\phi\rangle\langle\phi|$, we have $\rho^2 = \rho$. Also, we can get the eigenvalues and eigenvectors of ρ (Assume $|\Phi\rangle$ is orthogonal of $|\phi\rangle$):

$$\begin{aligned} \rho|\phi\rangle &= |\phi\rangle\langle\phi|\phi\rangle = |\phi\rangle. \\ \rho|\Phi\rangle &= |\phi\rangle\langle\phi|\Phi\rangle = 0. \end{aligned} \tag{1}$$

8. $\langle\psi|M|\psi\rangle = Tr(\langle\psi|M|\psi\rangle) = Tr(M|\psi\rangle\langle\psi|) = Tr(M\rho)$.[4]

II.   NOTATIONS

Matrix tensor product $\otimes$, the tensor product of two matrices is

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,2}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \\ a_{2,1}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{2,2}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}. \tag{2}$$

Here, we simplify denote it as

$$|\phi\rangle \otimes |\psi\rangle = |\phi\rangle|\psi\rangle \tag{3}$$

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A.   Trace

Trace $Tr(\rho)$: summation of the diagonal terms.

Partial trace $Tr_B(\rho_{AB})$: Assume A is n-dimension state, B is m-dimension state, then ρ_{AB} can be regarded as a $nm \times nm$ matrix. $Tr_B(\rho_{AB})$ is a $n \times n$ matrix, where $(Tr_B(\rho_{AB}))_{ij} = \sum_{k=1}^{m}(\rho_{AB})_{ik,jk}$.s Particularly, $Tr_B(A \otimes B) = Tr(B)A$.

B.   Schmidt decomposition

For any pure state $|\psi\rangle_{AB}$ of a bipartite system, there are orthonormal bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ such that:

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}|i\rangle_A \otimes |i'\rangle_B. \tag{5}$$

The subsystems A and B have the same eigenvalues, p_is. The number of p_is is called the Schmidt number of $|\psi\rangle_{AB}$. We denote that the pure state is a entangled state when the the Schmidt number is greater than one. It is easy to see that the Bell states are entangled states.

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[1] No standard way to visualize two or more qubits system until now.
[2] why we can't distinguish $|u\rangle$ and $e^{i\theta}|u\rangle$? because $|u\rangle\langle u| = e^{i\theta}|u\rangle\langle u|e^{-i\theta}$. Their density matrices are always the same.
[3] By physical meaning of the measurement
[4] $Tr(AB) = \sum_i \sum_j a_{ij}b_{ji} = \sum_i \sum_j b_{ij}a_{ji} = Tr(BA)$.

III. TWO-QUBIT SYSTEM

A. an interesting "paradox"

When considering a two qubits state which is written as

$$|\psi\rangle_{AB} = |0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B. \tag{6}$$

When measure the system A in the Z basis, with probability $1/2$, the measurement result is $|0\rangle$ and the prepared state is $|0\rangle_A|0\rangle_B$. With probability $1/2$, the measurement result is $|1\rangle$ and the prepared state is $|1\rangle_A|1\rangle_B$.

That is:

$$|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle.$$

Consider another two qubits state :

$$|\psi\rangle_{AB} = |+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B. \tag{7}$$

By similar induction way, we have

$$|\psi\rangle_A = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = |0\rangle.$$

But actually,

$$|\psi\rangle_{AB} = |0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B \tag{8}$$
$$= 1/2(|+\rangle + |-\rangle)(|+\rangle + |-\rangle) + 1/2(|+\rangle - |-\rangle)(|+\rangle - |-\rangle) \tag{9}$$
$$= |+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B. \tag{10}$$

So now we have: $|0\rangle = |+\rangle$. What's wrong?

Explanation: The key of the above paradox is that $|\psi\rangle_A$ isn't a pure state any more, we only can write it as a density matrix. The density operator of subsystem A is given by

$$\rho_A = tr_B(|\psi\rangle_{AB}\langle\psi|_{AB}) = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 1|). \tag{11}$$

B. EPR paradox

Alice and Bob have a two qubits system:

$$|\psi\rangle_{AB} = |0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B.$$

Alice gets qubit A, Bob gets qubit B. Then Alice goes to a planet which far from Bob. Then if Alice wants to tell 0 to B, she measures A in $|+/-\rangle$ basis; if she wants to tell 1 to B, she measures A with $|0/1\rangle$ basis.

WOLG, We assume that after measurement for A, Alice gets $|0\rangle$. Then for Bob, he measures B after a while with basis $|0/1\rangle$, if he gets $|0\rangle$ with probability 1, he can say Alice wants to tell him 0.

So by this process, we transform information which is faster than light. What's the problem?

Explanation: For Bob, if he gets $|0\rangle$, he doesn't know which basis Alice uses. Because if Alice chooses $|0/1\rangle$ basis, Bob has 0.5 probability to get $|0\rangle$; if Alice chooses $|+/-\rangle$ basis, Bob also has 0.5 probability to get $|0\rangle$.

This experiments tells us, quantum has the **non-locality** and also **no-signalling** properties.

Something are changed, but we don't know.

C. The properties for general density matrix

In general, the state is represented by a density operator. In the case where the state of the subsystem is a ray, and we say that the state is pure. Otherwise the state is mixed. If $\rho_A = \rho_A^2$, then $|\psi_A\rangle$ is a pure state. Otherwise, the density matrix of A is $\rho_A = \sum_a p_a|a\rangle\langle a|$, where $\sum_a p_a = 1$, $0 < p_a < 1$. The trace distance $tr\rho_A^2 = \sum_a p_a^2 < \sum_a p_a = 1$.

Properties of a general density matrix

1. self-adjoint: $\rho_A = \rho_A^\dagger$

2. positivity: $\forall |\psi\rangle$, $\langle \psi | \rho_A | \psi \rangle \geq 0$

3. completeness: $Tr(\rho_A) = 1$

D. Schmidt decomposition

For any pure state $|\psi\rangle_{AB}$ of a bipartite system, there are orthonormal bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ such that:

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |i\rangle_A \otimes |i'\rangle_B. \tag{12}$$

The subsystems A and B have the same eigenvalues, p_is. The number of p_is is called the Schmidt number of $|\psi\rangle_{AB}$. We denote that the pure state is a entangled state when the the Schmidt number is greater than one. It is easy to see that the Bell states are entangled states.

E. Bell basis

The dimension of the 2-qubit Hilbert space is 4. Thus, there are 4 basis states. Bell state basis is widely used, especially for the case involving entanglement. The 4 Bell states in the Z basis are,

$$\begin{aligned}
\Phi^+ &= |00\rangle + |11\rangle \\
\Phi^- &= |00\rangle - |11\rangle \\
\Psi^+ &= |01\rangle + |10\rangle \\
\Psi^- &= |01\rangle - |10\rangle
\end{aligned} \tag{13}$$

in the X basis are

$$\begin{aligned}
\Phi^+ &= |++\rangle + |--\rangle \\
\Phi^- &= |-+\rangle + |+-\rangle \\
\Psi^+ &= |++\rangle - |--\rangle \\
\Psi^- &= |-+\rangle - |+-\rangle
\end{aligned} \tag{14}$$

in the Y basis are

$$\begin{aligned}
\Phi^+ &= |+i-i\rangle + |-i+i\rangle \\
\Phi^- &= |+i+i\rangle + |-i-i\rangle \\
\Psi^+ &= -i(|+i+i\rangle - |-i-i\rangle) \\
\Psi^- &= i(|+i-i\rangle - |-i+i\rangle)
\end{aligned} \tag{15}$$

Many interesting simple quantum information phenomenons come with Bell states, such as Bell's inequality, Teleportation, super dense coding, quantum key distribution, and Deutsch's algorithm.

F. Bloch sphere and qubit tomography

A useful representation of the state of a single qubit is the Bloch sphere representation. Since the overall phase is irrelevant, a pure state of a qubit can be written as

$$|u\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi} \sin\frac{\theta}{2}|1\rangle. \tag{16}$$

Therefore, it is convenient to represent it as a vector living the surface of a unit sphere with the spherical coordinate $(r = 1, \theta, \varphi)$.
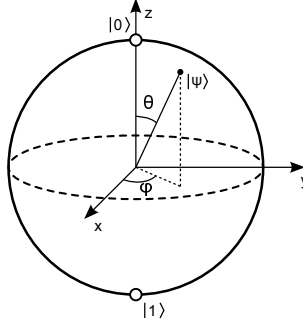
FIG. 1. Bloch sphere.

In general, a qubit might be in a mixed state, and then it will be within the Bloch sphere instead on the surface. In general, the density matrix of a qubit can be written as

$$\rho = \frac{1}{2}(\sigma_0 + \vec{P} \cdot \vec{\sigma})$$
$$= \frac{1}{2}\begin{pmatrix} 1 + P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix} \tag{17}$$

where $\vec{P} = (P_x, P_y, P_z)$ is a vector and $|\vec{P}| \leq 1$. When $|\vec{P}| = 1$, Eq. (17) represents a pure qubit.

Quantum state tomography is the process of reconstructing the quantum state for a quantum system by proper measurements. The quantum state can be pure (vector) or in general mixed (density matrix). A set of measurements is called tomographically complete if it can uniquely identify the state. That is, the measurement outcomes are able to provide all the information about the state. In the classical physics, it corresponds to system calibration.

Let us take qubit tomography for example. As given in Eq. (17), the Z basis measure provides the information on P_z. By changing the Z and X basis, by the Hadamard transformation, we can conclude that the X basis measure provides the information on P_x. Similarly, the Y basis measure provides the information on P_y. Thus, X, Y, and Z measurements is tomographically complete for a qubit. Another way to put this is,

$$\rho = \frac{1}{2}[tr(\rho)\sigma_0 + tr(\sigma_x\rho)\sigma_x + tr(\sigma_y\rho)\sigma_y + tr(\sigma_z\rho)\sigma_z]. \tag{18}$$

Now, let us move a bit further, tomography for n qubits,

$$\rho = 2^{-n} \sum_{v_1,v_2,\ldots,v_n \in \{0,x,y,z\}} tr(\sigma_{v1} \otimes \sigma_{v2} \otimes \cdots \otimes \sigma_{vn}\rho)\sigma_{v1} \otimes \sigma_{v2} \otimes \cdots \otimes \sigma_{vn}, \tag{19}$$

where the sum is over all possible the identity and Pauli matrices.

In another related concept, quantum process tomography, known quantum states are used to probe a quantum process to find out how the process can be described. Similarly, quantum measurement tomography works to find out what measurement is being performed.

The general principle behind quantum state tomography is that by repeatedly performing many different measurements on quantum systems described by identical density matrices, frequency counts can be used to infer probabilities, and these probabilities are combined with Born's rule to determine a density matrix which fits the best with the observations.

IV. POSITIVE OPERATOR-VALUED MEASURE

For the bipartite state $|\psi\rangle_{AB}$ given in Eq. (??), if we only measure system A, the observable can be expressed as

$$M_A \otimes I_B \tag{20}$$

where M_A is a self-adjoint operator acting on system A. Then the expectation value of the measurement outcome is,

$$\langle\psi|M_A \otimes I_B|\psi\rangle = (a^*\langle 0|_A\langle 0|_B + b^*\langle 1|_A\langle 1|_B)(M_A \otimes I_B)(a|0\rangle_A|0\rangle_B + b|1\rangle_A|1\rangle_B)$$
$$= |a|^2\langle 0|M_A|0\rangle + |b|^2\langle 1|M_A|1\rangle. \tag{21}$$

Since we know that the density matrix of $\rho_A = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|$, Eq. (21) can be written as,

$$\langle M_A \rangle = tr(M_A \rho_A) \tag{22}$$

We can understands the difference between the Positive Operator-Valued Measure (POVM) and PVM with the same sense that a density matrix is to a pure state. The POVM can be used to describe the effect of PVM acts on a large system. The operators $\{E_a\}$ form a complete set of Hermitian nonnegative operators and satisfy that:

1. Hermiticity: $E_a = E_a^\dagger$.

2. Positivity : $\langle \psi|E_a|\psi \rangle \geq 0$.

3. Completeness: $\sum_a E_a = I$.

V. QUANTUM CHANNEL

The quantum channel is also called "super operator". The super means that the map takes operators to operators, rather than vectors to vectors. Unitary evolution on $\mathcal{H}_A \otimes \mathcal{H}_B$ will not in general appear to be unitary if we restrict our attention to $\mathcal{H}_A$ alone. Rather, evolution in HA will be described by a quantum channel, (which can be inverted by another channel only if unitary). A general channel $\mathcal{E}$ has an operator-sum representation:

$$\mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger,$$
$$\sum_a M_a^\dagger M_a = I. \tag{23}$$

VI. ENSEMBLES

A mixed state of a system A can be prepared as an ensemble of pure states in many different ways, all of which are experimentally indistinguishable if we observe system A alone.

$$\rho_A = \sum_i p_i |\psi_i\rangle\langle \psi_i|, \sum p_i = 1. \tag{24}$$

For any such ρ_A, we can construct a purification of ρ_A, $|\Psi_1\rangle_{AB}$,

$$|\Psi_1\rangle_{AB} = \sum_i \sqrt{p_i}|\psi_i\rangle_A |\alpha_i\rangle_B, \tag{25}$$

where $\{|\alpha_i\rangle_B\}$ are mutually orthogonal and normalized. The relation between two purifications, $|\Psi_1\rangle_{AB}$ and $|\Psi_2\rangle_{AB}$, is given by,

$$|\Psi_1\rangle_{AB} = (I_A \otimes U_B)|\Psi_2\rangle_{AB}, \tag{26}$$

the two states differ by an unitary change of basis acting in $\mathcal{H}_B$ alone.

VII. QUANTUM TELEPORTATION

The seminal work by C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters is published in 1993 [?]. Quantum teleportation is a process by which quantum information can be transmitted (exactly, in principle) from one location to another, with the help of classical communication and previously shared quantum entanglement between the sending and receiving location. Because it depends on classical communication, which can proceed no faster than the speed of light, it cannot be used for superluminal transport or communication of classical bits. It also cannot be used to make copies of a system, as this violates the no-cloning theorem.

Quantum teleportation can be regarded as a secure way to transfer information [?].
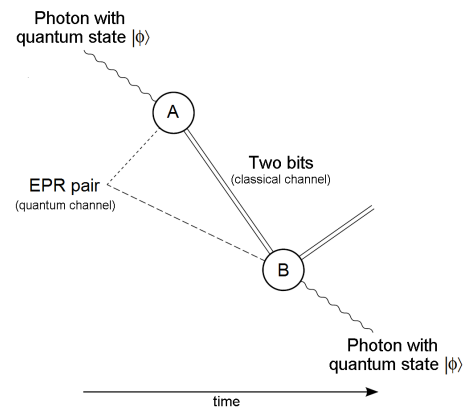
[1] Wiki, "https://en.wikipedia.org/wiki/epr_paradox,".

FIG. 2. Schematic diagram for quantum teleportation.