Lecture 1: one-qubit system

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In this chapter, we shall learn about the simplest system in quantum information — qubit. A qubit can be represented by a quantum state, processed by operators, and read out by measurement. Here, we focus on one qubit system: **pure state**, **projection-valued measure (PVM)**, and **unitary transformation**. We shall also cover some related concepts: **Born's law**, **density matrix**, **Bloch sphere**, **Pauli matrices**, **quantum no-cloning theorem**, **no-deletion theorem**.

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I.   NOTATIONS AND DEFINITIONS

A **state** is a complete description of a physical system.

A.   Dirac's bra-ket notations

1. Ket: $|\psi\rangle$ denotes a quantum state ψ, $|0\rangle$ denotes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle$ denotes $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$|\psi\rangle = a|0\rangle + b|1\rangle, \tag{1}$$

   where $a, b \in \mathcal{C}$.

2. Bra: $\langle\psi|$ denotes $\psi^\dagger$, the Hermitian conjugate of $|\psi\rangle$, $\langle 0|$ denotes $(1, 0)$ and $\langle 1|$ denotes $(0, 1)$.

$$\psi^\dagger = \begin{pmatrix} a^* & b^* \end{pmatrix} = a^* \begin{pmatrix} 1 & 0 \end{pmatrix} + b^* \begin{pmatrix} 0 & 1 \end{pmatrix}$$
$$\langle\psi| = a^*\langle 0| + b^*\langle 1|, \tag{2}$$

3. Inner product of two quantum states

$$\langle\psi_1|\psi_2\rangle = \begin{pmatrix} a_1^* & b_1^* \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = a_1^* a_2 + b_1^* b_2. \tag{3}$$

B.   Hilbert space

Quantum states form a Hilbert space $\mathcal{H}$. A Hilbert space is a vector space (with an inner product) over the complex numbers $\mathcal{C}$. Vectors will be denoted $|\psi\rangle$ (Dirac's bra-ket notation). In quantum mechanics, a state is a ray in a Hilbert space. A Hilbert space has the properties:

i) Positivity: $\langle\psi|\psi\rangle > 0$ for $|\psi\rangle \neq 0$.

ii) Linearity: $\langle\phi|(a|\psi_1\rangle + b|\psi_2\rangle) = a\langle\phi|\psi_1\rangle + b\langle\phi|\psi_2\rangle$.

iii) Skew symmetry: $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$.

C.   ray

· It is an equivalence class of vectors that differ by multiplication by a nonzero complex scalar.

· For any nonzero ray, we can by convention choose a representative of the class, denoted $|\psi\rangle$, that has unit norm:

$$\langle\psi|\psi\rangle = 1. \tag{4}$$

· Thus states correspond to normalized vectors, and the overall phase of the vector has no physical significance:$|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ describe the same state, where $|e^{i\alpha}| = 1$.

· Since every ray corresponds to a possible state, given two states $|\phi\rangle$, $|\psi\rangle$, another state can be constructed as the linear superposition of the two, $a|\phi\rangle + b|\psi\rangle$. The relative phase in this superposition is physically significant.

· We identify $a|\phi\rangle + b|\psi\rangle$ with $e^{i\alpha}(a|\phi\rangle + b|\psi\rangle)$ but not with $a|\phi\rangle + e^{i\alpha}b|\psi\rangle$.

II.   A PURE QUBIT

A bit in information theory can be represented by 0/1, which could be used to represent two states of a capacitor — high/low voltage. We assume these two states 0/1 can be unambiguously distinguished.

A qubit is an analogy of bit. Similarly, a qubit represents the state of a two-level system by a ket $|0\rangle$ and $|1\rangle$. The difference is that a qubit can be prepared in a superposition of the two states $|0\rangle$ and $|1\rangle$, which is indicated as a linear combination of the two states. If the two states $|0\rangle$ and $|1\rangle$ can be treated as two orthogonal vectors,

$$|\phi\rangle = a|0\rangle + b|1\rangle, \tag{5}$$

where $a, b \in \mathcal{C}$. Notes on the state representation:

1. Born's probability interpretation: if one measures[1] the state Eq. (5), there is a probability $|a|^2/(|a|^2 + |b|^2)$ to get 0 and a probability $|b|^2/(|a|^2 + |b|^2)$ to get 1.

2. From Born's law, the overall factor in Eq. (5), $|a|^2 + |b|^2$, does not affect the state. One can pick $|a|^2 + |b|^2 = 1$ as normalized state representation.

3. Strictly speaking, $|\phi\rangle$ represents a *ray* instead of a vector, same for $|0\rangle$ and $|1\rangle$.

4. The outcome is intrinsically random according Born's law. Einstein, "God does not play dice with the universe". In fact, we can experimentally verify this[2]!

Define density matrix of a quantum state,

$$\begin{aligned}
\rho &= |\phi\rangle\langle\phi| \\
&= (a|0\rangle + b|1\rangle)(a^*\langle 0| + b^*\langle 1|) \\
&= |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1| + ab^*|0\rangle\langle 1| + a^*b|1\rangle\langle 0| \\
&= \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}.
\end{aligned} \tag{6}$$

A density matrix (of a pure qubit, which has to be normalized) has following properties:

1. self-adjoint: $\rho^\dagger = \rho$

2. trace one: $Tr(\rho) = 1$

3. positivity: $\rho \geq 0$

4. $\rho^2 = \rho$

Question: Born's law is reflected in the diagonal terms of Eq. (6). Are the off-diagonal terms useless?

A.   Bloch sphere

When a, b are real numbers, the state in Eq. (5) can be represented by rays in x-y plane. Of course, for any state such that a/b or b/a is a real number, the rays form a plane. What if a, b are general complex numbers? We need to think how many free (real) parameters existing in Eq. (5). Since $|a|^2 + |b|^2$ does not affect the state, we can always set it to be 1 (normalized state representation). Thus, there are two free (real) parameters.

A normalized pure state of a qubit can be written as

$$|u\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle. \tag{7}$$

Therefore, it is convenient to represent it as a vector living the surface of a unit sphere with the spherical coordinate $(r = 1, \theta, \varphi)$.

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[1] Very soon, we shall learn how to do "measure".

[2] Later, we shall cover this in Bell's inequality. You are encouraged to read more interesting materials on the debate between Einstein and Bohr.
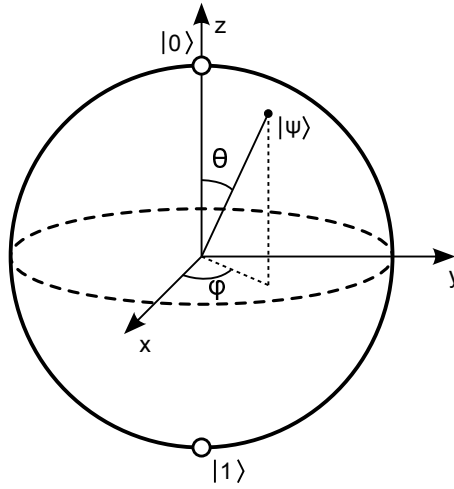
FIG. 1. Bloch sphere.

Exercise: write down the states of $|x+\rangle$, $|x-\rangle$, $|y+\rangle$, $|y-\rangle$, $|z+\rangle$, $|z-\rangle$, according to Eq. (7).

Answer: $|x+\rangle = |+\rangle$, $|x-\rangle = |-\rangle$, $|y+\rangle = |+i\rangle$, $|y-\rangle = |-i\rangle$, $|z+\rangle = |0\rangle$, $|z-\rangle = |1\rangle$.

Exercise: write down the states of $|+45°\rangle$, $|-45°\rangle$, according to Eq. (7); are they orthogonal?

Answer: $|+45°\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i\frac{\pi}{4}}\sin\frac{\pi}{4}|1\rangle = \frac{\sqrt{2}}{2}\left(|0\rangle + \frac{1+i}{\sqrt{2}}|1\rangle\right)$, $|-45°\rangle = \cos\frac{\pi}{4}|0\rangle + e^{-i\frac{\pi}{4}}\sin\frac{\pi}{4}|1\rangle = \frac{\sqrt{2}}{2}\left(|0\rangle + \frac{1-i}{\sqrt{2}}|1\rangle\right)$.

$\langle +45°|-45°\rangle = \frac{1}{2}\left(1 + \frac{1+i}{\sqrt{2}}\frac{1-i}{\sqrt{2}}\right) = 1 \neq 0$, so they are not orthogonal.

Question: how a pair of orthogonal qubits represented in the Bloch sphere?

Answer: same line, opposite orientation.

B. Pauli matrix and qubit basis

Pauli matrices are a set of three 2×2 complex matrices which are Hermitian unitary,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

For those who are interested in where these Pauli matrices come from, you are encouraged to read Chapter 2 of John Preskill's lecture notes[3].

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \tag{9}$$

A qubit can be prepared in three bases: X, Y, Z, which are corresponding to the eigenstates of Pauli matrices: σ_x, σ_y, σ_z, respectively. Denote the eigenstates of the Z basis by $|0\rangle$ and $|1\rangle$, then the other eigenstates of the X and Y bases are

$$|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2},$$
$$|\pm i\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}, \tag{10}$$

respectively. These three bases are mutually unbiased bases (MUB), that is,

$$\langle \psi_i | \phi_j \rangle = 1/\sqrt{d}, \tag{11}$$

where $|\psi_i\rangle$ and $|\phi_j\rangle$ are basis states from two different MUB and d is the dimension.

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[3] http://quantum.phys.cmu.edu/CQT/chaps/cqt15.pdf

C.   Qubit tomography

In general, a qubit might be in a mixed state, and then it will be within the Bloch sphere instead on the surface. In general, the density matrix of a qubit can be written as

$$
\begin{aligned}
\rho &= \frac{1}{2}(\sigma_0 + \vec{P} \cdot \vec{\sigma}) \\
&= \frac{1}{2}\begin{pmatrix} 1 + P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix}
\end{aligned}
\tag{12}
$$

where $\vec{P} = (P_x, P_y, P_z)$ is a vector and $|\vec{P}| \leq 1$. When $|\vec{P}| = 1$, Eq. (12) represents a pure qubit.

Quantum state tomography is the process of reconstructing the quantum state for a quantum system by proper measurements. The quantum state can be pure (vector) or in general mixed (density matrix). A set of measurements is called tomographically complete if it can uniquely identify the state. That is, the measurement outcomes are able to provide all the information about the state. In the classical physics, it corresponds to system calibration.

Let us take qubit tomography for example. As given in Eq. (12), the Z basis measure provides the information on P_z. From Eq. (24), we know that the X basis measure provides the information on P_x. Similarly, the Y basis measure provides the information on P_y. Thus, X, Y, and Z measurements is tomographically complete for a qubit. Another way to put this is,

$$
\rho = \frac{1}{2}[tr(\rho)\sigma_0 + tr(\sigma_x \rho)\sigma_x + tr(\sigma_y \rho)\sigma_y + tr(\sigma_z \rho)\sigma_z].
\tag{13}
$$

Now, let us move a bit further, tomography for n qubits,

$$
\rho = 2^{-n} \sum_{v_1, v_2, \ldots, v_n \in \{0,x,y,z\}} tr(\sigma_{v1} \otimes \sigma_{v2} \otimes \cdots \otimes \sigma_{vn}\rho)\sigma_{v1} \otimes \sigma_{v2} \otimes \cdots \otimes \sigma_{vn},
\tag{14}
$$

where the sum is over all possible the identity and Pauli matrices.

In another related concept, quantum process tomography, known quantum states are used to probe a quantum process to find out how the process can be described. Similarly, quantum measurement tomography works to find out what measurement is being performed.

The general principle behind quantum state tomography is that by repeatedly performing many different measurements on quantum systems described by identical density matrices, frequency counts can be used to infer probabilities, and these probabilities are combined with Born's rule to determine a density matrix which fits the best with the observations.

III.   MEASUREMENT

Physics is an experimental science. If we observe (or measure) a quantum state in lab, what kind of outcomes do we expect and how are they related to the quantum state?

A measurement is a process in which information about the state of a physical system is acquired by an observer. In quantum mechanics, the measurement of an observable A prepares an eigenstate of A, and the observer learns the value of the corresponding eigenvalue.

A.   Observables

An observable is a self-adjoint operator on $\mathcal{H}$: an observable is a property of a physical system that in principle can be measured. In quantum mechanics, an observable is a self-adjoint operator.

1. Operator: An operator is a linear map taking vectors to vectors,

$$
A : |\psi\rangle \mapsto A|\psi\rangle, A(a|\phi\rangle + b|\psi\rangle) = aA|\phi\rangle + bA|\psi\rangle.
\tag{15}
$$

2. Adjoint: The adjoint $A^\dagger$ of the operator A is defined by

$$
\langle\psi|A\phi\rangle = \langle A^\dagger\psi|\phi\rangle,
\tag{16}
$$

for all vectors $|\psi\rangle$, $|\phi\rangle$.

3. Self-adjoint: A is self-adjoint if $A = A^\dagger$, or in other words, if $\langle\psi|A|\phi\rangle = \langle\phi|A|\psi\rangle$ for all vectors $|\psi\rangle$, $|\phi\rangle$.

   i) If A and B are self adjoint, then so is $A + B$ (because $(A+B)^\dagger = A^\dagger + B^\dagger$).

   ii) $(AB)^\dagger = B^\dagger A^\dagger$, so that AB is self adjoint only if A and B commute.

   iii) Note that $AB + BA$ and $i(AB - BA)$ are always self-adjoint if A and B are.

4. A self-adjoint operator in a Hilbert space H has a spectral representation-its eigenstates form a complete orthonormal basis in $\mathcal{H}$. We can express a self-adjoint operator A as

$$A = \sum_n a_n E_n. \tag{17}$$

Here each a_n is an eigenvalue of A, and E_n is the corresponding orthogonal projection onto the space of eigenvectors with eigenvalue a_n. The E_n is satisfy,

$$E_n E_m = \delta_{n,m} E_n,$$
$$E_n^\dagger = E_n. \tag{18}$$

5. The orthogonal projector onto the one-dimensional space spanned by the vector $|\psi\rangle$ may be expressed as $|\psi\rangle\langle\psi|$, where $\langle\psi|$ is the bra that annihilates vectors orthogonal to $|\psi\rangle$. Therefore, an alternative notation for the spectral representation of A is

$$A = \sum_n |n\rangle a_n \langle n|, \tag{19}$$

where $\{|n\rangle\}$ is the orthonormal basis of eigenstates of A, with $A|n\rangle = a_n|n\rangle$.

B.   Projection-valued measure

A projection-valued measure (PVM) is an orthogonal projection.

1. If the quantum state just prior to the measurement is $|\psi\rangle$, then the outcome a_n is obtained with a priori probability,

$$\mathbf{Prob}(a_n) = ||E_n|\psi\rangle||^2 = \langle\psi|E_n|\psi\rangle. \tag{20}$$

2. if the outcome a_n is attained, then the (normalized) quantum state just after the measurement is

$$\frac{E_n|\psi\rangle}{||E_n|\psi\rangle||}. \tag{21}$$

3. If the measurement is immediately repeated, then according to this rule the same outcome is obtained again, with probability one.

4. If many identically prepared systems are measured, each described by the state $|\psi\rangle$, then the expectation value of the outcomes is,

$$\langle a\rangle \equiv \sum_n a_n \mathbf{Prob}(a_n) = \sum_n \langle\psi|E_n|\psi\rangle = \langle\psi|A|\psi\rangle. \tag{22}$$

IV.   UNITARY TRANSFORMATION

In quantum mechanics, the state evolution through time follows Schrodinger's equation, shown in Fig. 2. For the moment, we do not want to going to any details for the equation. All we need to know is that the evolution is unitary. (In this class, evolution = operation = transformation.)
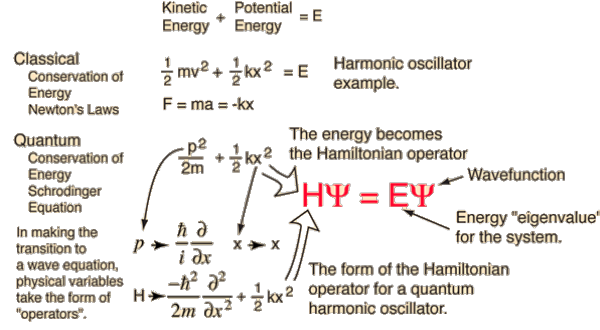
FIG. 2. Schrodinger's equation

A widely used single qubit operation, Hadamard transformation,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{23}$$

rotates between the X and Z bases, because

$$H\sigma_x H^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z. \tag{24}$$

A spin-$\frac{1}{2}$ system can be treated as a simple qubit system. Lots of properties of qubit is derived from the spin-$\frac{1}{2}$ system. It turns out that the polarization of a single photon can be well described by a qubit. The X, Y and Z measurements correspond to the polarization along that axis. This is nontrivial. Photons are massless and have spin-1. From now on, we will pretend a single photon as a spin-$\frac{1}{2}$ system, which is easier to understand for a physicist.

V.   SOME INTERESTING RESULTS

A.   No-cloning theorem[4]

It is impossible to make a copy of an unknown quantum state. Otherwise, we assume that there is a unitary evolution U such that,

$$|\psi\rangle|0\rangle \to U(|\psi\rangle|0\rangle) = |\psi\rangle|\psi\rangle. \tag{25}$$

Then for any two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$,

$$\begin{aligned} U(|\psi_1\rangle|0\rangle) &= |\psi_1\rangle|\psi_1\rangle, \\ U(|\psi_2\rangle|0\rangle) &= |\psi_2\rangle|\psi_2\rangle. \end{aligned} \tag{26}$$

The inner product of this two equations is given by,

$$\begin{aligned} (\langle 0|\langle\psi_1|U^*)(U|\psi_2\rangle|0\rangle) &= (\langle 0|\langle\psi_1|)(|\psi_2\rangle|0\rangle) = \langle\psi_1|\psi_2\rangle. \\ (\langle 0|\langle\psi_1|U^*)(U|\psi_2\rangle|0\rangle) &= (\langle\psi_1|\langle\psi_1|)(|\psi_2\rangle|\psi_2\rangle) = \langle\psi_1|\psi_2\rangle^2. \end{aligned} \tag{27}$$

Then $\langle\psi_1|\psi_2\rangle = \langle\psi_1|\psi_2\rangle^2$, $\langle\psi_1|\psi_2\rangle$ is equal to 0 or 1. Then $|\psi_1\rangle$ and $|\psi_2\rangle$ are same or orthogonal.

Question: is the no-cloning theorem consistent with Heisenberg's uncertainty relation?

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[4] http://iiis.tsinghua.edu.cn/ RC3/liyuanzhi/nocloning.pdf

B. No-deletion theorem[5]

Given two copies of some arbitrary quantum state, it is impossible to delete one of the copies. For an unknown state $|\psi\rangle = a|0\rangle + b|1\rangle$, there is no linear isometric transformation such that

$$U|\psi\rangle|\psi\rangle|A\rangle = U|\psi\rangle|0\rangle|A'\rangle \tag{28}$$

where the final state of the ancilla qubit is independent of $|\psi\rangle$.

Suppose

$$\begin{aligned} U|0\rangle|0\rangle|A\rangle &= U|0\rangle|0\rangle|A_0\rangle \\ U|1\rangle|1\rangle|A\rangle &= U|1\rangle|0\rangle|A_1\rangle \end{aligned} \tag{29}$$

For an unknown state $|\psi\rangle$,

$$\begin{aligned} U|\psi\rangle|\psi\rangle|A\rangle &= U(a^2|0\rangle|0\rangle|A\rangle + b^2|1\rangle|1\rangle|A\rangle + ab|1\rangle|0\rangle|A\rangle + ab|0\rangle|1\rangle|A\rangle)) \\ &= a^2|0\rangle|0\rangle|A_0\rangle + b^2|1\rangle|0\rangle|A_1\rangle + ab|\Psi\rangle \end{aligned} \tag{30}$$

We also know that:

$$U|\psi\rangle|\psi\rangle|A\rangle = |\psi\rangle|0\rangle|A'\rangle = a|0\rangle|0\rangle|A'\rangle + b|1\rangle|0\rangle|A'\rangle. \tag{31}$$

Then Eq. (30) is equal to Eq. (31). Thus we have

$$\begin{aligned} |\Psi\rangle &= |0\rangle|0\rangle|A_1\rangle + |1\rangle|0\rangle|A_0\rangle. \\ |A'\rangle &= a|A_0\rangle + b|A_1\rangle \end{aligned} \tag{32}$$

From the normalized requirement of $|A'\rangle$, $|A_0\rangle$ and $|A_1\rangle$ are orthogonal ($a^2 + b^2 = 1$). This means that although $|\psi\rangle$ is deleted, but its information is contained by $|A'\rangle$.

Question: no-cloning plus no-deletion form an information conservation law?

VI. PRACTICAL SYSTEMS

Photon polarizations: horizontal and vertical polarizations are represented as $|H\rangle$ and $|V\rangle$; diagonal polarizations are represented as $|+45^o\rangle$ and $|-45^o\rangle$ or just $|+\rangle$ and $|-\rangle$; circular polarizations are represented as $|R\rangle$ and $|L\rangle$. If we denote $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$,

$$\begin{aligned} |R\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \\ |L\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \end{aligned} \tag{33}$$

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[5] https://en.wikipedia.org/wiki/ Quantum_no-deleting_theorem