Contents

1 Schema

1.1 Entity Sets

function(function-id, oddCoefficient, oddAddend, evenDivisor)
evaluation(eval-id, *function-id*, value)
chain(chain-id, *eval-id*, chain)
loop(loop-id, *chain-id*, loop, stabilization)

1.2 Relations

evaluation(function, chain) | $[1..1] \rightarrow [0..n], [0..m]$

1.3 Tables

evaluation(function-id, chain-id, value)

2 Lemmas

$$Converge(x_n, L) :\equiv (\forall\varepsilon > 0)(\exists N)[n > N \to |x_n - L| < \varepsilon]$$
$$Diverge(x_n) :\equiv (\forall L)(\exists\varepsilon > 0)(\forall N)[n > N \to |x_n - L| \geq \varepsilon]$$

Lemma 1. $(2^k, 2^k m + 2^k - \ell)$, $2 \leq \ell \leq 2^k - 2$

Proof. $\oint$ $x \equiv 1 \pmod 2$.
x,
$2^k x + 2^k m + 2^k - \ell$,
$2^{k-1}x + 2^{k-1}m + 2^{k-1} - \frac{\ell}{2}$,
$2^k(2^{k-1}x + 2^{k-1}m + 2^{k-1} - \frac{\ell}{2}) + 2^k m + 2^k - \ell$
$2^{2k-1}x + 2^{2k-1}m + 2^{2k-1} - \ell 2^{k-1} + 2^k m + 2^k - \ell$ ∎

Lemma 2. $(a, b, 2)$ *goes to infinity if and only if* $(b, a, 2)$ *does*

Proof. $\oint$ $(a, b, 2)$ goes to infinity.
$\oint$ $x \equiv 1 \pmod 2$
$\oint$ a, b even
$x, ax + b, a'x + b', a(a'x + b') + b = aa'x + ab' + b, a'^2 x + a'b' + b', \ldots$
$x, bx + a, b'x + a', bb'x + ba' + a, b'^2 x + b'a' + a'$
$\oint$ a, b odd
$x, ax + b, (ax + b)', a(ax + b)' + b, (a(ax + b)' + b)', a(a(ax + b)' + b)' + b, \ldots$
$x, bx + a, (bx + a)', b(bx + a)' + a, (b(bx + a)' + a)', b(b(bx + a)' + a)' + a, \ldots$ ∎

Lemma 3. $(4, 4k + 2, 2)$, $k = 1, \ldots$

Proof. $\oint$ $x \equiv 1 \pmod 2$. The iterations of this function are $x, 4x+4k+2, 2x+2k+1, 8x+12k+6, 4x+6k+3, 16x+28k+14, 8x+14k+7, 32x+60k+30, 16x+30k+15, 64x+124k+62, 32x+62k+31, 128x+252k+126, 64x+126k+63, 256x+508k+254, 128x+254k+127, 512x+1020k+510, \ldots$. We can rewrite these as $x, 2x+2k+1, 4x+4k+2+2k+1, 8x+8k+4+4k+2+2k+1, \ldots$, or even $x, 2x+2k+1, 2(2x+2k+1)+2k+1, 4(2x+2k+1)+2(2k+1)+2k+1, \ldots$. Hence, we can see that the sequence of odd numbers in this sequence is $\left\{2^\ell(2x + 2k + 1) + \sum_{m=0}^{\ell} m(2k + 1)\right\}_{\ell=1}^{\infty}$, and the sequence of even numbers is double this. Since both sequences approach infinity as $\ell \to \infty$, this sequence of iterations goes to infinity.

$\oint$ $x \equiv 0 \pmod 2$. Then, x reduces to an odd factor that initiates the above sequence. ∎

Lemma 4. $(4k+2,4,2)$

Proof. $\oint$ $x \equiv 1 \pmod 2$
$x, 4xk+2x+4,$
$2xk+x+2, 8k^2x+(8x+8)k+2x+8,$
$4k^2x+(4x+4)k+x+4, 16k^3x+(24x+16)k^2+(12x+24)k+2x+12,$
$8k^3x+(12x+8)k^2+(6x+12)k+x+6, 32k^4x+(64x+32)k^3+(48x+64)k^2+(16x+48)k+2x+16,$
$16k^4x+(32x+16)k^3+(24x+32)k^2+(8x+24)k+x+8, 64k^5x+(160x+64)k^4+(160x+160)k^3+(80x+160)k^2+(20x+80)k+2x+20,$
$32*k^5*x+(80*x+32)*k^4+(80*x+80)*k^3+(40*x+80)*k^2+(10*x+40)*k+x+10, \dots$

$x,$
$2kx+x+2,$
$4k^2x+(4x+4)k+x+4,$
$8k^3x+(12x+8)k^2+(6x+12)k+x+6,$
$16k^4x+(32x+16)k^3+(24x+32)k^2+(8x+24)k+x+8,$
$32k^5x+(80x+32)k^4+(80x+80)k^3+(40x+80)k^2+(10x+40)k+x+10, \dots$
$x,$
$2kx+x+2,$
$4k^2x+(4x+4)k+x+4,$
$8k^3x+(12x+8)k^2+(4x+4)k+(2x+8)k+x+6,$
$16k^4x+(32x+16)k^3+(12x+8)k^2+(12x+24)k^2+(4x+4)k+(2x+8)k+(2x+12)k+x+8,$
$32k^5x+(80x+32)k^4+(32x+16)k^3+(48x+54)k^3+(12x+8)k^2+(12x+24)k^2+(16x+48)k^2+$

$(4x+4)k+(2x+8)k+(2x+12)k+(2x+16k)+x+10, \dots$

$(2k)^\ell x+(\textit{something})+x+2\ell$ ■

Lemma 5. $(2k+1, 2(k-1)+1)$, $x=2k+1$, $k \pmod 2 = 0$*???*

Proof. $2k+1, (2k+1)^2+2k-1=4k$ ■

Lemma 6. *We can say nothing of $(a,a,2)$.*

Proof. $\oint$ $a \pmod 2 \equiv 1$, and $x \pmod 2 \equiv 1$. The chain of this function at x is $x, ax+a$. Since $ax+a \pmod 2 \equiv 0$, we can say nothing more here, nor can we say anything when we start with an x for which $x \mod 2 \equiv 0$, since it merely reduces to the odd case.

$\oint$ $a \pmod 2 \equiv 0$ and $x \pmod 2 \equiv 1$. The chain of this function at x starts as $x, ax + a$. Since $a \pmod 2 \equiv 0$, this becomes $a'x + a'$, where a' is a stripped of its factors of 2. Then, $a'x + a' \pmod 2 \equiv 0$, and, again, we can do nothing more.

$\oint$ $x \pmod 2 \equiv 0$. Then, x reduces to an odd factor that initiates the above process and of which we can say nothing. ■

Lemma 7. $(o, e, 2) \to \infty\, \forall x$

Proof. $\oint$ $x \pmod 2 \equiv 1$. Then, $ox + e \pmod 2 \equiv 1$. We continue to feed these odd numbers back into the function, and the sequence goes to infinity.

$\oint$ $x \pmod 2 \equiv 0$. Then, x reduces to an odd factor x' that initiates the above sequence, taking it to infinity. ■

Lemma 8. $(e, eh, 2)$, *where e is an even positive integer and $h \pmod 2 \equiv 0$, goes to infinity.*

Proof. $\oint$ $x \pmod 2 \equiv 1$. The chain of this function starts as $x, ex+eh, x+h$. If $h \pmod 2 \equiv 1$, then $x + h \pmod 2 \equiv 0$, and we can't continue the sequence. If $h \pmod 2 \equiv 0$, then $x + h$ is odd, and we can continue the sequences as $e(x+h)+eh, x+2h, e(x+2h)+eh, \ldots$, which can be split into the two sequences $\{x + kh\}_{k=0}^{\infty}$ and $\{e(x + kh) + eh\}_{k=0}^{\infty}$, which both diverge.

$\oint$ $x \pmod 2 \equiv 0$. Then, x reduces to an odd factor that initiates the above sequences. ■

Example 1. $(2, 8, 2) \to \infty \forall x$ and $(2, 12, 2) \to \infty \forall x$

Lemma 9. $(eh, e, 2)$, *where e is an even positive integer and $h \pmod 2 \equiv 0$, goes to infinity.*

Proof. $\oint$ $x \pmod 2 \equiv 1$. This function's chain at x begins as $x, ehx+e, hx+1$. If $h \pmod 2 \equiv 1$, then $hx + 1 \pmod 2 \equiv 0$, and we can't continue the sequence. If $h \pmod 2 \equiv 0$, then $hx + 1 \pmod 2 \equiv 1$, and the sequence continues as $eh^2x + eh + e, h^2x + h + 1, eh^3x + eh^2 + eh + e, \ldots$, which can be split into the two sequences $\left\{h^k x + \sum_{\ell=0}^{k-1} h^\ell\right\}_{k=1}^{\infty}$ and $\left\{eh^k x + \sum_{\ell=0}^{k-1} eh^\ell\right\}_{k=1}^{\infty}$, both of which go to infinity.

$\oint x \pmod 2 \equiv 0$. Then, x reduces to an odd factor that initiates the above sequences.

Let $(x_n) = \left\{ h^k x + \sum_{\ell=0}^{k-1} h^\ell \right\}_{k=1}^{\infty}$ and $\oint$ $Converge(x_n, L)$. Then, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that $|x_n - L| < \varepsilon$ when $n > N$.

Let $L \in \mathbb{R}$ and let $\varepsilon = \frac{1}{2}\left(h^k x + \sum_{\ell=0}^{k-1} h^\ell \right)$. ■

Example 2. $(12, 2, 2) \to \infty \forall x$ and $(12, 6, 2) \to \infty \forall x$

FIX: ...We could just look at the result of stripping an even number of all its factors of 2.

3 $x \bmod evenDivisor \neq 0$

$(3, 7, 3)$
$3, 16, 16/3$

$c > 1$

$$f(x) = \begin{cases} e_1(x) & x \pmod c = c - 1 \\ e_2(x) & x \pmod c = c - 2 \\ \vdots & \\ e_{n-1}(x) & x \pmod c = 2 \\ e_n(x) & x \pmod c = 1 \\ x/c & x \pmod c = 0 \end{cases}$$

c = 2

$$f(x) = \begin{cases} 3n + 1 & x \pmod 2 \equiv 1 \\ x/2 & x \pmod 2 \equiv 0 \end{cases}$$

c = 2

$$f(x) = \begin{cases} 2n + 1 & x \pmod 2 \equiv 1 \\ x/2 & x \pmod 2 \equiv 0 \end{cases}$$

c = 3

$$f(x) = \begin{cases} 3n + 2 & x \pmod 3 \equiv 2 \\ 3n + 1 & x \pmod 3 \equiv 1 \\ x/3 & x \pmod 3 \equiv 0 \end{cases}$$

or

$$f(x) = \begin{cases} 3n + 1 & x \pmod 3 \equiv 2 \\ 3n + 2 & x \pmod 3 \equiv 1 \\ x/3 & x \pmod 3 \equiv 0 \end{cases}$$

This would have only one value $a = 3$. The others are determined from a.

4 Entity Logic

STUFF TO READ

4.1 Entity Set

Let S be a set. $s \in S$ is an instance of S. There is a set of attributes $A(S)$ containing functions $f_i : E \to \mathbb{R}$ and relations $R_i : E \to \{True, False\}$, $i \in \mathbb{N}$.

e.g., $s \in S$, or: $John(s) = True, Smith(s) = True, \ldots$
But then is *John* the attribute, or *name*?

What is $\mathbb{N}$ as an entity set or as a database?
$A(\mathbb{N}) = \{prime, composite, even, odd, numPrimeFactors, f_1(x, t_2, \ldots, t_n), f_2(x, t_2, \ldots, t_n), \ldots, y \text{ s.t. } R(x, y), \ldots\}$

$A(\text{MOVEMENT}) = \{velocity, momentum, friction, force, normal\_force, \ldots\}$

What about constants of physics? What is there to be said *about* them? $A(\{G, c, h, k, \ldots\}) = \{\ldots\}$

An entity set is what we say about a thing. A database is what we must say about at least two things.

4.2 Relationship

Definition 1. Let E be an entity set, $e_1, e_2 \in E$, and $a \in A(E)$. We call a the **primary key** of E if $a(e_1) = a(e_2) \longleftrightarrow e_1 = e_2$.

Theorem 1. *Let $f : A \to B$ be a function. Then, f is injective if and only if f is a primary key for the entity set of A.*

Proof. Let $f : A \rightarrowtail B$ be an injective function. Then, $\forall x, y \in A$, $f(x) = f(y) \to x = y$. Since f is a function, we have that $x = y \to f(x) = f(y)$, and, hence, f is a primary key of A.

Let $f \in A(A)$ be a primary key. So, $\forall x, y \in A, f(x) = f(y) \longleftrightarrow x = y$. Put $B = \mathbb{R}$. Then, $f : A \rightarrowtail B$ is an injective function. ∎

Definition 2. Let $f : A \to B$ be a mapping. We call f **ill-defined** if for some $x = y \in A$, $f(x) \neq f(y)$.

Definition 3. Let E_1 and E_2 be entity sets, and let P the primary key of E_1. We call P a **foreign key** of E_2 if $Px \, \forall x \in E_1$ and $Py \, \forall y \in E_2$.

Definition 4. An ER-diagram is a graph with entity sets $E_1, \ldots, E_n$ as vertices and with foreign keys $P_1, \ldots, P_m$, $m \leq \binom{n}{2}$ as edges.

5 Case Theory

Definition 5. Let $f : A \to B$ be a function and $c \in B$ a constant. We call a formula of the form $P(x) \to f(x) = c$ a **case** of f. If P is a modulus over some $n \in \mathbb{N}$, we call such a case a **modular case** or a **modular n-case**. Moreover, we call a function defined only by modular cases a **modular function**.

Definition 6. Let $f : A \to B$ be a function. If f can be defined by an atomic formula, then, we call f a **whole function**. If f can defined by n cases, then we call f a **piecewise function** or an **n-piece function**.

Definition 7. Let $f : A \to B$ be an n-piece function defined by cases $C_1, \ldots, C_n$. If one case C_i, $1 \leq i \leq n$, is a modular m-case defined as $x \equiv 0 \pmod{m} \to f(x) = \frac{x}{m}$, then we call f a **stripping function**.

Example 3. The function $f : \mathbb{N} \to \mathbb{N}$ given by

$$f(x) = \begin{cases} 3x + 1 & x \equiv 1 \pmod{2} \\ \frac{x}{2} & x \equiv 0 \pmod{2} \end{cases}$$

is a modular 2-piece stripping function defined by the cases $x \equiv 1 \pmod{2} \to f(x) = 3x + 1$ and $x \equiv 0 \pmod{2} \to f(x) = \frac{x}{2}$.

Theorem 2. *Let $f : A \to B$ be a whole function, where A and B are finite. Then, f is an n-piece function.*

Proof. Let $f : A \to B$, where $|A| = n \leq m = |B|$, be a function given by the formula $f(x) = b$ for some $b \in B$. We have $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$. Then,

$$f(x) = \begin{cases} f(x) = b_1 & x = a_1 \\ f(x) = b_2 & x = a_2 \\ \vdots & \\ f(x) = b_n & x = a_n \end{cases}.$$

That is, we may define f as $\bigwedge_{i=1}^{n} x = a_i \to f(x) = b_i$.

Now, let $f : A \to B$, where $|A| = n > m = |B|$, be a function given by the formula $f(x) = b$, where $b \in B$. We have $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$. Then,

$$f(x) = \begin{cases} f(x) = b_1 & x = a_1 \\ f(x) = b_2 & x = a_2 \\ \vdots & \\ f(x) = b_m & x = a_m \\ f(x) = b_{i_1} & x = a_{m+1} \\ f(x) = b_{i_2} & x = a_{m+2} \\ \vdots & \\ f(x) = b_{i_{n-m}} & x = a_n \end{cases},$$

where $1 \leq i_j \leq m$ for $j = 1, \ldots, n-m$. That is, we may define f as

$$\left(\bigwedge_{i=1}^{m} x = a_i \to f(x) = b_i \right) \wedge \left(\bigwedge_{i=m+1}^{n} \left(\bigwedge_{j=1}^{n-m} x = a_i \to f(x) = b_{i_j} \right) \right).$$

■

Theorem 3. *Let $f : A \to B$ be a whole function, where $|A| < \aleph_0 \leq |B|$. Then, f is an n-piece function.*

Proof. Let f, A, and B be as such. Then, $A = \{a_1, \ldots, a_n\}$ and we may define f with the formula $f(x) = b$ for some $b \in B$. Then, for some $b_1, \ldots, b_n \in B$,

$$f(x) = \begin{cases} f(x) = b_1 & x = a_1 \\ f(x) = b_2 & x = a_2 \\ \vdots & \\ f(x) = b_n & x = a_n \end{cases},$$

and we may define f as $\bigwedge_{i=1}^{n} x = a_i \to f(x) = b_i$. ■

Definition 8. Let $f : A \to B$ be a function. If f is not piecewise, then a definition of f in cases requires an infinite number of cases. We say that such a function is **∞-piece**.

Theorem 4. *Let $f : A \to B$ be a whole function, where A is infinite. Then, f is an ∞-piece function.*

Proof. Let $f : A \to B$ be as such, and suppose A is infinite. We have $A = \{a_0, a_1, \dots\}$. Suppose we define f with the cases $f(a_0) = b_0$, $f(a_1) = b_1, \dots, f(a_n) = b_n$ for some $b_1, \dots, b_n \in B$. Then, there is an $a_{n+1} \in A$ at which f is not defined. Thus, we are no longer talking about f but rather a function $g : S \to B$, where $S \subset A$ is finite. Therefore, f is ∞-piece. ■

Examples. The successor function $S : \mathbb{N} \to \mathbb{N}$ given by $S(x) = x + 1$ is ∞-piece.

$f : \mathbb{N} \to \mathbb{R}, f(x) = \sum\limits_{n=1}^{x} \frac{1}{n}$

Any function whose domain contains $\mathbb{N}$.

$\zeta : \{s \in \mathbb{C} \mid \Re(s) > 1\} \to \mathbb{C}$, $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$

6 Buchberger

$S_{ij} = \frac{\operatorname{lcm}\{g_i, g_j\}}{g_i} f_i - \frac{\operatorname{lcm}\{g_i, g_j\}}{g_j} f_j$

$f_i = p_1, \dots, p_n$, $f_j = p_1, \dots, p_m$

p_i, p_j

$S_{ij} = \frac{\operatorname{lcm}\{p_i, p_j\}}{p_i} f_i - \frac{\operatorname{lcm}\{p_i, p_j\}}{g_j} f_j$

$6 = 2 \cdot 3$, $15 = 3 \cdot 5$

$S_{1,2} = (2 \cdot 3/2)6 - (2 \cdot 3/3)15 = 18 - 30 = -12$

$$f^{-1}(x) = \begin{cases} \{2x\} & x \equiv 0, 1, 2, 3, 5 \\ \{2x, \frac{x-1}{3}\} & x \equiv 4 \end{cases} \pmod 6$$

$\oint x \equiv 4 \pmod 6$

$2x \equiv 2 \pmod 6$, $x \equiv 1 \pmod 6$

$x - 1 \equiv 3 \pmod 6$, $\frac{x-1}{3} \equiv 1 \mod 6$

$\oint x \equiv 0 \pmod 6$

$2x \equiv 0 \pmod 6$, $\frac{x-1}{3} \equiv 2 \pmod 6$

$\not\mathcal{S}\ x \equiv 1 \pmod 6$

$2x \equiv 2 \pmod 6, \frac{x-1}{3} \equiv 0 \pmod 6$

$\not\mathcal{S}\ x \equiv 2 \pmod 6$

$2x \equiv 4 \pmod 6, \frac{x-1}{3} \equiv 1 \pmod 6$

$\not\mathcal{S}\ x \equiv 3 \pmod 6$

$2x \equiv 0 \pmod 6, \frac{x-1}{3} \equiv 2 \pmod 6$

$\not\mathcal{S}\ x \equiv 5 \pmod 6$

$2x \equiv 4 \pmod 6, \frac{x-1}{3} \equiv 4 \pmod 6$