Chapter 6
Pontryagin Minimum Principle

6.2 Pontryagin Minimum Principle

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$||\mathbf{u}(t)|| \leq \mathbf{U}$$

$$\begin{aligned}\delta J = &\int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}} \right)_* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_* \right]' \delta \mathbf{x}(t) dt \\ &+ \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}} \right)_*' \delta \mathbf{u}(t) dt \\ &+ \mathcal{L}\big|_{t_f} \delta t_f + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_*' \delta \mathbf{x}(t) \right] \Bigg|_{t_f} . \end{aligned} \tag{2.7.18}$$

$$\begin{aligned}\mathcal{L}^* &= \mathcal{L}^*(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &= \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &\quad + \left(\frac{\partial S}{\partial \mathbf{x}} \right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t} \right)_* - \boldsymbol{\lambda}^{*'}(t) \dot{\mathbf{x}}^*(t). \end{aligned} \tag{2.7.28}$$

$$\delta J(\mathbf{u}^*(t), \delta\mathbf{u}(t)) = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial \mathcal{H}}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}(t)\right]_* \delta\mathbf{x}(t) \right.$$

$$\left. + \underline{\left[\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right]_*' \delta\mathbf{u}(t)} + \left[\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}} - \dot{\mathbf{x}}(t)\right]_*' \delta\boldsymbol{\lambda}(t) \right\} dt$$

$$+ \left[\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}(t)\right]_{*t_f}' \delta\mathbf{x}_f + \left[\mathcal{H} + \frac{\partial S}{\partial t}\right]_{*t_f} \delta t_f. \tag{6.2.6}$$

$$\delta J(\mathbf{u}^*(t), \delta\mathbf{u}(t)) = \int_{t_0}^{t_f} \left[\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right]' \delta\mathbf{u}(t) dt. \tag{6.2.7}$$

$$= \int_{t_0}^{t_f} [\mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \boldsymbol{\lambda}^*(t), t)$$

$$- \mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t)]\, dt.$$

$$\mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \boldsymbol{\lambda}^*(t), t) \geq \mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t).$$

$$\mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \le \mathcal{H}(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}^*(t), t)$$

$$\min_{|\mathbf{u}(t)| \le \mathbf{U}} \{\mathcal{H}\left(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}^*(t), t\right)\} = \mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t)$$

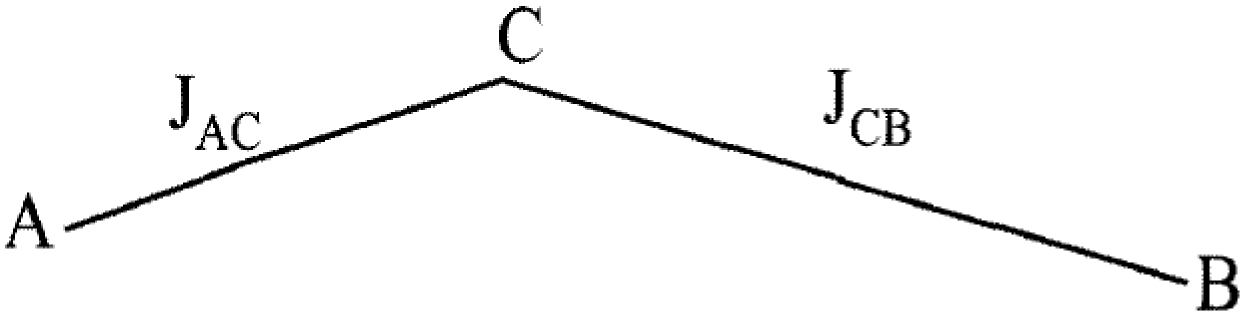
Table 6.1 Summary of Pontryagin Minimum Principle

A. Statement of the Problem
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$, the performance index as $J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$, and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0$ and t_f and $\mathbf{x}(t_f) = \mathbf{x}_f$ are free, find the optimal control.

B. Solution of the Problem	
Step 1	Form the Pontryagin $\mathcal{H}$ function $\mathcal{H}(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t) = V(\mathbf{x}(t),\mathbf{u}(t),t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t)$
Step 2	Minimize $\mathcal{H}$ w.r.t. $\mathbf{u}(t)(\leq \mathbf{U})$ $\mathcal{H}(\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t),t) \leq \mathcal{H}(\mathbf{x}^*(t),\mathbf{u}(t),\boldsymbol{\lambda}^*(t),t)$
Step 3	Solve the set of 2n state and costate equations $\dot{\mathbf{x}}^*(t) = \left(\frac{\partial\mathcal{H}}{\partial\boldsymbol{\lambda}}\right)_*$ and $\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial\mathcal{H}}{\partial\mathbf{x}}\right)_*$ with boundary conditions $\mathbf{x}_0$ and $\left[\mathcal{H} + \frac{\partial S}{\partial t}\right]_{*t_f} \delta t_f + \left[\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}\right]'_{*t_f} \delta\mathbf{x}_f = 0.$

6.3 Dynamic Programming

6.3.1 Principle of Optimality



$$J_{AB}^{*} = J_{AC} + J_{CB}^{*}$$

An optimal policy has the property that whatever the previous state and decision (i.e., control), the remaining decisions must constitute an optimal policy with regard to the state resulting from the previous decision.

6.3.2 Optimal Control Using Dynamic Programming

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k), k)$$

$$J_i(\mathbf{x}(k_i)) = J = S(\mathbf{x}(k_f), k_f) + \sum_{k=i}^{k_f-1} V(\mathbf{x}(k), \mathbf{u}(k))$$

Principle of optimality

$$J_k^*(\mathbf{x}(k)) = \min_{\mathbf{u}(k)} \left[V[\mathbf{x}(k), \mathbf{u}(k)] + J_{k+1}^*(\mathbf{x}^*(k+1))\right]$$

Example 6.2

$$x(k+1) = x(k) + u(k)$$

$$J = \frac{1}{2}x^2(k_f) + \frac{1}{2}\sum_{k=k_0}^{k_f-1}\left[x^2(k) + u^2(k)\right] \qquad k_f = 2$$

$$-1.0 \le u(k) \le +1.0, \quad k = 0, 1, 2 \quad \text{or}$$
$$u(k) = -1.0, \quad -0.5, \quad 0, \quad +0.5, \quad +1.0$$

$$0 \le x(k) \le +2.0, \quad k = 0, 1 \quad \text{or}$$
$$x(k) = 0, \quad 0.5, \quad 1.0 \quad 1.5 \quad 2.0.$$

Solution:

$$J_k(x(k)) = \min_{u(k)} \left[\frac{1}{2} u^2(k) + \frac{1}{2} x^2(k) + J^*_{k+1} \right]$$

Principle of optimality

$$\underbrace{J^*(x(0))}_{J_{02}} = \min_{u(0)} [V(x(0), u(0)) + \underbrace{J^*(x(1))}_{J_{12}}]$$

Table 6.2 Computation of Cost during the Last Stage $k = 2$

Current State $x(1)$	Current Control $u(1)$	Next State $x(2)$	Cost J_{12}	Optimal Cost $J^*_{12}(x(1))$	Optimal Control $u^*(x(1), 1)$
	-1.0	1.0	3.0		
	-0.5	1.5	2.25	$J^*_{12}(2.0)$=2.25	u^*(1.5,1) = -0.5
2.0	0	2.0	4.0		
	0.5	~~2.5~~			
	1.0	~~3.0~~			
	-1.0	0.5	1.75	$J^*_{12}(1.5)$=1.75	u^*(1.5,1) = -1.0
	-0.5	1.0	1.75	$J^*_{12}(1.5)$=1.75	u^*(1.5,1) = -0.5
1.5	0	1.5	2.25		
	0.5	2.0	3.25		
	1.0	~~2.5~~			
	-1.0	0	1.0		
	-0.5	0.5	0.75	$J^*_{12}(1.0) = 0.75$	u^*(1,1) = -0.5
1.0	0	1.0	1.0		
	0.5	1.5	1.75		
	1.0	2.0	3.0		

Use these to calculate the above: $x(2) = x(1) + u(1)$;
$J_{12} = 0.5x^2(2) + 0.5u^2(1) + 0.5x^2(1)$
A strikeout ($\longrightarrow$) indicates the value is not admissible.

Current State $x(1)$	Current Control $u(1)$	Next State $x(2)$	Cost J_{12}	Optimal Cost $J^*_{12}(x(1))$	Optimal Control $u^*(x(1),1)$
	-1.0	~~-0.5~~			
	-0.5	0	0.25	$J^*_{12}(0.5)=0.25$	$u^*(0.5,1)=-0.5$
0.5	0	0.5	0.25	$J^*_{12}(0.5)=0.25$	$u^*(0.5,1)=0$
	0.5	1.0	0.75		
	1.0	1.5	1.75		
	-1.0	~~-1.0~~			
	-0.5	~~-0.5~~			
0	0	0	0	$J^*_{12}(0)=0$	$u^*(0,1)=0$
	0.5	0.5	0.25		
	1.0	1.0	1.0		

Use these to calculate the above: $x(2) = x(1) + u(1)$;
$J_{12} = 0.5x^2(2) + 0.5u^2(1) + 0.5x^2(1)$
A strikeout ($\longrightarrow$) indicates the value is not admissible.

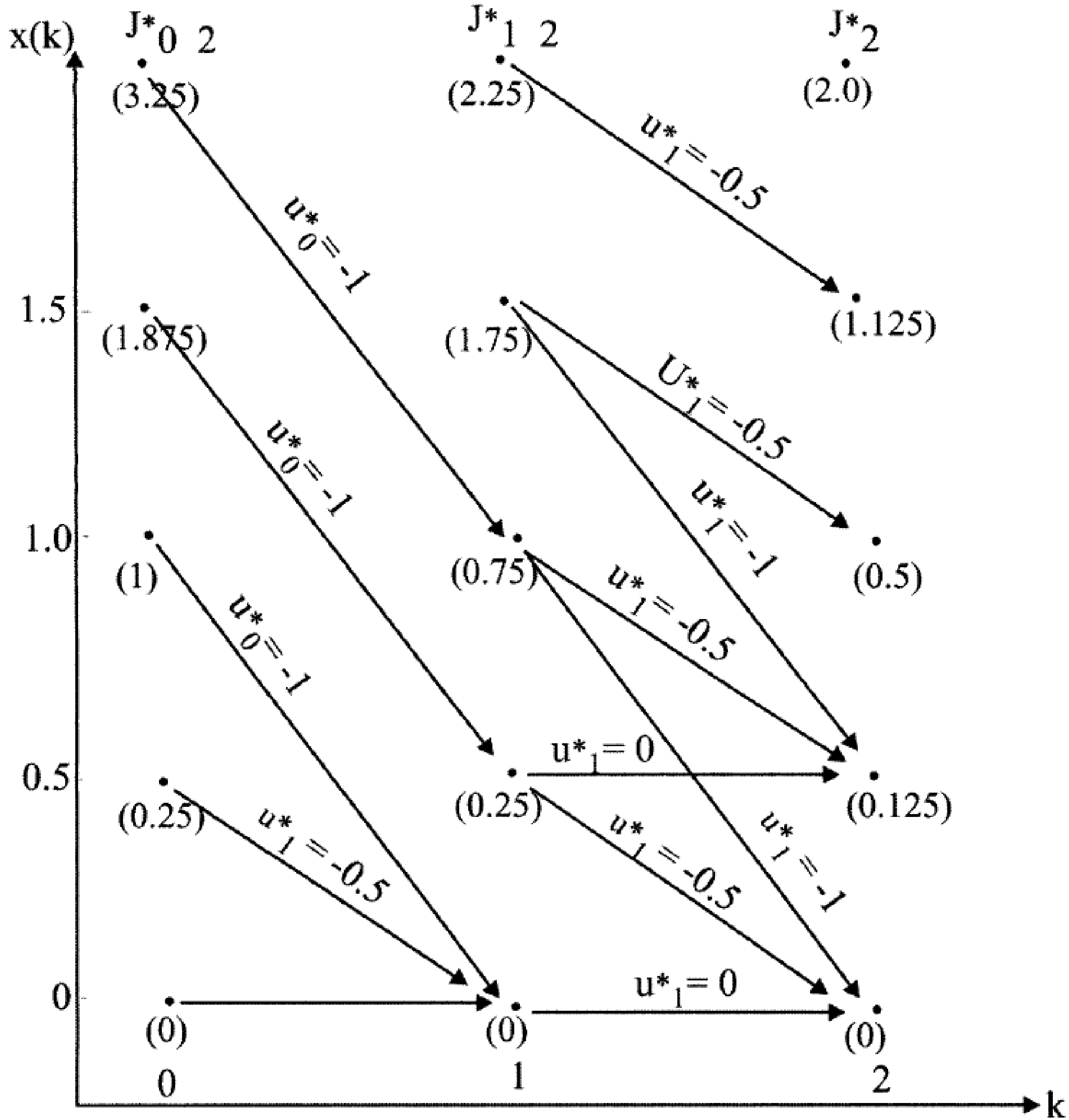
Table 6.3 Computation of Cost during the Stage $k = 1, 0$

Current State $x(0)$	Current Control $u(0)$	Next State $x(1)$	Cost J_{02}	Optimal Cost $J^*_{02}(x(0))$	Optimal Control $u^*(x(0), 0)$
	-1.0	1.0	3.25	$J^*_{02}(2.0) = 3.25$	$u^*(2.0, 0) = -1.0$
	-0.5	1.5	3.875		
2.0	0	2.0	4.25		
	0.5	~~2.5~~			
	1.0	~~3.0~~			
	-1.0	0.5	1.875	$J^*_{02}(1.5) = 1.875$	$u^*(1.5, 0) = -1.0$
	-0.5	1.0	2.0		
1.5	0	1.5	2.875		
	0.5	2.0	3.25		
	1.0	~~2.5~~			
	-1.0	0	1.0	$J^*_{02}(1)=1$	$u^*(1,0)$=-1.0
	-0.5	0.5	0.875		
1.0	0	1.0	1.25		
	0.5	1.5	2.375		
	1.0	2.0	3.0		

Use these to calculate the above: $x(1) = x(0) + u(0)$;
$J_{02} = 0.5u^2(0) + 0.5x^2(0) + J^*_{12}(x(1))$
A strikeout ($\longrightarrow$) indicates the value is not admissible.

Current State $x(0)$	Current Control $u(0)$	Next State $x(1)$	Cost J_{02}	Optimal Cost $J^*_{02}(x(0))$	Optimal Control $u^*(x(0), 0)$
0.5	-1.0	~~-0.5~~			
	-0.5	0	0.25	$J^*_{02}(0.5)$=0.25	u*(1,0)=-0.5
	0	0.5	0.375		
	0.5	1.0	1.0		
	1.0	1.5	2.375		
0	-1.0	~~-1.0~~			
	-0.5	~~-0.5~~			
	0	0	0	$J^*_{02}(0)$=0	$u^*(0,0) = 0$
	0.5	0.5	0.375		
	1.0	1.0	1.25		

Use these to calculate the above: $x(1) = x(0) + u(0)$;
$J_{02} = 0.5u^2(0) + 0.5x^2(0) + J^*_{12}(x(1))$
A strikeout ($\longrightarrow$) indicates the value is not admissible.


x(k)
J*0 2
J*1 2
J*2
(3.25)
(2.25)
(2.0)
u*1 = -0.5
u*0 = -1
1.5
(1.875)
(1.75)
(1.125)
U*1 = -0.5
u*0 = -1
u*1 = -1
1.0
(1)
(0.75)
(0.5)
u*0 = -1
u*1 = -0.5
u*1 = 0
0.5
(0.25)
(0.25)
(0.125)
u*1 = -0.5
u*1 = -0.5
u*1 = -1
0
u*1 = 0
(0)
(0)
(0)
0
1
2
k

6.4 The Hamilton-Jacobi-Bellman Equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J(\mathbf{x}(t_0), t_0) = \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

Define

$$J^*(\mathbf{x}^*(t), t) = \int_{t}^{t_f} V(\mathbf{x}^*(\tau), \mathbf{u}^*(\tau), \tau)d\tau$$

$$\Rightarrow \quad \frac{dJ^*(\mathbf{x}^*(t), t)}{dt} = -V(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$$

$$\underline{\frac{dJ^*(\mathbf{x}^*(t),t)}{dt}} = \left(\frac{\partial J^*(\mathbf{x}^*(t),t)}{\partial \mathbf{x}^*}\right)' \dot{\mathbf{x}}^*(t) + \frac{\partial J^*(\mathbf{x}^*(t),t)}{\partial t},$$

$$= \left(\frac{\partial J^*(\mathbf{x}^*(t),t)}{\partial \mathbf{x}^*}\right)' \mathbf{f}(\mathbf{x}^*(t),\mathbf{u}^*(t),t) + \frac{\partial J^*(\mathbf{x}^*(t),t)}{\partial t}.$$

$$\Rightarrow \quad \frac{\partial J^*(\mathbf{x}^*(t),t)}{\partial t} + \underline{V(\mathbf{x}^*(t),\mathbf{u}^*(t),t)} + \left(\frac{\partial J^*(\mathbf{x}^*(t),t)}{\partial \mathbf{x}^*}\right)' \mathbf{f}(\mathbf{x}^*(t),\mathbf{u}^*(t),t) = 0.$$

Define

$$\mathcal{H} = V(\mathbf{x}(t), \mathbf{u}(t), t) + \left(\frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial \mathbf{x}^*}\right)' \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$\boxed{\frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t} + \mathcal{H}\left(\mathbf{x}^*(t), \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial \mathbf{x}^*}, \mathbf{u}^*(t), t\right) = 0;\ \forall\ t \in [t_0, t_f)}$$

with boundary condition

$$J^*(\mathbf{x}^*(t_f), t_f) = 0$$

$$\text{or}\quad J^*(\mathbf{x}^*(t_f), t_f) = S(\mathbf{x}^*(t_f), t_f)$$

Table 6.4 Procedure Summary of Hamilton-Jacobi-Bellman (HJB) Approach

A. Statement of the Problem
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$, the performance index as $J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$, and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0; \quad \mathbf{x}(t_f)$ is free find the optimal control.

B. Solution of the Problem	
Step 1	Form the Pontryagin $\mathcal{H}$ function $\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), J^*_{\mathbf{x}}, t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + J^{*\prime}_{\mathbf{x}} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$.
Step 2	Minimize $\mathcal{H}$ w.r.t. $\mathbf{u}(t)$ as $\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$ and obtain $\mathbf{u}^*(t) = \mathbf{h}(\mathbf{x}^*(t), J^*_{\mathbf{x}}, t)$.
Step 3	Using the result of Step 2, find the optimal $\mathcal{H}^*$ function $\mathcal{H}^*(\mathbf{x}^*(t), \mathbf{h}(\mathbf{x}^*(t), J^*_{\mathbf{x}}, t), J^*_{\mathbf{x}}, t) = \mathcal{H}^*(\mathbf{x}^*(t), J^*_{\mathbf{x}}, t)$ and obtain the HJB equation.
Step 4	Solve the HJB equation $J^*_t + \mathcal{H}(\mathbf{x}^*(t), J^*_{\mathbf{x}}, t) = 0$. with boundary condition $J^*(\mathbf{x}^*(t_f), t_f) = S(\mathbf{x}(t_f), t_f)$.
Step 5	Use the solution J^*, from Step 4 to evaluate $J^*_{\mathbf{x}}$ and substitute into the expression for $\mathbf{u}^*(t)$ of Step 2, to obtain the optimal control.

Example 6.3

$$\dot{x}(t) = -2x(t) + u(t)$$

$$J = \frac{1}{2}x^2(t_f) + \frac{1}{2}\int_0^{t_f}[x^2(t) + u^2(t)]dt$$

Solution:

$$V(\mathbf{x}(t), \mathbf{u}(t), t) = \frac{1}{2}u^2(t) + \frac{1}{2}x^2(t); \quad S(\mathbf{x}(t_f), t_f) = \frac{1}{2}x^2(t_f)$$

$$f(\mathbf{x}(t), \mathbf{u}(t), t) = -2x(t) + u(t). \tag{6.4.19}$$

$$\mathcal{H}\left[\mathbf{x}^*(t), J_{\mathbf{x}}, \mathbf{u}^*(t), t\right] = V(\mathbf{x}(t), \mathbf{u}(t), t) + J_{\mathbf{x}}\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$
$$= \frac{1}{2}u^2(t) + \frac{1}{2}x^2(t) + J_{\mathbf{x}}(-2x(t) + u(t)).$$

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \longrightarrow u(t) + J_x = 0 \quad \Rightarrow \quad u^*(t) = -J_x$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}(-J_x)^2 + \frac{1}{2}x^2(t) + J_x(-2x(t) - J_x)$$
$$= -\frac{1}{2}J_x^2 + \frac{1}{2}x^2(t) - 2x(t)J_x.$$

HJB Eq.

$$\Rightarrow \quad J_t - \frac{1}{2}J_x^2 + \frac{1}{2}x^2(t) - 2x(t)J_x = 0$$

BC: $J(x(t_f), t_f) = S(x(t_f), t_f) = \frac{1}{2}x^2(t_f)$

Assume $J(x(t)) = \frac{1}{2}p(t)x^2(t)$

$$J(x(t_f)) = \frac{1}{2}x^2(t_f) = \frac{1}{2}p(t_f)x^2(t_f) \Rightarrow p(t_f) = 1$$

$$\Rightarrow J_x = p(t)x(t); \qquad J_t = \frac{1}{2}\dot{p}(t)x^2(t)$$

$$\Rightarrow u^*(t) = -p(t)x^*(t)$$

HJB Eq.

$$\Rightarrow \left(\frac{1}{2}\dot{p}(t) - \frac{1}{2}p^2(t) - 2p(t) + \frac{1}{2}\right)x^{*^2}(t) = 0$$

$$\Rightarrow \frac{1}{2}\dot{p}(t) - \frac{1}{2}p^2(t) - 2p(t) + \frac{1}{2} = 0$$

$$\Rightarrow p(t) = \frac{(\sqrt{5}-2) + (\sqrt{5}+2)\left[\frac{3-\sqrt{5}}{3+\sqrt{5}}\right]e^{2\sqrt{5}(t-t_f)}}{1 - \left[\frac{3-\sqrt{5}}{3+\sqrt{5}}\right]e^{2\sqrt{5}(t-t_f)}}$$

Note: Let us note that as $t_f \rightarrow \infty$, $p(t)$ in (6.4.33) becomes $p(\infty) = \bar{p} = \sqrt{5} - 2$, and the optimal control (6.4.30) is

$$\underline{u(t) = -(\sqrt{5} - 2)x(t)}. \tag{6.4.34}$$

6.5 LQR System Using H-J-B Equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$J = \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt$$

$$\begin{aligned}\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), J^*_{\mathbf{x}}, t) = {} & \frac{1}{2}\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \frac{1}{2}\mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t) \\ & + J^{*\prime}_{\mathbf{x}}(\mathbf{x}(t), t)[\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)]\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0 & \longrightarrow \mathbf{R}(t)\mathbf{u}(t) + \mathbf{B}'(t)J^{*\prime}_{\mathbf{x}}(\mathbf{x}(t), t) = 0 \\ & \longrightarrow \mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)J^*_{\mathbf{x}}(\mathbf{x}(t), t)\end{aligned}$$

note $\quad \dfrac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} = \mathbf{R}(t)$ ⇨ *minimum* control

$$\begin{aligned}\mathcal{H}(\mathbf{x}(t),\mathbf{u}(t),J_{\mathbf{x}}^*,t) &= \frac{1}{2}\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t)+\frac{1}{2}J_{\mathbf{x}}^{*\prime}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)J_{\mathbf{x}}^*\\ &\quad +J_{\mathbf{x}}^{*\prime}\mathbf{A}(t)\mathbf{x}(t)-J_{\mathbf{x}}^{*\prime}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)J_{\mathbf{x}}^*\\ &= \frac{1}{2}\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t)-\frac{1}{2}J_{\mathbf{x}}^{*\prime}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)J_{\mathbf{x}}^*\\ &\quad +J_{\mathbf{x}}^{*\prime}\mathbf{A}(t)\mathbf{x}(t). \qquad (6.5.7)\end{aligned}$$

HJB eq: $J_t^*+\mathcal{H}(\mathbf{x}^*(t),\mathbf{u}^*(t),J_{\mathbf{x}}^*,t)=0$

$$\Rightarrow \quad J_t^*+\frac{1}{2}\mathbf{x}^{*\prime}(t)\mathbf{Q}(t)\mathbf{x}^*(t)-\frac{1}{2}J_{\mathbf{x}}^{*\prime}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)J_{\mathbf{x}}^* +J_{\mathbf{x}}^{*\prime}\mathbf{A}(t)\mathbf{x}^*(t)=0$$

with BC $\quad J^*(\mathbf{x}^*(t_f),t_f)=\frac{1}{2}\mathbf{x}^{*\prime}(t_f)\mathbf{F}(t_f)\mathbf{x}^*(t_f)$

assume $$J^*(\mathbf{x}(t),t) = \frac{1}{2}\mathbf{x}'(t)\mathbf{P}(t)\mathbf{x}(t)$$

where, $\mathbf{P}(t)$ is a real, symmetric, positive-definite matrix

$$\begin{cases} \dfrac{\partial J^*}{\partial t} = J_t = \dfrac{1}{2}\mathbf{x}(t)\dot{\mathbf{P}}(t)\mathbf{x}(t) \\ \dfrac{\partial J^*}{\partial \mathbf{x}} = J_{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}(t) \end{cases}$$

HJB eq

$$\begin{aligned} \frac{1}{2}\mathbf{x}'(t)\dot{\mathbf{P}}(t)\mathbf{x}(t) + \frac{1}{2}&\mathbf{x}(t)\mathbf{Q}(t)\mathbf{x}(t) \\ &- \frac{1}{2}\mathbf{x}'(t)\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}(t) \\ &+\mathbf{x}'(t)\mathbf{P}(t)\mathbf{A}(t)\mathbf{x}(t) = 0. \end{aligned} \tag{6.5.13}$$

$$\boxed{\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t) - \mathbf{Q}(t).}$$

with BC $\boxed{\mathbf{P}(t_f) = \mathbf{F}(t_f).}$

Also, $\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}^*(t)$

Example 6.4

$$\dot{x}(t) = -2x(t) + u(t)$$

$$J = \int_0^{\infty} \left[x^2(t) + u^2(t)\right] dt$$

Solution:

$$V(x(t), u(t)) = x^2(t) + u^2(t),$$
$$f(x(t), u(t)) = -2x(t) + u(t).$$

$$\begin{aligned}\mathcal{H}(x(t), u(t), J_x^*) &= V(x(t), u(t)) + J_x^* f(x(t), u(t)) \\ &= x^2(t) + u^2(t) + 2fx(t)\left[-2x(t) + u(t)\right] \\ &= x^2(t) + u^2(t) - 4fx^2(t) + 2fx(t)u(t)\end{aligned}$$

Assume $J^* = fx^2(t)$

$$\frac{\partial \mathcal{H}}{\partial u} = 2u^*(t) + 2fx^*(t) = 0 \longrightarrow u^*(t) = -fx^*(t).$$

$$\mathcal{H}^*(x^*(t), J_x^*, t) = x^{*^2}(t) - 4fx^{*^2}(t) - f^2x^{*^2}(t)$$

HJB eq $\quad \mathcal{H}^*(x^*(t), J_x^*) + J_t^* = 0$

$$\longrightarrow \quad x^{*^2}(t) - 4fx^{*^2}(t) - f^2x^{*^2}(t) = 0$$

$$\longrightarrow \quad f^2 + 4f - 1 = 0$$

$$\longrightarrow \quad f = -2 \pm \sqrt{5}.$$

$$u^*(t) = -fx^*(t) = -(\sqrt{5} - 2)x^*(t)$$