Chapter 2

Calculus of Variations and Optimal Control

- Calculus of variations (CoV) or variational calculus deals with finding the *optimum* (maximum or minimum) value of a functional.
- In this chapter, we start with some basic definitions and a simple variational problem of extremizing a functional.
- We then incorporate the plant as a conditional optimization problem and discuss various types of problems based on the boundary conditions.
- We briefly mention both the Lagrangian and Hamiltonian formalisms for optimization.

2.1 Basic Concepts

2.1.1 Function and Functional

Function

- A variable x is a function of a variable quantity t, (written as $x(t) = f(t)$), if to every value of t over a certain range of t there corresponds a value x;
- i.e., we have a correspondence: to a number t there corresponds a number x. Note that here t need not be always time but any independent variable.

Example 2.1

Consider

$$x(t) = 2t^2 + 1. \tag{2.1.1}$$

For $t = 1$, $x = 3, t = 2$, $x = 9$ and so on. Other functions are $x(t) = 2t$; $x(t_1, t_2) = t_1^2 + t_2^2$.

Functional

- A variable quantity J is a functional dependent on a function $f(x)$, written as $J = J(f(x))$, if to each function $f(x)$, there corresponds a value J,
- i.e., we have a correspondence: to the function $f(x)$ there corresponds a number J. Functional depends on several functions.

- Loosely speaking, a functional can be thought of as a "function of a function."

Example 2.2

Let $x(t) = 2t^2 + 1$. Then

$$J(x(t)) = \int_0^1 x(t)dt = \int_0^1 (2t^2 + 1)dt = \frac{2}{3} + 1 = \frac{5}{3} \tag{2.1.2}$$

is the area under the curve $x(t)$. If $v(t)$ is the velocity of a vehicle, then

$$J(v(t)) = \int_{t_0}^{t_f} v(t)dt \tag{2.1.3}$$

is the path traversed by the vehicle. Thus, here $x(t)$ and $v(t)$ are functions of t, and J is a functional of $x(t)$ or $v(t)$.

2.1.2 Increment

Increment of a Function

- In order to consider optimal values of a function, we need the definition of an increment.

DEFINITION 2.1 *The increment of the function f, denoted by Δf, is defined as*

$$\boxed{\Delta f \triangleq f(t + \Delta t) - f(t)}. \tag{2.1.4}$$

It is easy to see from the definition that Δf depends on both the independent variable t and the increment of the independent variable Δt, and hence strictly speaking, we need to write the increment of a function as $\Delta f(t, \Delta t)$.

Example 2.3

If

$$f(t) = (t_1 + t_2)^2 \tag{2.1.5}$$

find the increment of the function $f(t)$.

Solution: The increment Δf becomes

$$\begin{aligned}
\Delta f &\triangleq f(t + \Delta t) - f(t) \\
&= (t_1 + \Delta t_1 + t_2 + \Delta t_2)^2 - (t_1 + t_2)^2 \\
&= (t_1 + \Delta t_1)^2 + (t_2 + \Delta t_2)^2 + 2(t_1 + \Delta t_1)(t_2 + \Delta t_2) - \\
&\quad (t_1^2 + t_2^2 + 2t_1 t_2) \\
&= 2\underline{(t_1 + t_2)}\underline{\Delta t_1} + 2\underline{(t_1 + t_2)}\underline{\Delta t_2} + (\Delta t_1)^2 + (\Delta t_2)^2 \\
&\quad + 2\Delta t_1 \Delta t_2.
\end{aligned} \tag{2.1.6}$$

Increment of a Functional

DEFINITION 2.2 *The increment of the functional J, denoted by ΔJ, is defined as*

$$\boxed{\Delta J \triangleq J(x(t) + \delta x(t)) - J(x(t)).} \tag{2.1.7}$$

Here $\delta x(t)$ is called the *variation* of the function $x(t)$. Since the increment of a functional is dependent upon the function $x(t)$ and its variation $\delta x(t)$, strictly speaking, we need to write the increment as $\Delta J(x(t), \delta x(t))$.

Example 2.4

Find the increment of the functional

$$J = \int_{t_0}^{t_f} \left[2x^2(t) + 1\right] dt. \tag{2.1.8}$$

Solution: The increment of J is given by

$$\begin{aligned}
\Delta J &\triangleq J(x(t) + \delta x(t)) - J(x(t)), \\
&= \int_{t_0}^{t_f} \left[2(x(t) + \delta x(t))^2 + 1\right] dt - \int_{t_0}^{t_f} \left[2x^2(t) + 1\right] dt, \\
&= \int_{t_0}^{t_f} \left[4x(t)\delta x(t) + 2(\delta x(t))^2\right] dt.
\end{aligned} \tag{2.1.9}$$

Calculus		Variation of calculus	
Independent variable	t	Function	$x(t)$
Function	$f(t)$	Functional	$J(x(t))$
Increment of independent variable	Δt	Variation of function	δx
Increment of function	Δf	Increment of functional	ΔJ
Differential of function	df	Variation of functional	δJ

2.1.3 Differential and Variation

Differential of a Function

(a) Differential of a Function: Let us define at a point t^* the increment of the function f as

$$\Delta f \triangleq \underline{f(t^* + \Delta t)} - f(t^*). \tag{2.1.10}$$

By expanding $f(t^* + \Delta t)$ in a Taylor series about t^*, we get

$$\Delta f = \underline{f(t^*) + \boxed{\left(\frac{df}{dt}\right)_* \Delta t} + \frac{1}{2!}\left(\frac{d^2 f}{dt^2}\right)_* (\Delta t)^2 + \cdots} - f(t^*). \tag{2.1.11}$$

Neglecting the higher order terms in Δt,

$$\Delta f \approx \left(\frac{df}{dt}\right)_* \Delta t = \dot{f}(t^*)\Delta t = df. \tag{2.1.12}$$

Here, *df* is called the *differential* of f at the point t^*. $\dot{f}(t^*)$ is the *derivative* or slope of f at t^*. In other words, the differential df is the first order approximation to increment Δt. Figure 2.1 shows the relation between increment, differential and derivative.

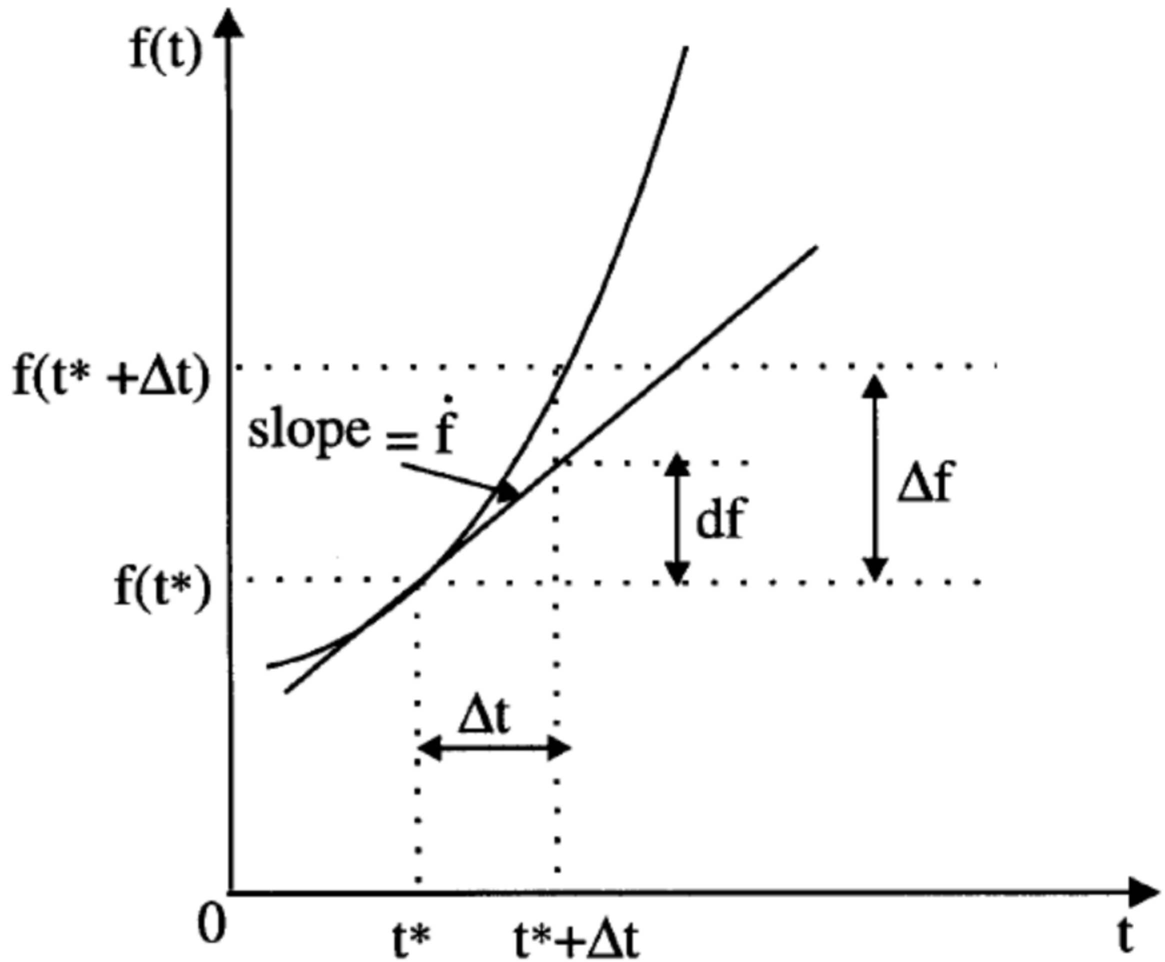


Figure 2.1 Increment Δf, Differential df, and Derivative $\dot{f}$ of a Function $f(t)$

- Δf: Increment of a function $f(t)$
- df: Differential of a function $f(t)$

  $\approx$ First-order approximation of increment Δf
- $\dot{f}$: Derivative of a function $f(t)$

Example 2.5

Let $f(t) = t^2 + 2t$. Find the increment and the derivative of the function $f(t)$.

Solution: By definition, the increment Δf is

$$
\begin{aligned}
\Delta f &\triangleq f(t + \Delta t) - f(t), \\
&= (t + \Delta t)^2 + 2(t + \Delta t) - (t^2 + 2t), \\
&= 2t\Delta t + 2\Delta t + \Delta t^2 \\
&= 2t\Delta t + 2\Delta t + \cdots + \text{higher order terms}, \\
&\approx 2(t + 1)\Delta t, \\
&= \dot{f}(t)\Delta t. \qquad (2.1.13)
\end{aligned}
$$

Here, $\dot{f}(t) = 2(t + 1)$.

Variation of a Functional

(b) Variation of a Functional: Consider the increment of a functional

$$\Delta J \triangleq \underline{J(x(t) + \delta x(t))} - J(x(t)). \qquad (2.1.14)$$

Expanding $J(x(t) + \delta x(t))$ in a Taylor series, we get

$$\begin{aligned}
\Delta J &= \underline{J(x(t)) + \frac{\partial J}{\partial x}\delta x(t) + \frac{1}{2!}\frac{\partial^2 J}{\partial x^2}(\delta x(t))^2 + \cdots} - J(x(t)) \\
&= \boxed{\frac{\partial J}{\partial x}\delta x(t)} + \boxed{\frac{1}{2!}\frac{\partial^2 J}{\partial x^2}(\delta x(t))^2} + \cdots \\
&= \underline{\delta J} + \underline{\delta^2 J} + \cdots, \qquad (2.1.15)
\end{aligned}$$

Derivatives of a functional *J*(*x*(*t*) w.r.t. a function *x*(*t*) where,

$$\delta J = \frac{\partial J}{\partial x}\delta x(t) \quad \text{and} \quad \delta^2 J = \frac{1}{2!}\frac{\partial^2 J}{\partial x^2}(\delta x(t))^2 \tag{2.1.16}$$

are called the *first variation* (or simply the *variation*) and the *second variation* of the functional J, respectively. The variation δJ of a functional J is the *linear* (or first order approximate) part (in $\delta x(t)$) of the increment ΔJ. Figure 2.2 shows the relation between increment and the first variation of a functional.

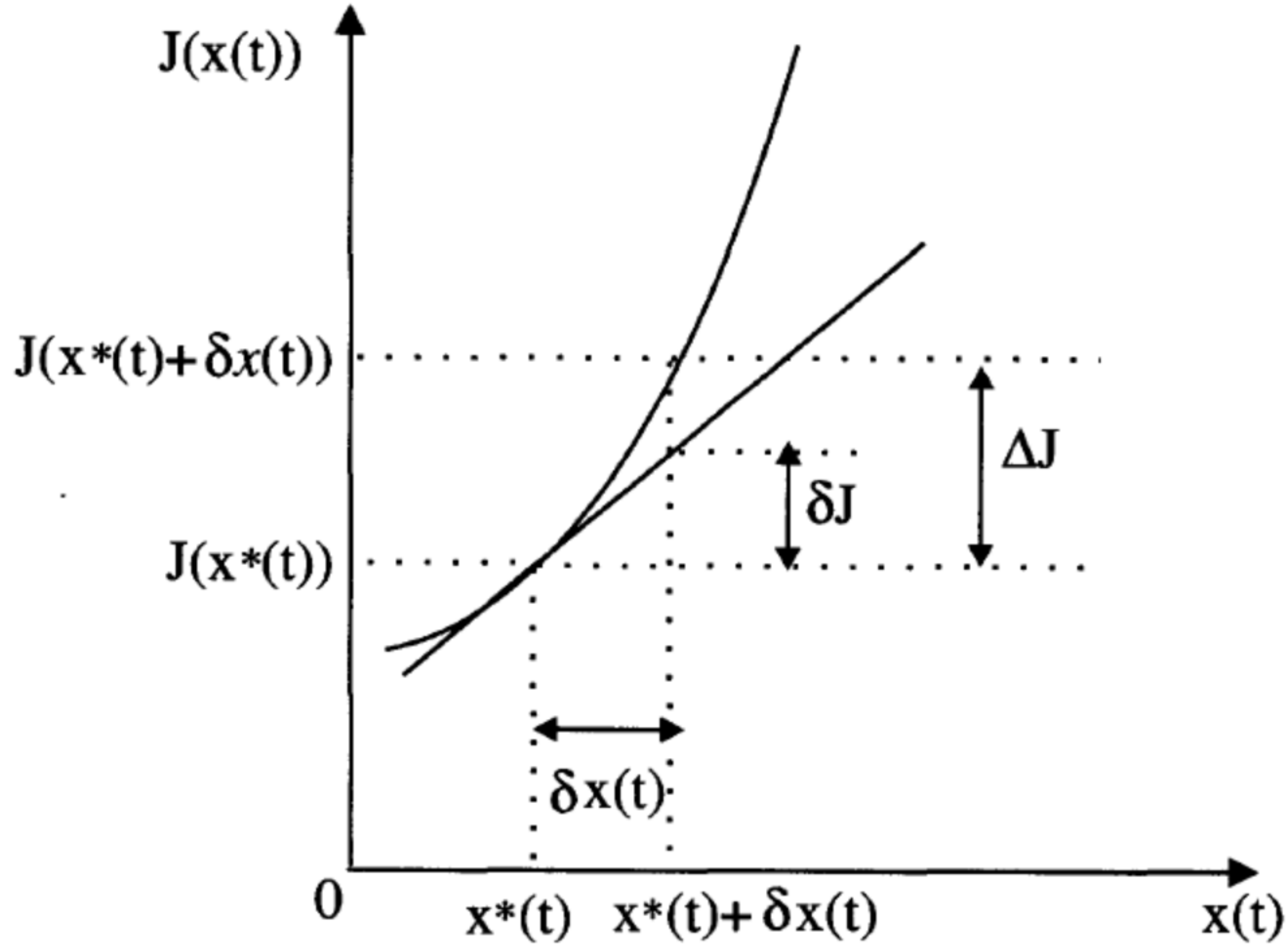


Figure 2.2 Increment ΔJ and the First Variation δJ of the Functional J

Example 2.6

Given the functional

$$J(x(t)) = \int_{t_0}^{t_f} [2x^2(t) + 3x(t) + 4]dt, \tag{2.1.17}$$

evaluate the variation of the functional.

Solution: First, we form the increment and then extract the variation as the first order approximation. Thus

$$\begin{aligned}
\Delta J &\triangleq J(x(t) + \delta x(t)) - J(x(t)), \\
&= \int_{t_0}^{t_f} \Big[2(x(t) + \delta x(t))^2 + 3(x(t) + \delta x(t)) + 4) \\
&\qquad -(2x^2(t) + 3x(t) + 4)\Big] dt, \\
&= \int_{t_0}^{t_f} \Big[4x(t)\delta x(t) + 2(\delta x(t))^2 + 3\delta x(t)\Big] dt. \qquad (2.1.18)
\end{aligned}$$

$$= \int_{t_0}^{t_f} [4x(t)+3]\delta x(t)dt + \int_{t_0}^{t_f} 2(\delta x(t))^2 dt$$

Considering only the first order terms, we get the (first) variation as

$$\delta J(x(t), \delta x(t)) = \int_{t_0}^{t_f} (4x(t) + 3)\delta x(t) dt. \tag{2.1.19}$$

$$\delta^2 J(x(t), \delta x(t)) = \int_{t_0}^{t_f} 2(\delta x(t))^2 dt$$

2.2 Optimum of a Function and a Functional

We give some definitions for optimum or extremum (maximum or minimum) of a function and a functional [47, 46, 79]. The *variation* plays the same role in determining optimal value of a functional as the *differential* does in finding extremal or optimal value of a function.

DEFINITION 2.3 Optimum of a Function: *A function $f(t)$ is said to have a relative optimum at the point t^* if there is a positive parameter ϵ such that for all points t in a domain $\mathcal{D}$ that satisfy $|t - t^*| < \epsilon$, the increment of $f(t)$ has the same sign (positive or negative).*

In other words, if

$$\Delta f = f(t) - f(t^*) \geq 0, \tag{2.2.1}$$

then, $f(t^*)$ is a relative local *minimum*. On the other hand, if

$$\Delta f = f(t) - f(t^*) \leq 0, \tag{2.2.2}$$

then, $f(t^*)$ is a relative local *maximum*. If the previous relations are valid for arbitrarily large ϵ, then, $f(t^*)$ is said to have a *global* absolute optimum. Figure 2.3 illustrates the (a) minimum and (b) maximum of a function.

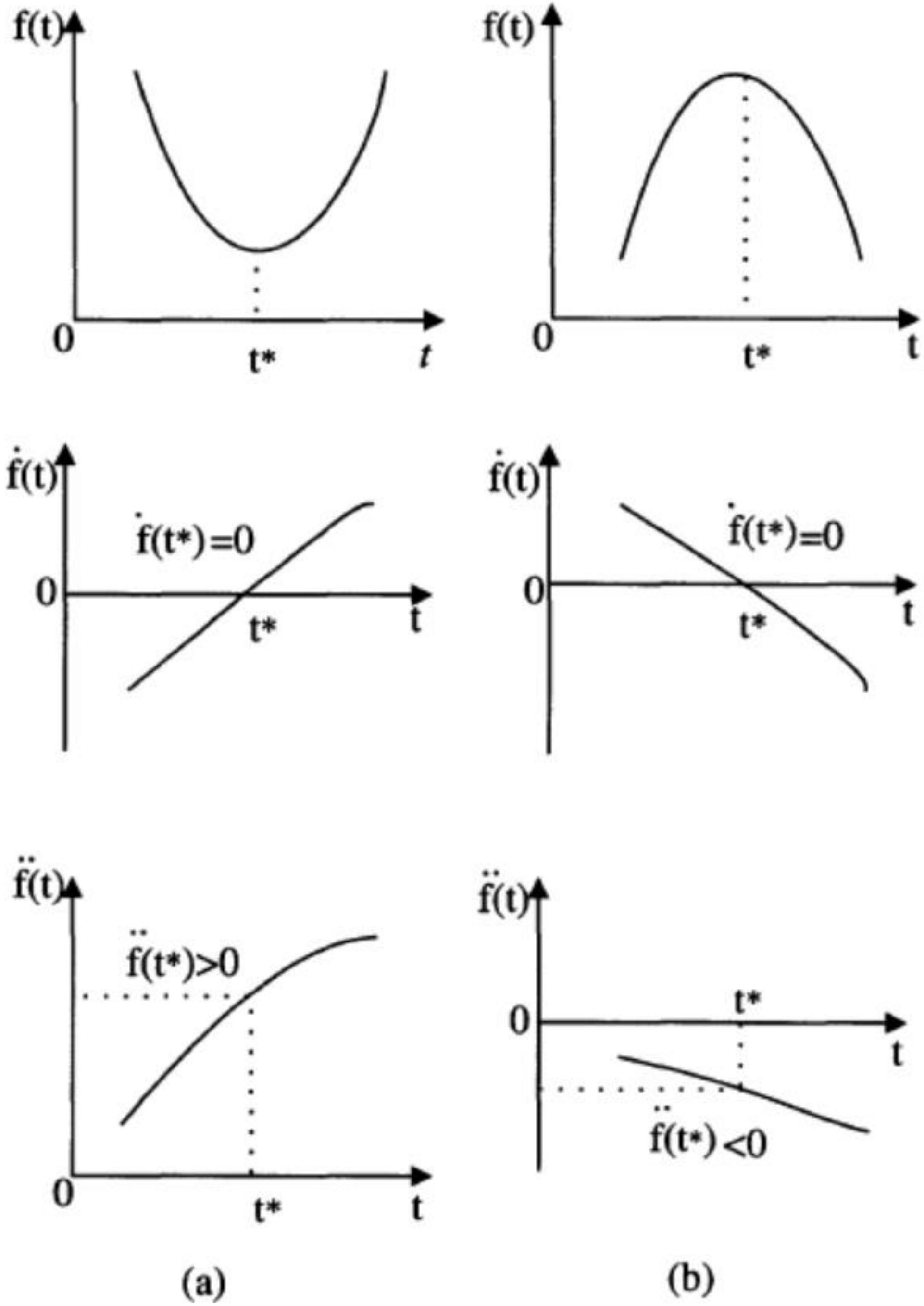


Figure 2.3 (a) Minimum and (b) Maximum of a Function $f(t)$

It is well known that the *necessary* condition for optimum of a function is that the (first) differential vanishes, i.e., $df = 0$. The *sufficient* condition

1. for *minimum* is that the second differential is positive, i.e., $d^2 f > 0$, and

2. for *maximum* is that the second differential is negative, i.e., $d^2 f < 0$.

If $d^2 f = 0$, it corresponds to a *stationary* (or inflection) point.

Necessary condition

- If an optimum exists, then $df = 0$.

$\Rightarrow$ If $df \neq 0$, then an optimum does not exist.

- If $d^2f > 0$, then a minimum exists.
- If $d^2f < 0$, then a maximum exists.
- If $d^2f = 0$, then a stationary point exists.

Sufficient conditions

- $df = 0$

$\Rightarrow d^2f > 0$ or $d^2f < 0$

$\Rightarrow$ minimum or maximum

$\Rightarrow$ optimum

DEFINITION 2.4 Optimum of a Functional: *A functional J is said to have a relative optimum at x^* if there is a positive ϵ such that for all functions x in a domain Ω which satisfy $|x - x^*| < \epsilon$, the increment of J has the same sign.*

In other words, if

$$\Delta J = J(x) - J(x^*) \geq 0, \tag{2.2.3}$$

then $J(x^*)$ is a relative *minimum*. On the other hand, if

$$\Delta J = J(x) - J(x^*) \leq 0, \tag{2.2.4}$$

then, $J(x^*)$ is a relative *maximum*. If the above relations are satisfied for arbitrarily large ϵ, then, $J(x^*)$ is a *global* absolute optimum.

Analogous to finding extremum or optimal values for *functions*, in variational problems concerning *functionals*, the result is that the variation must be zero on an optimal curve. Let us now state the result in the form of a theorem, known as *fundamental theorem of the calculus of variations*, the proof of which can be found in any book on calculus of variations [47, 46, 79].

THEOREM 2.1

For $x^*(t)$ to be a candidate for an optimum, the (first) variation of J must be zero on $x^*(t)$, i.e., $\delta J(x^*(t), \delta x(t)) = 0$ for all admissible values of $\delta x(t)$. This is a necessary condition. As a sufficient condition for minimum, the second variation $\delta^2 J > 0$, and for maximum $\delta^2 J < 0$.

Necessary condition

- If $x^*(t)$ is an optimum, then $\delta f = 0$.

$\Rightarrow$ If $\delta f \neq 0$, then an optimum does not exist.

- If $\delta^2 f > 0$, then a minimum exists.
- If $\delta^2 f < 0$, then a maximum exists.
- If $\delta^2 f = 0$, then a stationary point exists.

Sufficient conditions

- $\delta f = 0$

  $\Rightarrow \delta^2 f > 0$ or $\delta^2 f < 0$

  $\Rightarrow$ minimum or maximum

  $\Rightarrow$ optimum

2.3 The Basic Variational Problem

2.3.1 Fixed-End Time and Fixed-End State System

t_0, t_f fixed

x_0, x_f fixed

- We address a fixed-end time and fixed-end state problem, where both the *initial* time and state and the *final* time and state are fixed or given *a priori*.
- Let $x(t)$ be a scalar function with continuous first derivatives and the vector case can be similarly dealt with.
- The problem is to find the *optimal* function $x^*(t)$ for which the functional

$$J(x(t)) = \int_{t_0}^{t_f} V(x(t), \dot{x}(t), t)dt \tag{2.3.1}$$

  has a relative *optimum*.
- It is assumed that the integrand V has continuous first and second partial derivatives w.r.t. all its arguments; t_0 and t_f are fixed (or given *a priori*) and the end points are fixed, i.e.,

$$x(t = t_0) = x_0; \quad x(t = t_f) = x_f. \tag{2.3.2}$$

Steps to find the optimal solution to the fixed-end time and fixed-end state system

- Step 1: *Assumption of an Optimum*
- Step 2: *Variations and Increment*
- Step 3: *First Variation*
- Step 4: *Fundamental Theorem*
- Step 5: *Fundamental Lemma*
- Step 6: *Euler-Lagrange Equation*

Step 1: *Assumption of an Optimum*

- Let us assume that $x^*(t)$ is the optimum attained for the function $x(t)$.
- Take some admissible function $x_a(t) = x^*(t) + \delta x(t)$
- The function $x_a(t)$ should also satisfy the boundary conditions

$$\delta x(t_0) = \delta x(t_f) = 0. \tag{2.3.3}$$

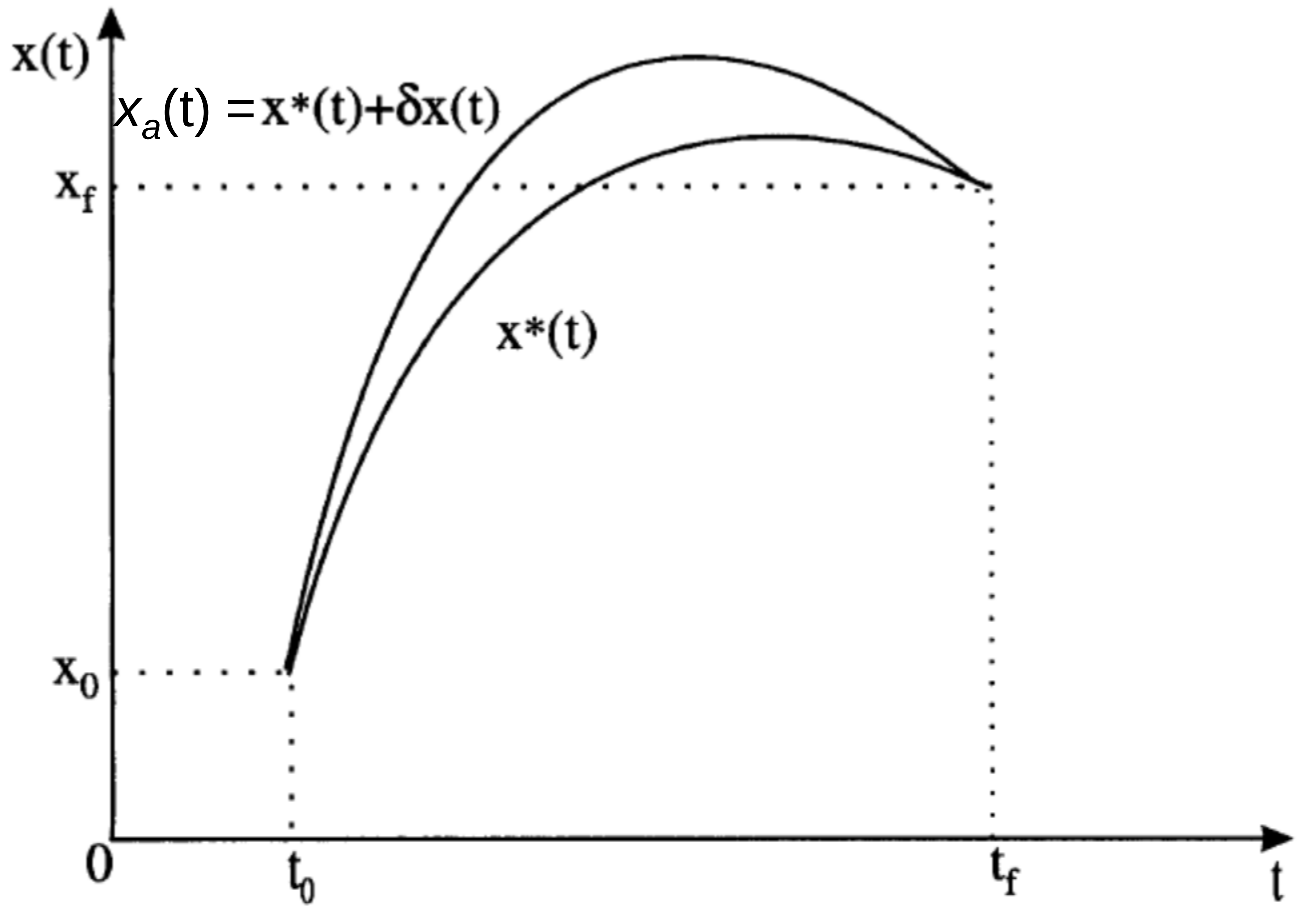


Figure 2.4 Fixed-End Time and Fixed-End State System

Step 2: *Variations and Increment*

$$\Delta J(x^*(t), \delta x(t)) \triangleq J(\underline{x^*(t) + \delta x(t)}, \underline{\dot{x}^*(t) + \delta \dot{x}(t)}, t)$$
$$- J(x^*(t), \dot{x}^*(t), t)$$
$$= \int_{t_0}^{t_f} V\left(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t\right) dt$$
$$- \int_{t_0}^{t_f} V(x^*(t), \dot{x}^*(t), t) dt. \qquad (2.3.4)$$

which by combining the integrals can be written as

$$\Delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} \left[V\left(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t\right)\right.$$
$$\left. - V(x^*(t), \dot{x}^*(t), t)\right] dt. \qquad (2.3.5)$$

where,

$$\dot{x}(t) = \frac{dx(t)}{dt} \quad \text{and} \quad \underline{\delta \dot{x}(t) = \frac{d}{dt}\{\delta x(t)\}} \qquad (2.3.6)$$

Expanding V in the increment (2.3.5) in a Taylor series

$$\begin{aligned}
\Delta J &= \Delta J(x^*(t), \delta x(t)) \\
&= \int_{t_0}^{t_f} \left[\underline{\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial x} \delta x(t) + \frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial \dot{x}} \delta \dot{x}(t)} \right. \\
&\quad + \frac{1}{2!} \left\{ \frac{\partial^2 V(\ldots)}{\partial x^2} (\delta x(t))^2 + \frac{\partial^2 V(\ldots)}{\partial \dot{x}^2} (\delta \dot{x}(t))^2 + \right. \\
&\quad \left. \left. + 2 \frac{\partial^2 V(\ldots)}{\partial x \partial \dot{x}} \delta x(t) \delta \dot{x}(t) \right\} + \cdots \right] dt. \qquad (2.3.7)
\end{aligned}$$

Here, the partial derivatives are w.r.t. $x(t)$ and $\dot{x}(t)$ at the optimal condition (*) and * is omitted for simplicity.

Step 3: *First Variation*

$$\boxed{\delta J(x^*(t), \delta x(t))} = \int_{t_0}^{t_f} \left[\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial x} \delta x(t) \right.$$
$$\left. + \underline{\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial \dot{x}} \delta \dot{x}(t)} \right] dt. \qquad (2.3.8)$$

Integration by parts

$$\begin{aligned}
\underline{\int_{t_0}^{t_f} \left(\frac{\partial V}{\partial \dot{x}}\right)_* \delta \dot{x}(t) dt} &= \int_{t_0}^{t_f} \left(\frac{\partial V}{\partial \dot{x}}\right)_* \frac{d}{dt}(\delta x(t)) dt \\
&= \int_{t_0}^{t_f} \left(\frac{\partial V}{\partial \dot{x}}\right)_* d(\delta x(t)), \\
&= \left[\left(\frac{\partial V}{\partial \dot{x}}\right)_* \delta x(t)\right]_{t_0}^{t_f} \\
&\quad - \int_{t_0}^{t_f} \delta x(t) \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{x}}\right)_* dt. \qquad (2.3.9)
\end{aligned}$$

$$\delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} \left(\frac{\partial V}{\partial x}\right)_* \delta x(t) dt + \left[\left(\frac{\partial V}{\partial \dot{x}}\right)_* \delta x(t)\right]_{t_0}^{t_f}$$

$$- \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* \delta x(t) dt,$$

$$= \int_{t_0}^{t_f} \left[\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_*\right] \delta x(t) dt$$

$$+ \left[\left(\frac{\partial V}{\partial \dot{x}}\right)_* \underline{\delta x(t)}\right]\Bigg|_{t_0}^{t_f}. \qquad (2.3.10)$$

Using the relation (2.3.3) for boundary variations in (2.3.10), we get

$$\delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} \left[\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_*\right] \delta x(t) dt. \quad (2.3.11)$$

Step 4: *Fundamental Theorem*

- Apply the fundamental theorem of the calculus of variations (Theorem 2.1)

$\delta J(x^*(t), \delta x(t)) = 0.$

$$\int_{t_0}^{t_f} \left[\left(\frac{\partial V}{\partial x} \right)_* - \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{x}} \right)_* \right] \delta x(t) dt = 0. \qquad (2.3.12)$$

Note that the function $\delta x(t)$ must be zero at t_0 and t_f, but for this, it is completely arbitrary.

Step 5: *Fundamental Lemma*

- Take advantage of the following lemma called *the fundamental lemma of the calculus of variations*

LEMMA 2.1

If for every function $g(t)$ which is continuous,

$$\int_{t_0}^{t_f} g(t)\delta x(t)dt = 0 \qquad (2.3.13)$$

where the function $\delta x(t)$ is continuous in the interval $[t_0, t_f]$, then the function $g(t)$ must be zero everywhere throughout the interval $[t_0, t_f]$. (see Figure 2.5.)

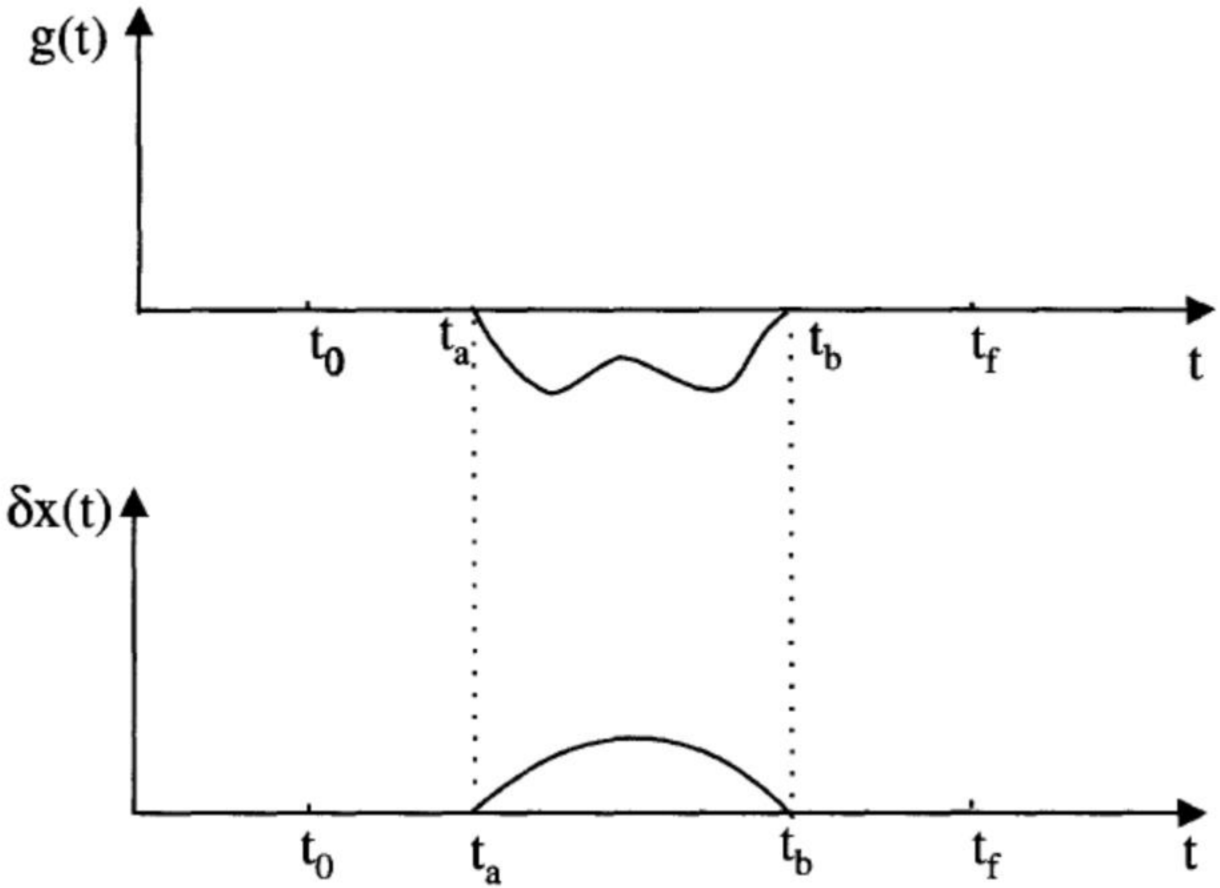


Figure 2.5 *A Nonzero $g(t)$ and an Arbitrary $\delta x(t)$*

Proof

- Assume that $g(t)$ is nonzero (positive or negative) during a short interval $[t_a, t_b]$.
- Let us select $\delta x(t)$, which is arbitrary, to be positive (or negative) throughout the interval where $g(t)$ has a nonzero value.
- By this selection of $\delta x(t)$, the value of the integral in (2.3.13) will be nonzero.
- This contradicts our assumption that $g(t)$ is non-zero during the interval.
- Thus $g(t)$ must be identically zero everywhere during the entire interval $[t_o, t_f]$ in (2.3.13).

Step 6: *Euler-Lagrange Equation*

- Applying the previous lemma

$$\left(\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial \dot{x}}\right)_* = 0 \quad (2.3.14)$$

or

$$\boxed{\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* = 0} \quad (2.3.15)$$

for all $t \in [t_0, t_f]$. This equation is called Euler equation, first published in 1741 [126].

A historical note is worthy of mention.

> *Euler obtained the equation (2.3.14) in 1741 using an elaborate and cumbersome procedure. Lagrange studied Euler's results and wrote a letter to Euler in 1755 in which he obtained the previous equation by a more elegant method of "variations" as described above. Euler recognized the simplicity and generality of the method of Lagrange and introduced the name* **calculus of variations**. *The all important fundamental equation (2.3.14) is now generally known as Euler-Lagrange (E.-L.) equation after these two great mathematicians of the 18th century. Lagrange worked further on optimization and came up with the well-known Lagrange multiplier rule or method.*

2.3.2 Discussion on Euler-Lagrange Equation

Some comments on the Euler-Lagrange equation

1. $$V_x - \frac{d}{dt}(V_{\dot{x}}) = 0 \qquad (2.3.16)$$

where,

$$V_x = \frac{\partial V}{\partial x} = V_x(x^*(t), \dot{x}^*(t), t); \qquad V_{\dot{x}} = \frac{\partial V}{\partial \dot{x}} = V_{\dot{x}}(x^*(t), \dot{x}^*(t), t). \qquad (2.3.17)$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* &= \frac{d}{dt}\left(\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial \dot{x}}\right)_*, \\
&= \frac{1}{dt}\left(\frac{\partial^2 V}{\partial x \partial \dot{x}} dx + \frac{\partial^2 V}{\partial \dot{x} \partial \dot{x}} d\dot{x} + \frac{\partial^2 V}{\partial t \partial \dot{x}} dt\right)_*, \\
&= \left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)_* \left(\frac{dx}{dt}\right)_* + \left(\frac{\partial^2 V}{\partial \dot{x} \partial \dot{x}}\right)_* \left(\frac{d^2 x}{dt^2}\right)_* + \left(\frac{\partial^2 V}{\partial t \partial \dot{x}}\right)_* \\
&= V_{x\dot{x}} \dot{x}^*(t) + V_{\dot{x}\dot{x}} \ddot{x}^*(t) + V_{t\dot{x}}. \qquad (2.3.18)
\end{aligned}$$

$$\boxed{V_x - V_{t\dot{x}} - V_{x\dot{x}} \dot{x}^*(t) - V_{\dot{x}\dot{x}} \ddot{x}^*(t) = 0.} \qquad (2.3.19)$$

2. The presence of $\frac{d}{dt}$ and/or $\dot{x}^*(t)$ in the EL equation (2.3.14) means that it is a *differential* equation.

3. In the EL equation (2.3.14), the term $\frac{\partial V(x^*(t),\dot{x}^*(t),t)}{\partial \dot{x}}$ is in general a function of $x^*(t)$, $\dot{x}^*(t)$, and t. Thus when this function is differentiated w.r.t. t, $\ddot{x}^*(t)$ may be present. This means that the differential equation (2.3.14) is in general of *second* order. This is also evident from the alternate form (2.3.19) for the EL equation.

4. There may also be terms involving products or powers of $\ddot{x}^*(t)$, $\dot{x}^*(t)$, and $x^*(t)$, in which case, the differential equation becomes *nonlinear*.

5. The explicit presence of t in the arguments indicates that the coefficients may be *time-varying*.

6. The conditions at initial point $t = t_0$ and final point $t = t_f$ leads to a *boundary value* problem.

7. Thus, the Euler-Lagrange equation (2.3.14) is, in general, a nonlinear, time-varying, two-point boundary value, second order, ordinary differential equation. Thus, we often have a nonlinear *two-point boundary value problem* (TPBVP). The solution of the nonlinear TPBVP is quite a formidable task and often done using numerical techniques. This is the price we pay for demanding optimal performance!

8. Compliance with the Euler-Lagrange equation is only a *necessary* condition for the optimum. Optimal may sometimes not yield either a maximum or a minimum; just as inflection points where the derivative vanishes in differential calculus. However, if the Euler-Lagrange equation is not satisfied for any function, this indicates that the optimum does not exist for that functional.

Summary

$$\text{Min} \quad J(x(t)) = \int_{t_0}^{t_f} V(x(t), \dot{x}(t), t)\,dt$$

$$x(t = t_0) = x_0; \quad x(t = t_f) = x_f.$$

CoV

$$\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* = 0$$

(Eular-Lagrange equation)

$$x(t = t_0) = x_0; \quad x(t = t_f) = x_f.$$

Nonlinear, time-varying ODE
with two boundary conditions

Solution obtained by numerical techniques

Example: Spring-mass system

Lagrangian

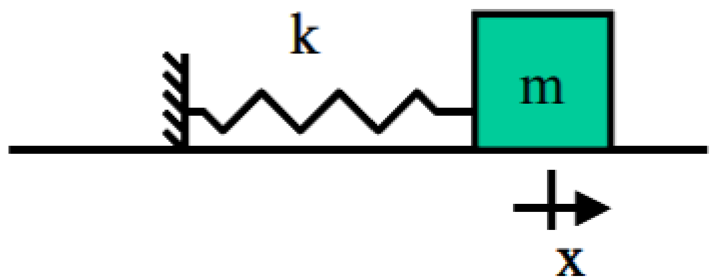
$$L = T - V = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2$$

$$V = L(t, x, \dot{x})$$

$$x(t_0) = x_0, \qquad x(t_f) = x_f$$

Spring mass system

- Linear spring
- Frictionless table



Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \xRightarrow{q = x} \quad \begin{array}{l} m\ddot{x} + kx = 0 \\ x(t_0) = x_0, \qquad x(t_f) = x_f \end{array}$$

2.3.3 Different Cases for Euler-Lagrange Equation

Euler-Lagrange equation

$$\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* = 0$$

$$V_x - \frac{d}{dt}\left(V_{\dot{x}}\right) = 0$$

$$V_x - V_{t\dot{x}} - V_{x\dot{x}}\dot{x}^*(t) - V_{\dot{x}\dot{x}}\ddot{x}^*(t) = 0$$

Case 1: $V = V(\dot{x}(t), t)$

Case 2: $V = V(\dot{x}(t))$

Case 3: $V = V(x(t), \dot{x}(t))$

Case 4: $V = V(x(t), t)$

Case 1: V is dependent of $\dot{x}(t)$, and t. That is, $V = V(\dot{x}(t), t)$. Then $V_x = 0$. The Euler-Lagrange equation (2.3.16) becomes

$$\frac{d}{dt}(V_{\dot{x}}) = 0. \tag{2.3.20}$$

This leads us to

$$\boxed{V_{\dot{x}} = \frac{\partial V(\dot{x}^*(t), t)}{\partial \dot{x}} = C} \tag{2.3.21}$$

where, C is a constant of integration.

Case 2: V is dependent of $\dot{x}(t)$ only. That is, $V = V(\dot{x}(t))$. Then $V_x = 0$. The Euler-Lagrange equation (2.3.16) becomes

$$\frac{d}{dt}(V_{\dot{x}}) = 0 \longrightarrow V_{\dot{x}} = C. \tag{2.3.22}$$

In general, the solution of either (2.3.21) or (2.3.22) becomes

$$\boxed{\dot{x}^*(t) = C_1 \longrightarrow x^*(t) = C_1 t + C_2.} \tag{2.3.23}$$

This is simply an equation of a straight line.

Case 3: V is dependent of $x(t)$ and $\dot{x}(t)$. That is, $V = V(x(t), \dot{x}(t))$. Then $V_{t\dot{x}} = 0$. Using the other form of the Euler-Lagrange equation (2.3.19), we get

$$V_x - V_{x\dot{x}}\dot{x}^*(t) - V_{\dot{x}\dot{x}}\ddot{x}^*(t) = 0. \qquad (2.3.24)$$

Multiplying the previous equation by $\dot{x}^*(t)$, we have

$$\dot{x}^*(t)\left[V_x - V_{x\dot{x}}\dot{x}^*(t) - V_{\dot{x}\dot{x}}\ddot{x}^*(t)\right] = 0. \qquad (2.3.25)$$

$$\Rightarrow V_x\dot{x} - V_{x\dot{x}}\dot{x}^2 - V_{\dot{x}\dot{x}}\dot{x}\ddot{x} = 0$$

$$\Rightarrow V_x\dot{x} + V_{\dot{x}}\ddot{x} - V_{\dot{x}}\ddot{x} - V_{x\dot{x}}\dot{x}^2 - V_{\dot{x}\dot{x}}\dot{x}\ddot{x} = 0$$

$$\Rightarrow (V_x\dot{x} + V_{\dot{x}}\ddot{x}) - (V_{\dot{x}}\ddot{x} + V_{x\dot{x}}\dot{x}^2 + V_{\dot{x}\dot{x}}\dot{x}\ddot{x}) = 0$$

$$\Rightarrow \frac{dV}{dt} - \frac{d(V_{\dot{x}}\dot{x})}{dt} = 0$$

This can be rewritten as

$$\frac{d}{dt}\left(V - \dot{x}^*(t)V_{\dot{x}}\right) = 0 \longrightarrow V - \dot{x}^*(t)V_{\dot{x}} = C. \qquad (2.3.26)$$

This can be rewritten as

$$\frac{d}{dt}\left(V - \dot{x}^*(t)V_{\dot{x}}\right) = 0 \longrightarrow V - \dot{x}^*(t)V_{\dot{x}} = C. \qquad (2.3.26)$$

The previous equation can be solved using any of the techniques such as, separation of variables.

For the mass-spring system, $V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$V - \dot{x}^*(t)V_{\dot{x}} = C. \Rightarrow -\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 = C$$

Constant total energy

Case 4: V is dependent of $x(t)$, and t, i.e., $V = V(x(t), t)$. Then, $V_{\dot{x}} = 0$ and the Euler-Lagrange equation (2.3.16) becomes

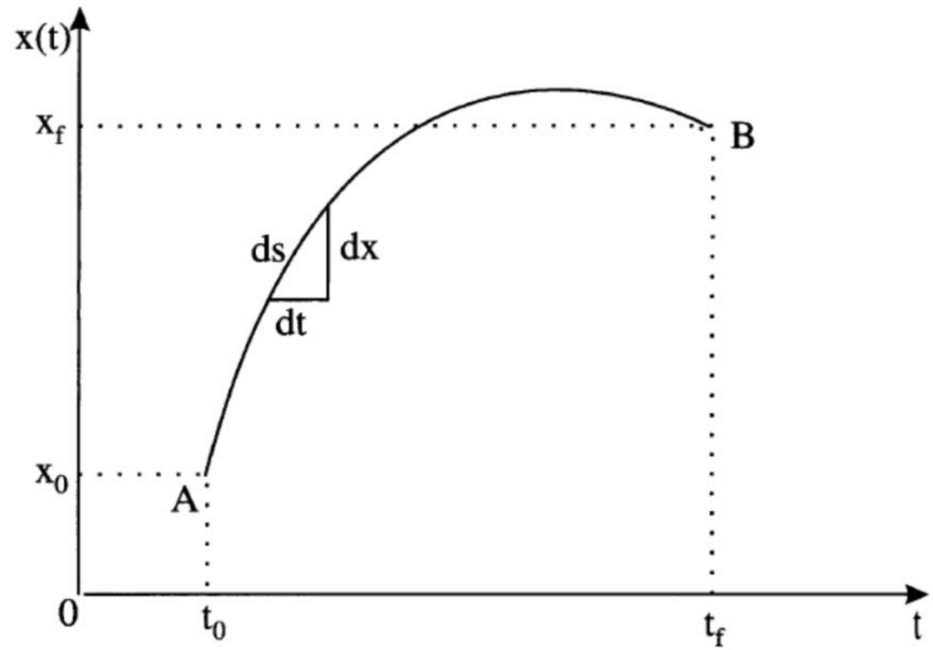
$$\frac{\partial V(x^*(t), t)}{\partial x} = 0. \tag{2.3.27}$$

The solution of this equation does not contain any arbitrary constants and therefore generally speaking does not satisfy the boundary conditions $x(t_0)$ and $x(t_f)$. Hence, in general, no solution exists for this variational problem. Only in rare cases, when the function $x(t)$ satisfies the given boundary conditions $x(t_0)$ and $x(t_f)$, it becomes an optimal function.

Example 2.7

Find the minimum length between any two points.

Solution:



$$(ds)^2 = (dx)^2 + (dt)^2$$

$$ds = \sqrt{1 + \dot{x}^2(t)}dt$$

where $\dot{x}(t) = \dfrac{dx}{dt}$

$$S = J = \int ds = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}^2(t)}dt = \int_{t_0}^{t_f} V(\dot{x}(t))dt$$

Case 2: a straight line

Applying the Euler-Lagrange equation

$$\Rightarrow \quad \frac{\dot{x}^*(t)}{\sqrt{1+\dot{x}^{*2}(t)}} = C$$

$$\Rightarrow \quad x^*(t) = C_1 t + C_2$$

if $x(0) = 1$ and $x(2) = 5$, $C_1 = 2$ and $C_2 = 1$

$$\Rightarrow \quad x^*(t) = 2t + 1$$

Although the previous example is a simple one,

1. it illustrates the formulation of a performance index from a given simple specification or a statement, and

2. the solution is well known *a priori* so that we can easily verify the application of the Euler-Lagrange equation.

In the previous example, we notice that the integrand V in the functional (2.3.30), is a function of $\dot{x}(t)$ only. Next, we take an example, where, V is a function of $x(t)$, $\dot{x}(t)$ and t.

Example 2.8

Find the optimum of

$$J = \int_0^2 \left[\dot{x}^2(t) - 2tx(t)\right] dt \tag{2.3.33}$$

that satisfy the boundary (initial and final) conditions

$$x(0) = 1 \quad \text{and} \quad x(2) = 5. \tag{2.3.34}$$

Solution: $V = \dot{x}^2(t) - 2tx(t)$

$$\frac{\partial V}{\partial x} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) = 0 \longrightarrow -2t - \frac{d}{dt}\left(2\dot{x}(t)\right) = 0$$

$$\longrightarrow \ddot{x}(t) = t. \tag{2.3.35}$$

$$x^*(t) = \frac{t^3}{6} + C_1 t + C_2 \tag{2.3.36}$$

2.4 The Second Variation

- In the study of extrema of functionals, we have so far considered only the *necessary* condition for a functional to have a relative or *weak extremum*,
- i.e., the condition that the first variation vanish leading to the classic *Euler-Lagrange equation.*
- To establish the nature of optimum (maximum or minimum), it is required to examine the *second variation.*
- In the relation (2.3.7) for the increment consider the terms corresponding to the second variation [120],

$$
\begin{aligned}
\Delta J &= \Delta J(x^*(t), \delta x(t)) \\
&= \int_{t_0}^{t_f} \left[\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial x} \delta x(t) + \frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial \dot{x}} \delta \dot{x}(t) \right. \\
&\quad + \frac{1}{2!} \left\{ \frac{\partial^2 V(\ldots)}{\partial x^2} (\delta x(t))^2 + \frac{\partial^2 V(\ldots)}{\partial \dot{x}^2} (\delta \dot{x}(t))^2 + \right. \\
&\quad \left. \left. + 2 \frac{\partial^2 V(\ldots)}{\partial x \partial \dot{x}} \delta x(t) \delta \dot{x}(t) \right\} + \cdots \right] dt. \qquad (2.3.7)
\end{aligned}
$$

$$
\begin{aligned}
\delta^2 J = \int_{t_0}^{t_f} \frac{1}{2!} &\left[\left(\frac{\partial^2 V}{\partial x^2} \right)_* (\delta x(t))^2 + \left(\frac{\partial^2 V}{\partial \dot{x}^2} \right)_* (\delta \dot{x}(t))^2 \right. \\
&\left. + 2 \left(\frac{\partial^2 V}{\partial x \partial \dot{x}} \right)_* \delta x(t) \delta \dot{x}(t) \right] dt. \qquad (2.4.1)
\end{aligned}
$$

$$\delta^2 J = \int_{t_0}^{t_f} \frac{1}{2!}\left[\left(\frac{\partial^2 V}{\partial x^2}\right)_* (\delta x(t))^2 + \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* (\delta \dot{x}(t))^2 \right.$$

$$\left. + 2\underline{\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)_* \delta x(t)\delta \dot{x}(t)}\right] dt. \qquad (2.4.1)$$

only using integration by parts $\int u dv = uv - \int v du$

$u = \frac{\partial^2 V}{\partial x \partial \dot{x}} \delta x(t)$ and $v = \delta x(t)$

using $\delta x(t_0) = \delta x(t_f) = 0$ for fixed-end conditions

$$\delta^2 J = \frac{1}{2}\int_{t_0}^{t_f}\left[\left\{\left(\frac{\partial^2 V}{\partial x^2}\right)_* \underline{- \frac{d}{dt}\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)_*}\right\}(\delta x(t))^2 \right.$$

$$\left. + \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* (\delta \dot{x}(t))^2\right] dt. \qquad (2.4.2)$$

According to Theorem 2.1, the fundamental theorem of the calculus of variations, the sufficient condition for a *minimum* is $\delta^2 J > 0$. This, for arbitrary values of $\delta x(t)$ and $\delta \dot{x}(t)$, means that

$$\boxed{\begin{aligned} \left(\frac{\partial^2 V}{\partial x^2}\right)_* - \frac{d}{dt}\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)_* &> 0, \qquad (2.4.3) \\ \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* &> 0. \qquad (2.4.4) \end{aligned}}$$

For *maximum*, the signs of the previous conditions are reversed.

- Alternatively, we can rewrite the second variation (2.4.1) in matrix form as

$$\delta^2 J = \int_{t_0}^{t_f} \frac{1}{2!} \left[\left(\frac{\partial^2 V}{\partial x^2} \right)_* (\delta x(t))^2 + \left(\frac{\partial^2 V}{\partial \dot{x}^2} \right)_* (\delta \dot{x}(t))^2 + 2 \left(\frac{\partial^2 V}{\partial x \partial \dot{x}} \right)_* \delta x(t) \delta \dot{x}(t) \right] dt. \qquad (2.4.1)$$

$$\begin{aligned} \delta^2 J &= \frac{1}{2} \int_{t_0}^{t_f} [\delta x(t) \;\; \delta \dot{x}(t)] \begin{bmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial \dot{x}} \\ \frac{\partial^2 V}{\partial x \partial \dot{x}} & \frac{\partial^2 V}{\partial \dot{x}^2} \end{bmatrix}_* \begin{bmatrix} \delta x(t) \\ \delta \dot{x}(t) \end{bmatrix} dt \\ &= \frac{1}{2} \int_{t_0}^{t_f} [\delta x(t) \;\; \delta \dot{x}(t)] \Pi \begin{bmatrix} \delta x(t) \\ \delta \dot{x}(t) \end{bmatrix} dt \end{aligned} \qquad (2.4.5)$$

where,

$$\Pi = \begin{bmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial \dot{x}} \\ \frac{\partial^2 V}{\partial x \partial \dot{x}} & \frac{\partial^2 V}{\partial \dot{x}^2} \end{bmatrix}_* . \qquad (2.4.6)$$

If the matrix $\mathbf{\Pi}$ in the previous equation is positive (negative) definite, we establish a minimum (maximum). In many cases since $\delta x(t)$ is arbitrary, the coefficient of $(\delta \dot{x}(t))^2$, i.e., $\partial^2 V/\partial \dot{x}^2$ determines the sign of $\delta^2 J$. That is, the sign of second variation agrees with the sign of $\partial^2 V/\partial \dot{x}^2$. Thus, for *minimization* requirement

$$\boxed{\left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* > 0.} \tag{2.4.7}$$

For *maximization*, the sign of the previous equation reverses. In the literature, this condition is called *Legendre condition* [126].

> *In 1786, Legendre obtained this result of deciding whether a given optimum is maximum or minimum by examining the second variation. The second variation technique was further generalized by Jacobi in 1836 and hence this condition is usually called Legendre-Jacobi condition.*

For the mass-spring system, $V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

$$\left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* > 0. \quad \Rightarrow \quad m > 0$$

Example 2.9

Verify that the straight line represents the minimum distance between two points.

Solution:

$$J = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}^2(t)}dt = \int_{t_0}^{t_f} V(\dot{x}(t))dt \qquad (2.4.8)$$

$$x^*(t) = C_1 t + C_2.$$

$$\left(\frac{\partial V}{\partial \dot{x}}\right)_* = \frac{\dot{x}^*(t)}{\sqrt{1 + \dot{x}^{*^2}(t)}} \quad \text{and} \quad \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* = \frac{1}{\left[1 + \dot{x}^{*^2}(t)\right]^{3/2}}. \qquad (2.4.9)$$

Since $\dot{x}^*(t)$ is a constant (+ve or -ve), the previous equation satisfies the condition (2.4.7). Hence, the distance between two points as given by $x^*(t)$ (straight line) is minimum.

2.5 Extrema of Functions with Conditions

- We begin with an example of finding the extrema of a function under a condition (or constraint).
- We solve this example with two methods, first by *direct* method and then by Lagrange *multiplier* method.
- Let us note that we consider this simple example only to illustrate some basic concepts associated with conditional extremization [120].

Example 2.10

- A manufacturer wants to maximize the volume of the material stored in a circular tank subject to the condition that the material used for the tank is limited (constant).

- Thus, for a constant thickness of the material, the manufacturer wants to minimize the volume of the material used and hence part of the cost for the tank.

Solution:

Then the volume contained by the tank is

$$V(d,h) = \pi d^2 h/4 \tag{2.5.1}$$

and the cross-sectional surface area (upper, lower and side) of the tank is

$$A(d,h) = 2\pi d^2/4 + \pi dh = A_0. \tag{2.5.2}$$

Our intent is to maximize $V(d,h)$ keeping $A(d,h) = A_0$, where A_0 is a given constant.

Max $\quad V(d,h) = \pi d^2 h/4$

Subjected to $\quad A(d,h) = 2\pi d^2/4 + \pi dh = A_0 =$ Constant

$\Rightarrow \quad d, h$

We discuss two methods:

1. First one is called the *Direct Method* using simple calculus

2. Second one is called *Lagrange Multiplier Method* using the Lagrange multiplier method.

1 Direct Method:

From (2.5.2),
$$h = \frac{A_0 - \pi d^2/2}{\pi d}. \tag{2.5.3}$$

(2.5.1) $\Rightarrow$
$$V(d) = A_0 d/4 - \pi d^3/8. \tag{2.5.4}$$

Differentiating (2.5.4) w.r.t. *d* and set it to zero to get

$$\frac{A_0}{4} - \frac{3}{8}\pi d^2 = 0. \tag{2.5.5}$$

$$\Rightarrow \quad d^* = \sqrt{\frac{2A_0}{3\pi}}. \tag{2.5.6}$$

- The second derivative of *V* w.r.t. *d* in (2.5.4) is *negative* for *maximum* by Definition 2.3.

Substituting (2.5.6) into (2.5.3) leads to

$$h^* = \sqrt{\frac{2A_0}{3\pi}}. \tag{2.5.7}$$

2 Lagrange Multiplier Method

Let the original volume relation (2.5.1) to be extremized be rewritten as

$$f(d,h) = \pi d^2 h/4 \tag{2.5.8}$$

and the condition (2.5.2) to be satisfied as

$$g(d,h) = 2\pi d^2/4 + \pi dh - A_0 = 0. \tag{2.5.9}$$

Then a new adjoint function $\mathcal{L}$ (called Lagrangian) is formed as

$$\begin{aligned} \mathcal{L}(d,h,\lambda) &= \underline{f(d,h) + \lambda g(d,h)} \\ &= \pi d^2 h/4 + \lambda(2\pi d^2/4 + \pi dh - A_0) \end{aligned} \tag{2.5.10}$$

where λ, a parameter yet to be determined, is called the *Lagrange multiplier*.

- The Lagrangian $\mathcal{L}$ is a function of three optimal variables *d*, *h*, and *A*. Thus,

Constraint (2.5.9)

$$\frac{\partial \mathcal{L}}{\partial d} = \pi dh/2 + \lambda(\pi d + \pi h) = 0 \tag{2.5.11}$$

$$\frac{\partial \mathcal{L}}{\partial h} = \pi d^2/4 + \lambda(\pi d) = 0 \tag{2.5.12}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 2\pi d^2/4 + \pi dh - A_0 = 0. \tag{2.5.13}$$

$$\Rightarrow \quad d^* = \sqrt{\frac{2A_0}{3\pi}}; \quad h^* = \sqrt{\frac{2A_0}{3\pi}}; \quad \lambda^* = -\sqrt{\frac{A_0}{24\pi}}. \tag{2.5.14}$$

Once again, to maximize the volume of a cylindrical tank, we need to have the height (h^*) equal to the diameter (d^*) of the tank. Note that we need to take the negative value of the square root function for λ in (2.5.14) in order to satisfy the physical requirement that the diameter d obtained from (2.5.12) as

$$d = -4\lambda \quad > 0 \tag{2.5.15}$$

is a positive value.

2.5.1 Direct Method

Consider the extrema of a function $f(x_1, x_2)$ with two *interdependent* variables x_1 and x_2, subject to the condition

$$g(x_1, x_2) = 0. \tag{2.5.16}$$

As a necessary condition for extrema, we have

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0. \tag{2.5.17}$$

However, since dx_1 and dx_2 are not *arbitrary*, but related by the condition

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0, \tag{2.5.18}$$

it is not possible to conclude as in the case of extremization of functions without conditions that

$$\frac{\partial f}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = 0 \tag{2.5.19}$$

in the necessary condition (2.5.17). This is easily seen, since if the set of extrema conditions (2.5.19) is solved for optimal values x_1^* and x_2^*, there is no guarantee that these optimal values, would, in general satisfy the given condition (2.5.16).

Now, assuming that $\partial g/\partial x_2 \neq 0$, (2.5.18) becomes

$$dx_2 = -\left\{\frac{\partial g/\partial x_1}{\partial g/\partial x_2}\right\} dx_1 \tag{2.5.20}$$

(2.5.17) $\Rightarrow$

$$df = \left[\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\left\{\frac{\partial g/\partial x_1}{\partial g/\partial x_2}\right\}\right] dx_1 = 0. \tag{2.5.21}$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial g}{\partial x_2}\right) - \left(\frac{\partial f}{\partial x_2}\right)\left(\frac{\partial g}{\partial x_1}\right) = 0. \tag{2.5.22}$$

Now, the relation (2.5.22) and the condition (2.5.16) are solved simultaneously.

$$\Rightarrow \quad \text{Jacobian} = \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{vmatrix} = 0. \tag{2.5.23}$$

This is also, as we know, the Jacobian of f and g w.r.t. x_1 and x_2. This method of elimination of the dependent variables is quite tedious for higher order problems.

2.5.2 Lagrange Multiplier Method

Consider again the extrema of the function $f(x_1, x_2)$ subject to the condition

$$g(x_1, x_2) = 0. \tag{2.5.24}$$

In this method, we form an *augmented* Lagrangian function

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2) \tag{2.5.25}$$

where, λ, a parameter (multiplier) yet to be determined, is the Lagrange multiplier.

$$\Rightarrow \quad d\mathcal{L} = df + \lambda dg = 0. \tag{2.5.28}$$

$$\Rightarrow \quad \left[\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1}\right] dx_1 + \left[\frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2}\right] dx_2 = 0. \tag{2.5.29}$$

$$\Rightarrow \quad \left\{ \begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial f}{\partial x_1} + \lambda^* \frac{\partial g}{\partial x_1} = 0 && (2.5.34) \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\partial f}{\partial x_2} + \lambda^* \frac{\partial g}{\partial x_2} = 0 && (2.5.35) \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= g(x_1^*, x_2^*) = 0. && (2.5.36) \end{aligned} \right.$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda^* \frac{\partial g}{\partial x_1} = 0 \qquad (2.5.34)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda^* \frac{\partial g}{\partial x_2} = 0 \qquad (2.5.35)$$

By eliminating λ^* between (2.5.34) and (2.5.35)

$$\left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial g}{\partial x_2}\right) - \left(\frac{\partial f}{\partial x_2}\right)\left(\frac{\partial g}{\partial x_1}\right) = 0 \qquad (2.5.37)$$

which is the same condition as (2.5.22) obtained by the direct method

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda^* \frac{\partial g}{\partial x_1} = 0 \quad (2.5.34)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda^* \frac{\partial g}{\partial x_2} = 0 \quad (2.5.35)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) = 0. \quad (2.5.36)$$

Necessary conditions

3 equations solved for 3 unknowns, x_1, x_2, λ

THEOREM 2.2

Consider the extrema of a continuous, real-valued function $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$ *subject to the conditions*

$$
\begin{aligned}
g_1(\mathbf{x}) &= g_1(x_1, x_2, \cdots, x_n) = 0 \\
g_2(\mathbf{x}) &= g_2(x_1, x_2, \cdots, x_n) = 0 \\
&\cdots \\
g_m(\mathbf{x}) &= g_m(x_1, x_2, \cdots, x_n) = 0
\end{aligned}
\tag{2.5.38}
$$

where, f *and* $\mathbf{g}$ *have continuous partial derivatives, and* $m < n$*. Let* $\lambda_1, \lambda_2, \cdots, \lambda_m$ *be the Lagrange multipliers corresponding to* m *conditions, such that the augmented Lagrangian function is formed as*

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}'\mathbf{g}(\mathbf{x}), \tag{2.5.39}$$

where, $\boldsymbol{\lambda}'$ *is the transpose of* $\boldsymbol{\lambda}$*. Then, the optimal values* $\mathbf{x}^*$ *and* $\boldsymbol{\lambda}^*$ *are the solutions of the following* $n + m$ *equations*

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}} + \boldsymbol{\lambda}' \frac{\partial \mathbf{g}}{\partial \mathbf{x}} = 0 \tag{2.5.40}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}} = \mathbf{g}(\mathbf{x}) = 0. \tag{2.5.41}$$

2.6 Extrema of Functionals with Conditions

$$J(x_1(t), x_2(t), t) = J = \int_{t_0}^{t_f} V(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), t)dt \quad (2.6.1)$$

subject to the condition (plant or system equation)

$$g(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)) = 0 \quad (2.6.2)$$

with fixed-end-point conditions

$$\begin{aligned} x_1(t_0) &= x_{10}; \quad x_2(t_0) = x_{20} \\ x_1(t_f) &= x_{1f}; \quad x_2(t_f) = x_{2f}. \end{aligned} \quad (2.6.3)$$

Now we address this problem under the following step

- **Step 1:** *Lagrangian*
- **Step 2:** *Variations and Increment*
- **Step 3:** *First Variation*
- **Step 4:** *Fundamental Theorem*
- **Step 5:** *Fundamental Lemma*
- **Step 6:** *Euler-Lagrange Equation*

- **Step 1:** *Lagrangian:* We form an *augmented* functional

$$J_a = \int_{t_0}^{t_f} \mathcal{L}(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), \lambda(t), t)dt \tag{2.6.4}$$

where, $\lambda(t)$ is the Lagrange multiplier, and the Lagrangian $\mathcal{L}$ is defined as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), \lambda(t), t) \\ &= V(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), t) \\ &\quad +\lambda(t)g(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t))\end{aligned} \tag{2.6.5}$$

Note from the performance index (2.6.1) and the augmented performance index (2.6.4) that $J_a = J$ if the condition (2.6.2) is satisfied for any $\lambda(t)$.

- **Step 2:** *Variations and Increment:* Next, assume optimal values and then consider the *variations* and *increment* as

$$
\begin{aligned}
&x_i(t) = x_i^*(t) + \delta x_i(t), \qquad \dot{x}_i(t) = \dot{x}_i^*(t) + \delta \dot{x}_i(t), \quad i = 1, 2 \\
&\Delta J_a = J_a(x_i^*(t) + \delta x_i(t), \dot{x}_i^*(t) + \delta \dot{x}_i(t), t) - J_a(x_i^*(t), \dot{x}_i^*(t), t),
\end{aligned}
\tag{2.6.6}
$$

  for $i = 1, 2$.

- **Step 3:** *First Variation:* Then using the Taylor series expansion and retaining linear terms only, the *first* variation of the functional J_a becomes

$$\delta J_a = \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_1} \right)_* \delta x_1(t) + \left(\frac{\partial \mathcal{L}}{\partial x_2} \right)_* \delta x_2(t) + \underline{\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right)_* \delta \dot{x}_1(t)} + \underline{\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \right)_* \delta \dot{x}_2(t)} \right] dt. \tag{2.6.7}$$

As before in the section on CoV, we rewrite the terms containing $\delta \dot{x}_1(t)$ and $\delta \dot{x}_2(t)$ in terms of those containing $\delta x_1(t)$ and $\delta x_2(t)$ only (using integration by parts, $\int u dv = uv - \int v du$). Thus

$$\begin{aligned}\int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial x_1} \right)_* \delta \dot{x}_1(t) dt &= \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right)_* \frac{d}{dt} (\delta x_1(t)) dt \\ &= \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right)_* d(\delta x_1(t)) \\ &= \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right)_* \delta x_1(t) \right] \Bigg|_{t_0}^{t_f} \\ &\quad - \int_{t_0}^{t_f} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right)_* \delta x_1(t) dt.\end{aligned} \tag{2.6.8}$$

Using the above, we have the first variation (2.6.7) as

$$\begin{aligned}\delta J_a = &\int_{t_0}^{t_f}\left[\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* \delta x_1(t) + \left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* \delta x_2(t)\right] dt \\ &+ \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \delta x_1(t)\right]\Bigg|_{t_0}^{t_f} + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* \delta x_2(t)\right]\Bigg|_{t_0}^{t_f} \\ &- \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \delta x_1(t) dt - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* \delta x_2(t) dt.\end{aligned} \tag{2.6.9}$$

Since this is a fixed-final time and fixed-final state problem as given by (2.6.3), no variations are allowed at the final point. This means

$$\delta x_1(t_0) = \delta x_2(t_0) = \delta x_1(t_f) = \delta x_2(t_f) = 0. \tag{2.6.10}$$

Using the boundary variations (2.6.10) in the augmented first variation (2.6.9), we have

$$\begin{aligned}\delta J_a = &\int_{t_0}^{t_f}\left[\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_*\right] \delta x_1(t) dt \\ &+ \int_{t_0}^{t_f}\left[\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_*\right] \delta x_2(t) dt.\end{aligned} \tag{2.6.11}$$

- **Step 4:** *Fundamental Theorem*

  1. We invoke the fundamental theorem of the calculus of variations (Theorem 2.1) and make the first variation (2.6.11) equal to zero. $\delta J = 0$

  2. Remembering that *both* $\delta x_1(t)$ and $\delta x_2(t)$ are not independent, because $x_1(t)$ and $x_2(t)$ are related by the condition (2.6.2), we choose $\delta x_2(t)$ as the *independent* variation and $\delta x_1(t)$ as the *dependent* variation.

  3. Let us choose the multiplier $\lambda^*(t)$ which is arbitrarily introduced and is at our disposal, in such a way that the coefficient of the *dependent* variation $\delta x_1(t)$ in (2.6.11) vanish. That is

$$\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* = 0. \tag{2.6.12}$$

With these choices, the first variation (2.6.11) becomes

$$\int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_*\right] \delta x_2(t) dt = 0. \tag{2.6.13}$$

Step 5: *Fundamental Lemma:* Using the fundamental lemma of CoV (Lemma 2.1) and noting that since $\delta x_2(t)$ has been chosen to be *independent* variation and hence *arbitrary*, the only way (2.6.13) can be satisfied, in general, is that the coefficient of $\delta x_1(t)$ also vanish. That is

$$\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* = 0. \tag{2.6.14}$$

Also, from the Lagrangian(2.6.5) note that

$$\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_* = 0 \tag{2.6.15}$$

yields the constraint relation (2.6.2).

$$\delta J_a = \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_1} \right)_* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right)_* \right] \delta x_1(t) dt + \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_2} \right)_* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \right)_* \right] \delta x_2(t) dt. \quad (2.6.11)$$

Theorem 2.1

$$\delta J_a = 0$$

δx_2 independent variable

$$\left(\frac{\partial \mathcal{L}}{\partial x_1} \right)_* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right)_* = 0. \qquad (2.6.12)$$

$$\int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_2} \right)_* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \right)_* \right] \delta x_2(t) dt = 0. \qquad (2.6.13)$$

Lemma 2.1

$$\left(\frac{\partial \mathcal{L}}{\partial x_2} \right)_* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \right)_* = 0. \qquad (2.6.14)$$

Step 6: *Euler-Lagrange Equation:* Combining the various relations (2.6.12), (2.6.14), and (2.6.15), the *necessary* conditions for extremization of the functional (2.6.1) subject to the condition (2.6.2) (according to Euler-Lagrange equation) are

$$\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* = 0 \tag{2.6.16}$$

$$\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* = 0 \tag{2.6.17}$$

$g = 0$
constraint

$$\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\lambda}}\right)_* = 0. \tag{2.6.18}$$

Now, we generalize the preceding procedure for an nth order system.

$$J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt \qquad (2.6.19)$$

$$g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0; \quad i = 1, 2, \ldots, m \qquad (2.6.20)$$

and boundary conditions, $\mathbf{x}(0)$ and $\mathbf{x}(t_f)$.

$$J_a = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t)dt \qquad (2.6.21)$$

$$\boxed{\mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \boldsymbol{\lambda}'(t) g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)} \quad (2.6.22)$$

and the Lagrange multiplier $\boldsymbol{\lambda}(t) = [\lambda_1(t),\ \lambda_2(t),\ \ldots,\ \lambda_m(t)]'$.

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0,} \qquad (2.6.23)$$

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\lambda}}}\right)_* = 0 \longrightarrow g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0.} \qquad (2.6.24)$$

Example 2.11

Minimize $\quad J = \int_0^1 \left[x^2(t) + u^2(t)\right] dt \qquad (2.6.25)$

Boundary conditions: $\quad x(0) = 1; \quad x(1) = 0 \qquad (2.6.26)$

Plant/Constraint: $\quad \dot{x}(t) = -x(t) + u(t). \qquad (2.6.27)$

1 Direct Method:

$$
\begin{aligned}
J &= \int_0^1 [x^2(t) + (\underbrace{\dot{x}(t) + x(t)}_{u(t)})^2]dt \\
&= \int_0^1 [2x^2(t) + \dot{x}^2(t) + 2x(t)\dot{x}(t)]dt. \qquad (2.6.28)
\end{aligned}
$$

Applying the Euler-Lagrange equation

$$4x^*(t) + 2\dot{x}^*(t) - \frac{d}{dt}(2\dot{x}^*(t) + 2x^*(t)) = 0. \qquad (2.6.31)$$

$$\Longrightarrow \quad \ddot{x}^*(t) - 2x^*(t) = 0 \qquad (2.6.32)$$

$$\Longrightarrow \quad x^*(t) = C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t} \qquad (2.6.33)$$

Using the given boundary conditions

$$C_1 = 1/(1 - e^{-2\sqrt{2}}); \qquad C_2 = 1/(1 - e^{2\sqrt{2}}). \qquad (2.6.34)$$

$$
\begin{aligned}
\Longrightarrow \quad u^*(t) &= \dot{x}^*(t) + x^*(t) \\
&= C_1(1 - \sqrt{2})e^{-\sqrt{2}t} + C_2(1 + \sqrt{2})e^{\sqrt{2}t}. \qquad (2.6.35)
\end{aligned}
$$

2 Lagrange Multiplier Method

$$g(x(t), \dot{x}(t), u(t)) = \dot{x}(t) + x(t) - u(t) = 0. \qquad (2.6.36)$$

$$\begin{aligned} J &= \int_0^1 \left[x^2(t) + u^2(t) + \lambda(t) \left\{ \dot{x}(t) + x(t) - u(t) \right\} \right] dt \\ &= \int_0^1 \mathcal{L}(x(t), \dot{x}(t), u(t), \lambda(t)) dt \qquad (2.6.37) \end{aligned}$$

Applying the Euler-Lagrange equation

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_* = 0 \longrightarrow 2x^*(t) + \lambda^*(t) - \dot{\lambda}^*(t) = 0 \quad (2.6.39)$$

$$\left(\frac{\partial \mathcal{L}}{\partial u}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right)_* = 0 \longrightarrow 2u^*(t) - \lambda^*(t) = 0 \quad (2.6.40)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\lambda}}\right)_* = 0 \longrightarrow \dot{x}^*(t) + x^*(t) - u^*(t) = 0 \quad (2.6.41)$$

and solve for optimal $x^*(t)$, $u^*(t)$, and $\lambda^*(t)$. We get first from (2.6.40) and (2.6.41)

$$\lambda^*(t) = 2u^*(t) = 2(\dot{x}^*(t) + x^*(t)). \tag{2.6.42}$$

Using the equation (2.6.42) in (2.6.39)

$$2x^*(t) + 2(\dot{x}^*(t) + x^*(t)) - 2(\ddot{x}^*(t) + \dot{x}^*(t)) = 0. \tag{2.6.43}$$

Solving the previous equation, we get

$$\ddot{x}^*(t) - 2x^*(t) = 0 \longrightarrow x^*(t) = C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t}. \tag{2.6.44}$$

Once we know $x^*(t)$, we get $\lambda^*(t)$ and hence $u^*(t)$ from (2.6.42) as

$$\begin{aligned} u^*(t) &= \dot{x}^*(t) + x^*(t) \\ &= C_1(1 - \sqrt{2})e^{-\sqrt{2}t} + C_2(1 + \sqrt{2})e^{\sqrt{2}t}. \end{aligned} \tag{2.6.45}$$

The constants C_1 and C_2 are evaluated using the boundary conditions (2.6.26)

The ODE Eq. (2.6.32) with B.C., Eq. (2.6.26) for Example 2.11 using Symbolic Toolbox of the MATLAB.

```
x=dsolve('D2x-2*x=0','x(0)=1,x(1)=0')

x =

-(exp(2^(1/2))^2+1)/(exp(2^(1/2))^2-1)*sinh(2^(1/2)*t)+
cosh(2^(1/2)*t)

           1/2 2              1/2
    (exp(2    )  + 1) sinh(2     t)              1/2
  - ------------------------------ + cosh(2     t)
          1/2 2
    exp(2    )  - 1
```

```
u =

-(exp(2^(1/2))^2+1)/(exp(2^(1/2))^2-1)*cosh(2^(1/2)*t)*2^(1/2)+
sinh(2^(1/2)*t)*2^(1/2)-(exp(2^(1/2))^2+1)/(exp(2^(1/2))^2-
1)*sinh(2^(1/2)*t)+cosh(2^(1/2)*t)

           1/2 2                1/2       1/2
     (exp(2    )  + 1) cosh(2     t) 2                1/2       1/2
  - ---------------------------------- + sinh(2     t) 2
                    1/2 2
               exp(2    )  - 1

                  1/2 2                1/2
            (exp(2    )  + 1) sinh(2     t)           1/2
          - ----------------------------- + cosh(2     t)
                        1/2 2
                   exp(2    )  - 1
```

Summary

Min $\quad J(x_1(t), x_2(t), t) = J = \int_{t_0}^{t_f} V(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), t)dt \quad (2.6.1)$

subject to $\quad g(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)) = 0 \qquad (2.6.2)$

$$\text{B.C.} \quad \begin{aligned} x_1(t_0) &= x_{10}; \quad x_2(t_0) = x_{20} \\ x_1(t_f) &= x_{1f}; \quad x_2(t_f) = x_{2f}. \end{aligned} \qquad (2.6.3)$$

Lagrangian:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), \lambda(t), t) \\ &= V(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), t) \\ &\quad + \lambda(t) g(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)) \end{aligned} \qquad (2.6.5)$$

Euler-Lagrange Equation:

$$\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* = 0 \qquad (2.6.16)$$

$$\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* = 0 \qquad (2.6.17)$$

$$g = 0 \longleftarrow \left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\lambda}}\right)_* = 0. \qquad (2.6.18)$$

2.7 Variational Approach to Optimal Control Systems

- In this section, we approach the optimal control system by variational techniques,
- and in the process introduce the Hamiltonian function,
- which was used by Pontryagin and his associates to develop the famous *Minimum Principle* [109].

2.7.1 Terminal Cost Problem

Consider the plant as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{2.7.1}$$

the performance index as

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.7.2}$$

and given boundary conditions as

$$\mathbf{x}(t_0) = \mathbf{x}_0; \quad \underline{\mathbf{x}(t_f) \text{ is free and } t_f \text{ is free}} \tag{2.7.3}$$

where, $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are n- and r- dimensional state and control vectors respectively. This *problem of Bolza* is the one with the most general form of the performance index.

$$J_2(\mathbf{u}(t)) = \int_{t_0}^{t_f} \left[V(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{dS}{dt} \right] dt$$

$$= \underbrace{\int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t) dt + S(\mathbf{x}(t_f), t_f)}_{J(u(t))} - \overset{\text{Fixed}}{S(\mathbf{x}(t_0), t_0)}. \quad (2.7.5)$$

$$\frac{d[S(\mathbf{x}(t), t)]}{dt} = \left(\frac{\partial S}{\partial \mathbf{x}} \right)' \dot{\mathbf{x}}(t) + \frac{\partial S}{\partial t}. \quad (2.7.6)$$

Min $J(\mathbf{u}(t))$ $\Rightarrow$ Min $J_2(\mathbf{u}(t))$

- **Step 1:** *Assumption of Optimal Conditions*
- **Step 2:** *Variations of Control and State Vectors*
- **Step 3:** *Lagrange Multiplier*
- **Step 4:** *Lagrangian*
- **Step 5:** *First Variation*
- **Step 6:** *Condition for Extrema*
- **Step 7:** *Hamiltonian*

- **Step 1:** *Assumptions of Optimal Conditions:* We assume optimum values $\mathbf{x}^*(t)$ and $\mathbf{u}^*(t)$ for state and control, respectively. Then

$$
\begin{aligned}
J(\mathbf{u}^*(t)) &= \int_{t_0}^{t_f} \left[\underline{V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{dS(\mathbf{x}^*(t), t)}{dt}} \right] dt \\
\dot{\mathbf{x}}^*(\mathbf{t}) &= \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t). \qquad (2.7.7)
\end{aligned}
$$

- **Step 2:** *Variations of Controls and States:* We consider the variations (perturbations) in control and state vectors as (see Figure 2.7)

$$\mathbf{x}(t) = \mathbf{x}^*(t) + \delta\mathbf{x}(t); \qquad \mathbf{u}(t) = \mathbf{u}^*(t) + \delta\mathbf{u}(t). \tag{2.7.8}$$

Then, the state equation (2.7.1) and the performance index (2.7.5) become

$$\begin{aligned}
&\dot{\mathbf{x}}^*(\mathbf{t}) + \delta\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{f}(\mathbf{x}^*(\mathbf{t}) + \delta\mathbf{x}(\mathbf{t}), \mathbf{u}^*(\mathbf{t}) + \delta\mathbf{u}(\mathbf{t}), \mathbf{t}) \\
&J(\mathbf{u}(t)) = \int_{t_0}^{t_f+\delta t_f} \left[V(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t) + \frac{dS}{dt} \right] dt
\end{aligned} \tag{2.7.9}$$

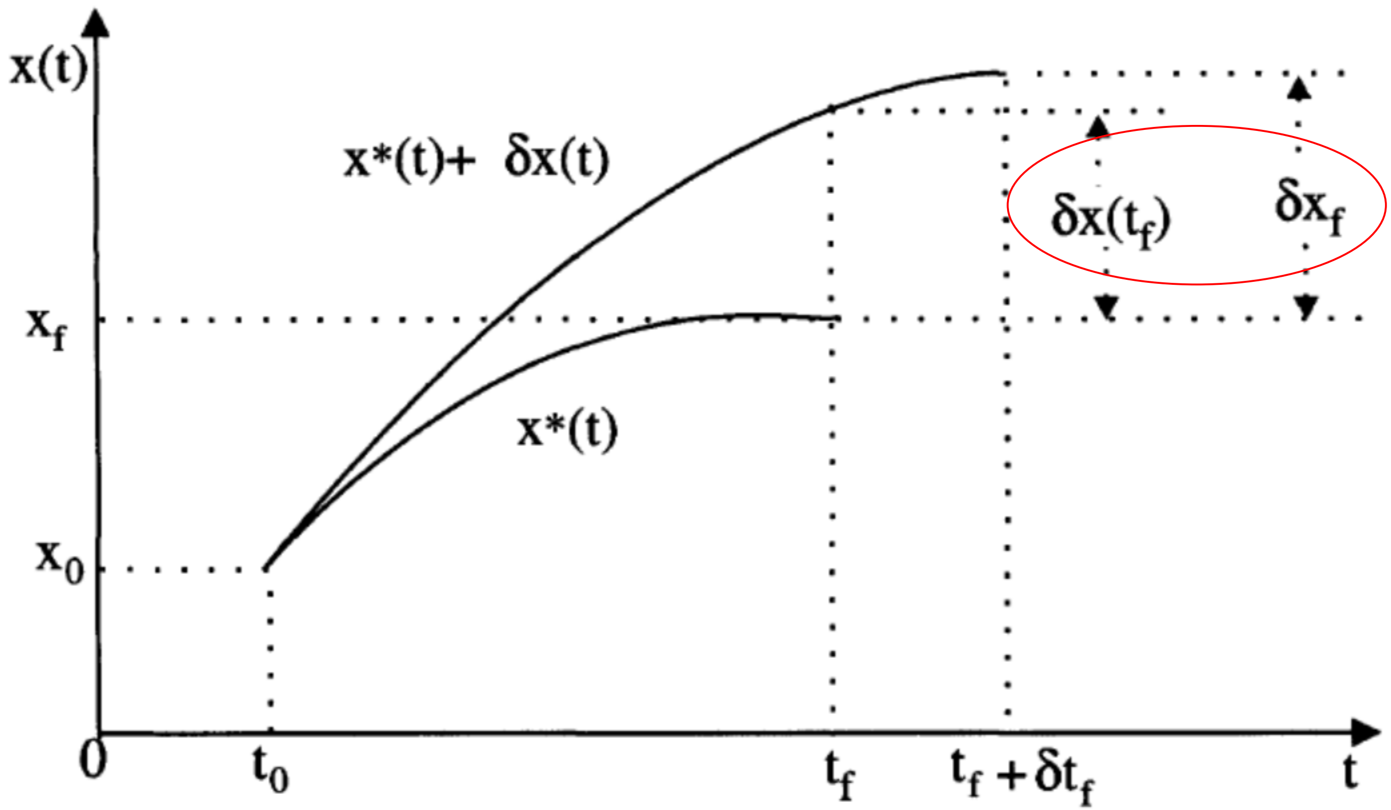


Figure 2.7 Free-Final Time and Free-Final State System

- **Step 3:** *Lagrange Multiplier:* Introducing the Lagrange multiplier vector $\boldsymbol{\lambda}(t)$ (also called costate vector) and using (2.7.6), we introduce the augmented performance index at the optimal condition as

$$J_a(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} \left[V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_* + \boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}\right] dt \tag{2.7.10}$$

and at any other (perturbed) condition as

$$\begin{aligned} J_a(\mathbf{u}(t)) = \int_{t_0}^{t_f+\delta t_f} &[V(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t) \\ &+ \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' [\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)] + \left(\frac{\partial S}{\partial t}\right)_* \\ &+ \boldsymbol{\lambda}'(t)\,[\mathbf{f}(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t) \\ &- \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}]]dt. \end{aligned} \tag{2.7.11}$$

- **Step 4:** *Lagrangian:* Let us define the Lagrangian function at optimal condition as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\ &\quad + \boldsymbol{\lambda}'(t) \left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\} \end{aligned} \tag{2.7.12}$$

and at any other condition as

$$\begin{aligned}\mathcal{L}^\delta &= \mathcal{L}^\delta(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t) \\ &\quad + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' [\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)] + \left(\frac{\partial S}{\partial t}\right)_* \\ &\quad + \boldsymbol{\lambda}'(t) \left[\mathbf{f}(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t)\right. \\ &\quad \left. - \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}\right]. \end{aligned} \tag{2.7.13}$$

With these, the augmented performance index at the optimal and any other condition becomes

$$
\begin{aligned}
J_a(\mathbf{u}^*(t)) &= \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t)dt = \int_{t_0}^{t_f} \mathcal{L}dt \\
J_a(\mathbf{u}(t)) &= \int_{t_0}^{t_f+\delta t_f} \mathcal{L}^\delta dt = \int_{t_0}^{t_f} \mathcal{L}^\delta dt + \underline{\int_{t_f}^{t_f+\delta t_f} \mathcal{L}^\delta dt}. \quad (2.7.14)
\end{aligned}
$$

Using mean-value theorem and Taylor series, and retaining the *linear* terms only, we have

$$
\begin{aligned}
\int_{t_f}^{t_f+\delta t_f} \mathcal{L}^\delta dt &\approx \mathcal{L}^\delta\Big|_{t_f} \delta t_f \\
&\approx \left\{\mathcal{L} + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_*^{\prime} \delta\mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*^{\prime} \delta\dot{\mathbf{x}}(t) \right. \\
&\qquad \left. + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*^{\prime} \delta\mathbf{u}(t)\right\}\Bigg|_{t_f} \delta t_f \\
&\approx \underline{\mathcal{L}|_{t_f} \delta t_f}. \qquad (2.7.15)
\end{aligned}
$$

- **Step 5:** *First Variation:* Defining increment ΔJ, using Taylor series expansion, extracting the first variation δJ by retaining only the first order terms, we get the first variation as

$$
\begin{aligned}
\Delta J &= \underline{J_a(\mathbf{u}(t))} - J_a(\mathbf{u}^*(t)) \\
&= \boxed{\int_{t_0}^{t_f} (\underline{\mathcal{L}^\delta} - \mathcal{L})dt} + \underline{\mathcal{L}|_{t_f}\,\delta t_f}
\end{aligned}
$$

$$
\begin{aligned}
\delta J = \int_{t_0}^{t_f} \left\{ \boxed{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_*' \delta \mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta \mathbf{u}(t)} \right\} dt \\
+ \mathcal{L}|_{t_f}\,\delta t_f. \qquad (2.7.16)
\end{aligned}
$$

Considering the $\delta\dot{\mathbf{x}}(t)$ term in the first variation (2.7.16) and integrating by parts (using $\int u dv = uv - \int v du$),

$$
\begin{aligned}
\int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) dt &= \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \frac{d}{dt}\left(\delta \mathbf{x}(t)\right) dt \\
&= \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \mathbf{x}(t)\right]\Bigg|_{t_0}^{t_f} \\
&\quad - \int_{t_0}^{t_f} \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*\right]' \delta \mathbf{x}(t) dt. \qquad (2.7.17)
\end{aligned}
$$

Also note that since $\mathbf{x}(t_0)$ is specified, $\delta\mathbf{x}(t_0) = 0$. Thus, using (2.7.17) the first variation δJ in (2.7.16) becomes

$$\begin{aligned}\delta J = &\int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*\right]' \delta\mathbf{x}(t)dt \\ &+ \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta\mathbf{u}(t)dt \\ &+ \mathcal{L}|_{t_f}\, \delta t_f + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta\mathbf{x}(t)\right]\Bigg|_{t_f}. \qquad (2.7.18)\end{aligned}$$

- **Step 6:** *Condition for Extrema*

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0 \tag{2.7.19}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0. \tag{2.7.20}$$

$$\mathcal{L}^*|_{t_f}\,\delta t_f + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_* \delta \mathbf{x}(t)\right]\Bigg|_{t_f} = 0. \tag{2.7.21}$$

Plant/Constraint ←

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* = 0. \tag{2.7.22}$$

$\delta x(t_f) = ?$

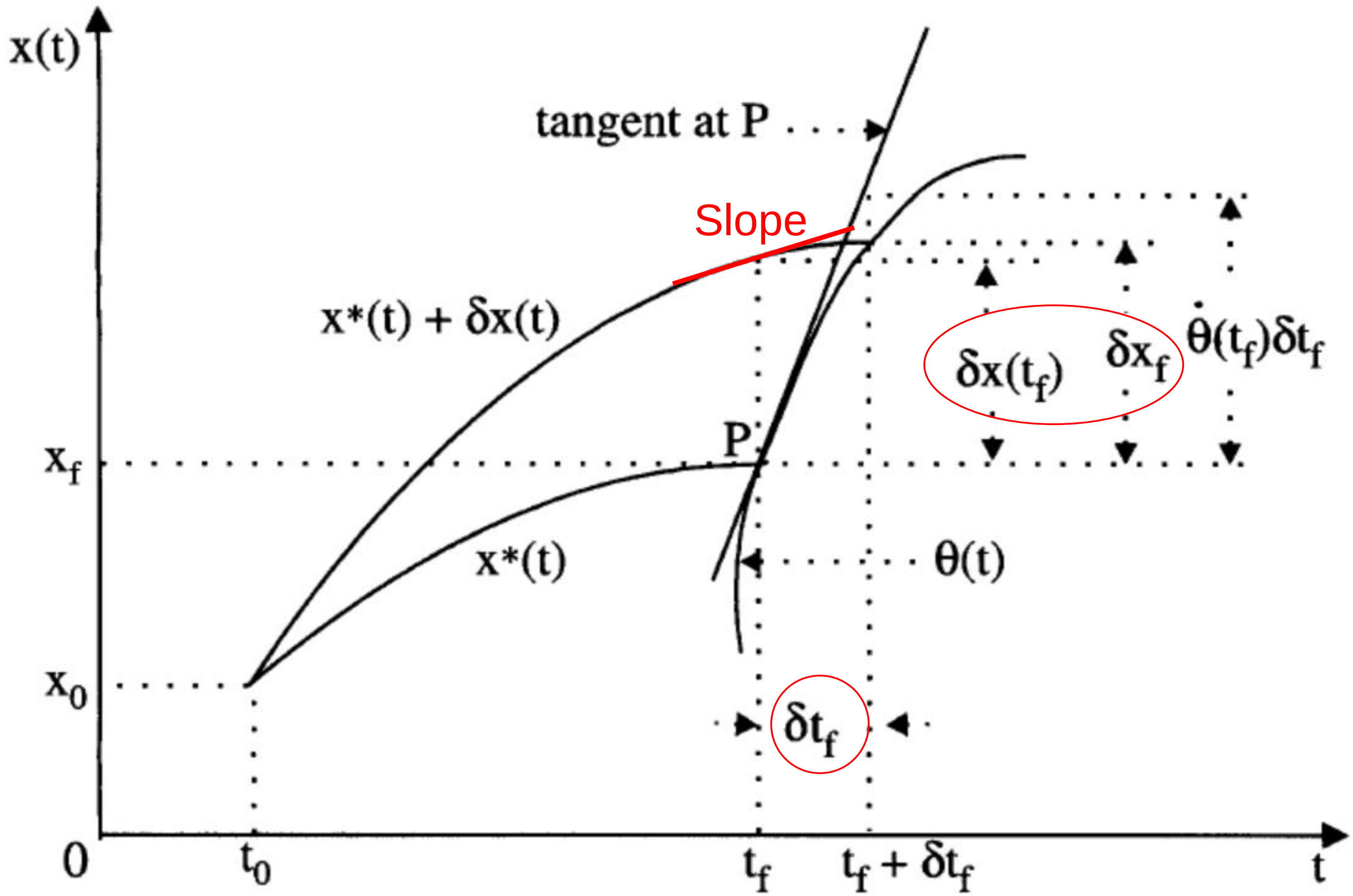


Figure 2.8 Final-Point Condition with a Moving Boundary $\theta(t)$

$$\dot{\mathbf{x}}^*(t_f) + \delta\dot{\mathbf{x}}(t_f) \approx \frac{\delta \mathbf{x}_f - \delta \mathbf{x}(t_f)}{\delta t_f} \qquad (2.7.23)$$

$$\dot{\mathbf{x}}^*(t_f) + \delta\dot{\mathbf{x}}(t_f) \approx \frac{\delta\mathbf{x}_f - \delta\mathbf{x}(t_f)}{\delta t_f} \tag{2.7.23}$$

which is rewritten as

$$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) + \{\dot{\mathbf{x}}^*(t_f) + \delta\dot{\mathbf{x}}(t_f)\}\,\delta t_f \tag{2.7.24}$$

and retaining only the linear (in δ) terms in the relation (2.7.24), we have

$$\delta\mathbf{x}(t_f) = \delta\mathbf{x}_f - \dot{\mathbf{x}}^*(t_f)\delta t_f. \tag{2.7.25}$$

Using (2.7.25) in the boundary condition (2.7.21), we have the general boundary condition in terms of the Lagrangian as

$$\left[\mathcal{L}^* - \left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}}\right)_*' \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}}\right)_*'\Bigg|_{t_f} \delta\mathbf{x}_f = 0. \tag{2.7.26}$$

$$\mathcal{L}^*|_{t_f}\,\delta t_f + \left[\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}}\right)_*' \delta\mathbf{x}(t)\right]\Bigg|_{t_f} = 0. \tag{2.7.21}$$

- **Step 7:** *Hamiltonian:* We define the Hamiltonian $\mathcal{H}^*$ (also called the Pontryagin $\mathcal{H}$ function) at the optimal condition as

$$\boxed{\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*'}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t),} \quad (2.7.27)$$

where,

$$\mathcal{H}^* = \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t).$$

Note: no $\dot{x}(t)$

Then from (2.7.12) the Lagrangian $\mathcal{L}^*$ in terms of the Hamiltonian $\mathcal{H}^*$ becomes

$$\begin{aligned}\mathcal{L}^* &= \mathcal{L}^*(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &= \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &\quad + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_* - \boldsymbol{\lambda}^{*'}(t)\dot{\mathbf{x}}^*(t). \end{aligned} \quad (2.7.28)$$

(from Step 4)

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\ &= \underline{V(\mathbf{x}^*(t), \mathbf{u}^*(t), t)} + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\ &\quad + \underline{\boldsymbol{\lambda}'(t)\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)} - \dot{\mathbf{x}}^*(t)\} \end{aligned} \quad (2.7.12)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0 \longrightarrow \boxed{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0} \tag{2.7.29}$$

(2.7.20)

(2.7.19)

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}^*}{\partial \dot{\mathbf{x}}}\right)_* = 0 \longrightarrow$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* + \left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial^2 S}{\partial \mathbf{x}\partial t}\right)_* - \frac{d}{dt}\left\{\left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' - \boldsymbol{\lambda}^*(t)\right\} = 0 \longrightarrow$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* + \left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial^2 S}{\partial \mathbf{x}\partial t}\right)_* - \left[\left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial^2 S}{\partial \mathbf{x}\partial t}\right)_* - \dot{\boldsymbol{\lambda}}^*(t)\right] = 0$$

$$\boxed{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)} \tag{2.7.30}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* = 0 \longrightarrow \boxed{\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).} \tag{2.7.31}$$

(2.7.22)

Looking at the similar structure of the relation (2.7.30) for the optimal costate $\boldsymbol{\lambda}^*(t)$ and (2.7.31) for the optimal state $\mathbf{x}^*(t)$ it is clear why $\boldsymbol{\lambda}(t)$ is called the *costate* vector. Finally, using the relation (2.7.28), the boundary condition (2.7.26) at the optimal condition reduces to

$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f}' \delta \mathbf{x}_f = 0.} \tag{2.7.32}$$

This is the general boundary condition for free-end point system in terms of the Hamiltonian.

Summary

Min $\quad J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \qquad (2.7.2)$

s.t. $\quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \qquad (2.7.1)$

B.C. $\quad \mathbf{x}(t_0) = \mathbf{x}_0; \quad \mathbf{x}(t_f)$ is free and t_f is free $\qquad (2.7.3)$

Min $\quad J(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} \left[V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{dS(\mathbf{x}^*(t), t)}{dt} \right] dt$

s.t. $\quad \dot{\mathbf{x}}^*(\mathbf{t}) = \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t). \qquad (2.7.7)$

B.C. $\quad \mathbf{x}(t_0) = \mathbf{x}_0; \quad \mathbf{x}(t_f)$ is free and t_f is free $\qquad (2.7.3)$

Lagrangian:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\ &\quad + \boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\} \qquad (2.7.12)\end{aligned}$$

CoV

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0 \qquad (2.7.19)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0. \qquad (2.7.20)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* = 0. \qquad (2.7.22)$$

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*'\Bigg|_{t_f} \delta \mathbf{x}_f = 0. \qquad (2.7.26)$$

Hamiltonian: $\boxed{\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*\prime}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t),}$ (2.7.27)

$$\boxed{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)} \quad (2.7.30)$$

$$\boxed{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0} \quad (2.7.29)$$

$$\boxed{\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).} \quad (2.7.31)$$

$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.} \quad (2.7.32)$$

$L = V + (\frac{\partial S}{\partial x})'\dot{x} + \frac{\partial S}{\partial t}$ $+\lambda'(f - \dot{x})$	$H = V$ $+\lambda' f$
$\frac{\partial L}{\partial x} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = 0$ $\frac{\partial L}{\partial u} = 0$ $\frac{\partial L}{\partial \lambda} = 0$	$\frac{\partial H}{\partial x} = -\dot{\lambda}$ $\frac{\partial H}{\partial u} = 0$ $\frac{\partial H}{\partial \lambda} = \dot{x}$
$[L - (\frac{\partial L}{\partial \dot{x}})'\dot{x}]_{t_f}\,\delta t_f$ $+(\frac{\partial L}{\partial \dot{x}})'_{t_f}\,\delta x_f = 0$	$[H + \frac{\partial S}{\partial t}]_{t_f}\,\delta t_f$ $+[\frac{\partial S}{\partial x} - \lambda]'_{t_f}\,\delta x_f = 0$

2.7.2 *Different Types of Systems*

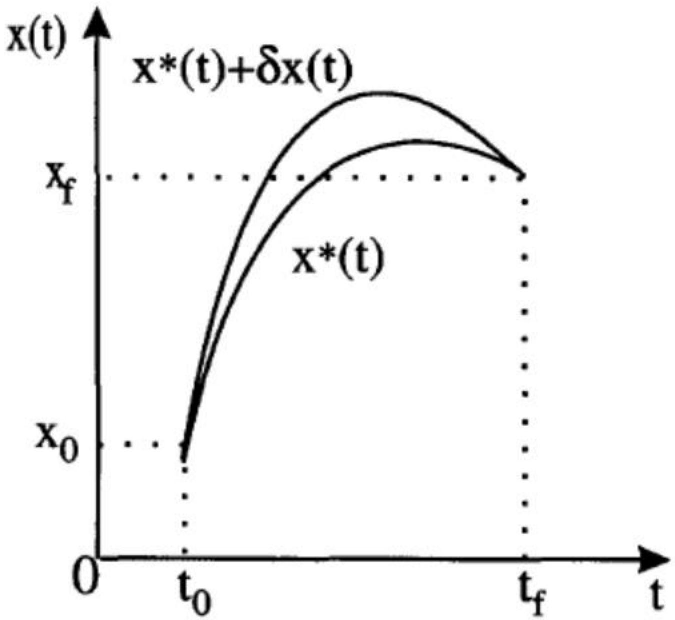
- We now obtain different cases depending on the statement of the problem regarding the final time t_f and the final state $x(t_f)$ (see Figure 2.9).
  - (a) Fixed-Final Time and Fixed-Final State System
  - (b) Free-Final Time and Fixed-Final State System
  - (c) Fixed-Final Time and Free-Final State System
  - (d) Free-Final Time and Free-Final State System
    - *Free-Final Time and Dependent Free-Final State System*
    - *Free-Final Time and Independent Free-Final State*

$$\boxed{\underline{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f}} \underline{\delta t_f} + \underline{\left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f}'} \underline{\delta \mathbf{x}_f} = 0.} \tag{2.7.32}$$

- **Type (a):** *Fixed-Final Time and Fixed-Final State System:* Here, since t_f and $\mathbf{x}(t_f)$ are fixed or specified (Figure 2.9(a)), both δt_f and $\delta \mathbf{x}_f$ are zero in the general boundary condition (2.7.32), and there is no extra boundary condition to be used other than those given in the problem formulation.

$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f}' \delta \mathbf{x}_f = 0.} \tag{2.7.32}$$

$$\Rightarrow \quad \delta t_f = \delta x_f = 0$$

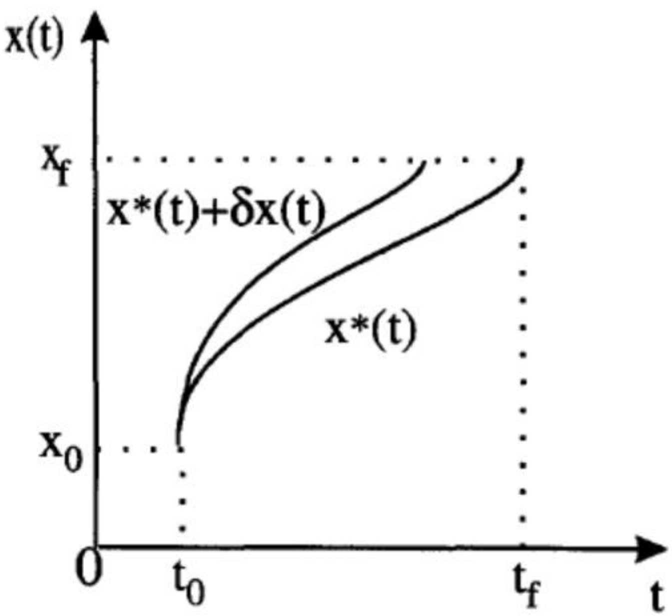


- **Type (b):** *Free-Final Time and Fixed-Final State System:* Since t_f is free or not specified in advance, δt_f is arbitrary, and since $\mathbf{x(t_f)}$ is fixed or specified, $\delta \mathbf{x}_f$ is zero as shown in Figure 2.9(b). Then, the coefficient of the arbitrary δt_f in the general boundary condition (2.7.32) is zero resulting in

$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.} \qquad (2.7.32)$$

$$\boxed{\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{*t_f} = 0.} \qquad (2.7.33)$$

$$\delta x_f = 0$$



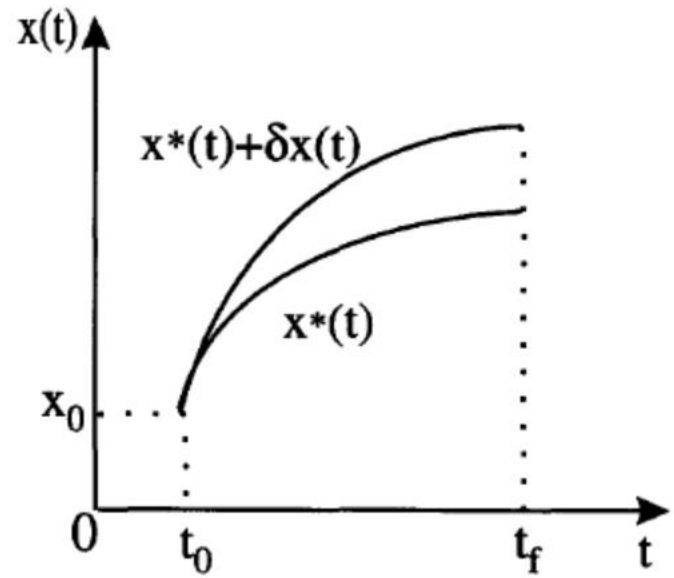
- **Type (c):** *Fixed-Final Time and Free-Final State System:* Here t_f is specified and $\mathbf{x}(t_f)$ is free (see Figure 2.9(c)). Then δt_f is zero and $\delta \mathbf{x}_f$ is arbitrary, which in turn means that the coefficient of $\delta \mathbf{x}_f$ in the general boundary condition (2.7.32) is zero. That is

$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.} \tag{2.7.32}$$

$$\left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)_{*t_f} = 0$$

$$\longrightarrow \boxed{\boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f}.} \tag{2.7.34}$$

$$\delta t_f = 0$$



$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.}$$

(2.7.32)

- **Type (d):** *Free-Final Time and Dependent Free-Final State System:* If t_f and $\mathbf{x}(t_f)$ are *related* such that $\mathbf{x}(t_f)$ lies on a moving curve $\boldsymbol{\theta}(t)$ as shown in Figure 2.8, then

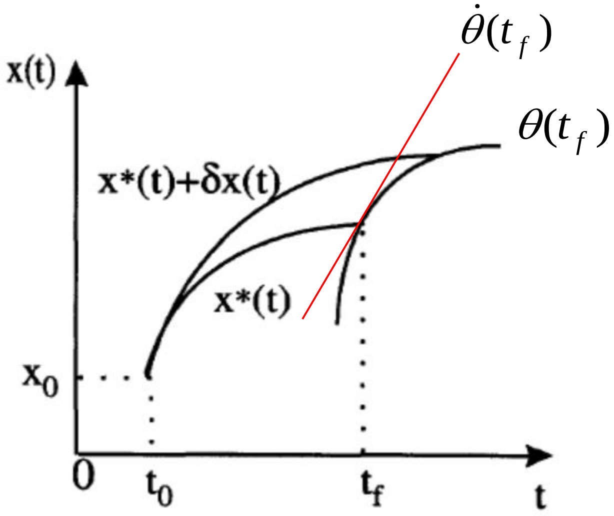
$$\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f) \quad \text{and} \quad \delta \mathbf{x}_f \approx \dot{\boldsymbol{\theta}}(t_f)\delta t_f. \tag{2.7.35}$$

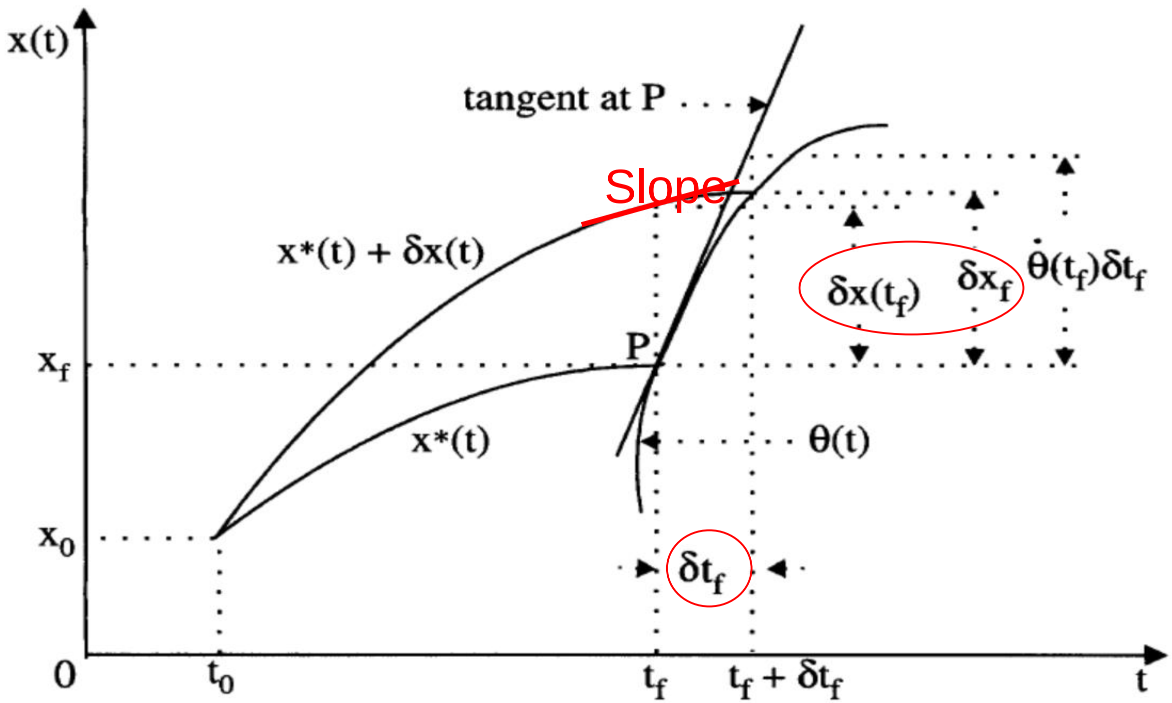
Using (2.7.35), the boundary condition (2.7.32) for the optimal condition becomes

$$\left[\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_* + \left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)'_* \dot{\boldsymbol{\theta}}(t)\right]_{t_f} \delta t_f = 0. \tag{2.7.36}$$

Since t_f is free, δt_f is arbitrary and hence the coefficient of δt_f in (2.7.36) is zero. That is

$$\boxed{\left[\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_* + \left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)'_* \dot{\boldsymbol{\theta}}(t)\right]_{t_f} = 0.} \tag{2.7.37}$$


$\dot{\theta}(t_f)$
$\theta(t_f)$
x(t)
x*(t)+δx(t)
x*(t)
x_0
0
t_0
t_f
t


x(t)
tangent at P
Slope
x*(t) + δx(t)
$\delta x(t_f)$
δx_f
$\dot{\theta}(t_f)\delta t_f$
x_f
P
x*(t)
θ(t)
x_0
δt_f
0
t_0
t_f
$t_f + \delta t_f$
t

$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.} \tag{2.7.32}$$

- **Type (e):** *Free-Final Time and Independent Free-Final State:* If t_f and $\mathbf{x}(t_f)$ are *not related*, then δt_f and $\delta \mathbf{x}_f$ are unrelated, and the boundary condition (2.7.32) at the optimal condition becomes

$$\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{*t_f} = 0 \tag{2.7.38}$$

$$\left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)_{*t_f} = 0. \tag{2.7.39}$$

2.7.3 Sufficient Condition

- In order to determine the nature of optimization, i.e., whether it is minimum or maximum, we need to consider the second variation and examine its sign.

$$\begin{aligned}\Delta J &= J_a(\mathbf{u}(t)) - J_a(\mathbf{u}^*(t)) \\ &= \int_{t_0}^{t_f} (\mathcal{L}^\delta - \mathcal{L})dt + \mathcal{L}|_{t_f} \delta t_f\end{aligned}$$

$$\begin{aligned}\delta J = \int_{t_0}^{t_f} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_*' \delta \mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta \mathbf{u}(t) \right\} dt \\ + \mathcal{L}|_{t_f} \delta t_f. \qquad (2.7.16)\end{aligned}$$

$$\begin{aligned}\mathcal{L}^* &= \mathcal{L}^*(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &= \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &\quad + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_* - \boldsymbol{\lambda}^{*\prime}(t)\dot{\mathbf{x}}^*(t). \qquad (2.7.28)\end{aligned}$$

$$\begin{aligned}
\delta^2 J &= \int_{t_0}^{t_f} \left[\frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} (\delta \mathbf{x}(t))^2 + \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} (\delta \mathbf{u}(t))^2 + 2 \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u} \partial \mathbf{x}} (\delta \mathbf{u}(t) \delta \mathbf{x}(t)) \right]_* dt \\
&= \int_{t_0}^{t_f} \left[\delta \mathbf{x}'(t) \;\; \delta \mathbf{u}'(t) \right] \begin{bmatrix} \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} \\ \\ \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \end{bmatrix}_* \begin{bmatrix} \delta \mathbf{x}(t) \\ \delta \mathbf{u}(t) \end{bmatrix} dt \\
&= \int_{t_0}^{t_f} \left[\delta \mathbf{x}'(t) \;\; \delta \mathbf{u}'(t) \right] \Pi \begin{bmatrix} \delta \mathbf{x}(t) \\ \delta \mathbf{u}(t) \end{bmatrix} dt. \qquad (2.7.40)
\end{aligned}$$

For the *minimum*, the second variation $\delta^2 J$ must be *positive*. This means that the matrix Π in (2.7.40)

$$\Pi = \begin{bmatrix} \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} \\ \\ \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \end{bmatrix}_* \qquad (2.7.41)$$

must be *positive definite.*

But the important condition is that the second partial derivative of $\mathcal{H}^*$ w.r.t. $\mathbf{u}(t)$ must be positive. That is

$$\boxed{\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* > 0} \tag{2.7.42}$$

and for the *maximum*, the sign of (2.7.42) is reversed.

2.7.4 *Summary of Pontryagin Procedure*

Consider a free-final time and free-final state problem with general cost function (Bolza problem), where we want to minimize the performance index

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.7.43}$$

for the plant described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.7.44}$$

with the boundary conditions as

$$\mathbf{x}(t = t_0) = \mathbf{x}_0; \quad t = t_f \text{ is free and } \mathbf{x}(t_f) \text{ is free.} \tag{2.7.45}$$

Here, $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are $n-$ and $r-$ dimensional state and control vectors respectively. Let us note that $\mathbf{u}(t)$ is *unconstrained*. The entire procedure (called Pontryagin Principle) is now summarized in Table 2.1.

Table 2.1 Procedure Summary of Pontryagin Principle for Bolza Problem

A. Statement of the Problem
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$, the performance index as $J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$, and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0$ and t_f and $\mathbf{x}(t_f) = \mathbf{x}_f$ are free, find the optimal control.

B. Solution of the Problem	
Step 1	Form the Pontryagin $\mathcal{H}$ function $\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t).$
Step 2	Minimize $\mathcal{H}$ w.r.t. $\mathbf{u}(t)$ $\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$ and obtain $\mathbf{u}^*(t) = \mathbf{h}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t).$
Step 3	Using the results of Step 2 in Step 1, find the optimal $\mathcal{H}^*$ $\mathcal{H}^*(\mathbf{x}^*(t), \mathbf{h}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t), \boldsymbol{\lambda}^*(t), t) = \mathcal{H}^*(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t).$
Step 4	Solve the set of 2n differential equations $\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*$ and $\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*$ with initial conditions $\mathbf{x}_0$ and the final conditions $\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.$
Step 5	Substitute the solutions of $\mathbf{x}^*(t)$, $\boldsymbol{\lambda}^*(t)$ from Step 4 into the expression for the optimal control $\mathbf{u}^*(t)$ of Step 2.

C. Types of Systems		
(a). Fixed-final time and fixed-final state system, Fig. 2.9(a)		
(b). Free-final time and fixed-final state system, Fig. 2.9(b)		
(c). Fixed-final time and free-final state system, Fig. 2.9(c)		
(d). Free-final time and dependent free-final state system, Fig. 2.9(d).		
(e). Free-final time and independent free-final state system		
Type	**Substitutions**	**Boundary Conditions**
(a)	$\delta t_f = 0, \delta \mathbf{x}_f = 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \mathbf{x}_f$
(b)	$\delta t_f \neq 0, \delta \mathbf{x}_f = 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \mathbf{x}_f, \left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0$
(c)	$\delta t_f = 0, \delta \mathbf{x}_f \neq 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f}$
(d)	$\delta \mathbf{x}_f = \dot{\boldsymbol{\theta}}(t_f)\delta t_f$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ $\left[\mathcal{H}^* + \frac{\partial S}{\partial t} + \left\{\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right\}' \dot{\boldsymbol{\theta}}(t)\right]_{t_f} = 0$
(e)	$\delta t_f \neq 0$ $\delta \mathbf{x}_f \neq 0$	$\delta \mathbf{x}(t_0) = \mathbf{x}_0$ $\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0, \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f} = 0$

Example 2.12

Given a second order (double integrator) system as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t)\end{aligned} \qquad (2.7.46)$$

and the performance index as

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt \qquad (2.7.47)$$

find the optimal control and optimal state, given the boundary (initial and final) conditions as

$$\underline{\mathbf{x}(0) = [1 \;\; 2]'; \qquad \mathbf{x}(2) = [1 \;\; 0]'.} \quad \textbf{Type (a)} \qquad (2.7.48)$$

Assume that the control and state are unconstrained.

Solution:

$$
\begin{aligned}
V(\mathbf{x}(t), \mathbf{u}(t), t) &= V(u(t)) = \frac{1}{2}u^2(t) \\
\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) &= [f_1,\ f_2]',
\end{aligned}
\tag{2.7.49}
$$

where, $f_1 = x_2(t), \quad f_2 = u(t).$

- **Step 1:** Form the Hamiltonian function as

$$
\begin{aligned}
\mathcal{H} &= \mathcal{H}(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)) \\
&= V(u(t)) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\
&= \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t).
\end{aligned}
\tag{2.7.50}
$$

- **Step 2:** Find $u^*(t)$ from

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \longrightarrow u^*(t) + \lambda_2^*(t) = 0 \longrightarrow u^*(t) = -\lambda_2^*(t). \quad (2.7.51)$$

- **Step 3:** Using the results of Step 2 in Step 1, find the optimal $\mathcal{H}^*$ as

$$\begin{aligned}\mathcal{H}^*(x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)) &= \frac{1}{2}\lambda_2^{*^2}(t) + \lambda_1^*(t)x_2^*(t) - \lambda_2^{*^2}(t) \\ &= \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*^2}(t). \quad (2.7.52)\end{aligned}$$

- **Step 4:** Obtain the state and costate equations from

$$
\begin{aligned}
\dot{x}_1^*(t) &= +\left(\frac{\partial \mathcal{H}}{\partial \lambda_1}\right)_* = x_2^*(t) \\
\dot{x}_2^*(t) &= +\left(\frac{\partial \mathcal{H}}{\partial \lambda_2}\right)_* = -\lambda_2^*(t) \\
\dot{\lambda}_1^*(t) &= -\left(\frac{\partial \mathcal{H}}{\partial x_1}\right)_* = 0 \\
\dot{\lambda}_2^*(t) &= -\left(\frac{\partial \mathcal{H}}{\partial x_2}\right)_* = -\lambda_1^*(t).
\end{aligned}
\tag{2.7.53}
$$

Solving the previous equations, we have the optimal state and costate as

$$
\begin{aligned}
x_1^*(t) &= \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1 \\
x_2^*(t) &= \frac{C_3}{2}t^2 - C_4 t + C_2 \\
\lambda_1^*(t) &= C_3 \\
\lambda_2^*(t) &= -C_3 t + C_4.
\end{aligned}
\tag{2.7.54}
$$

- **Step 5:** Obtain the optimal control from

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4 \tag{2.7.55}$$

where, C_1, C_2, C_3, and C_4 are constants evaluated using the given boundary conditions (2.7.48). These are found to be

$$C_1 = 1, \quad C_2 = 2, \quad C_3 = 3, \quad \text{and} \quad C_4 = 4. \tag{2.7.56}$$

Finally, we have the optimal states, costates and control as

$$\begin{aligned}
x_1^*(t) &= 0.5t^3 - 2t^2 + 2t + 1, \\
x_2^*(t) &= 1.5t^2 - 4t + 2, \\
\lambda_1^*(t) &= 3, \\
\lambda_2^*(t) &= -3t + 4, \\
u^*(t) &= 3t - 4.
\end{aligned} \tag{2.7.57}$$

The system with the optimal controller is shown in Figure 2.10.

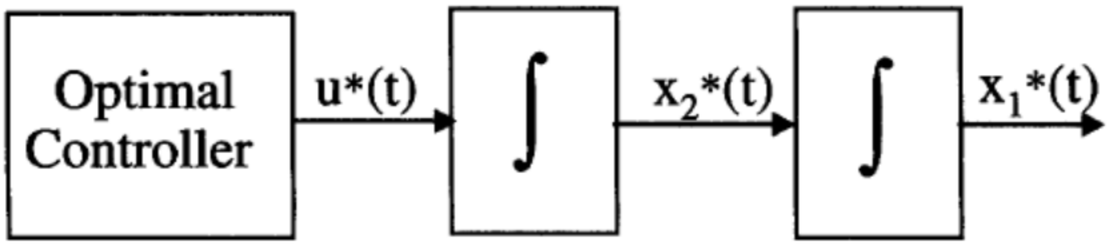


Figure 2.10 Optimal Controller for Example 2.12

The solution for the set of differential equations (2.7.53) with the boundary conditions (2.7.48) for Example 2.12 using Symbolic Toolbox of the MATLAB©, Version 6, is shown below.

```
*******************************************************************
%% Solution Using Symbolic Toolbox (STB) in
%% MATLAB Version 6.0
%%
S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1,...
x1(0)=1,x2(0)=2,x1(2)=1,x2(2)=0')
```

```
S.x1
S.x2
S.lambda1
S.lambda2

S =


    lambda1: [1x1 sym]
    lambda2: [1x1 sym]
         x1: [1x1 sym]
         x2: [1x1 sym]
S.x1

ans =

1+2*t-2*t^2+1/2*t^3

S.x2

ans =
2-4*t+3/2*t^2
```

```
S.lambda1

ans =
3
S.lambda2

ans =

4-3*t
```

```
Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
*********************************************************
```

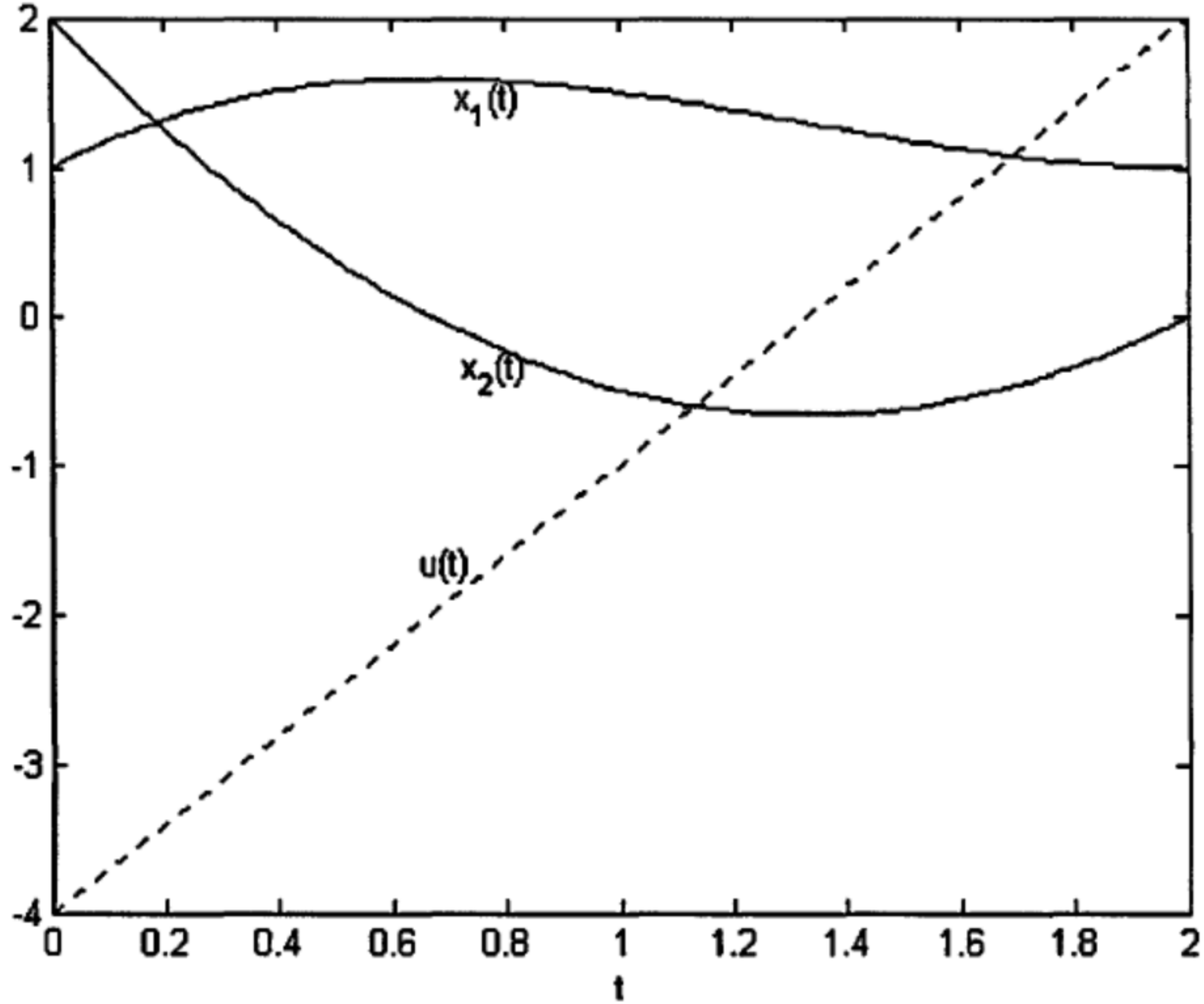


Figure 2.11 Optimal Control and States for Example 2.12

Example 2.13 $J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt$ => $S = 0$

Consider the same Example 2.12 with changed boundary conditions as

$$\mathbf{x}(0) = [1 \;\; 2]'; \quad x_1(2) = 0; \quad x_2(2) \text{ is free.} \tag{2.7.58}$$

Find the optimal control and optimal states.

(Type (c))

Solution:

$$\begin{aligned}
x_1^*(t) &= \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1, \\
x_2^*(t) &= \frac{C_3}{2}t^2 - C_4 t + C_2, \\
\lambda_1^*(t) &= C_3, \\
\lambda_2^*(t) &= -C_3 t + C_4, \\
u^*(t) &= -\lambda_2^*(t) = C_3 t - C_4.
\end{aligned} \tag{2.7.59}$$

$$\lambda_2(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{*t_f} = 0 \tag{2.7.60}$$

(since $S = 0$). Thus we have the four boundary conditions as

$$x_1(0) = 1; \quad x_2(0) = 2; \quad x_1(2) = 0; \quad \lambda_2(2) = 0. \tag{2.7.61}$$

With these boundary conditions substituted in (2.7.59), the constants are found to be

$$C_1 = 1; \quad C_2 = 2; \quad C_3 = 15/8; \quad C_4 = 15/4. \tag{2.7.62}$$

$$\left\{\begin{aligned} x_1^*(t) &= \frac{5}{16}t^3 - \frac{15}{8}t^2 + 2t + 1, \\ x_2^*(t) &= \frac{15}{16}t^2 - \frac{15}{4}t + 2, \\ \lambda_1^*(t) &= \frac{15}{8}, \\ \lambda_2^*(t) &= -\frac{15}{8}t + \frac{15}{4}, \\ u^*(t) &= \frac{15}{8}t - \frac{15}{4}. \end{aligned}\right. \tag{2.7.63}$$

```
*************************************************************************
%% Solution Using Symbolic Toolbox (STB) in
%% MATLAB Version 6.0
%%
S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1,
x1(0)=1,x2(0)=2,x1(2)=0,lambda2(2)=0')

S =

    lambda1: [1x1 sym]
    lambda2: [1x1 sym]
         x1: [1x1 sym]
         x2: [1x1 sym]
S.x1

ans =

5/16*t^3+2*t+1-15/8*t^2

S.x2

ans =

15/16*t^2+2-15/4*t
```

```
S.lambda1

ans =

15/8

S.lambda2

ans =

-15/8*t+15/4
```

```
%% Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
```

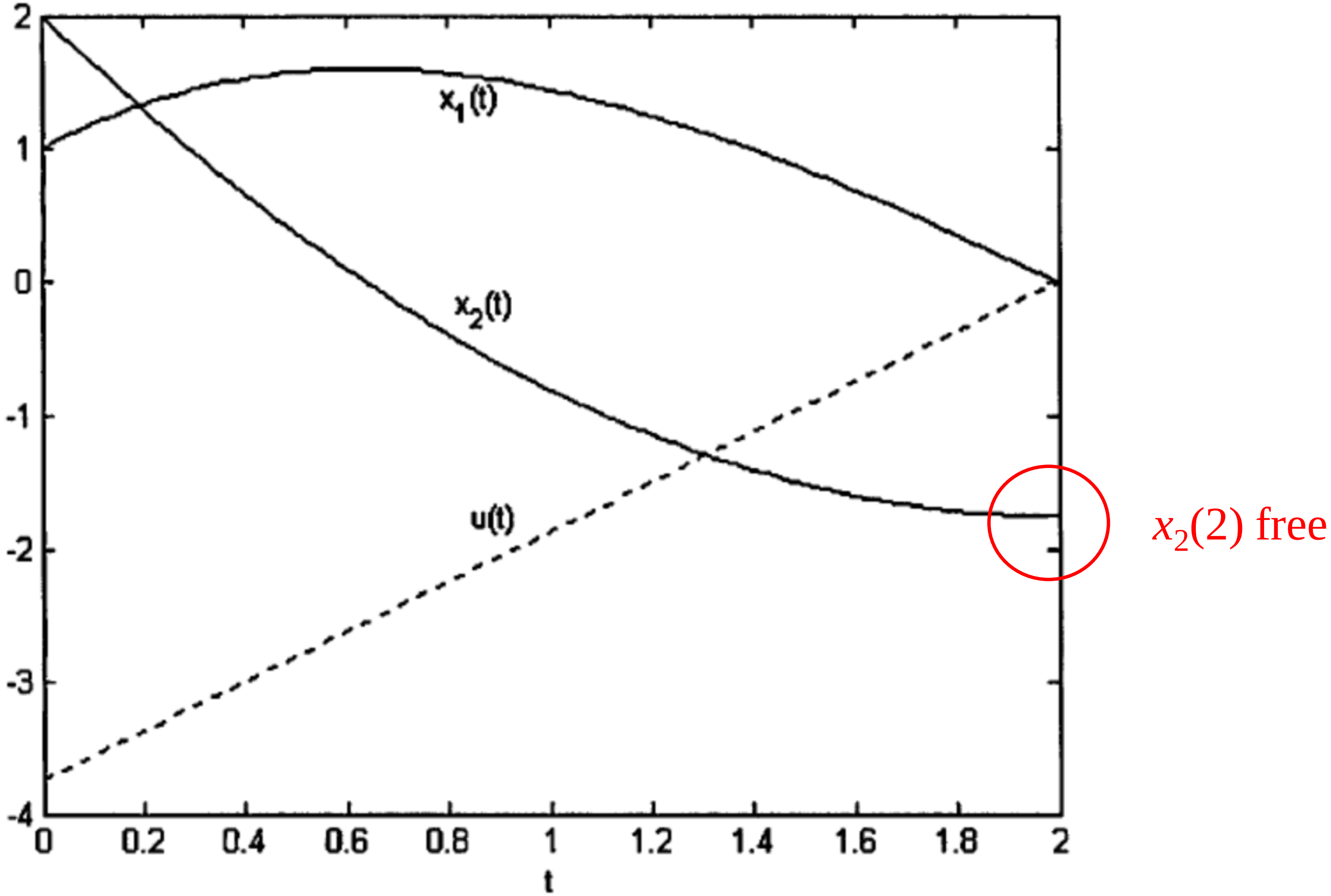


Figure 2.12 Optimal Control and States for Example 2.13

Example 2.14

Consider the same Example 2.12 with changed boundary conditions as

$$\mathbf{x}(0) = [1 \;\; 2]'; \quad \cancel{x_1(2) = 0}; \quad x_1(t_f) = 3; \quad x_2(t_f) \text{ is free.} \tag{2.7.64}$$

Find the optimal control and optimal state.

(Type (e))

Solution:

$$\begin{aligned}
x_1^*(t) &= \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1, \\
x_2^*(t) &= \frac{C_3}{2}t^2 - C_4 t + C_2, \\
\lambda_1^*(t) &= C_3, \\
\lambda_2^*(t) &= -C_3 t + C_4, \\
u^*(t) &= -\lambda_2^*(t) = C_3 t - C_4.
\end{aligned} \tag{2.7.65}$$

$$\left\{\begin{aligned} &\left(\mathcal{H}+\frac{\partial S}{\partial t}\right)_{t_f}=0 \longrightarrow \lambda_1(t_f)x_2(t_f)-0.5\lambda_2^2(t_f)=0 \qquad (2.7.66)\\ &\lambda_2(t_f)=\left(\frac{\partial S}{\partial x_2}\right)=0 \qquad (2.7.67)\end{aligned}\right.$$

$$\Rightarrow \left\{\begin{aligned} &x_1(0)=1; \quad x_2(0)=2; \quad x_1(t_f)=3;\\ &\lambda_2(t_f)=0; \quad \lambda_1(t_f)x_2(t_f)-0.5\lambda_2^2(t_f)=0.\end{aligned}\right. \qquad (2.7.68)$$

$$\Rightarrow C_1=1; \quad C_2=2; \quad C_3=4/9; \quad C_4=4/3; \quad \underline{t_f=3.} \qquad (2.7.69)$$

$$\Rightarrow \left\{\begin{aligned} x_1^*(t)&=\frac{4}{54}t^3-\frac{2}{3}t^2+2t+1,\\ x_2^*(t)&=\frac{4}{18}t^2-\frac{4}{3}t+2,\\ \lambda_1^*(t)&=\frac{4}{9},\\ \lambda_2^*(t)&=-\frac{4}{9}t+\frac{4}{3},\\ u^*(t)&=\frac{4}{9}t-\frac{4}{3}.\end{aligned}\right. \qquad (2.7.70)$$

```
*********************************************************
%% Solution Using Symbolic Toolbox (STB) in
%% of MATLAB Version 6
%%
clear all
S=dsolve('Dx1=x2,Dx2=-lam2,Dlam1=0,Dlam2=-lam1,x1(0)=1,
    x2(0)=2,x1(tf)=3,lam2(tf)=0')
t='tf';
eq1=subs(S.x1)-'x1tf';
eq2=subs(S.x2)-'x2tf';
eq3=S.lam1-'lam1tf';
eq4=subs(S.lam2)-'lam2tf';
eq5='lam1tf*x2tf-0.5*lam2tf^2';
S2=solve(eq1,eq2,eq3,eq4,eq5,'tf,x1tf,x2tf,lam1tf,
    lam2tf','lam1tf<>0')
%% lam1tf<>0 means lam1tf is not equal to 0;
%% This is a condition derived from eq5.
%% Otherwise, without this condition in the above
%% SOLVE routine, we get two values for tf (1 and 3 in this case)
%%
```

```
tf=S2.tf
x1tf=S2.x1tf;
x2tf=S2.x2tf;
clear t
x1=subs(S.x1)
x2=subs(S.x2)
lam1=subs(S.lam1)
lam2=subs(S.lam2)
%% Convert the symbolic values to
%% numerical values as shown below.
j=1;
tf=double(subs(S2.tf))
%% coverts tf from symbolic to numerical
for tp=0:0.05:tf
t=sym(tp);
%% coverts tp from numerical to symbolic
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lam2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
```

```
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
*********************************************************
```

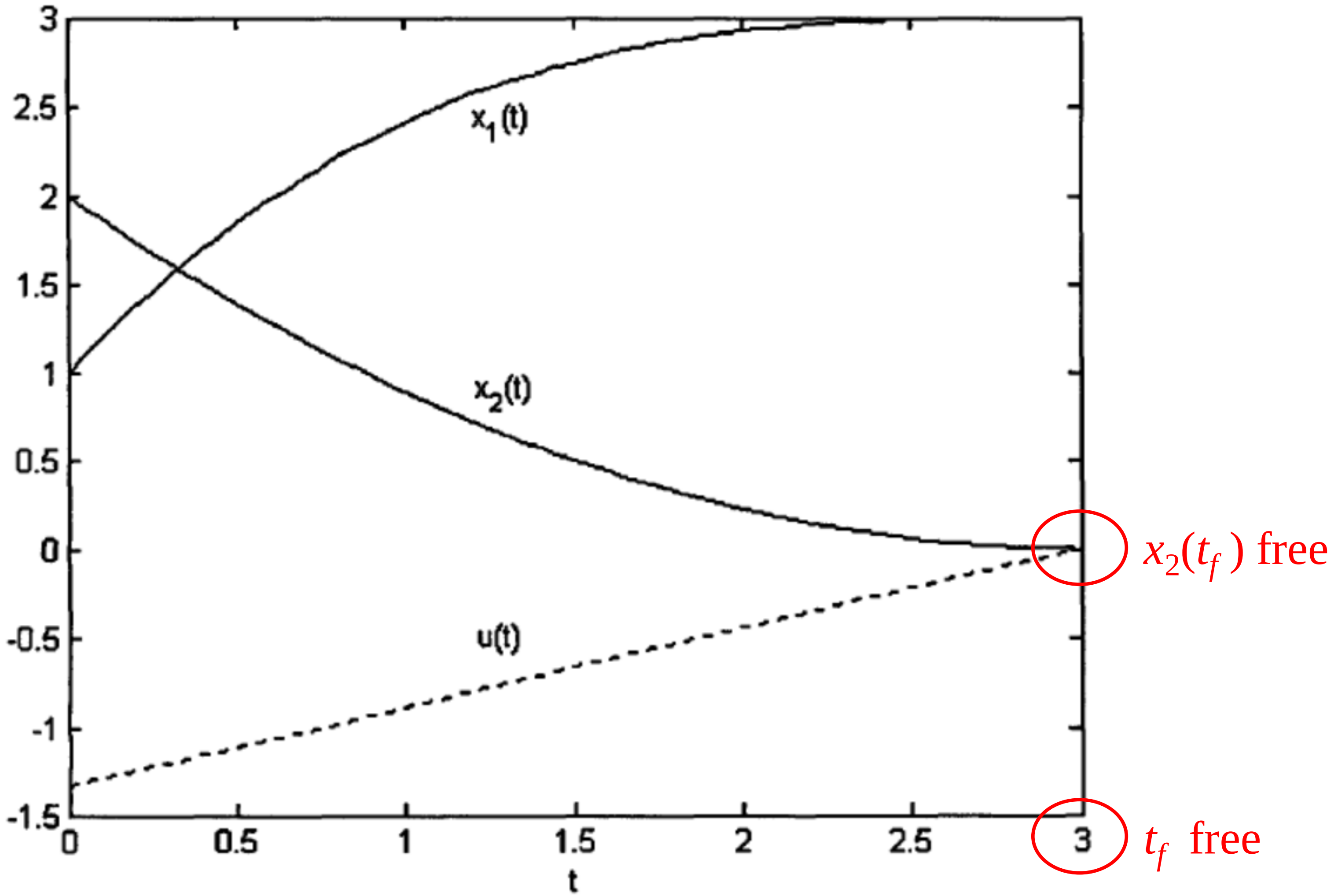


Figure 2.13 Optimal Control and States for Example 2.14

Example 2.15

We consider the same Example 2.12 with changed performance index

$$J = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2 + \frac{1}{2}\int_0^2 u^2 dt \qquad (2.7.71)$$

and boundary conditions as

$$\mathbf{x}(0) = [1 \;\; 2]; \qquad \mathbf{x}(2) = \text{is free.} \qquad (2.7.72)$$

(Type (c))

$$\begin{aligned} x_1^*(t) &= \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1, \\ x_2^*(t) &= \frac{C_3}{2}t^2 - C_4 t + C_2, \\ \lambda_1^*(t) &= C_3, \\ \lambda_2^*(t) &= -C_3 t + C_4, \\ u^*(t) &= -\lambda_2(t) = C_3 t - C_4. \end{aligned} \qquad (2.7.73)$$

$$\boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f} \tag{2.7.74}$$

where,

$$S(\mathbf{x}(t_f)) = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2. \tag{2.7.75}$$

$$\left\{\begin{aligned} \lambda_1^*(t_f) &= \left(\frac{\partial S}{\partial x_1}\right)_{t_f} \longrightarrow \lambda_1^*(2) = x_1(2) - 4 \\ \lambda_2^*(t_f) &= \left(\frac{\partial S}{\partial x_2}\right)_{t_f} \longrightarrow \lambda_2^*(2) = x_2(2) - 2. \end{aligned}\right. \tag{2.7.76}$$

$$C_1 = 1, \quad C_2 = 2, \quad C_3 = \frac{3}{7}, \quad C_4 = \frac{4}{7}. \tag{2.7.77}$$

$$
\begin{aligned}
x_1^*(t) &= \frac{1}{14}t^3 - \frac{2}{7}t^2 + 2t + 1, \\
x_2^*(t) &= \frac{3}{14}t^2 - \frac{4}{7}t + 2, \\
\lambda_1^*(t) &= \frac{3}{7}, \\
\lambda_2^*(t) &= -\frac{3}{7}t + \frac{4}{7}, \\
u^*(t) &= \frac{3}{7}t - \frac{4}{7}.
\end{aligned} \tag{2.7.78}
$$

```
*******************************************************************
%% Solution Using Symbolic Math Toolbox (STB) in
%%  MATLAB Version 6
%%
S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1,
x1(0)=1,x2(0)=2,lambda1(2)=x12-4,lambda2(2)=x22-2')
t='2';
S2=solve(subs(S.x1)-'x12',subs(S.x2)-'x22','x12,x22');
%% solves for x1(t=2) and x2(t=2)
x12=S2.x12;
x22=S2.x22;
clear t

S =


      lambda1: [1x1 sym]
      lambda2: [1x1 sym]
           x1: [1x1 sym]
           x2: [1x1 sym]
```

```
x1=subs(S.x1)

x1 =

1-2/7*t^2+1/14*t^3+2*t


x2=subs(S.x2)

x2 =

-4/7*t+3/14*t^2+2

lambda1=subs(S.lambda1)

lambda1 =

3/7

 lambda2=subs(S.lambda2)

 lambda2 =

 4/7-3/7*t
```

```
%% Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end

plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
*******************************************************************
```

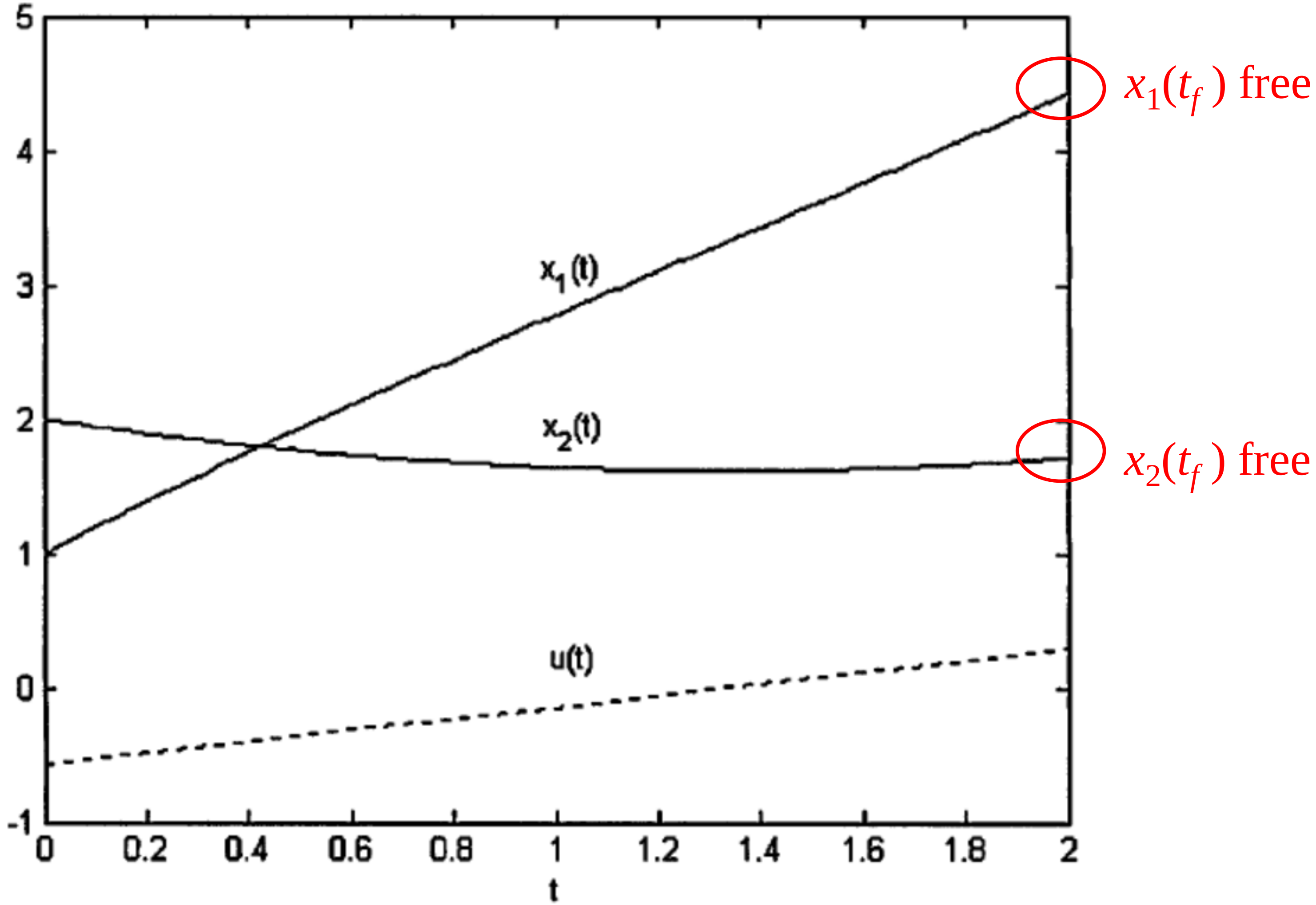


Figure 2.14 Optimal Control and States for Example 2.15

2.8 *Summary of Variational Approach*

2.8.1 Stage I: Optimization of a Functional

$$J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\,dt \tag{2.8.1}$$

$$\mathbf{x}(t_0) \text{ fixed and } \underline{\mathbf{x}(t_f) \text{ free}}. \tag{2.8.2}$$

necessary condition:

$$\boxed{\left(\frac{\partial V}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{\mathbf{x}}}\right)_* = 0.} \tag{2.8.3}$$

$$\boxed{\left[V - \dot{\mathbf{x}}'(t)\left(\frac{\partial V}{\partial \dot{\mathbf{x}}}\right)\right]_{*t_f} \delta t_f + \left(\frac{\partial V}{\partial \dot{\mathbf{x}}}\right)'_{*t_f} \delta \mathbf{x}_f = 0.} \tag{2.8.4}$$

This boundary condition is to be modified depending on the nature of the given t_f and $x(t_f)$.

Sufficient condition:

$$\boxed{\left(\frac{\partial^2 V}{\partial \dot{\mathbf{x}}^2}\right)_* > 0} \quad \text{for minimum} \tag{2.8.5}$$

$$\boxed{\left(\frac{\partial^2 V}{\partial \dot{\mathbf{x}}^2}\right)_* < 0} \quad \text{for maximum.} \tag{2.8.6}$$

2.8.2 Stage II: Optimization of a Functional with Condition

$$J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt \tag{2.8.7}$$

$$\mathbf{x}(t_0) \text{ fixed and } \underline{\mathbf{x}(t_f) \text{ free},} \tag{2.8.8}$$

State equation: $$\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0. \tag{2.8.9}$$

(2.8.7) and (2.8.9) $\Rightarrow$ $$J_a = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \underline{\boldsymbol{\lambda}(t)}, t)dt \tag{2.8.10}$$

$$\boxed{\mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t).} \tag{2.8.11}$$

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0} \quad \text{\underline{state} equation and} \tag{2.8.12}$$

$g = 0 \Leftarrow$ $$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\lambda}}}\right)_* = 0} \quad \text{\underline{costate} equation.} \tag{2.8.13}$$

$$\boxed{\left[\mathcal{L} - \dot{\mathbf{x}}'(t)\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)\right]_{*t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_{*t_f} \delta \mathbf{x}_f = 0.} \tag{2.8.14}$$

B.C.

2.8.3 Stage III: Optimal Control System with Lagrangian Formalism

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{2.8.15}$$

$$\mathbf{x}(t_0) \text{ is fixed and } \underline{\mathbf{x}(t_f) \text{ is free}}, \tag{2.8.16}$$

$$J(\mathbf{u}(t)) = \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt. \tag{2.8.17}$$

State eq.: $$\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), t) = \underline{\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t) = 0.} \tag{2.8.18}$$

Augmented functional:

$$J_a(\mathbf{u}(t)) = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t)dt \tag{2.8.19}$$

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t) \left\{ \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t) \right\}. \end{aligned} \tag{2.8.20}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0$$ *state* equation, (2.8.21)

State eq.

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* - \underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\lambda}}}\right)_*}_{=0} = 0$$ *costate* equation, and (2.8.22)

***u* = ?**

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* - \underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{u}}}\right)_*}_{=0} = 0$$ *control* equation. (2.8.23)

$$\left[\mathcal{L} - \dot{\mathbf{x}}'(t)\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)\right]_{*t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_{*t_f} \delta \mathbf{x}_f = 0. \qquad (2.8.24)$$

2.8.4 Stage IV: Optimal Control System with Hamiltonian Formalism: Pontryagin Principle

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (2.8.25)$$

$$\mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = \mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) - \boldsymbol{\lambda}'(t)\dot{\mathbf{x}}(t). \quad (2.8.26)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}(-\boldsymbol{\lambda}^*) = 0 \quad (2.8.27)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* - \dot{\mathbf{x}}^*(t) - \frac{d}{dt}(0) = 0 \quad (2.8.28)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* - \frac{d}{dt}(0) = 0 \quad (2.8.29)$$

$$\boxed{\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*} \quad \textit{state} \text{ equation,} \quad (2.8.30)$$

$$\boxed{\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*} \quad \textit{costate} \text{ equation, and} \quad (2.8.31)$$

$$\boxed{0 = +\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*} \quad \textit{control} \text{ equation.} \quad (2.8.32)$$

$$[\mathcal{H} - \boldsymbol{\lambda}'(t)\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}'(t)(-\boldsymbol{\lambda}(t))]\big|_{*t_f}\delta t_f + [-\boldsymbol{\lambda}'(t)]\big|_{*t_f}\delta \mathbf{x}_f = 0 \quad (2.8.33)$$

$$\boxed{\mathcal{H}\big|_{*t_f}\delta t_f - \boldsymbol{\lambda}^{*\prime}(t_f)\delta\mathbf{x}_f = 0.} \quad (2.8.34)$$

Sufficient condition

$$\boxed{\left(\frac{\partial^2\mathcal{H}}{\partial\mathbf{u}^2}\right)_* > 0} \quad \text{for minimum and} \quad (2.8.35)$$

$$\boxed{\left(\frac{\partial^2\mathcal{H}}{\partial\mathbf{u}^2}\right)_* < 0} \quad \text{for maximum.} \quad (2.8.36)$$

Free-Final Point System with Final Cost Function

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.8.37}$$

$$J(\mathbf{u}(t)) = \underline{S(\mathbf{x}(t_f), t_f)} + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.8.38}$$

$$\mathbf{x}(t_0) \text{ is fixed and } \mathbf{x}(t_f) \text{ is free.} \tag{2.8.39}$$

$$J_1(\mathbf{u}(t)) = \int_{t_0}^{t_f} \left[V(\mathbf{x}(t), \mathbf{u}(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)' \dot{\mathbf{x}}(t) + \frac{\partial S}{\partial t} \right] dt. \tag{2.8.40}$$

$$\boxed{\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*} \quad \textit{state} \text{ equation} \tag{2.8.41}$$

$$\boxed{\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*} \quad \textit{costate} \text{ equation} \tag{2.8.42}$$

$$\boxed{0 = +\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*} \quad \textit{control} \text{ equation} \tag{2.8.43}$$

$$\boxed{\left[\mathcal{H} + \frac{\partial S}{\partial t}\right]_{*t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right) - \boldsymbol{\lambda}(t)\right]'_{*t_f} \delta \mathbf{x}_f = 0.} \tag{2.8.44}$$

$$\boxed{\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* > 0} \text{ for minimum and} \tag{2.8.45}$$

$$\boxed{\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* < 0} \text{ for maximum.} \tag{2.8.46}$$

2.8.5 *Salient Features*

1. Significance of Lagrange Multiplier (also called the costate (or adjoint) function):

a. It is imposed by the plant equation.

b. It enables us to use the Euler-Lagrange equation for each of the variables $x(t)$ and $u(t)$ separately.

2. Lagrangian and Hamiltonian:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), \mathbf{u}(t), t) \\ &= V(\mathbf{x}(t), \mathbf{u}(t), t) \\ &\quad + \boldsymbol{\lambda}'(t) \left\{ \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \underline{- \dot{\mathbf{x}}(t)} \right\} \qquad (2.8.47)\\ \mathcal{H} &= \mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}(t), \mathbf{u}(t), t) \\ &\quad + \boldsymbol{\lambda}'(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t). \qquad (2.8.48)\end{aligned}$$

Note: both are *scalar* functions only.

3. *Optimization of Hamiltonian*

a.

$$\text{Min } J(\mathbf{u}(t)) = \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt. \quad (2.8.17)$$

$$\text{s.t. } \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad (2.8.15)$$

$$\Rightarrow \boxed{0 = +\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*} \quad \textit{control} \text{ equation.} \quad (2.8.32)$$

We "reduced" our original *functional* optimization problem to an ordinary *function* optimization problem

b. We assumed unconstrained or unbounded control *u*(*t*) and obtained the control relation (2.8.32).

c. In practice, the control *u*(*t*) is always *limited* by such things as saturation of amplifiers, speed of a motor, or thrust of a rocket. The constrained optimal control systems are discussed in Chapter 7.

d. $$\boxed{\min_{\mathbf{u} \in \mathbf{U}} \mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \mathbf{u}(t), t) = \mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \mathbf{u}^*(t), t)} \tag{2.8.49}$$

or equivalently

$$\boxed{\mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \mathbf{u}^*(t), t) \leq \mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \mathbf{u}(t), t).} \tag{2.8.50}$$

4. ***Pontryagin Maximum Principle***

If we define the Hamiltonian as

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = -V(\mathbf{x}(t), \mathbf{u}(t), t) + \hat{\boldsymbol{\lambda}}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad (2.8.51)$$

we have *Maximum Principle*.

5. ***Hamiltonian at the Optimal Condition***

$$
\begin{aligned}
\mathcal{H}^* &= \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\
\frac{d\mathcal{H}^*}{dt} &= \frac{d\mathcal{H}^*}{dt} \\
&= \cancelto{-\dot{\lambda}}{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)'_*} \dot{\mathbf{x}}^*(t) + \cancelto{\dot{x}}{\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)'_*} \dot{\boldsymbol{\lambda}}^*(t) + \cancelto{0}{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)'_*} \dot{\mathbf{u}}^*(t) + \left(\frac{\partial \mathcal{H}}{\partial t}\right)_*
\end{aligned}
\tag{2.8.52}
$$

Using the state, costate and control equations (2.8.30) to (2.8.32) in the previous equation, we get

$$
\left(\frac{d\mathcal{H}}{dt}\right)_* = \left(\frac{\partial \mathcal{H}}{\partial t}\right)_* . \tag{2.8.53}
$$

If $\mathcal{H}$ does not depend on t explicitly, then

$$
\boxed{\left.\frac{d\mathcal{H}}{dt}\right|_* = 0} \tag{2.8.54}
$$

$\Rightarrow$ $\mathcal{H}$ is constant w.r.t. the time t

6. ***Two-Point Boundary Value Problem (TPBVP):***

a) The state and costate equations (2.8.30) and (2.8.32) are solved using the initial and final conditions. In general, these are nonlinear, time varying and we may have to resort to numerical methods for their solutions.

b) The state and costate equations are the same for any kind of boundary conditions.

c) Although obtaining the state and costate equations is very easy, the computation of their solutions is quite tedious.

7.

- We get only the open-loop optimal control (OLOC) as shown in Figure 2.15.
- In many cases it is very tedious.
- Changes in plant parameters are not taken into account by the OLOC.

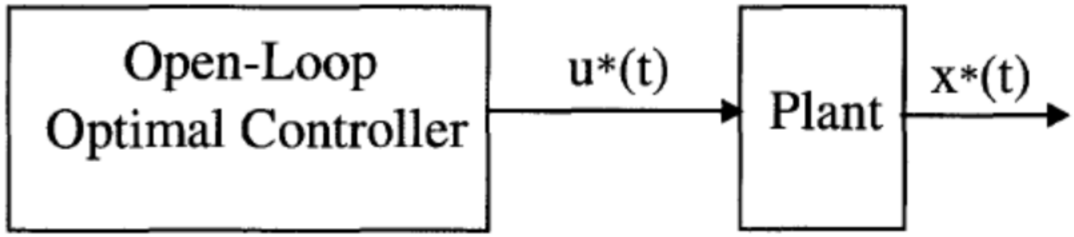


Figure 2.15 Open-Loop Optimal Control

- This prompts us to think a closed-loop optimal control (CLOC), i.e., to obtain optimal control $u^*(t)$ in terms of the state $x^*(t)$ as shown in Figure 2.16.
- This CLOC will have many advantages such as sensitive to plant parameter variations and simplified construction of the controller.
- The closed-loop optimal control systems are discussed in Chapter 7.

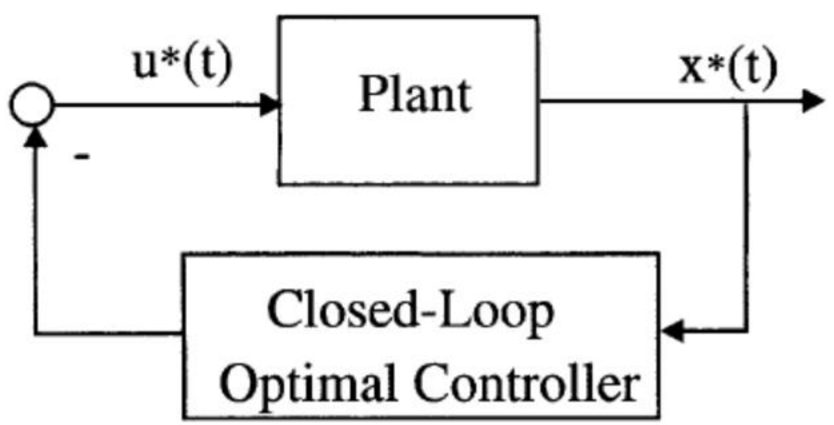


Figure 2.16 Closed-Loop Optimal Control