Chapter 3
Linear Quadratic Optimal Control Systems I

- In this chapter, we present the closed-loop *optimal control* of *linear* plants or systems with *quadratic* performance index or measure.
- This leads to the *linear quadratic regulator* (LQR) system dealing with state regulation, output regulation, and tracking.
- Broadly speaking, we are interested in the design of optimal linear systems with quadratic performance indices.
- It is suggested that the student reviews the material in Appendices A (Vectors and Matrices) and B (State Space Analysis) given at the end of the book.

3.1 Problem Formulation

Consider a linear, time-varying (LTV) system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{3.1.1}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{3.1.2}$$

with a cost functional (CF) or performance index (PI)

$$\begin{aligned} J(\mathbf{u}(t)) &= J(\mathbf{x}(t_0), \mathbf{u}(t), t_0) \\ &= \frac{1}{2}\left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right]' \mathbf{F}(t_f)\left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right] \\ &\quad + \frac{1}{2}\int_{t_0}^{t_f}\Big[\left[\mathbf{z}(t) - \mathbf{y}(t)\right]' \mathbf{Q}(t)\left[\mathbf{z}(t) - \mathbf{y}(t)\right] + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\Big]\, dt \end{aligned} \tag{3.1.3}$$

where, $\mathbf{x}(t)$ is nth *state* vector,

$\mathbf{y}(t)$ is mth *output* vector,

$\mathbf{z}(t)$ is mth reference or *desired* output vector

$\mathbf{u}(t)$ is rth *control* vector,

$\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t)$ (or $\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{x}(t)$

Assumptions

- The control $u(t)$ is unconstrained.
- All the states and/or outputs are completely measurable.
- Depending on the final time t_f being finite (infinite), the system is called finite- (infinite-) time horizon system.

System categories

- If our objective is to keep the *state* $x(t)$ near zero (i.e., $z(t) = 0$ and $C = I$), then we call it *state regulator* system.
- If our interest is to keep the *output* $y(t)$ near zero (i.e., $z(t) = 0$), then it is termed the *output regulator* system.
- If we try to keep the *output or state* near a *desired* state or output, then we are dealing with a *tracking* system.

Let us consider the various matrices in the cost functional (3.1.3) and their implications.

$$
\begin{aligned}
J(\mathbf{u}(t)) &= J(\mathbf{x}(t_0), \mathbf{u}(t), t_0) \\
&= \frac{1}{2}\left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right]' \mathbf{F}(t_f) \left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right] \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f} \Big[\left[\mathbf{z}(t) - \mathbf{y}(t)\right]' \mathbf{Q}(t) \left[\mathbf{z}(t) - \mathbf{y}(t)\right] + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\Big]\, dt
\end{aligned}
\tag{3.1.3}
$$

1. *The Error Weighted Matrix* $\mathbf{Q}(t)$: In order to keep the error $\mathbf{e}(t)$ small and error squared non-negative, the integral of the expression $\frac{1}{2}\mathbf{e}'(t)\mathbf{Q}(t)\mathbf{e}(t)$ should be nonnegative and small. Thus, the matrix $\mathbf{Q}(t)$ must be *positive semidefinite*. Due to the quadratic nature of the weightage, we have to pay more attention to large errors than small errors.

2. *The Control Weighted Matrix* $\mathbf{R}(t)$: The quadratic nature of the control cost expression $\frac{1}{2}\mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)$ indicates that one has to pay higher cost for larger control effort. Since the cost of the control has to be a positive quantity, the matrix $\mathbf{R}(t)$ should be *positive definite*.

$$
\begin{aligned}
J(\mathbf{u}(t)) &= J(\mathbf{x}(t_0), \mathbf{u}(t), t_0) \\
&= \frac{1}{2}\left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right]' \mathbf{F}(t_f) \left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right] \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f} \Big[\left[\mathbf{z}(t) - \mathbf{y}(t)\right]' \mathbf{Q}(t) \left[\mathbf{z}(t) - \mathbf{y}(t)\right] + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\Big]\, dt
\end{aligned}
\tag{3.1.3}
$$

3. *The Control Signal* $\mathbf{u}(t)$: The assumption that there are *no constraints* on the control $\mathbf{u}(t)$ is very important in obtaining the closed loop optimal configuration.

Combining all the previous assumptions, we would like on one hand, to keep the error small, but on the other hand, we must not pay higher cost to large controls.

$$
\begin{aligned}
J(\mathbf{u}(t)) &= J(\mathbf{x}(t_0), \mathbf{u}(t), t_0) \\
&= \frac{1}{2}\left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right]' \mathbf{F}(t_f) \left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right] \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f} \Big[\left[\mathbf{z}(t) - \mathbf{y}(t)\right]' \mathbf{Q}(t) \left[\mathbf{z}(t) - \mathbf{y}(t)\right] + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t) \Big] \, dt
\end{aligned}
\tag{3.1.3}
$$

4. *The Terminal Cost Weighted Matrix* $\mathbf{F}(t_f)$: The main purpose of this term is to ensure that the error $\mathbf{e}(t)$ at the final time t_f is as small as possible. To guarantee this, the corresponding matrix $\mathbf{F}(t_f)$ should be *positive semidefinite.*

Further, without loss of generality, we assume that the weighted matrices $\mathbf{Q}(t), \mathbf{R}(t)$, and $\mathbf{F}(t)$ are *symmetric.* The *quadratic* cost functional described previously has some attractive features:

(a) It provides an elegant procedure for the design of *closed-loop* optimal controller.

(b) It results in the optimal feed-back control that is *linear* in *state* function.

$$
\begin{aligned}
J(\mathbf{u}(t)) &= J(\mathbf{x}(t_0), \mathbf{u}(t), t_0) \\
&= \frac{1}{2}\left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right]' \mathbf{F}(t_f) \left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right] \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f} \Big[\left[\mathbf{z}(t) - \mathbf{y}(t)\right]' \mathbf{Q}(t) \left[\mathbf{z}(t) - \mathbf{y}(t)\right] + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\Big]\, dt
\end{aligned}
\tag{3.1.3}
$$

5. *Infinite Final Time*: When the final time t_f is infinity, the terminal cost term involving $\mathbf{F}(t_f)$ does not provide any realistic sense since we are always interested in the solutions over finite time. Hence, $\mathbf{F}(t_f)$ must be zero.

$$
\begin{aligned}
J(\mathbf{u}(t)) &= J(\mathbf{x}(t_0), \mathbf{u}(t), t_0) \\
&= \frac{1}{2}\left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right]' \mathbf{F}(t_f) \left[\mathbf{z}(t_f) - \mathbf{y}(t_f)\right] \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f} \Big[\left[\mathbf{z}(t) - \mathbf{y}(t)\right]' \mathbf{Q}(t) \left[\mathbf{z}(t) - \mathbf{y}(t)\right] + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\Big]\, dt
\end{aligned}
\tag{3.1.3}
$$

3.2 Finite-Time Linear Quadratic Regulator (LQR)

Consider a linear, time-varying plant described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{3.2.1}$$

with a cost functional

$$\begin{aligned} J(\mathbf{u}) &= J(\mathbf{x}(t_0), \mathbf{u}(t), t_0) \\ &= \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) \\ &\quad + \frac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt \\ &= \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) \\ &\quad + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}'(t)\ \mathbf{u}'(t)\right] \begin{bmatrix} \mathbf{Q}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{R}(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt \end{aligned} \tag{3.2.2}$$

Let us note that here, the reference or desired state $\mathbf{z}(t) = 0$ and hence the error $\underline{\mathbf{e}(t) = 0 - \mathbf{x}(t)}$

Imply a state regulator system!

We summarize again various assumptions as follows.

1. The control $\mathbf{u}(t)$ is *unconstrained*. However, in many physical situations, there are limitations on the control and state and the case of *unconstrained* control is discussed in a later chapter.
2. The initial condition $\mathbf{x}(t = t_0) = \mathbf{x}_0$ is given. The terminal time t_f is *specified*, and the final state $\mathbf{x}(t_f)$ is *not specified*.
3. The terminal cost matrix $\mathbf{F}(t_f)$ and the error weighted matrix $\mathbf{Q}(t)$ are nxn positive *semidefinite* matrices, respectively; and the control weighted matrix $\mathbf{R}(t)$ is an rxr positive *definite* matrix.
4. Finally, the fraction $\frac{1}{2}$ in the cost functional (3.2.2) is associated mainly to cancel a 2 that would have otherwise been carried on throughout the result, as seen later.

We follow the Pontryagin procedure described in Chapter 2 (Table 2.1) to obtain optimal solution and then propose the closed-loop configuration. First, let us list the various steps under which we present the method.

- **Step 1:** *Hamiltonian*
- **Step 2:** *Optimal Control*
- **Step 3:** *State and Costate System*
- **Step 4:** *Closed-Loop Optimal Control*
- **Step 5:** *Matrix Differential Riccati Equation*

- **Step 1:** *Hamiltonian*:

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)) = \frac{1}{2}\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \frac{1}{2}\mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t) + \boldsymbol{\lambda}'(t)\left[\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\right] \quad (3.2.3)$$

where, $\boldsymbol{\lambda}(t)$ is the costate vector of nth order.

- **Step 2:** *Optimal Control*:

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0 \longrightarrow \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}'(t)\boldsymbol{\lambda}^*(t) = 0 \qquad (3.2.4)$$

leading to $$\mathbf{u}^*(t) = -\underline{\mathbf{R}^{-1}(t)}\mathbf{B}'(t)\boldsymbol{\lambda}^*(t) \qquad (3.2.5)$$

Positive definite

where, we used

$$\frac{\partial}{\partial \mathbf{u}}\left\{\frac{1}{2}\mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right\} = \underline{\mathbf{R}(t)}\mathbf{u}(t) \quad \text{and}$$

Symmetric

$$\frac{\partial}{\partial \mathbf{u}}\{\boldsymbol{\lambda}'(t)\mathbf{B}(t)\mathbf{u}(t)\} = \mathbf{B}'(t)\boldsymbol{\lambda}(t).$$

- **Step 3:** *State and Costate System:*

$$\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* \longrightarrow \dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\underset{(3.2.5)}{\underline{\mathbf{u}^*(t)}} \qquad (3.2.6)$$

$$\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* \longrightarrow \dot{\boldsymbol{\lambda}}^*(t) = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}'(t)\boldsymbol{\lambda}^*(t). \quad (3.2.7)$$

$$\begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ \dot{\boldsymbol{\lambda}}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & -\mathbf{E}(t) \\ -\mathbf{Q}(t) & -\mathbf{A}'(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ \boldsymbol{\lambda}^*(t) \end{bmatrix} \qquad (3.2.8)$$

where $\mathbf{E}(t) = \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)$

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \underset{=0}{\underline{\delta t_f}} + \underline{\left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f}} \underset{\neq 0}{\underline{\delta \mathbf{x}_f}} = 0 \qquad (3.2.9)$$

$$\underline{\boldsymbol{\lambda}^*(t_f)} = \left(\frac{\partial S}{\partial \mathbf{x}(t_f)}\right)_*$$

$$= \frac{\partial\left[\frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f)\right]}{\partial \mathbf{x}(t_f)} = \underline{\mathbf{F}(t_f)\mathbf{x}^*(t_f)}. \qquad (3.2.10)$$

This final condition on the costate together with the initial condition on the state and the canonical system of equations (3.2.8) form a two-point, boundary value problem (TPBVP).

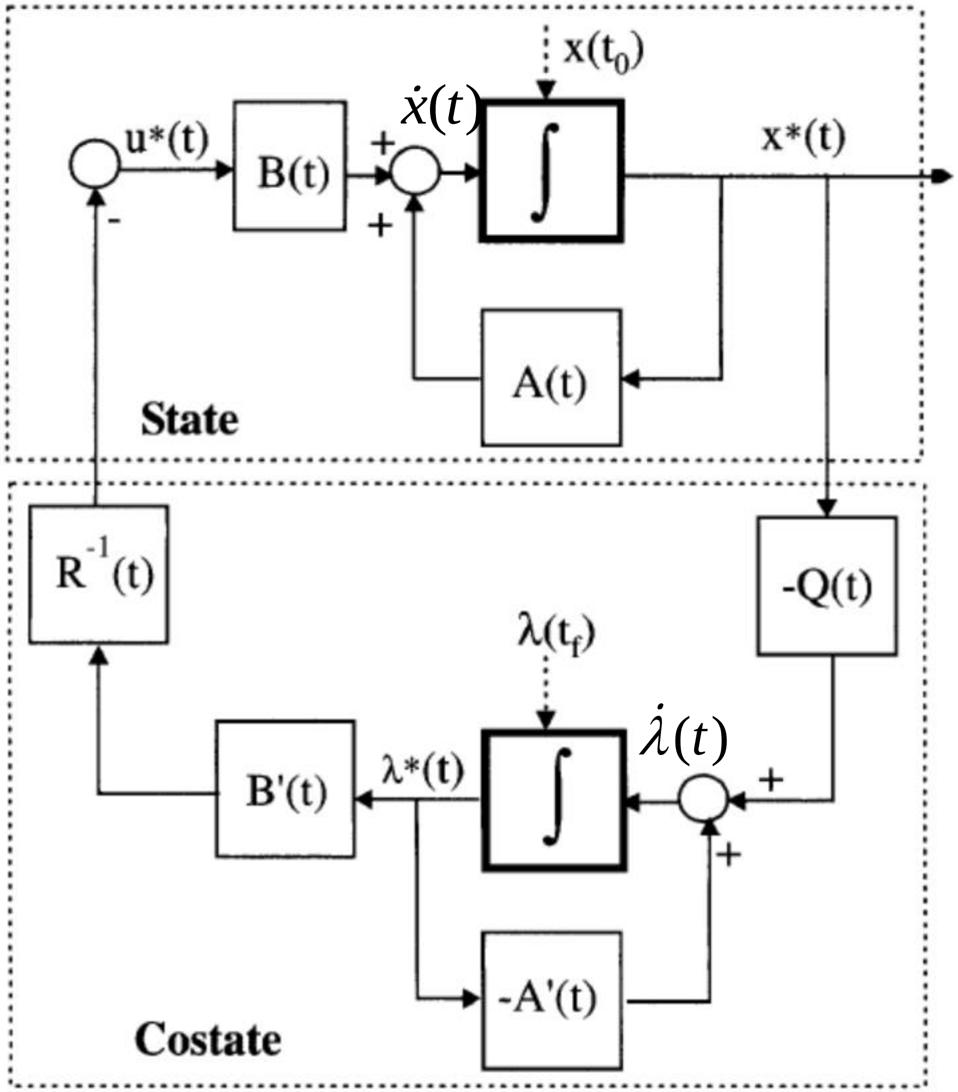


Figure 3.1 State and Costate System

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$\dot{\lambda}(t) = -Q(t)x(t) - A^T(t)\lambda(t)$$

$$u(t) = -R^{-1}(t)B^T(t)\lambda(t)$$

- **Step 4:** *Closed-Loop Optimal Control:*

$$\boldsymbol{\lambda}^*(t_f) = \mathbf{F}(t_f)\mathbf{x}^*(t_f). \qquad (3.2.10)$$

Assume

$$\boldsymbol{\lambda}^*(t) = \mathbf{P}(t)\mathbf{x}^*(t) \qquad (3.2.11)$$

where, $\mathbf{P}(t)$ is yet to be determined.

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}^*(t) \qquad (3.2.12)$$

$$\begin{cases} \dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) - \mathbf{B}(t)\underbrace{\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}^*(t)}_{u(t)}, & (3.2.14) \\ \dot{\boldsymbol{\lambda}}^*(t) = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}'(t)\underbrace{\mathbf{P}(t)\mathbf{x}^*(t)}_{\lambda(t)}. & (3.2.15) \end{cases}$$

Differentiating (3.2.11) w.r.t. time t, we get

$$\dot{\boldsymbol{\lambda}}^*(t) = \dot{\mathbf{P}}(t)\mathbf{x}^*(t) + \mathbf{P}(t)\dot{\mathbf{x}}^*(t). \qquad (3.2.13)$$

Now, substituting state and costate relations (3.2.14) and (3.2.15) in (3.2.13), we have

$$
\begin{aligned}
-\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}'(t)\mathbf{P}(t)\mathbf{x}^*(t) = \dot{\mathbf{P}}(t)\mathbf{x}^*(t) + \\
\mathbf{P}(t)\left[\mathbf{A}(t)\mathbf{x}^*(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}^*(t)\right] \longrightarrow \\
\Big[\underline{\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{Q}(t)-} \\
\underline{\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)}\Big]\,\mathbf{x}^*(t) = 0
\end{aligned}
\tag{3.2.16}
$$

- **Step 5:** *Matrix Differential Riccati Equation:*

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{Q}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t) = 0. \qquad (3.2.17)$$

- This is the matrix differential equation of the *Riccati type*.
- And, it often called the matrix *differential Riccati equation* (DRE).
- Also, the transformation (3.2.11) is called the *Riccati transformation*, *P*(*t*) is called the Riccati coefficient matrix or simply Riccati matrix or Riccati coefficient.
- And, (3.2.12) is the *optimal control (feedback) law*.

$$\boldsymbol{\lambda}^*(t) = \mathbf{P}(t)\mathbf{x}^*(t) \qquad (3.2.11)$$

$$\mathbf{u}^*(t) = \underbrace{-\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)}_{K(t)\ \text{feedback gain}}\underline{\mathbf{x}^*(t)} \qquad (3.2.12)$$

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) - \mathbf{Q}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) \quad (3.2.18)$$

where $\mathbf{E}(t) = \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)$.

Comparing the boundary condition(3.2.10) and the Riccati transformation (3.2.11), we have the final condition on $\mathbf{P}(t)$ as

$$\boldsymbol{\lambda}^*(t_f) = \mathbf{P}(t_f)\mathbf{x}^*(t_f) = \mathbf{F}(t_f)\mathbf{x}^*(t_f) \longrightarrow$$
$$\boxed{\mathbf{P}(t_f) = \mathbf{F}(t_f).} \quad (3.2.19)$$

Thus, the matrix DRE (3.2.17) or (3.2.18) is to be solved *backward in time* using the final condition (3.2.19) to obtain the solution $\mathbf{P}(t)$ for the entire interval $[t_0, t_f]$.

Summary

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad (3.2.1)$$

$$J(\mathbf{u}) = \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \qquad (3.2.2)$$

$\mathbf{x}(t = t_0) = \mathbf{x}_0$ is given. t_f is *specified*, $\mathbf{x}(t_f)$ is *not specified.*

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{Q}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t) = 0 \qquad (3.2.17)$$

$$\mathbf{P}(t_f) = \mathbf{F}(t_f)$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}^*(t) \qquad (3.2.12)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad (3.2.1)$$

$\mathbf{x}(t = t_0) = \mathbf{x}_0$ is given.
t_f is *specified*,

3.2.1 Symmetric Property of the Riccati Coefficient Matrix

Matrix differential Riccati equation (DRE)

$$\dot{\mathbf{P}}(t) = \underline{-\mathbf{P}(t)\mathbf{A}(t)}\ \underline{-\ \mathbf{A}'(t)\mathbf{P}(t)} - \mathbf{Q}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) \quad (3.2.18)$$

where $\mathbf{E}(t) = \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)$.

Transpose both sides of the matrix DRE

$$\dot{\mathbf{P}}^T(t) = \underline{-\ \mathbf{A}^T(t)\mathbf{P}^T(t)}\ \underline{-\mathbf{P}^T(t)\mathbf{A}(t)} - \mathbf{Q}^T(t) + \mathbf{P}^T(t)\mathbf{E}^T(t)\mathbf{P}^T(t)$$

where $\mathbf{E}^T(t) = \mathbf{B}(t)\mathbf{R}^{-T}(t)\mathbf{B}'(t)$.

Since $\mathbf{F}(t_f)$, $\mathbf{Q}(t)$, and $\mathbf{R}(t)$ are symmetric, the two DREs are the same.

Thus, $\underline{\mathbf{P}(t) = \mathbf{P}'(t)}$ (symmetric).

3.2.2 Optimal Control

- Is the optimal control $u^*(t)$ a minimum?
- This can be answered by considering the *second partials* of the Hamiltonian.
- Let us recall from Chapter 2 that this is done by examining the *second variation* of the cost functional.
- Thus, the condition for examining the nature of optimal control is that the matrix

$$\Pi = \begin{bmatrix} \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} \\ \\ \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u} \partial \mathbf{x}} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \end{bmatrix}_* \tag{3.2.20}$$

must be *positive definite* (*negative definite*) for *minimum* (*maximum*).

- In most of the cases this reduces to the condition that

$$\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* \tag{3.2.21}$$

- Now using the Hamiltonian (3.2.3) and calculating the various partials,

$$\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2}\right)_* = \mathbf{Q}(t), \quad \left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}\partial \mathbf{u}}\right)_* = 0,$$
$$\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}\partial \mathbf{x}}\right)_* = 0, \quad \left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* = \mathbf{R}(t). \tag{3.2.22}$$

- Substituting the previous partials in the condition (3.2.20), we have

$$\mathbf{\Pi} = \begin{bmatrix} \mathbf{Q}(t) & 0 \\ 0 & \mathbf{R}(t) \end{bmatrix}. \tag{3.2.23}$$

- Since $\mathbf{R}(t)$ is positive definite, and $\mathbf{Q}(t)$ is positive semidefinite, it follows that the preceding matrix (3.2.23) is only positive semidefinite.
- However, the condition that the second partial of $\boldsymbol{H}$ w.r.t. $\mathbf{u}^*(t)$, which is $\mathbf{R}(t)$, is positive definite, is enough to guarantee that the control $\mathbf{u}^*(t)$ is *minimum*.

3.2.3 Optimal Performance Index

- Here, we show how to obtain an expression for the optimal value of the performance index.

THEOREM 3.1

The optimal value of the PI (3.2.2) is given by

$$J^*(\mathbf{x}^*(t),t) = \frac{1}{2}\mathbf{x}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t). \tag{3.2.24}$$

Proof:

$$\frac{1}{2}\int_{t_0}^{t_f} \frac{d}{dt}\left(\mathbf{x}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t)\right) dt = -\frac{1}{2}\mathbf{x}^{*\prime}(t_0)\mathbf{P}(t_0)\mathbf{x}^*(t_0) + \underline{\frac{1}{2}\mathbf{x}^{*\prime}(t_f)\mathbf{P}(t_f)\mathbf{x}^*(t_f)}. \tag{3.2.25}$$

$$J(\mathbf{u}) = \underline{\frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f)} + \frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \tag{3.2.2}$$

$$\begin{aligned}
&= \underline{\frac{1}{2}\mathbf{x}^{*\prime}(t_0)\mathbf{P}(t_0)\mathbf{x}^*(t_0)} \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{x}^{*\prime}(t)\mathbf{Q}(t)\mathbf{x}^*(t) + \mathbf{u}^{*\prime}(t)\mathbf{R}(t)\mathbf{u}^*(t) + \underline{\frac{d}{dt}\left(\mathbf{x}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t)\right)}\right] dt \\
&= \frac{1}{2}\mathbf{x}^{*\prime}(t_0)\mathbf{P}(t_0)\mathbf{x}(t_0) \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{x}^{*\prime}(t)\mathbf{Q}(t)\mathbf{x}^*(t) + \mathbf{u}^{*\prime}(t)\mathbf{R}(t)\mathbf{u}^*(t)\right. \\
&\quad \left. + \underline{\dot{\mathbf{x}}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t) + \mathbf{x}^{*\prime}(t)\dot{\mathbf{P}}(t)\mathbf{x}^*(t) + \mathbf{x}^{*\prime}(t)\mathbf{P}(t)\boxed{\dot{\mathbf{x}}^*(t)}}\right] dt.
\end{aligned} \tag{3.2.26}$$

Now, using the state equation (3.2.14) for $\dot{\mathbf{x}}^*(t)$, we get

$$J^*(\mathbf{x}^*(t_0), t_0) = \frac{1}{2}\mathbf{x}^{*\prime}(t_0)\mathbf{P}(t_0)\mathbf{x}^*(t_0) + \frac{1}{2}\int_{t_0}^{t_f} \mathbf{x}^{*\prime}(t)\Big[\underbrace{\mathbf{Q}(t) + \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t) + \dot{\mathbf{P}}(t)}_{\text{DRE} = 0}\Big]\mathbf{x}^*(t)dt. \quad (3.2.27)$$

Note: $\dot{x} = Ax + Bu$
$= Ax - BR^{-1}B'Px$

$$\Rightarrow \boxed{J^*(\mathbf{x}(t_0), t_0) = \frac{1}{2}\mathbf{x}^{*\prime}(t_0)\mathbf{P}(t_0)\mathbf{x}^*(t_0).} \qquad (3.2.28)$$

Now, the previous relation is also valid for any $\mathbf{x}^*(t)$. *Thus,*

$$\boxed{J^*(\mathbf{x}^*(t),t) = \frac{1}{2}\mathbf{x}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t).} \tag{3.2.29}$$

In terms of the final time t_f, *the previous optimal cost becomes*

$$\boxed{J^*(\mathbf{x}(t_f),t_f) = \frac{1}{2}\mathbf{x}^{*\prime}(t_f)\mathbf{P}(t_f)\mathbf{x}^*(t_f).} \tag{3.2.30}$$

Since we are normally given the initial state $\mathbf{x}(t_0)$ *and the Riccati coefficient* $\mathbf{P}(t)$ *is solved for all time* t, *it is more convenient to use the relation (3.2.28)*.

3.2.4 Finite-Time Linear Quadratic Regulator: Time-Varying Case: Summary

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{3.2.31}$$

$$J = \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \tag{3.2.32}$$

where
$\mathbf{u}(t)$ is not constrained
t_f is specified
$\mathbf{x}(t_f)$ is not specified,
$\mathbf{F}(t_f)$ and $\mathbf{Q}(t)$ are $n \times n$ symmetric, positive semidefinite matrices
$\mathbf{R}(t)$ is $r \times r$ symmetric, positive definite matrix
the optimal control is given by

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}^*(t) = -\mathbf{K}(t)\mathbf{x}^*(t)} \tag{3.2.33}$$

where $\mathbf{K}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$ is called *Kalman gain* and $\mathbf{P}(t)$, the nxn symmetric, *positive definite* matrix (for all $t \in [t_0, t_f]$), is the solution of the matrix differential Riccati equation (DRE)

$$\boxed{\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) - \mathbf{Q}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)} \tag{3.2.34}$$

satisfying the final condition $\boxed{\mathbf{P}(t = t_f) = \mathbf{F}(t_f)}$ (3.2.35) the optimal state is the solution of

$$\boxed{\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\right]\mathbf{x}^*(t)} \tag{3.2.36}$$

and the optimal cost is

$$\boxed{J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t).} \tag{3.2.37}$$

The optimal control $\mathbf{u}^*(t)$, given by (3.2.33), is *linear* in the optimal state $\mathbf{x}^*(t)$. The entire procedure is now summarized in Table 3.1.

Table 3.1 Procedure Summary of Finite-Time Linear Quadratic Regulator System: Time-Varying Case

A. Statement of the Problem
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, the performance index as $J = \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt$, and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0, \quad t_f$ is fixed, and $\mathbf{x}(t_f)$ is free, find the optimal control, state and performance index.

B. Solution of the Problem	
Step 1	Solve the matrix differential Riccati equation $\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) - \mathbf{Q}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$ with final condition $\mathbf{P}(t = t_f) = \mathbf{F}(t_f)$.
Step 2	Solve the optimal state $\mathbf{x}^*(t)$ from $\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\right]\mathbf{x}^*(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.
Step 3	Obtain the optimal control $\mathbf{u}^*(t)$ as $\mathbf{u}^*(t) = -\mathbf{K}(t)\mathbf{x}^*(t)$ where, $\mathbf{K}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$.
Step 4	Obtain the optimal performance index from $J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t)$.

Note: It is simple to see that one can absorb the $\frac{1}{2}$ that is associated with J by redefining a performance measure as

$$J_2 = 2J = \mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt, \tag{3.2.38}$$

get the corresponding matrix differential Riccati equation for J_2 as

$$\frac{\dot{\mathbf{P}}_2(t)}{2} = -\frac{\mathbf{P}_2(t)}{2}\mathbf{A}(t) - \mathbf{A}'(t)\frac{\mathbf{P}_2(t)}{2} - \mathbf{Q}(t) + \frac{\mathbf{P}_2(t)}{2}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\frac{\mathbf{P}_2(t)}{2} \tag{3.2.39}$$

with final condition

$$\frac{\mathbf{P}_2(t = t_f)}{2} = \mathbf{F}(t_f). \tag{3.2.40}$$

Comparing the previous DRE for J_2 with the corresponding DRE (3.2.34) for J, we can easily see that $\boxed{\mathbf{P}_2(t) = 2\mathbf{P}(t)}$ and hence the optimal control becomes

$$\begin{aligned}\mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\frac{\mathbf{P}_2(t)}{2}\mathbf{x}^*(t) = -\frac{\mathbf{K}_2(t)}{2}\mathbf{x}^*(t) \\ &= -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}^*(t) = -\mathbf{K}(t)\mathbf{x}^*(t).\end{aligned} \tag{3.2.41}$$

Thus, using J_2 without the $\frac{1}{2}$ in the performance index, we get the same optimal control (3.2.41) for the original plant (3.2.31), but the only difference being that the Riccati coefficient matrix $\mathbf{P}_2(t)$ is *twice* that of $\mathbf{P}(t)$ and J_2 is *twice* that of J(for example, see [3, 42]).

3.2.5 Salient Features

1. *Riccati Coefficient:* The Riccati coefficient matrix $\mathbf{P}(t)$ is a time-varying matrix which depends upon the system matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$, the performance index (design) matrices $\mathbf{Q}(t)$, $\mathbf{R}(t)$ and $\mathbf{F}(t_f)$, and the terminal time t_f, but $\mathbf{P}(t)$ does not depend upon the initial state $\mathbf{x}(t_0)$ of the system.

2. $\mathbf{P}(t)$ is *symmetric* and hence it follows that the nxn order matrix DRE (3.2.18) represents a system of $n(n+1)/2$ *first order, nonlinear, time-varying, ordinary differential* equations.

3. *Optimal Control:* From (3.2.21), we see that the optimal control $\mathbf{u}^*(t)$ is minimum (maximum) if the control weighted matrix $\mathbf{R}(t)$ is *positive definite* (*negative definite*).

4. *Optimal State:* Using the optimal control (3.2.12) in the state equation (3.2.1), we have

$$\boxed{\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\right]\mathbf{x}^*(t) = \mathbf{G}(t)\mathbf{x}^*(t)} \tag{3.2.42}$$

where

$$\mathbf{G}(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t). \tag{3.2.43}$$

The solution of this state differential equation along with the initial condition $\mathbf{x}(t_0)$ gives the optimal state $\mathbf{x}^*(t)$. Let us note that there is no condition on the closed-loop matrix $\mathbf{G}(t)$ regarding *stability* as long as we are considering the *finite* final time (t_f) system.

5. *Optimal Cost:* It is shown in (3.2.29) that the minimum cost J^* is given by

$$J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t) \quad \text{for all} \quad t \in [t_0, t_f] \tag{3.2.44}$$

where, $\mathbf{P}(t)$ is the solution of the matrix DRE (3.2.18), and $\mathbf{x}^*(t)$ is the solution of the closed-loop optimal system (3.2.42).

6. *Definiteness of the Matrix* $\mathbf{P}(t)$:

$\mathbf{F}(t_f)$ is positive semidefinite and $\mathbf{P}(t_f) = \mathbf{F}(t_f)$

$\Rightarrow$ $\mathbf{P}(t_f)$ is *positive semidefinite*.

$\Rightarrow$ We can argue that $\mathbf{P}(t)$ is *positive definite* for all $t \in [t_0, t_f)$.

$\Rightarrow$ Suppose that $\mathbf{P}(t)$ is not positive definite for some $t = t_s < t_f$

$\Rightarrow$ there exists the corresponding state $\mathbf{x}^*(t_s)$

such that the cost function $\frac{1}{2}\mathbf{x}^{*\prime}(t_s)\mathbf{P}(t_s)\mathbf{x}^*(t_s) \leq 0$

$\Rightarrow$ which clearly violates that fact that *minimum cost has to be a positive quantity*.

$\Rightarrow$ $\mathbf{P}(t)$ is *positive definite* for all $t \in [t_0, t_f)$

$\Rightarrow$ Since we already know that $\mathbf{P}(t)$ is symmetric

we now have that $\mathbf{P}(t)$ is *positive definite, symmetric* matrix.

7. *Computation of Matrix DRE:* Under some conditions we can get analytical solution for the nonlinear matrix DRE as shown later. But in general, we may try to solve the matrix DRE (3.2.18) by integrating *backwards* from its known final condition (3.2.19).

8. *Independence of the Riccati Coefficient Matrix* $\mathbf{P}(t)$*:* The matrix $\mathbf{P}(t)$ is independent of the optimal state $\mathbf{x}^*(t)$, so that once the system and the cost are specified, that is, once we are given the system/plant matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$, and the performance index matrices $\mathbf{F}(t_f)$, $\mathbf{Q}(t)$, and $\mathbf{R}(t)$, we can independently compute the matrix $\mathbf{P}(t)$ before the optimal system operates in the forward direction from its initial condition. Typically, we compute (off-line) the matrix $\mathbf{P}(t)$ *backward* in the interval $t \in [t_f, t_0]$ and store them separately, and feed these stored values when the system is operating in the *forward* direction in the interval $t \in [t_0, t_f]$.

9. *Implementation of the Optimal Control:*

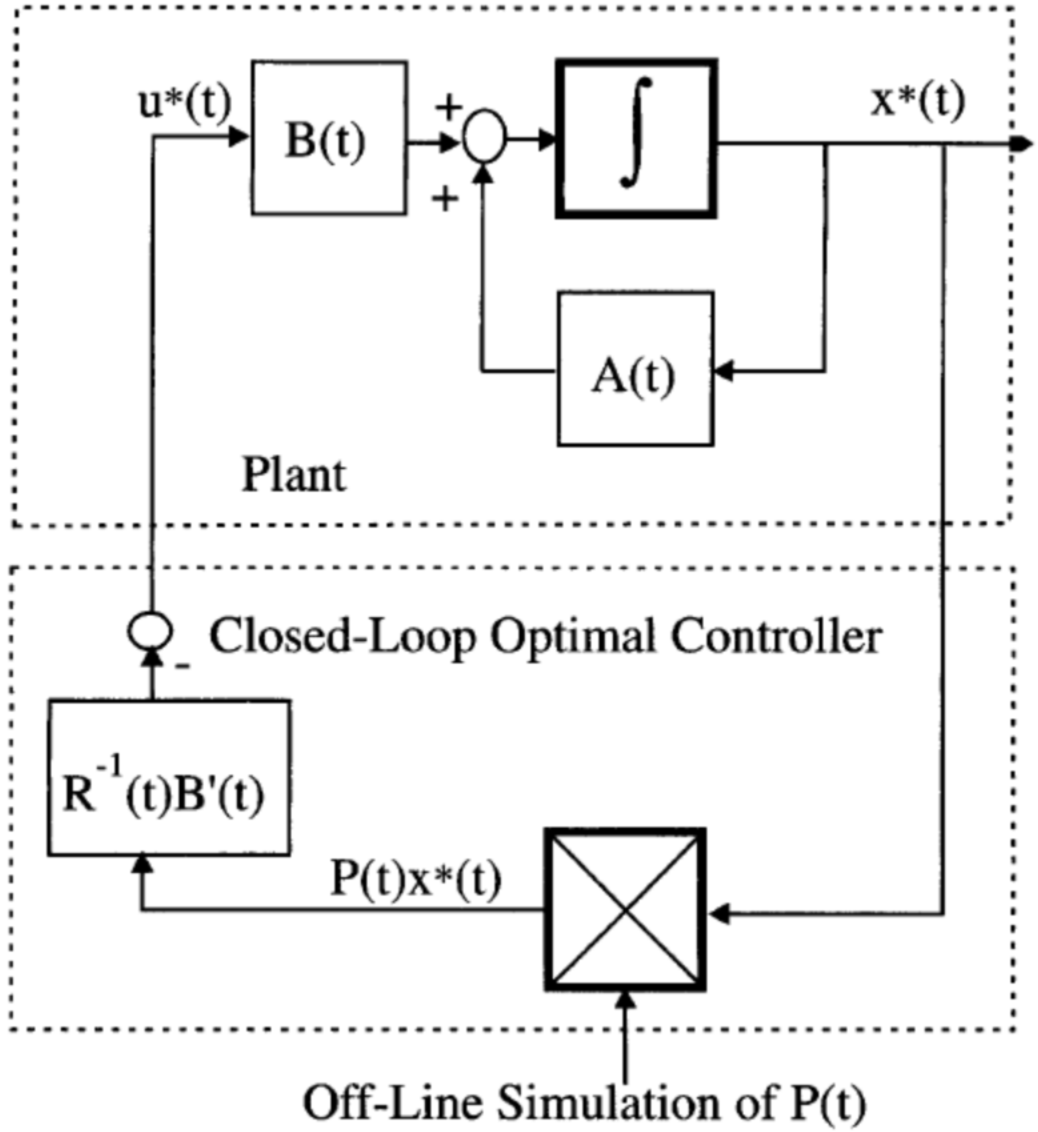


Figure 3.2 Closed-Loop Optimal Control Implementation

10. *Linear Optimal Control:* The optimal feedback control $\mathbf{u}^*(t)$ given by (3.2.12) is written as

$$\boxed{\mathbf{u}^*(t) = -\mathbf{K}(t)\mathbf{x}^*(t)} \tag{3.2.45}$$

where, the *Kalman gain* $\mathbf{K}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$. Or alternatively, we can write

$$\mathbf{u}^*(t) = -\mathbf{K}_a'(t)\mathbf{x}^*(t) \tag{3.2.46}$$

where, $\mathbf{K}_a(t) = \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)$. The previous optimal control is *linear* in state $\mathbf{x}^*(t)$. This is one of the nice features of the optimal control of linear systems with quadratic cost functionals. Also, note that the negative feedback in the optimal control relation (3.2.46) emerged from the *theory* of optimal control and was not introduced intentionally in our development.

11. *Controllability*:

- Do we need the controllability condition on the system for implementing the optimal feedback control?
- No, as long as we are dealing with a finite time (t_f) system, because the contribution of those uncontrollable states (which are also unstable) to the cost function is still a *finite* quantity only.
- However, if we consider an *infinite* time interval, we certainly need the controllability condition, as we will see in the next section.

3.2.6 LQR System for General Performance Index

- In this subsection, we address the state regulator system with a more general performance index than given by (3.2.2).
- Consider a linear, time-varying plant described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{3.2.48}$$

with a cost functional

$$\begin{aligned} J(\mathbf{u}) &= \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) \\ &\quad + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \underline{2\mathbf{x}'(t)\mathbf{S}\mathbf{u}(t)} + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \\ &= \frac{1}{2}\mathbf{x}'(t_f)\mathbf{F}(t_f)\mathbf{x}(t_f) \\ &\quad + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}'(t)\ \mathbf{u}'(t)\right] \begin{bmatrix} \mathbf{Q}(t) & \mathbf{S}(t) \\ \mathbf{S}'(t) & \mathbf{R}(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt, \end{aligned} \tag{3.2.49}$$

where, the various vectors and matrices are defined in earlier sections and the nxr matrix $\mathbf{S}(t)$ is only a positive definite matrix.

Using the identical procedure as for the LQR system, we get the matrix differential Riccati equation as

$$\begin{aligned}\dot{\mathbf{P}}(t) &= -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) - \mathbf{Q}(t) \\ &\quad + [\mathbf{P}(t)\mathbf{B}(t) + \mathbf{S}(t)]\,\mathbf{R}^{-1}(t)[\mathbf{B}'(t)\mathbf{P}(t) + \mathbf{S}'(t)]\end{aligned} \tag{3.2.50}$$

with the final condition on $\mathbf{P}(t)$ as

$$\mathbf{P}(t_f) = \mathbf{F}(t_f). \tag{3.2.51}$$

The optimal control is then given by

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\,[\mathbf{S}'(t) + \mathbf{P}(t)]\,\mathbf{x}(t). \tag{3.2.52}$$

Obviously, when $\mathbf{S}(t)$ is made zero in the previous analysis, we get the previous results shown in Table 3.1.

3.3 Analytical Solution to the Matrix Differential Riccati Equation

- In this section, we explore an analytical solution for the matrix differential Riccati equation (DRE).

(3.2.8) $\Rightarrow$

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\lambda}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{E} \\ -\mathbf{Q} & -\mathbf{A}' \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix} \qquad (3.3.1)$$

where, $\mathbf{E} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'$.

Let $$\Delta = \begin{bmatrix} \mathbf{A} & -\mathbf{E} \\ -\mathbf{Q} & -\mathbf{A}' \end{bmatrix}. \qquad (3.3.2)$$

Let us also recall that by the transformation $\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t)$, we get the differential matrix Riccati equation (3.2.18), rewritten for (time-invariant matrices $\mathbf{A}, \mathbf{B}, \mathbf{Q}$ and $\mathbf{R}$) as

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A} - \mathbf{A}'\mathbf{P}(t) - \mathbf{Q} + \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{P}(t), \qquad (3.3.3)$$

with the final condition

$$\mathbf{P}(t_f) = \mathbf{F}(t_f). \qquad (3.3.4)$$

In order to find analytical solution to the differential Riccati equation (3.3.3), it is necessary to show that if μ is an eigenvalue of the Hamiltonian matrix Δ in (3.3.2), then it implies that $-\mu$ is also the eigenvalue of Δ [89, 3].

define $$\Gamma = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \qquad (3.3.5)$$

so that $\Gamma^{-1} = -\Gamma$.

$$\Rightarrow \quad \Delta = \Gamma\Delta'\Gamma = -\Gamma\Delta'\Gamma^{-1}. \qquad (3.3.6)$$

Now, if μ is an eigenvalue of Δ with corresponding eigenvector $\mathbf{v}$,

$$\Delta\mathbf{v} = \mu\mathbf{v} \qquad (3.3.7)$$

then $$\Gamma\Delta'\Gamma\mathbf{v} = \mu\mathbf{v}, \qquad \Delta'\Gamma\mathbf{v} = -\mu\Gamma\mathbf{v} \qquad (3.3.8)$$

$$\Rightarrow \quad (\Gamma\mathbf{v})'\Delta = -\mu(\Gamma\mathbf{v})' \qquad (3.3.9)$$

Next, rearranging the eigenvalues of Δ as

$$\mathbf{D} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \tag{3.3.10}$$

where, $\mathbf{M}(-\mathbf{M})$ is a diagonal matrix with right-half-plane (left-half plane) eigenvalues. Let $\mathbf{W}$, the modal matrix of eigenvectors corresponding to $\mathbf{D}$, be defined as

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix}, \tag{3.3.11}$$

where, $[\mathbf{W}_{11} \quad \mathbf{W}_{21}]'$ are the n eigenvectors of the left-half-plane (stable) eigenvalues of Δ. Also,

$$\mathbf{W}^{-1}\Delta\mathbf{W} = \mathbf{D}. \tag{3.3.12}$$

Let us now define a state transformation

$$\begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix} = \mathbf{W} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{z}(t) \end{bmatrix}. \tag{3.3.13}$$

Then, using (3.3.12) and (3.3.13), the Hamiltonian system (3.3.1) becomes

$$\begin{aligned} \underline{\begin{bmatrix} \dot{\mathbf{w}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}} &= \mathbf{W}^{-1} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\lambda}}(t) \end{bmatrix} = \mathbf{W}^{-1}\Delta \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix} = \mathbf{W}^{-1}\Delta\mathbf{W} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{z}(t) \end{bmatrix} \\ &\underline{= \mathbf{D} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{z}(t) \end{bmatrix}.} \end{aligned} \tag{3.3.14}$$

Solving (3.3.14) in terms of the known final conditions, we have

$$\begin{bmatrix} \mathbf{w}(t) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} e^{-\mathbf{M}(t-t_f)} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{M}(t-t_f)} \end{bmatrix} \begin{bmatrix} \mathbf{w}(t_f) \\ \mathbf{z}(t_f) \end{bmatrix}. \tag{3.3.15}$$

Rewriting (3.3.15)

$$\begin{bmatrix} \mathbf{w}(t_f) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} e^{\mathbf{M}(t-t_f)} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{M}(t-t_f)} \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{z}(t_f) \end{bmatrix}. \tag{3.3.16}$$

Next, from (3.3.13) and using the final condition (3.3.4)

$$\begin{aligned}\boldsymbol{\lambda}(t_f) &= \underline{\mathbf{W}_{21}\mathbf{w}(t_f) + \mathbf{W}_{22}\mathbf{z}(t_f)} \\ &= \mathbf{F}\mathbf{x}(t_f) \\ &= \underline{\mathbf{F}\left[\mathbf{W}_{11}\mathbf{w}(t_f) + \mathbf{W}_{12}\mathbf{z}(t_f)\right]}. \end{aligned} \tag{3.3.17}$$

Solving the previous relation for $\mathbf{z}(t_f)$ in terms of $\mathbf{w}(t_f)$

$$\underline{\mathbf{z}(t_f) = \mathbf{T}(t_f)\mathbf{w}(t_f)}, \quad \text{where}$$
$$\mathbf{T}(t_f) = -\left[\mathbf{W}_{22} - \mathbf{F}\mathbf{W}_{12}\right]^{-1}\left[\mathbf{W}_{21} - \mathbf{F}\mathbf{W}_{11}\right]. \tag{3.3.18}$$

Again, from (3.3.16)

$$\begin{aligned}\mathbf{z}(t) &= e^{-\mathbf{M}(t_f - t)}\underline{\mathbf{z}(t_f)} \\ &= e^{-\mathbf{M}(t_f - t)}\underline{\mathbf{T}(t_f)\mathbf{w}(t_f)} \\ &= e^{-\mathbf{M}(t_f - t)}\mathbf{T}(t_f)\underline{e^{-\mathbf{M}(t_f - t)}\mathbf{w}(t)}. \end{aligned} \tag{3.3.19}$$

Rewriting the previous relation as

$$\mathbf{z}(t) = \mathbf{T}(t)\mathbf{w}(t), \quad \text{where,}$$
$$\mathbf{T}(t) = e^{-\mathbf{M}(t_f - t)}\mathbf{T}(t_f)e^{-\mathbf{M}(t_f - t)}. \tag{3.3.20}$$

Finally, to relate $\mathbf{P}(t)$ in (3.3.3) to the relation (3.3.20) for $\mathbf{T}(t)$, let us use (3.3.13) to write

$$\begin{aligned}\boldsymbol{\lambda}(t) &= \mathbf{W}_{21}\mathbf{w}(t) + \mathbf{W}_{22}\mathbf{z}(t) \\ &= \mathbf{P}(t)\mathbf{x}(t) \\ &= \mathbf{P}(t)\left[\mathbf{W}_{11}\mathbf{w}(t) + \mathbf{W}_{12}\mathbf{z}(t)\right]\end{aligned} \qquad (3.3.21)$$

and by (3.3.20), the previous relation can be written as

$$\Rightarrow \quad \left[\mathbf{W}_{21} + \mathbf{W}_{22}\mathbf{T}(t)\right]\mathbf{w}(t) = \mathbf{P}(t)\left[\mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{T}(t)\right]\mathbf{w}(t). \qquad (3.3.22)$$

Since the previous relation should hold good for all $\mathbf{x}(t_0)$ and hence for all states $\mathbf{w}(t)$, it implies that the analytical expression to the solution of $\mathbf{P}(t)$ is given by

$$\Rightarrow \quad \boxed{\mathbf{P}(t) = \left[\mathbf{W}_{21} + \mathbf{W}_{22}\mathbf{T}(t)\right]\left[\mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{T}(t)\right]^{-1}.} \qquad (3.3.23)$$

Summary

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$J(\mathbf{u}) = \frac{1}{2}\mathbf{x}^T(t_f)\mathbf{F}(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]dt$$

$$\Delta = \begin{bmatrix} \mathbf{A} & -\mathbf{E} \\ -\mathbf{Q} & -\mathbf{A}' \end{bmatrix} \quad \text{where, } \mathbf{E} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'.$$

the eigenvalues of Δ $\Rightarrow$ $\mathbf{D} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}$

$\mathbf{W}^{-1}\Delta\mathbf{W} = \mathbf{D}$ $\Rightarrow$ $\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix}$

$$\mathbf{T}(t) = e^{-\mathbf{M}(t_f - t)}\mathbf{T}(t_f)e^{-\mathbf{M}(t_f - t)}$$

where $\mathbf{T}(t_f) = -\left[\mathbf{W}_{22} - \mathbf{F}\mathbf{W}_{12}\right]^{-1}\left[\mathbf{W}_{21} - \mathbf{F}\mathbf{W}_{11}\right]$

$$\boxed{\mathbf{P}(t) = \left[\mathbf{W}_{21} + \mathbf{W}_{22}\mathbf{T}(t)\right]\left[\mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{T}(t)\right]^{-1}.}$$

3.3.1 MATLAB© Implementation of Analytical Solution to Matrix DRE

Example 3.1

Let us illustrate the previous procedure with a simple second order example. Given a double integral system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \qquad x_1(0) = 2 \\ \dot{x}_2(t) &= -2x_1(t) + x_2(t) + u(t), \qquad x_2(0) = -3, \end{aligned} \tag{3.3.24}$$

and the performance index (PI)

$$\begin{aligned} J = &\frac{1}{2}\left[x_1^2(5) + x_1(5)x_2(5) + 2x_2^2(5)\right] \\ &+\frac{1}{2}\int_0^5 \left[2x_1^2(t) + 6x_1(t)x_2(t) + 5x_2^2(t) + 0.25u^2(t)\right] dt, \end{aligned} \tag{3.3.25}$$

obtain the feedback control law.

Solution:

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}; \quad \mathbf{B}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{F}(t_f) = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$$

$$\mathbf{Q}(t) = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}; \quad \mathbf{R}(t) = r(t) = \frac{1}{4}; \quad t_0 = 0; \quad t_f = 5.$$

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}. \tag{3.3.26}$$

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\mathbf{x}^*(t) = -\mathbf{K}(t)\mathbf{x}^*(t)} \tag{3.2.33}$$

$$\begin{aligned} u^*(t) &= -4 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} x_1^*(t) \\ x_2^*(t) \end{bmatrix} \\ &= -4[p_{12}(t)x_1^*(t) + p_{22}(t)x_2^*(t)] \end{aligned} \tag{3.3.27}$$

the matrix DRE

$$\begin{bmatrix} \dot{p}_{11}(t) & \dot{p}_{12}(t) \\ \dot{p}_{12}(t) & \dot{p}_{22}(t) \end{bmatrix} = - \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 4 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \tag{3.3.28}$$

$$\text{final condition} \quad \begin{bmatrix} p_{11}(5) & p_{12}(5) \\ p_{12}(5) & p_{22}(5) \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}. \tag{3.3.29}$$

$$\Rightarrow \left\{ \begin{aligned} \dot{p}_{11}(t) &= 4p_{12}^2(t) + 4p_{12}(t) - 2, \\ & \qquad p_{11}(5) = 1, \\ \dot{p}_{12}(t) &= -p_{11}(t) - p_{12}(t) + 2p_{22}(t) + 4p_{12}(t)p_{22}(t) - 3, \\ & \qquad p_{12}(5) = 0.5 \\ \dot{p}_{22}(t) &= -2p_{12}(t) - 2p_{22}(t) + 4p_{22}^2(t) - 5, \\ & \qquad p_{22}(5) = 2. \end{aligned} \right. \tag{3.3.30}$$

```
****************************************************
%% Solution using Control System Toolbox of
%% the MATLAB. Version 6
%% The following file example.m requires
%% two other files lqrnss.m and lqrnssf.m
%% which are given in Appendix
clear all
A=[0.,1.;-2.,1.];
B=[0.;1.];
Q=[2.,3.;3.,5.];
F=[1.,0.5;0.5,2.];
R=[.25];
tspan=[0 5];
x0=[2.,-3.];
[x,u,K]=lqrnss(A,B,F,Q,R,x0,tspan);
******************************************************
```

given in Appendix C.

Matlab ODE Solver

Systems of ODEs

$$\begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, y_2, \ldots, y_n) \\ f_2(t, y_1, y_2, \ldots, y_n) \\ \vdots \\ f_n(t, y_1, y_2, \ldots, y_n) \end{pmatrix}$$

For example,

$$\begin{cases} y'_1 = y_2 \\ y'_2 = y_1 y_2 - 2 \end{cases}$$

Higher-Order ODEs

Define $\left[\begin{array}{l} y_1 = y \\ y_2 = y' \\ y_3 = y'' \\ \quad\vdots \\ y_n = y^{(n-1)} \end{array}\right. \Rightarrow \left\{\begin{array}{l} y'_1 = y_2 \\ y'_2 = y_3 \\ \quad\vdots \\ y'_n = f(t, y_1, y_2, \ldots, y_n) \end{array}\right.$

For example, $\quad y''' - y''y + 1 = 0$

Define $\left[\begin{array}{l} y_1 = y \\ y_2 = y' \\ y_3 = y'' \end{array}\right. \Rightarrow \left\{\begin{array}{l} y'_1 = y_2 \\ y'_2 = y_3 \\ y'_3 = y_1 y_3 - 1 \end{array}\right.$

Solver	Problem Type	Accuracy
ode45	Nonstiff	Medium
ode23		Low
ode113		Low to High
ode15s	Stiff	Low to Medium
ode23s		Low
ode23t		Low
ode23tb		Low
ode15i	Fully implicit	Low

ode45

Solve nonstiff differential equations — medium order method

Syntax

```
[t,y] = ode45(odefun,tspan,y0)
[t,y] = ode45(odefun,tspan,y0,options)
[t,y,te,ye,ie] = ode45(odefun,tspan,y0,options)
sol = ode45( ___ )
```

ODE function

Time span [0 tf]

I.C.

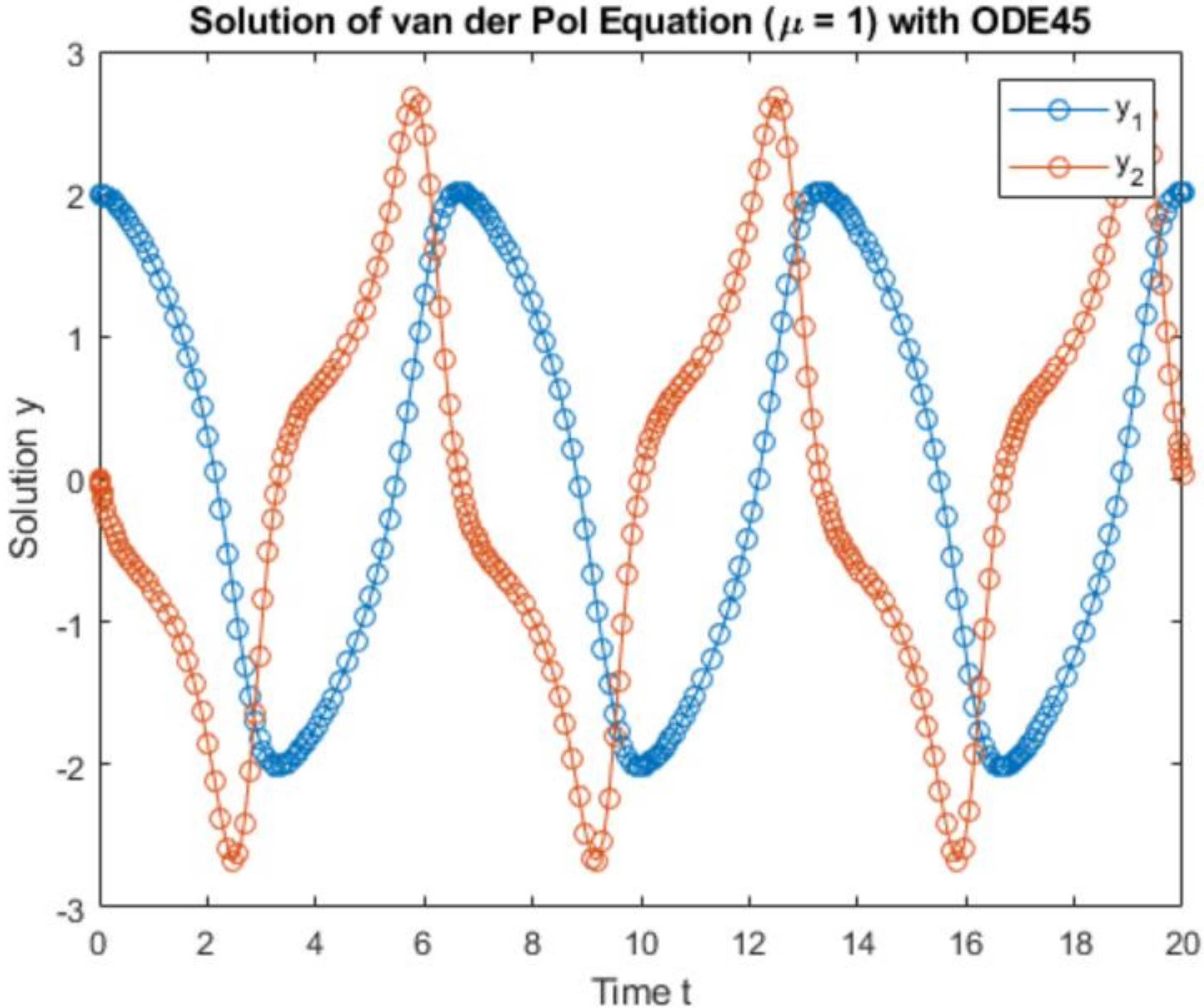
Examples

The van der Pol equation $$y_1'' - \mu\left(1 - y_1^2\right) y_1' + y_1 = 0,$$

Define $y_1' = y_2$ $\Rightarrow$ $$\begin{cases} y_1' &= y_2 \\ y_2' &= \mu(1 - y_1^2)y_2 - y_1 \end{cases}$$

```
function dydt = vdp1(t,y)
dydt = [y(2); (1-y(1)^2)*y(2)-y(1)];        % mu = 1
```

```
[t,y] = ode45(@vdp1,[0 20],[2; 0]);
plot(t,y(:,1),'-o',t,y(:,2),'-o')
title('Solution of van der Pol Equation (\mu = 1) with ODE45');
xlabel('Time t');
ylabel('Solution y');
legend('y_1','y_2')
```


Solution of van der Pol Equation (μ = 1) with ODE45
Solution y
Time t
y_1
y_2
3
2
1
0
-1
-2
-3
0
2
4
6
8
10
12
14
16
18
20

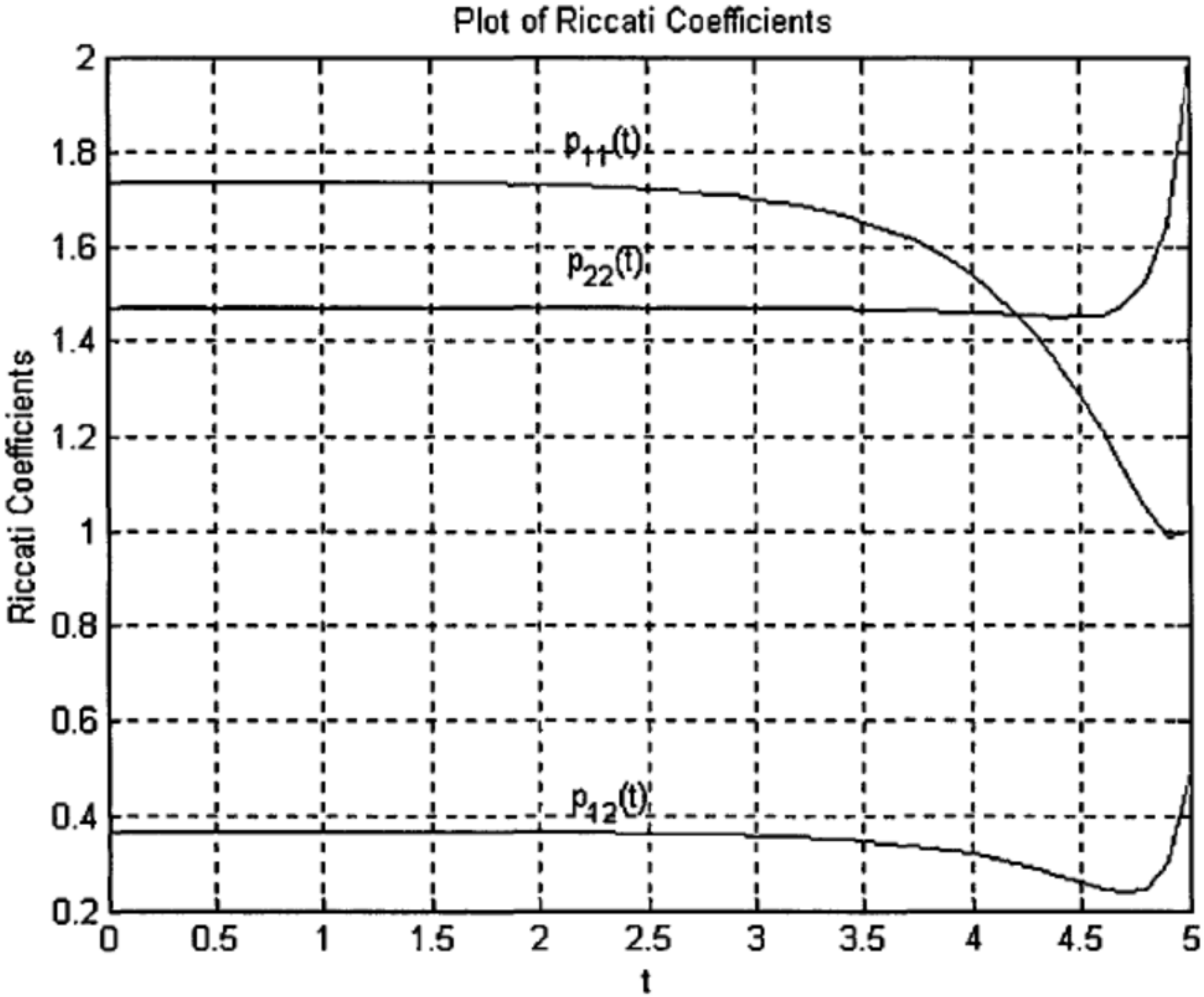
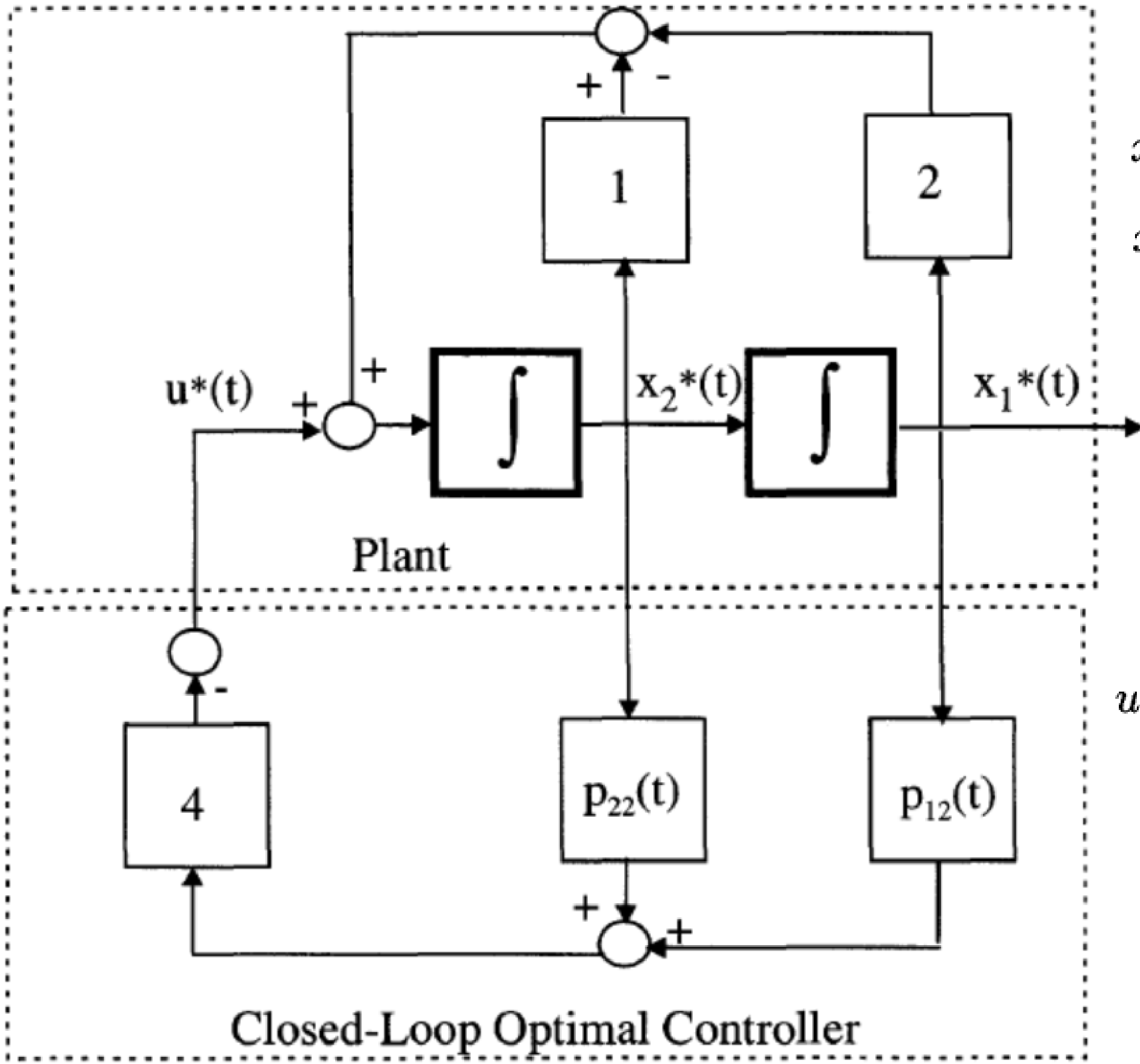


Figure 3.3 Riccati Coefficients for Example 3.1



$$\dot{x}_1(t) = x_2(t);$$
$$\dot{x}_2(t) = -2x_1(t) + x_2(t) + u(t)$$

$$u^*(t) = -4[p_{12}(t)x_1^*(t) + p_{22}(t)x_2^*(t)]$$

Figure 3.4 Closed-Loop Optimal Control System for Example 3.1

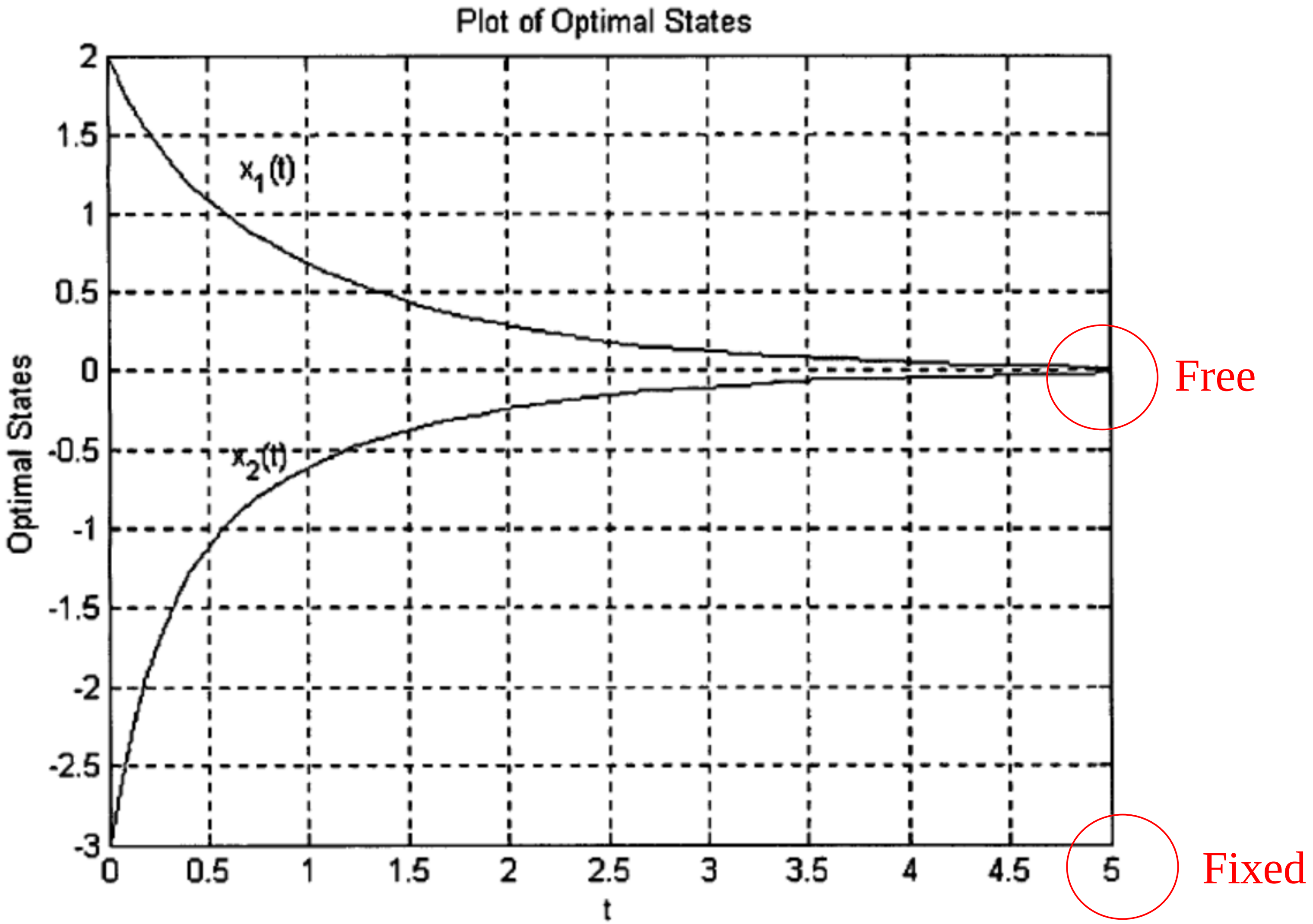


Figure 3.5 Optimal States for Example 3.1

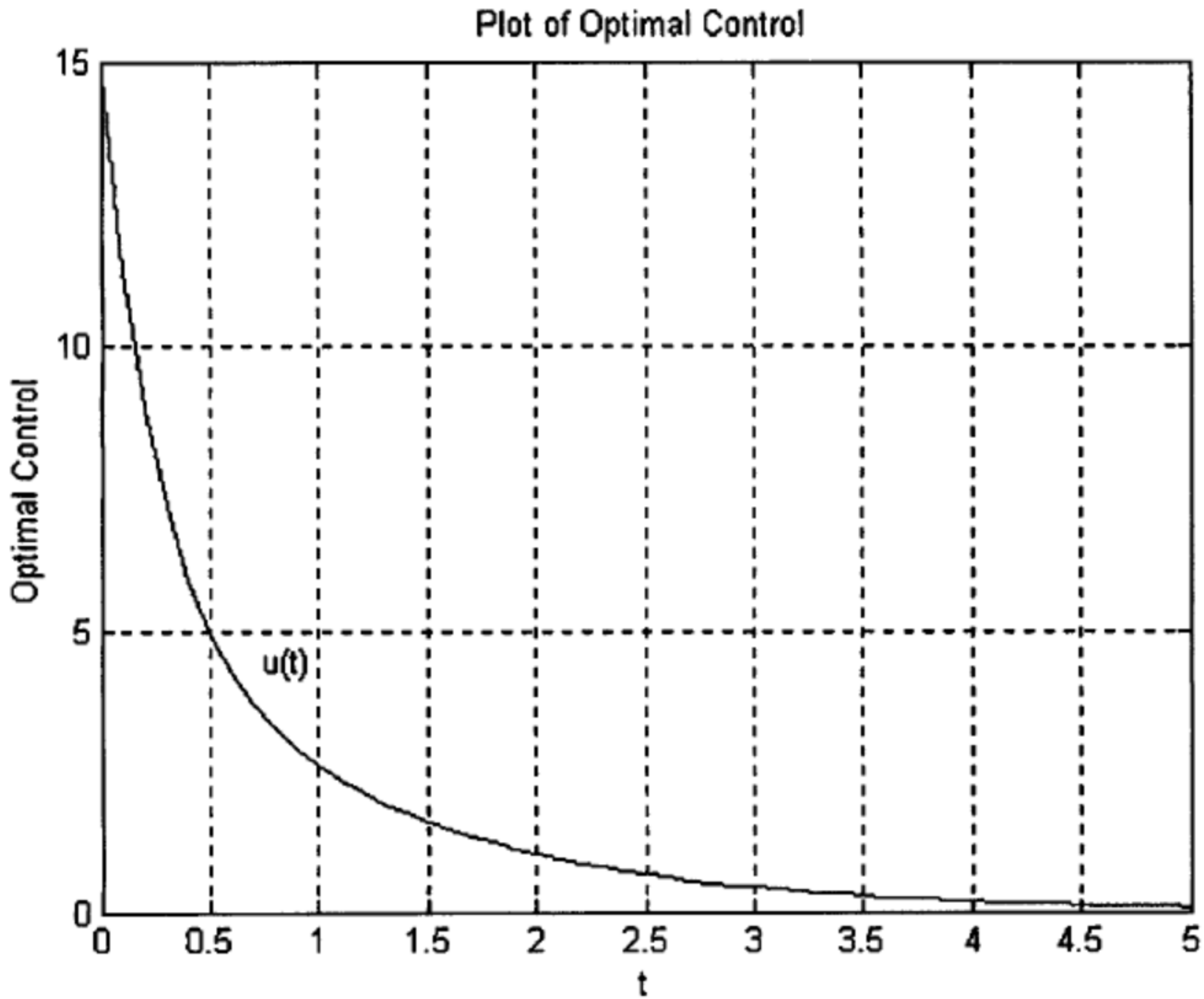


Figure 3.6 Optimal Control for Example 3.1

3.4 Infinite-Time LQR System I

In this section, let us make the terminal (final) time t_f to be infinite in the previous linear, *time-varying*, quadratic regulator system. Then, this is called the *infinite-time* (or infinite horizon) linear quadratic regulator system [6, 3].

Consider a linear, time-varying plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{3.4.1}$$

and a quadratic performance index

$$J = \frac{1}{2}\int_{t_0}^{\infty} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\,dt, \tag{3.4.2}$$

where, $\mathbf{u}(t)$ is not constrained. Also, $\mathbf{Q}(t)$ is nxn symmetric, positive *semidefinite* matrix, and $\mathbf{R}(t)$ is an rxr symmetric, positive *definite* matrix. Note, it makes no engineering sense to have a terminal cost term with terminal time being infinite.

Note: we need to impose the condition that the system (3.4.1) is ***completely controllable***.

Using results similar to the previous case of finite final time t_f (see Table 3.1), the optimal control for the infinite-horizon linear regulator system is obtained as

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\hat{\mathbf{P}}(t)\mathbf{x}^*(t), \tag{3.4.3}$$

where,

$$\hat{\mathbf{P}}(t) = \lim_{t_f \to \infty} \{\mathbf{P}(t)\}, \tag{3.4.4}$$

the nxn symmetric, positive definite matrix (for all $t \in [t_0, t_f]$) is the solution of the matrix differential Riccati equation (DRE)

$$\dot{\hat{\mathbf{P}}}(t) = -\hat{\mathbf{P}}(t)\mathbf{A}(t) - \mathbf{A}'(t)\hat{\mathbf{P}}(t) - \mathbf{Q}(t) + \hat{\mathbf{P}}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\hat{\mathbf{P}}(t), \tag{3.4.5}$$

satisfying the final condition $\lim_{t_f \to \infty} \hat{\mathbf{P}}(t_f) = 0.$ (3.4.6)

The optimal cost is given by

$$J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\hat{\mathbf{P}}(t)\mathbf{x}^*(t). \tag{3.4.7}$$

3.4.1 *Infinite-Time* Linear Quadratic Regulator: *Time Varying* Case: Summary

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \tag{3.4.8}$$

$$J = \frac{1}{2}\int_{t_0}^{\infty} \left[\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt, \tag{3.4.9}$$

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\hat{\mathbf{P}}(t)\mathbf{x}^*(t)} \tag{3.4.10}$$

$$\boxed{\dot{\hat{\mathbf{P}}}(t) = -\hat{\mathbf{P}}(t)\mathbf{A}(t) - \mathbf{A}'(t)\hat{\mathbf{P}}(t) - \mathbf{Q}(t) + \hat{\mathbf{P}}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\hat{\mathbf{P}}(t)} \tag{3.4.11}$$

$$\hat{\mathbf{P}}(t = t_f \to \infty) = 0. \tag{3.4.12}$$

$$\boxed{\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\hat{\mathbf{P}}(t)\right]\mathbf{x}^*(t)} \tag{3.4.13}$$

$$\boxed{J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\hat{\mathbf{P}}(t)\mathbf{x}^*(t).} \tag{3.4.14}$$

Table 3.2 Procedure Summary of Infinite-Time Linear Quadratic Regulator System: Time-Varying Case

A. Statement of the Problem	
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, the performance index as $J = \frac{1}{2}\int_{t_0}^{\infty} [\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt$, and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0$; $\quad \mathbf{x}(\infty)$ is free, find the optimal control, state and performance index.	
B. Solution of the Problem	
Step 1	Solve the matrix differential Riccati equation (DRE) $\dot{\hat{\mathbf{P}}}(t) = -\hat{\mathbf{P}}(t)\mathbf{A}(t) - \mathbf{A}'(t)\hat{\mathbf{P}}(t) - \mathbf{Q}(t) + \hat{\mathbf{P}}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\hat{\mathbf{P}}(t)$ with final condition $\hat{\mathbf{P}}(t = t_f) = 0$.
Step 2	Solve the optimal state $\mathbf{x}^*(t)$ from $\mathbf{x}^*(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\hat{\mathbf{P}}(t)\right]\mathbf{x}^*(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.
Step 3	Obtain the optimal control $\mathbf{u}^*(t)$ from $\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\hat{\mathbf{P}}(t)\mathbf{x}^*(t)$.
Step 4	Obtain the optimal performance index from $J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\hat{\mathbf{P}}(t)\mathbf{x}^*(t)$.

3.5 Infinite-Time LQR System II

- In this section, we examine the state regulator system with infinite time interval for a *linear* ***time-invariant*** (LTI) system.

Plant:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3.5.1}$$

and the cost functional as

$$J = \frac{1}{2}\int_0^\infty \left[\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right] dt \tag{3.5.2}$$

where, $\mathbf{x}(t)$ is nth order state vector; $\mathbf{u}(t)$ is rth order control vector; $\mathbf{A}$ is nxn-order state matrix; $\mathbf{B}$ is rxr-order control matrix; $\mathbf{Q}$ is nxn-order, symmetric, positive *semidefinite* matrix; $\mathbf{R}$ is rxr-order, symmetric, positive *definite* matrix.

Discussions

1. The infinite time interval is considered for the following reasons:
   - We wish to make sure that the state-regulator stays near zero state after the initial transient.
   - We want to include any special case of large final time.
2. With infinite final-time interval, to include the final cost function does not make any practical sense. Hence, the final cost term involving $\mathbf{F}(t_f)$ does not exist in the cost functional (3.5.2).

$$J = \frac{1}{2}\int_0^{\infty} \left[\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right] dt \qquad (3.5.2)$$

3. With infinite final-time interval, the system (3.5.1) has to be completely controllable.
   - Controllability matrix: $\left[\mathbf{B} \;\; \mathbf{AB} \cdots \mathbf{A}^{n-1}\mathbf{B}\right]$
   - If the system is not controllable and some or all of those uncontrollable states are unstable, then the cost functional would be infinite since the control interval is infinite. Alternatively, we can assume that the system (3.5.1) is completely stabilizable.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (3.5.1)$$

1. controllability and

2. $F(t_f) = 0$

imply that

$$\lim_{t_f \to \infty} \{\mathbf{P}(t)\} = \bar{\mathbf{P}} \tag{3.5.4}$$

where, $\bar{\mathbf{P}}$ is the *nxn positive definite*, symmetric, *constant* matrix. If $\bar{\mathbf{P}}$ is constant, then $\bar{\mathbf{P}}$ is the solution of the nonlinear, matrix, *algebraic Riccati equation* (ARE),

$$\frac{d\bar{\mathbf{P}}}{dt} = 0 = -\bar{\mathbf{P}}\mathbf{A} - \mathbf{A}'\bar{\mathbf{P}} + \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} - \mathbf{Q}. \tag{3.5.5}$$

Alternatively, we can write (3.5.5) as

$$\bar{\mathbf{P}}\mathbf{A} + \mathbf{A}'\bar{\mathbf{P}} + \mathbf{Q} - \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} = 0. \tag{3.5.6}$$

***Algebraic* Riccati equation**

- Note, for a time-varying system with finite-time interval, we have the differential Riccati equation (3.2.34),
- whereas for a linear time-invariant system with infinite-time horizon, we have the algebraic Riccati equation (3.5.6).

A historical note on R.E. Kalman

- Rudolph E. Kalman is best known for the linear filtering technique.
- The Kalman filter, which is based on the use of state-space techniques and recursive algorithms, revolutionized the field of estimation.
- The Kalman filter is widely used in navigational and guidance systems, radar tracking, sonar ranging, and satellite orbit determination, as well as in fields as diverse as seismic data processing, nuclear power plant instrumentation, and econometrics.

- Among Kalman's many outstanding contributions were the formulation and study of most fundamental state-space notions including controllability, observability, minimality, realizability from input and output data, matrix Riccati equations, linear-quadratic control, and the separation principle that are today ubiquitous in control.

- Born in Hungary, Kalman received BS and MS degrees from the Massachusetts Institute of Technology (MIT) and a Dsci in engineering from Columbia University in 1957.
- In the early years of his career he held research positions at International Business Machines (IBM) and at the Research Institute for Advanced Studies (RIAS) in Baltimore.
- From 1962 to 1971, he was at Stanford University. In 1971, he became a graduate research professor and director of the Center for Mathematical System Theory at the University of Florida, Gainesville, USA, and later retired with emeritus status.
- Kalman's contributions to control theory and to applied mathematics and engineering in general have been widely recognized with several honors and awards.

Rudolf E. Kálmán

Born	Rudolf Emil Kálmán[1] May 19, 1930 Budapest, Hungary
Died	July 2, 2016 (aged 86)[2] Gainesville, Florida
Citizenship	Hungary United States
Alma mater	Massachusetts Institute of Technology; Columbia University
Awards	IEEE Medal of Honor (1974) Rufus Oldenburger Medal (1976) Kyoto Prize (1985) Richard E. Bellman Control Heritage Award (1997) National Medal of Science (2009) Charles Stark Draper Prize

3.5.1 Meaningful Interpretation of Riccati Coefficient

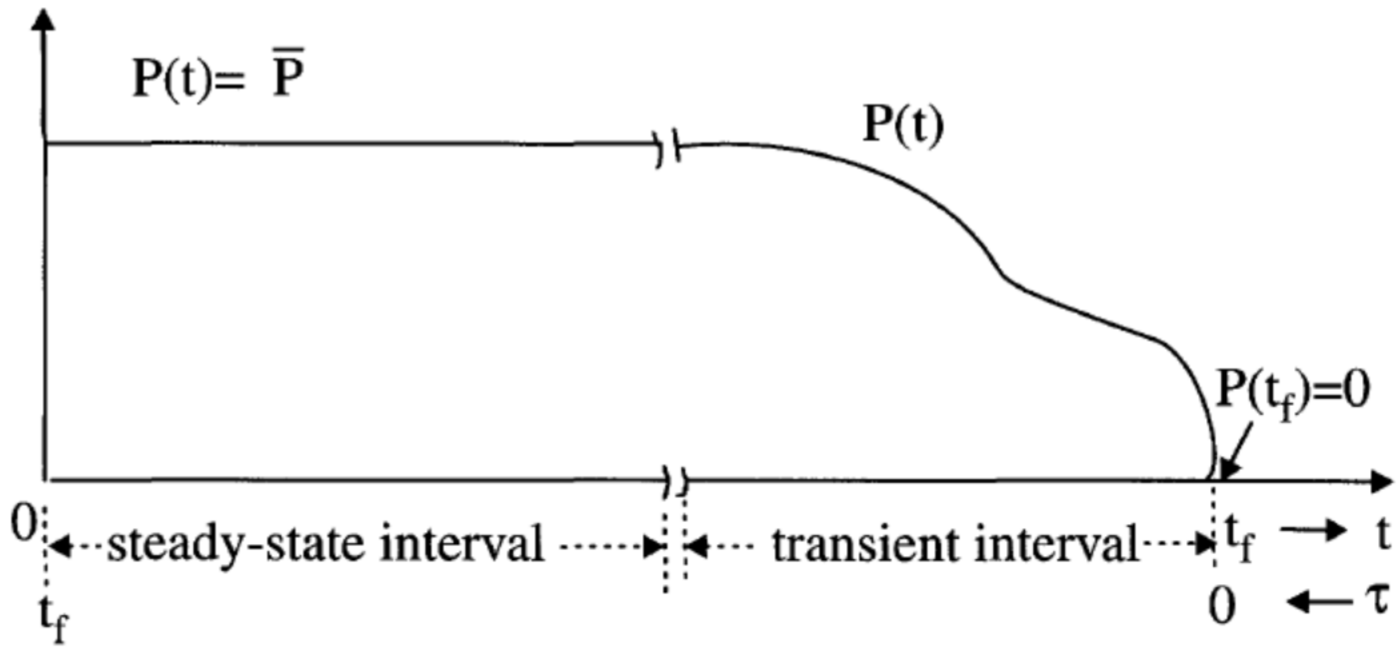


Figure 3.7 Interpretation of the Constant Matrix $\bar{\mathbf{P}}$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t) = -\bar{\mathbf{K}}\mathbf{x}^*(t), \qquad (3.5.7)$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t) = -\bar{\mathbf{K}}\mathbf{x}^*(t), \tag{3.5.7}$$

where, $\bar{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}$ is called the *Kalman gain*. Alternatively, we can write

$$\mathbf{u}^*(t) = -\bar{\mathbf{K}}_a'\mathbf{x}^*(t) \tag{3.5.8}$$

where, $\bar{\mathbf{K}}_a = \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}$. The optimal state is the solution of the system obtained by using the control (3.5.8) in the plant (3.5.1)

$$\dot{\mathbf{x}}^*(t) = \left[\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\right]\mathbf{x}^*(t) = \mathbf{G}\mathbf{x}^*(t), \tag{3.5.9}$$

where, the matrix $\mathbf{G} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}$ must have *stable* eigenvalues so that the closed-loop optimal system (3.5.9) is *stable*. This is required since any unstable states with infinite time interval would lead to an *infinite* cost functional J^*. Let us note that we have no constraint on the stability of the *original* system (3.5.1). This means that although the *original* system may be *unstable*, the *optimal* system must be definitely *stable*.

Finally, the minimum cost (3.2.29) is given by

$$J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\bar{\mathbf{P}}\mathbf{x}^*(t). \tag{3.5.10}$$

3.5.2 Analytical Solution of the Algebraic Riccati Equation

Finite-time LQR

$$\mathbf{T}(t) = e^{-\mathbf{M}(t_f - t)}\mathbf{T}(t_f)e^{-\mathbf{M}(t_f - t)}$$

$$\mathbf{T}(t_f) = -\left[\mathbf{W}_{22} - \mathbf{F}\mathbf{W}_{12}\right]^{-1}\left[\mathbf{W}_{21} - \mathbf{F}\mathbf{W}_{11}\right]$$

$$\boxed{\mathbf{P}(t) = \left[\mathbf{W}_{21} + \mathbf{W}_{22}\mathbf{T}(t)\right]\left[\mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{T}(t)\right]^{-1}.}$$

As $t_f \to \infty$, $e^{-\mathbf{M}(t_f - t)}$ goes to zero, which in turn makes $\mathbf{T}(t)$ tend to zero.

$$\Rightarrow \quad \boxed{\lim_{t_f \to \infty} \mathbf{P}(t, t_f) = \bar{\mathbf{P}} = \mathbf{W}_{21}\mathbf{W}_{11}^{-1}.} \qquad (3.5.11)$$

3.5.3 Infinite-Interval Regulator System: Time-Invariant Case: Summary

For a controllable, linear, time-invariant plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{3.5.12}$$

and the infinite interval cost functional

$$J = \frac{1}{2}\int_0^\infty \left[\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right] dt, \tag{3.5.13}$$

the optimal control is given by

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t)} \tag{3.5.14}$$

where, $\bar{\mathbf{P}}$, the nxn constant, *positive definite*, symmetric matrix, is the solution of the nonlinear, matrix *algebraic Riccati equation* (ARE)

$$\boxed{-\bar{\mathbf{P}}\mathbf{A} - \mathbf{A}'\bar{\mathbf{P}} + \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} - \mathbf{Q} = 0} \tag{3.5.15}$$

the optimal trajectory is the solution of

$$\boxed{\dot{\mathbf{x}}^*(t) = \left[\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\right]\mathbf{x}^*(t)} \tag{3.5.16}$$

and the optimal cost is given by

$$\boxed{J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\bar{\mathbf{P}}\mathbf{x}^*(t).} \tag{3.5.17}$$

Table 3.3 Procedure Summary of Infinite-Interval Linear Quadratic Regulator System: Time-Invariant Case

A. Statement of the Problem	
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, the performance index as $J = \frac{1}{2}\int_0^\infty [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]\, dt$, and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0; \quad \mathbf{x}(\infty) = 0$, find the optimal control, state and index.	
B. Solution of the Problem	
Step 1	Solve the matrix algebraic Riccati equation (ARE) $-\bar{\mathbf{P}}\mathbf{A} - \mathbf{A}'\bar{\mathbf{P}} - \mathbf{Q} + \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} = 0.$.
Step 2	Solve the optimal state $\mathbf{x}^*(t)$ from $\dot{\mathbf{x}}^*(t) = \left[\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\right]\mathbf{x}^*(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.
Step 3	Obtain the optimal control $\mathbf{u}^*(t)$ from $\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t)$.
Step 4	Obtain the optimal performance index from $J^* = \frac{1}{2}\mathbf{x}^{*\prime}(t)\bar{\mathbf{P}}\mathbf{x}^*(t)$.

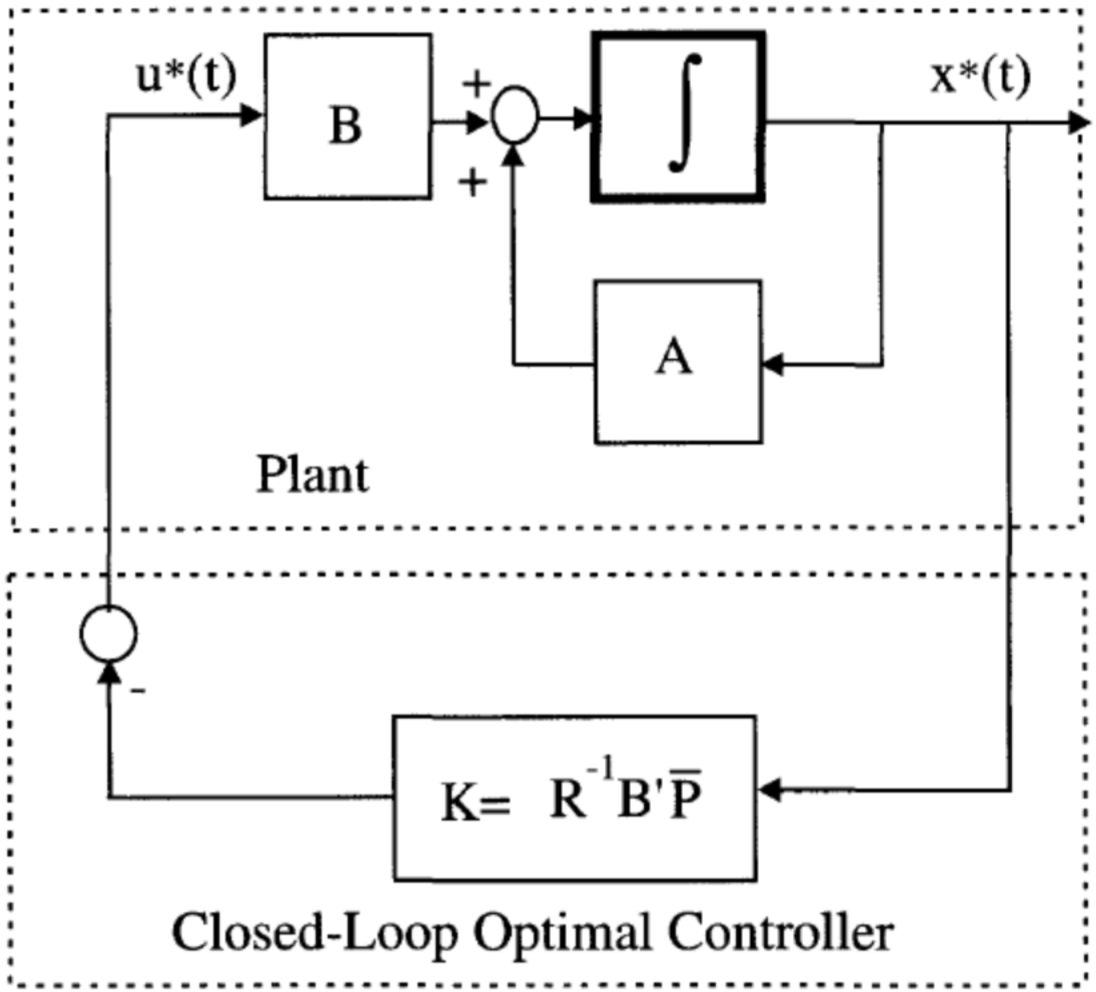


Figure 3.8 Implementation of the Closed-Loop Optimal Control: Infinite Final Time

Example 3.2

Given a second order plant

$$\dot{x}_1(t) = x_2(t), \qquad x_1(0) = 2$$
$$\dot{x}_2(t) = -2x_1(t) + x_2(t) + u(t), \qquad x_2(0) = -3 \tag{3.5.18}$$

and the performance index

$$J = \frac{1}{2}\int_0^{\infty} \left[2x_1^2(t) + 6x_1(t)x_2(t) + 5x_2^2(t) + 0.25u^2(t)\right] dt, \tag{3.5.19}$$

obtain the feedback optimal control law.

Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \tag{3.5.20}$$

$$\mathbf{Q} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}; \qquad \mathbf{R} = r = \frac{1}{4}; \qquad t_0 = 0; \qquad t_f = \infty. \tag{3.5.21}$$

Let $\bar{\mathbf{P}}$ be the 2x2 symmetric matrix

$$\bar{\mathbf{P}} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_{22} \end{bmatrix}. \tag{3.5.22}$$

Then, the optimal control (3.5.14) is given by

$$\begin{aligned} u^*(t) &= -4 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_{22} \end{bmatrix} \begin{bmatrix} x_1^*(t) \\ x_2^*(t) \end{bmatrix}, \\ &= -4[\bar{p}_{12}x_1^*(t) + \bar{p}_{22}x_2^*(t)], \end{aligned} \tag{3.5.23}$$

Matrix algebraic Riccati equation:

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = &- \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_{22} \end{bmatrix} + \\ & \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 4 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{12} & \bar{p}_{22} \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}. \end{aligned} \tag{3.5.24}$$

$$
\begin{cases}
4\bar{p}_{12}^2 + 4\bar{p}_{12} - 2 = 0 \\
-\bar{p}_{11} - \bar{p}_{12} + 2\bar{p}_{22} + 4\bar{p}_{12}\bar{p}_{22} - 3 = 0 \\
-2\bar{p}_{12} - 2\bar{p}_{22} + 4\bar{p}_{22}^2 - 5 = 0.
\end{cases} \tag{3.5.25}
$$

$$
\bar{\mathbf{P}} = \begin{bmatrix} 1.7363 & 0.3660 \\ 0.3660 & 1.4729 \end{bmatrix}. \tag{3.5.26}
$$

Using these Riccati coefficients (gains), the closed-loop optimal control (3.5.23) is given by

$$
\begin{aligned}
u^*(t) &= -4[0.366x_1^*(t) + 1.4729x_2^*(t)] \\
&= -[1.464x_1^*(t) + 5.8916x_2^*(t)].
\end{aligned} \tag{3.5.27}
$$

Using the closed-loop optimal control $u^*(t)$ from (3.5.27) in the original open-loop system (3.5.18), the closed-loop optimal system becomes

$$
\begin{aligned}
\dot{x}_1^*(t) &= x_2^*(t) \\
\dot{x}_2^*(t) &= -2x_1^*(t) + x_2^*(t) - 4[0.366x_1^*(t) + 1.4729x_2^*(t)]
\end{aligned} \tag{3.5.28}
$$

Using the initial conditions and the Riccati coefficient matrix (3.5.26), the optimal cost (3.5.17) is obtained as

$$
\begin{aligned}
J^* = \frac{1}{2}\mathbf{x}'(0)\bar{\mathbf{P}}\mathbf{x}(0) &= \frac{1}{2}\begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} 1.7363 & 0.3660 \\ 0.3660 & 1.4729 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \\
&= 7.9047. \qquad (3.5.29)
\end{aligned}
$$

lqr

Linear-Quadratic Regulator (LQR) design

Syntax

```
[K,S,P] = lqr(sys,Q,R,N)
[K,S,P] = lqr(A,B,Q,R,N)
```

$$\dot{x} = Ax + Bu$$

$$J = \frac{1}{2}\int_0^\infty [x^T Qx + 2x^T Su + u^T Ru]dt$$

$$= \frac{1}{2}\int_0^\infty \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

Kalman gain Poles

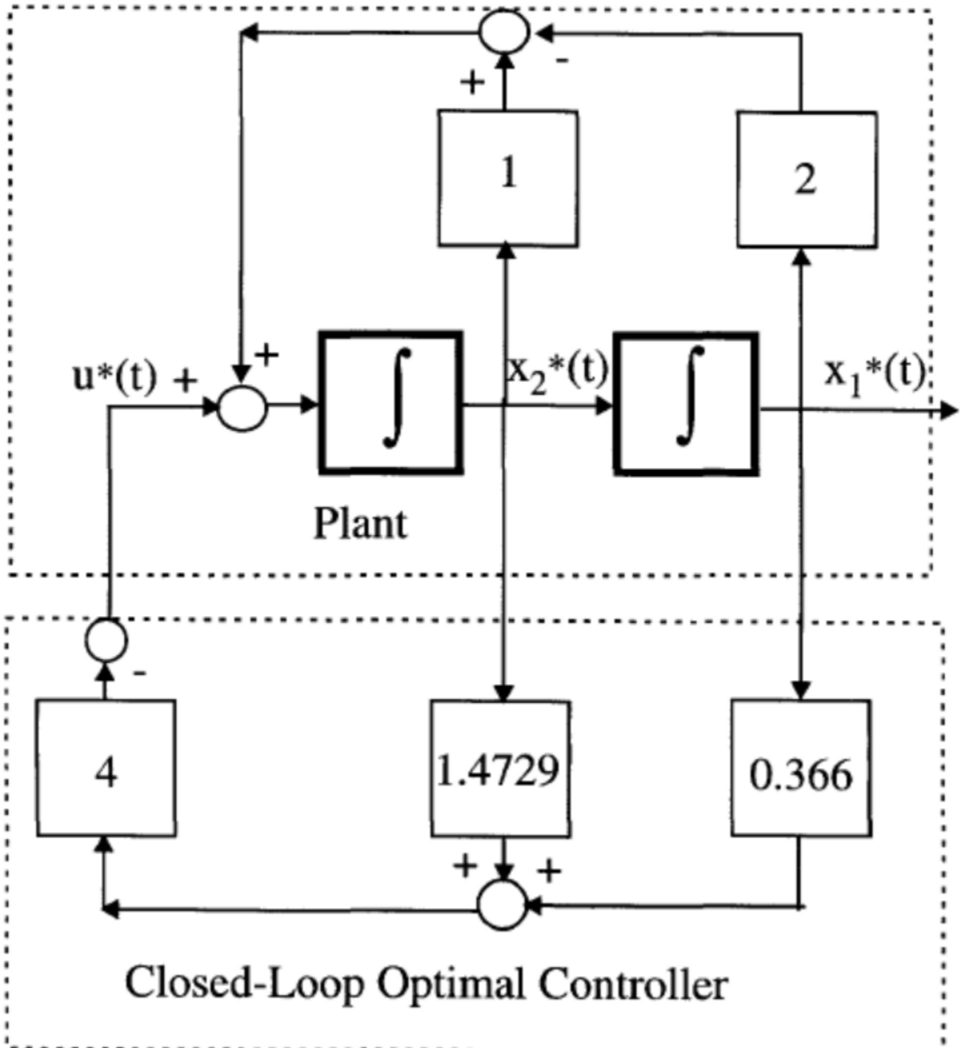
Eigenvalues

```
[X,K,L] = icare(A,B,Q,R,S,E,G)
```

$$A^T XE + E^T XA + E^T XGXE - (E^T XB + S)R^{-1}(B^T XE + S^T) + Q = 0$$

$E = I$; $G = 0$

$$A^T P + PA - [PB + S]R^{-1}[B^T P + S^T] + Q = 0$$



$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -2x_1(t) + x_2(t) + u(t)$$

$$u^*(t) = -4[0.366x_1^*(t) + 1.4729x_2^*(t)]$$

Figure 3.9 Closed-Loop Optimal Control System

```
%% Solution using Control System Toolbox in
%% The  MATLAB. Version 6
%% For Example:4-3
%%
 x10=2; %% initial condition on state x1
 x20=-3; %% initial condition on state x2
 X0=[x10;x20];
 A=[0 1;-2 1]; %% system matrix A
 B=[0;1]; %% system matrix B
 Q=[2 3;3 5]; %% performance index weighted matrix
 R=[0.25]; %% performance index weighted matrix
 [K,P,EV]=lqr(A,B,Q,R) %% K = feedback matrix;
%% P = Riccati matrix;
%% EV = eigenvalues of closed loop system A - B*K
K =

     1.4641     5.8916
 P =
    1.7363     0.3660
    0.3660     1.4729
 EV =
    -4.0326
    -0.8590
```

```
BIN=[0;0]; % dummy BIN for "initial" command
C=[1 1];
D=[1];
tfinal=10;
t=0:0.05:10;
[Y,X,t]=initial(A-B*K,BIN,C,D,X0,tfinal);
x1t=[1 0]*X'; %% extracting x1 from vector X
x2t=[0 1]*X'; %% extracting x2 from vector X
ut=-K*X';
plot(t,x1t,'k',t,x2t,'k')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
plot(t,ut,'k')
xlabel('t')
gtext('u(t)')
```

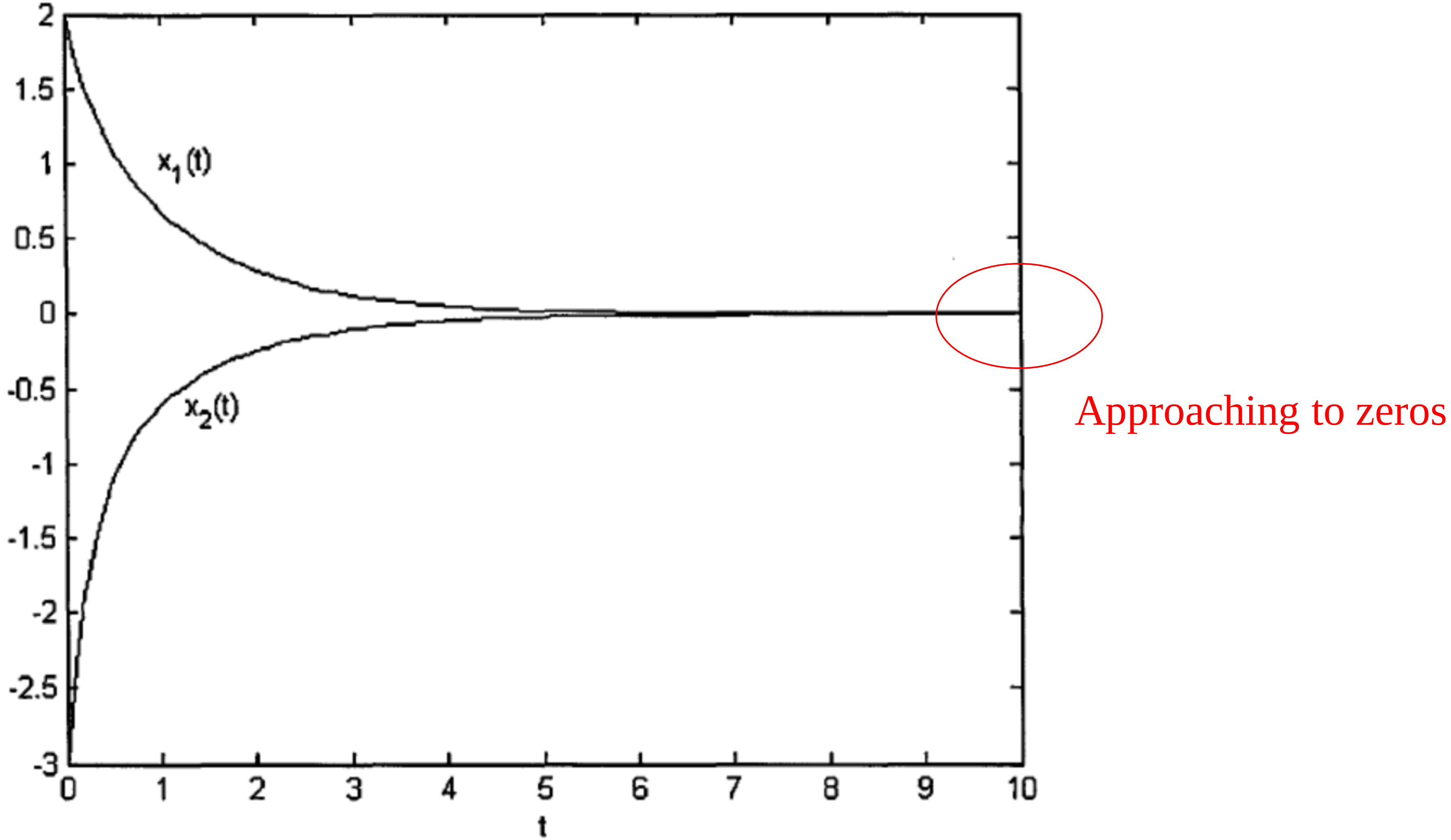


Figure 3.10 Optimal States for Example 3.2

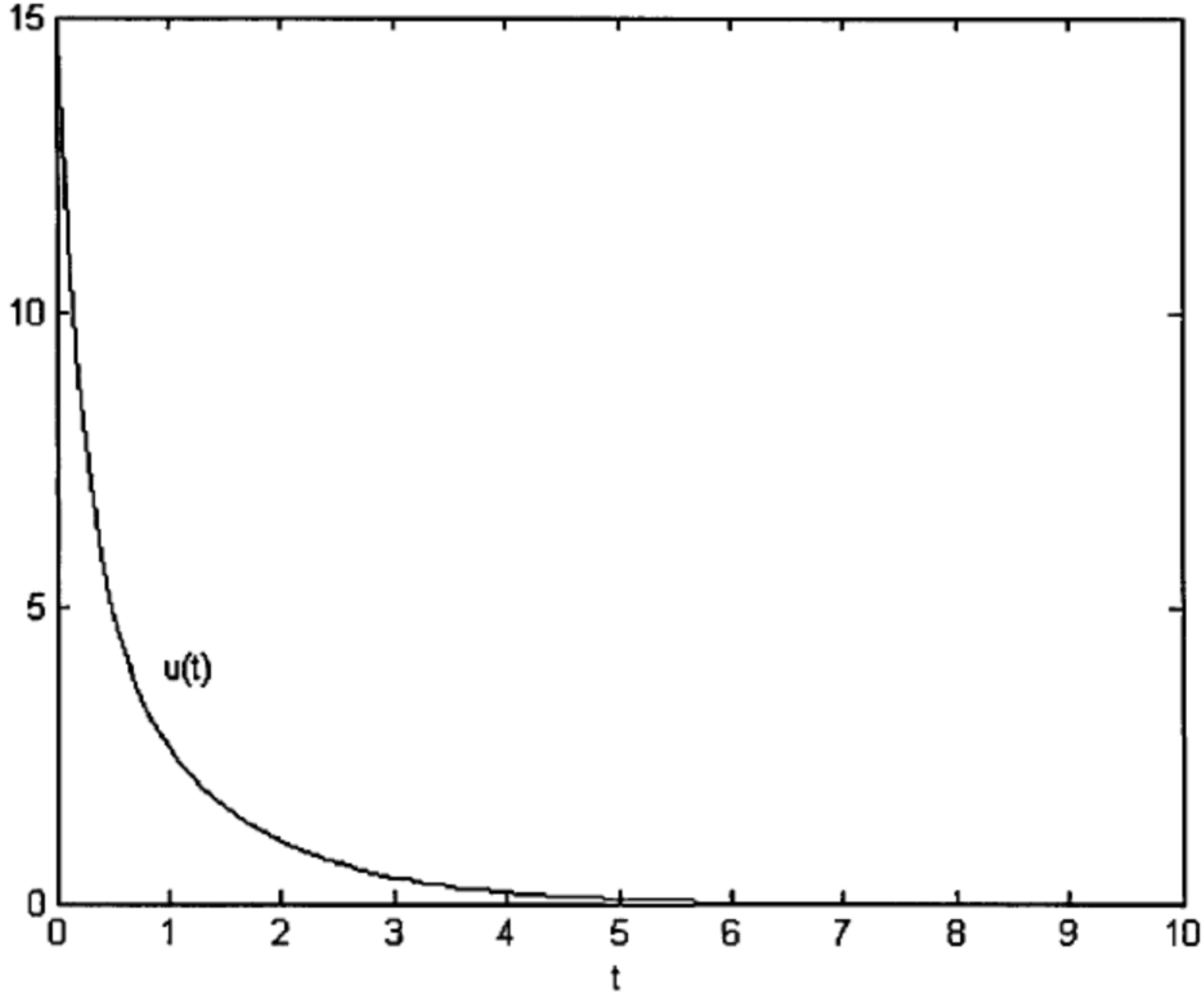


Figure 3.11 Optimal Control for Example 3.2

Using the optimal control $u^*(t)$ given by (3.5.23), the plant equations (3.5.18) are solved using MATLAB© to obtain the optimal states $x_1^*(t)$ and $x_2^*(t)$ and the optimal control $u^*(t)$ as shown in Figure 3.10 and Figure 3.11. Note that

1. the values of $\bar{\mathbf{P}}$ obtained in the example, are exactly the steady-state values of Example 3.1 and

   Finite-time LQR

2. the original plant (3.5.18) is *unstable* (eigenvalues at $2 \pm j1$) whereas the optimal closed-loop system (3.5.28) is *stable* (eigenvalues at $-4.0326, -0.8590$).

3.5.4 Stability Issues of Time-Invariant Regulator

1. The closed-loop optimal system (3.5.16) is not *always stable* especially when the original plant is unstable and these unstable states are not weighted in the PI (3.5.13). In order to prevent such a situation, we need the assumption that the pair $[\mathbf{A}, \mathbf{C}]$ is detectable, where C is any matrix such that $\mathbf{C}'\mathbf{C} = \mathbf{Q}$, which guarantees the stability of closed-loop optimal system. This assumption essentially ensures that all the potentially unstable states will show up in the $\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t)$ part of the performance measure.

2. The Riccati coefficient matrix $\bar{\mathbf{P}}$ is positive definite if and only if $[\mathbf{A}, \mathbf{C}]$ is completely observable.

3. The *detectability* condition is necessary for stability of the closed-loop optimal system.

4. Thus both *detectability* and *stabilizability* conditions are necessary for the existence of a stable closed-loop system.

3.5.5 Equivalence of Open-Loop and Closed-Loop Optimal Controls

Example 3.3

Consider a simple first order system

$$\dot{x}(t) = -3x(t) + u(t) \tag{3.5.30}$$

and the cost function (CF) as

$$J = \int_0^\infty [x^2(t) + u^2(t)]dt \tag{3.5.31}$$

where, $x(0) = 1$ and the final state $x(\infty) = 0$. Find the open-loop and closed-loop optimal controllers.

Solution: (a) Open-Loop Optimal Control:

$$\begin{aligned} V(x(t), u(t)) &= x^2(t) + u^2(t), \\ f(x(t), u(t)) &= -3x(t) + u(t). \end{aligned} \tag{3.5.32}$$

- **Step 1:**

$$\begin{aligned} \mathcal{H} &= V(x(t), u(t)) + \lambda(t) f(x(t), u(t)) \\ &= x^2(t) + u^2(t) + \lambda(t)[-3x(t) + u(t)]. \end{aligned} \tag{3.5.33}$$

- **Step 2:**

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \longrightarrow 2u^*(t) + \lambda^*(t) = 0 \longrightarrow u^*(t) = -\frac{1}{2}\lambda^*(t). \tag{3.5.34}$$

- **Step 3:**

$$\mathcal{H}^* = x^{*^2}(t) - \frac{1}{4}\lambda^{*^2}(t) - 3\lambda^*(t)x^*(t). \tag{3.5.35}$$

- **Step 4:**

$$\left\{\begin{aligned} \dot{x}^*(t) &= \frac{\partial \mathcal{H}^*}{\partial \lambda} \longrightarrow \dot{x}^*(t) = -\frac{1}{2}\lambda^*(t) - 3x^*(t), && (3.5.36) \\ \dot{\lambda}^*(t) &= -\frac{\partial \mathcal{H}^*}{\partial x} \longrightarrow \dot{\lambda}^*(t) = -2x^*(t) + 3\lambda^*(t), && (3.5.37) \end{aligned}\right.$$

$$\Rightarrow \quad \ddot{x}^*(t) - 10x^*(t) = 0, \qquad (3.5.38)$$

$$\Rightarrow \quad x^*(t) = C_1 e^{\sqrt{10}t} + C_2 e^{-\sqrt{10}t}. \qquad (3.5.39)$$

$$\Rightarrow \quad \begin{aligned} \lambda^*(t) &= 2\left[-\dot{x}^*(t) - 3x^*(t)\right] \\ &= -2C_1(\sqrt{10}+3)e^{\sqrt{10}t} + 2C_2(\sqrt{10}-3)e^{-\sqrt{10}t}. \end{aligned} \qquad (3.5.40)$$

$$\left.\begin{aligned} &x(0) = 1 \\ &\lambda(t_f = \infty) = 0 \end{aligned}\right\}$$

(Fixed final time, free final state)

$$\Rightarrow \quad \left\{\begin{aligned} x(0) = 1 &\longrightarrow C_1 + C_2 = 1 \\ \lambda(\infty) = 0 &\longrightarrow C_1 = 0. \end{aligned}\right. \qquad (3.5.41)$$

$$\Rightarrow \quad x^*(t) = e^{-\sqrt{10}t}; \qquad \lambda^*(t) = 2(\sqrt{10}-3)e^{-\sqrt{10}t}. \qquad (3.5.42)$$

- **Step 5:**

$$u^*(t) = -(\sqrt{10}-3)e^{-\sqrt{10}t}. \qquad (3.5.43)$$

(b) **Closed-Loop Optimal Control:**

$$\mathbf{A} = a = -3; \quad \mathbf{B} = b = 1;$$
$$\mathbf{Q} = q = 2; \quad \mathbf{R} = r = 2; \quad \bar{\mathbf{P}} = \bar{p}. \tag{3.5.44}$$

- **Step 1:**

$$\bar{p}(-3) + (-3)\bar{p} - \bar{p}(1)(\frac{1}{2})(1)\bar{p} + 2 = 0 \longrightarrow$$
$$\bar{p}^2 + 12\bar{p} - 4 = 0, \tag{3.5.45}$$

- **Step 2:**

$$u^*(t) = -r^{-1}b\bar{p}x^*(t) = -\frac{1}{2}(-6 + 2\sqrt{10})x^*(t)$$
$$= -(-3 + \sqrt{10})x^*(t). \tag{3.5.47}$$

- **Step 3:**

$$\dot{x}(t) = -3x^*(t) - (-3 + \sqrt{10})x^*(t) = -\sqrt{10}x^*(t). \quad (3.5.48) \qquad x(0) = 1$$

⇨ $x^*(t) = e^{-\sqrt{10}t}$ (3.5.49)

⇨ $u^*(t) = -(\sqrt{10} - 3)e^{-\sqrt{10}t}.$ (3.5.50)

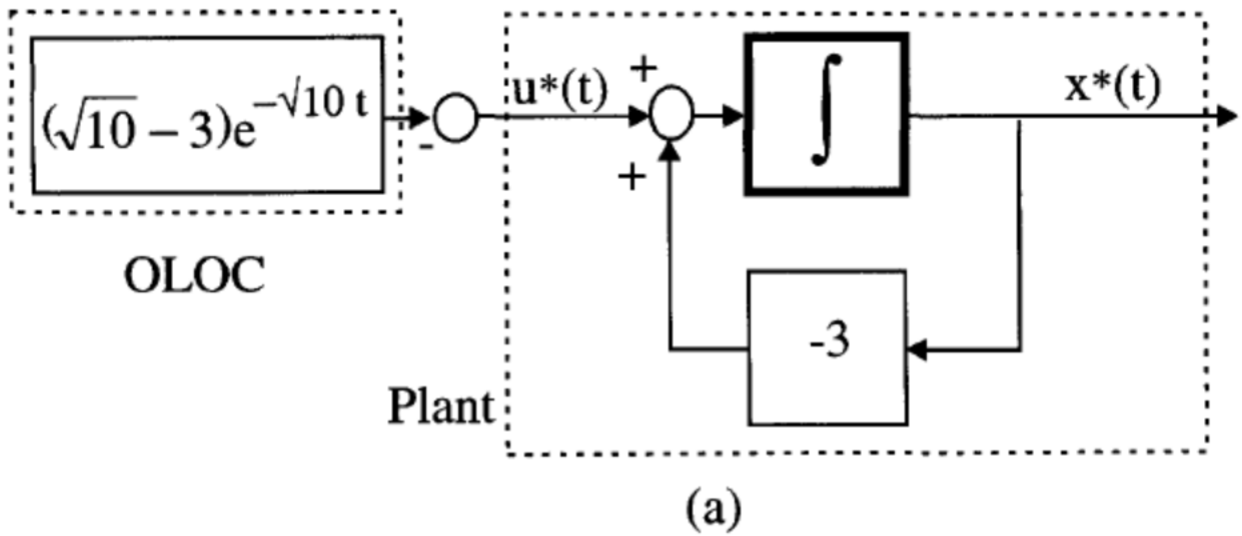


Figure 3.12 (a) Open-Loop Optimal Controller (OLOC)

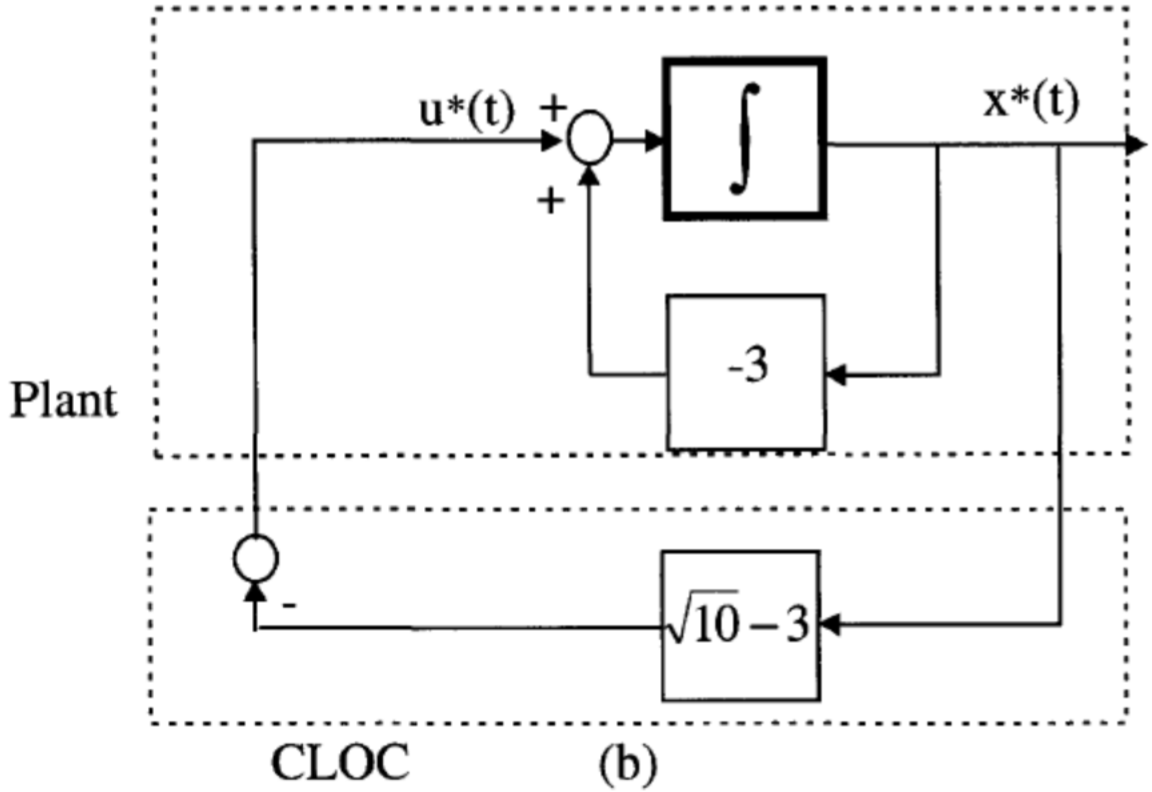


Figure 3.12 (b) Closed-Loop Optimal Controller (CLOC)

From the previous example, it is clear that

1. from the implementation point of view, the closed-loop optimal controller ($\sqrt{10} - 3$) is much *simpler* than the open-loop optimal controller ($(\sqrt{10}-3)e^{-\sqrt{10}t}$) which is an exponential time function and

2. with a closed-loop configuration, all the advantages of conventional feedback are incorporated.

3.6 Notes and Discussion

1. Many engineering and physical systems operate in *linear* range during normal operations.
2. There is a *wealth* of theoretical results available for *linear* systems which can be useful for linear quadratic methods.
3. The resulting optimal controller is *linear* in state and thus easy and simple for implementation purposes in real application of the LQ results.
4. Many (nonlinear) optimal control systems do not have solutions which can be easily computed. On the other hand, LQ optimal control systems have easily *computable* solutions.

5. As is well known, nonlinear systems can be examined for small variations from their normal operating conditions. Then, one can easily use to a first approximation a linear model and obtain linear optimal control to drive the original nonlinear system to its operating point.
6. Many of the concepts, techniques and computational procedures that are developed for linear optimal control systems in many cases many be carried on to nonlinear optimal control systems.
7. Linear optimal control designs for plants whose states are measurable possess a number of desirable robustness properties (such as good gain and phase margins and a good tolerance to nonlinearities) of classical control designs.