Chapter 1
Introduction

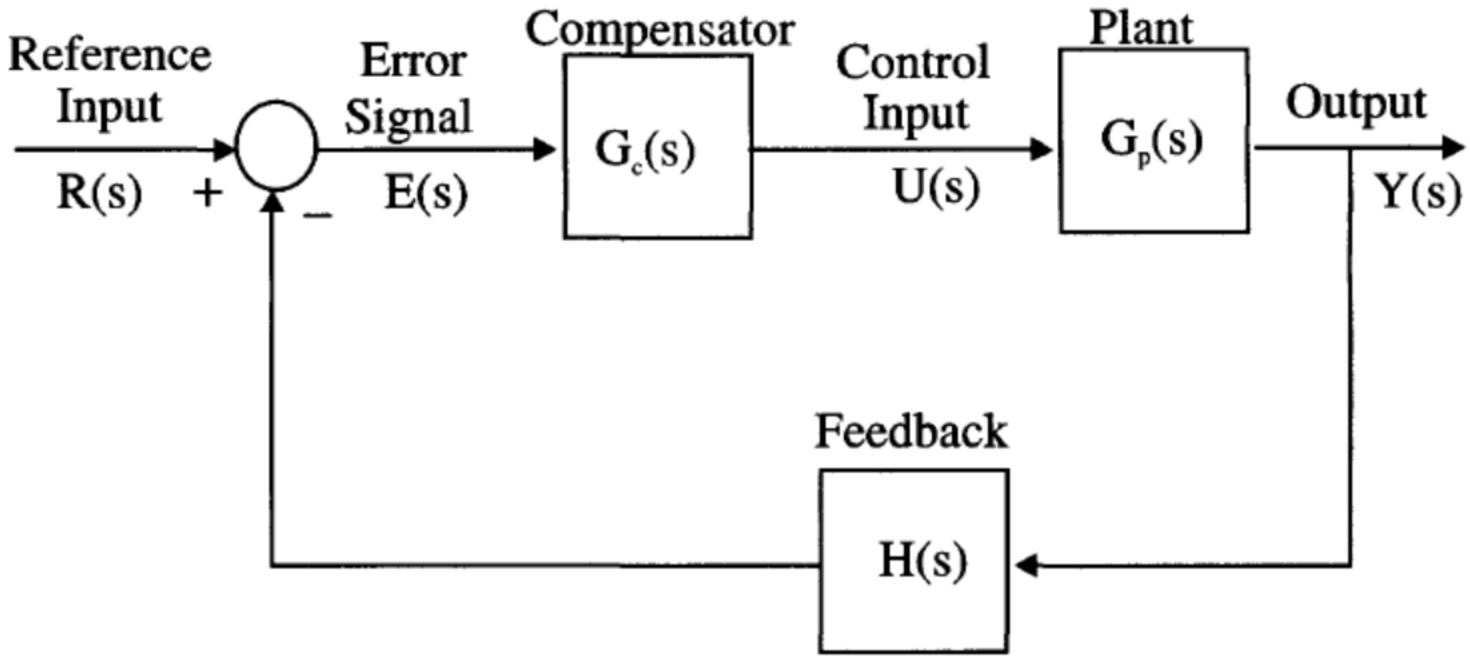
Introduction

- Optimization
- Optimal control
- Calculus of variations

1.1 Classical and Modern Control

Classical (conventional) control theory

- Single input and single output (SISO)
- Based on Laplace transform
- System representation in block diagram form



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad G(s) = G_c(s)G_p(s)$$

- The input $u(t)$ to the plant is determined by the error $e(t)$ and the compensator.
- All the variables are not readily available for feedback. In most cases only one output variable is available for feedback.

Modern control theory

- Multiple inputs and multiple outputs (MIMO)
- Based on state variable representation
- A set of first order differential (or difference) equations
- Linear, time-invariant form as

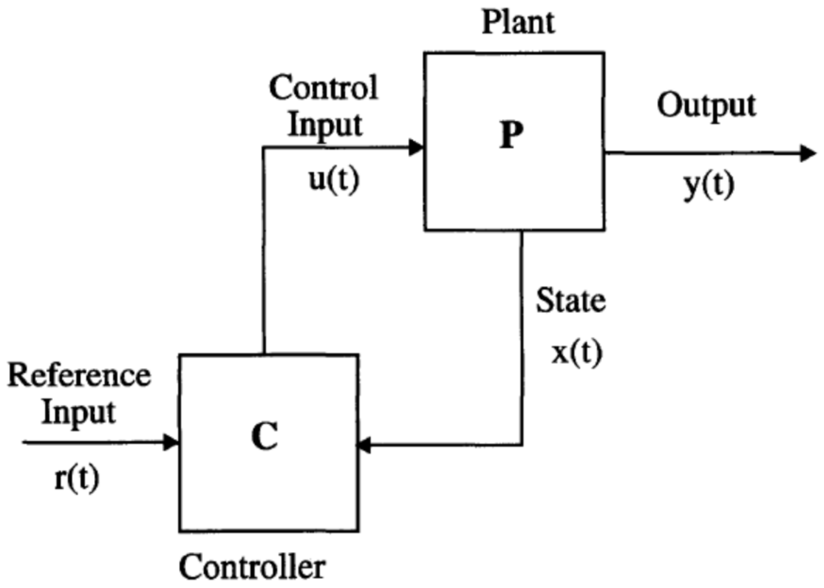
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- Nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$
$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)$$

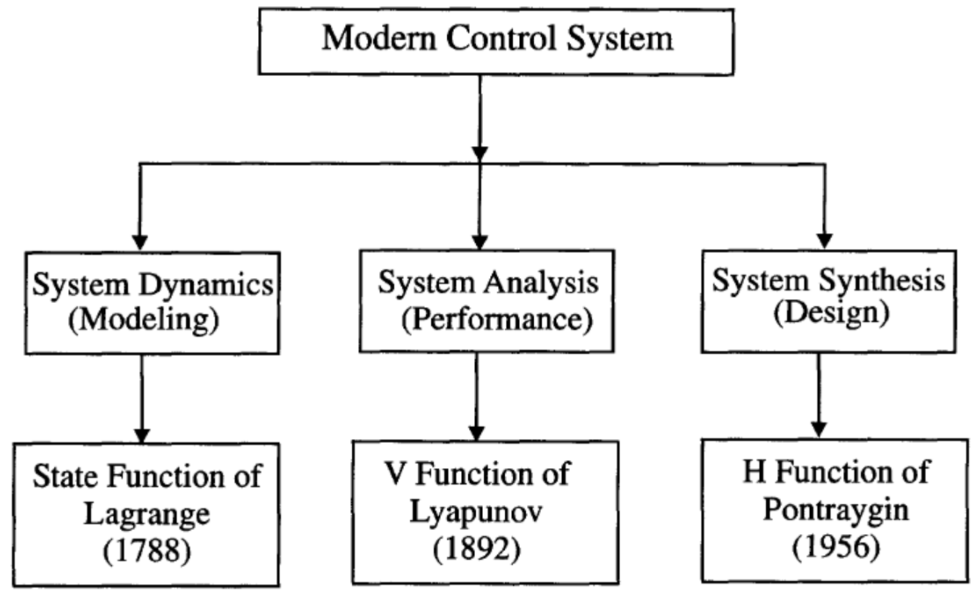
Modern Control configurations

- The input $u(t)$ is determined by the controller (consisting of error detector and compensator) driven by system states $x(t)$ and reference signal $r(t)$.
- All or most of the state variables are available for control.
- It depends on well-established matrix theory, which is amenable for large scale computer simulation.

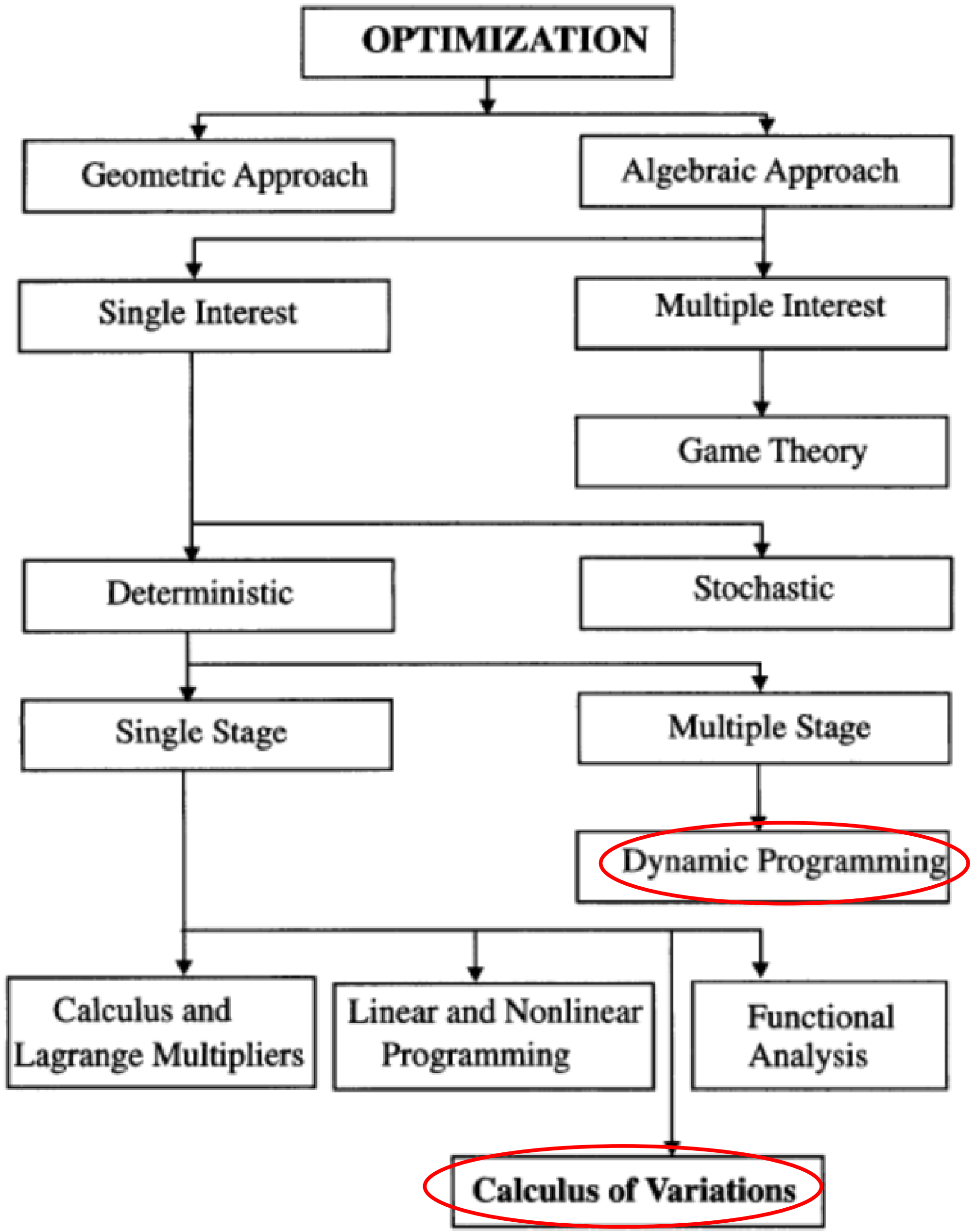


Components of a Modern Control System

- System dynamics (*modeling*) in terms of differential or difference equations based on the Lagrangian function
- Performance to find out mainly stability of the system by Lyapunov
- System synthesis (design) by applying control theory such as optimal control



1.2 Optimization


OPTIMIZATION
Geometric Approach
Algebraic Approach
Single Interest
Multiple Interest
Game Theory
Deterministic
Stochastic
Single Stage
Multiple Stage
Dynamic Programming
Calculus and
Lagrange Multipliers
Linear and Nonlinear
Programming
Functional
Analysis
Calculus of Variations

Optimization classification

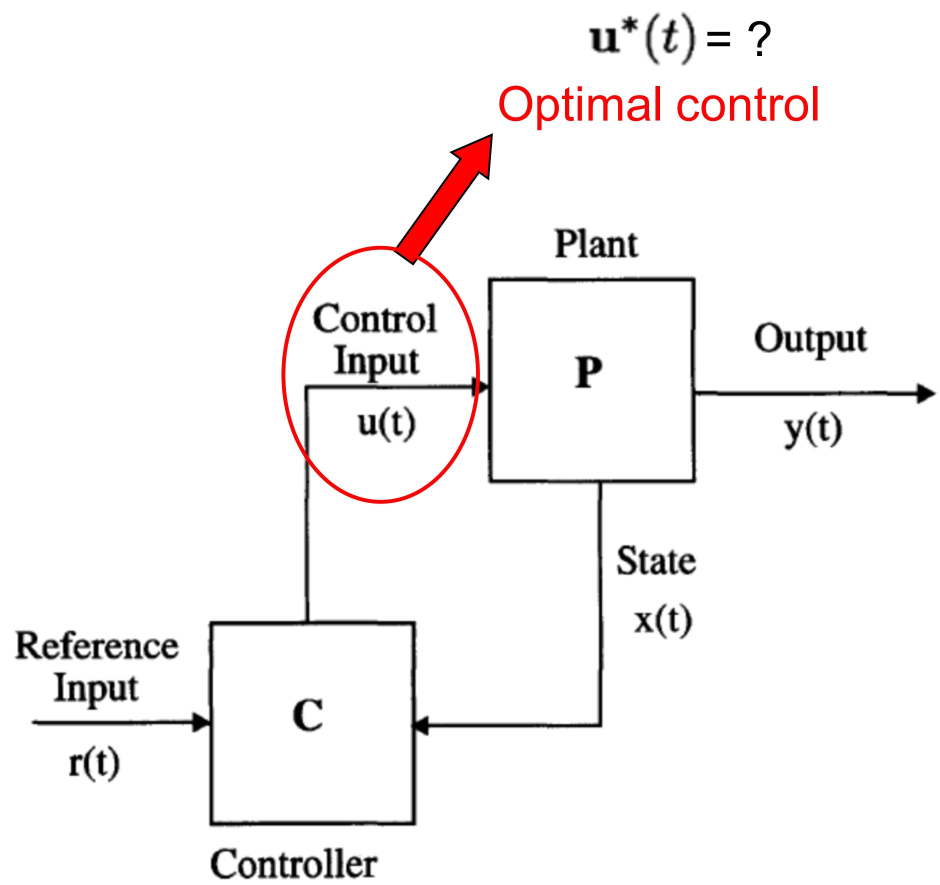
- Static Optimization is concerned with controlling a plant under *steady state* conditions.
  - The system variables are not changing with respect to time.
  - The plant is then described by *algebraic* equations.
  - Techniques used are ordinary calculus, Lagrange multipliers, linear and nonlinear programming.

- Dynamic Optimization concerns with the optimal control of plants under *dynamic* conditions.
  - The system variables are changing with respect to time and thus the time is involved in system description.
  - Then the plant is described by differential (or difference) equations.
  - Techniques used are search techniques, dynamic programming, variational calculus (or calculus of variations) and Pontryagin principle.

1.3 Optimal Control

- To determine control signals that will cause a process (plant)
- To satisfy some physical constraints and at the same time extremize (maximize or minimize) a chosen performance criterion (performance index or cost function)

- Find the optimal control $u^*(t)$ (* indicates optimal condition)
- Drive the plant P from initial state to final state
- With some constraints on controls and states and at the same time
- Extremizing the given performance index J.



The formulation of optimal control problem

- A mathematical description (or model) of the process to be controlled (generally in state variable form).
- A specification of the performance index.
- A statement of boundary conditions and the physical constraints on the states and/or controls.

1.3.1 Plant

- For the purpose of optimization, we describe a physical plant by a set of linear or nonlinear differential or difference equations.
- For example, a linear time-invariant system is described by the state and output relations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- A nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$
$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)$$

1.3.2 Performance Index

Classical control design

- Linear, time-invariant, single-input, single output (SISO) systems
- Typical performance criteria are system time response to step or ramp input characterized by rise time, settling time, peak overshoot, and steady state accuracy;
- And the frequency response of the system characterized by gain and phase margins, and bandwidth.

Optimal control problem

- To find a control which causes the dynamical system to reach a target or follow a state variable (or trajectory) and at the same time extremize a performance index

Performance Index

- Time-Optimal Control

$$J = \int_{t_0}^{t_f} dt = t_f - t_0 = t^*$$

- Fuel-Optimal Control System

$$J = \int_{t_0}^{t_f} |u(t)| dt$$

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} R_i |u_i(t)| dt$$

(for several controls, where R is a weighting factor)

- Minimum-Energy Control

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2(t) r_i dt$$ where r_i is the resistance and $u_i(t)$ is the current

or in general, $$J = \int_{t_0}^{t_f} \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)dt$$ where R is a *positive definite* matrix

Similarly, $$J = \int_{t_0}^{t_f} \mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t)dt$$

where, $\mathbf{x}_d(t)$ is the desired value, $\mathbf{x}_a(t)$ is the actual value, and $\mathbf{x}(t) = \mathbf{x}_a(t) - \mathbf{x}_d(t)$, is the error. Here, $\mathbf{Q}$ is a weighting matrix, which can be *positive semi-definite*.

- Terminal Control System

The terminal (final) error is $\mathbf{x}(t_f) = \mathbf{x}_a(t_f) - \mathbf{x}_d(t_f)$.

$$J = \mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f)$$

where F is a *positive semi-definite matrix.*

- General Optimal Control

$$J = \mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]dt$$

or, $$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

where, $\mathbf{R}$ is a positive definite matrix, and $\mathbf{Q}$ and $\mathbf{F}$ are positive semidefinite matrices, respectively. Note that the matrices $\mathbf{Q}$ and $\mathbf{R}$ may be time varying.

$$J = S(\mathbf{x}(t_f), t_f)$$ Mayer problem (terminal control)

$$J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$ Lagrange problem

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$ Bolza problem

1.3.3 Constraints

- The control $u(t)$ and state $x(t)$ vectors are either *unconstrained* or *constrained* depending upon the physical situation.
- Such as currents and voltages in an electrical circuit, speed of a motor, thrust of a rocket, constrained as

$$\mathbf{U}_{+} \leq \mathbf{u}(t) \leq \mathbf{U}_{-}, \quad \text{and} \quad \mathbf{X}_{-} \leq \mathbf{x}(t) \leq \mathbf{X}_{+}$$

where, $+$, and $-$ indicate the maximum and minimum values the variables can attain.

1.3.4 Formal Statement of Optimal Control System

- *Linear* time-invariant plant (system)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

- To optimize or extremize (minimize or maximize) a performance index

$$J = \mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]dt$$

- Nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

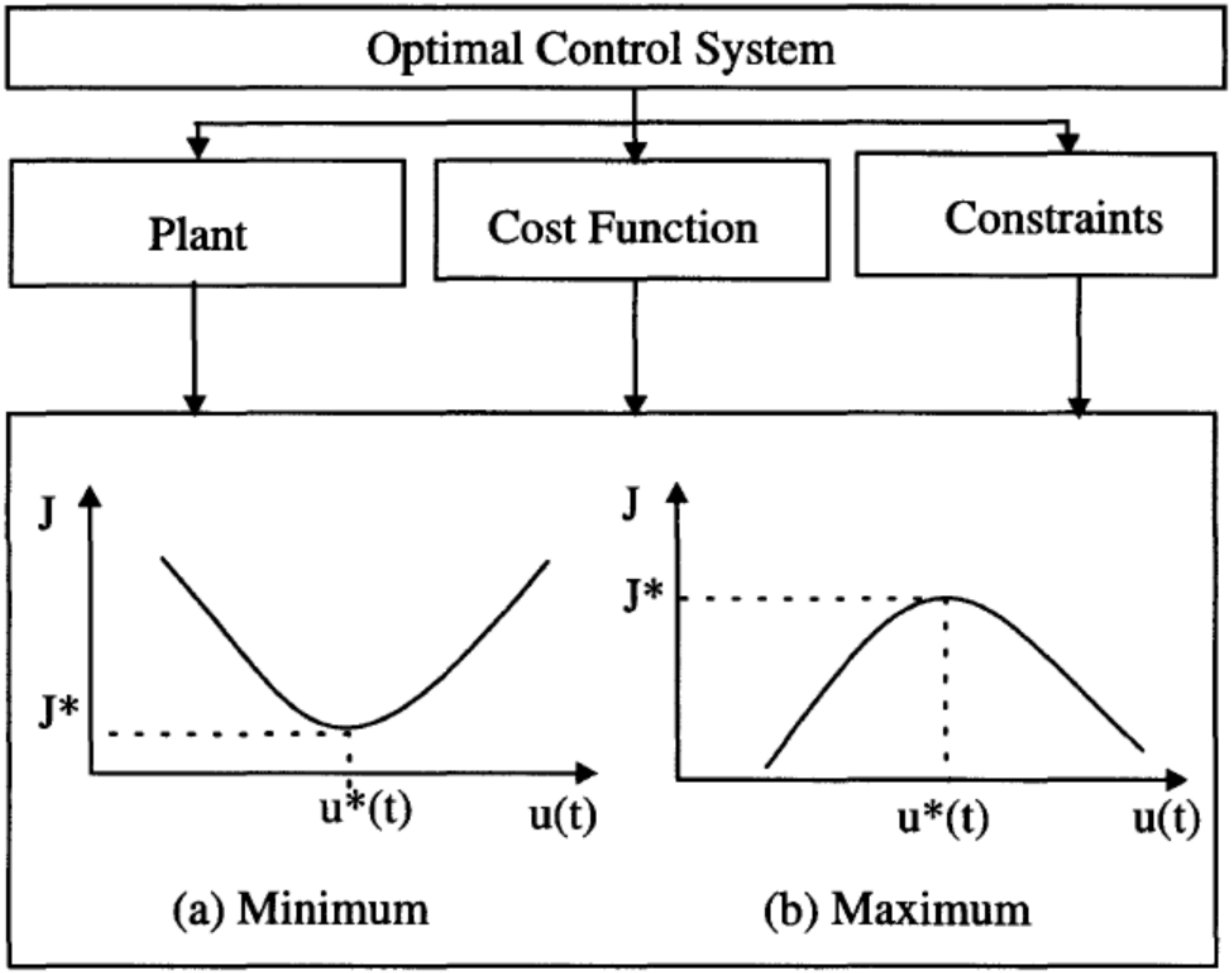
- To optimize the general performance index

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

- Constraints

$$\mathbf{U}_+ \leq \mathbf{u}(t) \leq \mathbf{U}_-, \quad \text{and} \quad \mathbf{X}_- \leq \mathbf{x}(t) \leq \mathbf{X}_+$$

- The final time t_f may be fixed, or free, and the final (target) state may be fully or partially fixed or free.


Optimal Control System
Plant
Cost Function
Constraints
J
J*
u*(t)
u(t)
(a) Minimum
J
J*
u*(t)
u(t)
(b) Maximum

Three stages of optimal control problems

- In the first stage, we just consider the performance index of the form (1.3.14)

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \qquad (1.3.14)$$

and use the well-known theory of calculus of variations to obtain optimal functions.

- In the second stage, we bring in the plant (1.3.11)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1.3.11}$$

and try to address the problem of finding optimal control $u^*(t)$ which will drive the plant and at the same time optimize the performance index (1.3.12).

$$J = \mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]dt \tag{1.3.12}$$

Next, the above topics are presented in discrete time domain.

- Finally, the topic of constraints on the controls and states (1.3.10) is considered along with the plant and performance index to obtain optimal control.

$$\mathbf{U}_+ \leq \mathbf{u}(t) \leq \mathbf{U}_-, \quad \text{and} \quad \mathbf{X}_- \leq \mathbf{x}(t) \leq \mathbf{X}_+ \qquad (1.3.10)$$

1.4 Historical Tour

  1.4.1 Calculus of Variations

  1.4.2 Optimal Control Theory

1.5 About This Book

1.6 Chapter Overview

Design procedure

- Define a performance index

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \qquad \text{(General form)}$$

or $$J = \mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]dt \qquad \text{(Quadratic form)}$$

- Select the weighting matrices F, Q, R

- Make sure the final time t_f fixed/free,
  and the final state x_f fully/partially fixed/free

- Establish a system model

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad \text{(Nonlinear eqs.)}$$

or $$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \text{(Linear eqs.)}$$

- Set constraints

$$\mathbf{U}_+ \leq \mathbf{u}(t) \leq \mathbf{U}_-, \quad \text{and} \quad \mathbf{X}_- \leq \mathbf{x}(t) \leq \mathbf{X}_+$$

- Solve the optimal control problem to obtain

  $u^*(t)$ as a function of $x(t)$ (state feedback)

- Substitute $u^*(t)$ back into the state equations to obtain

  $x(t)$

- Plot $x(t)$ and $u(t)$ vs. t

- Justify the performances based on $x(t)$ and $u(t)$

$\Rightarrow$ If the performance is not good, go back to modify the performance index until the performance is satisfied.

$\Rightarrow$ Or perform another optimization problem to obtain the optimal weighting matrices F, Q, R

$\Rightarrow$ Or apply another control theory

Exercises

- Prob. 1.1, 1.2, 1.3, 1.4, 1.5, 1.6