Chapter 4
Linear Quadratic Optimal Control Systems II

- In the previous chapter, we addressed the linear quadratic *regulator* system, where the aim was to obtain the optimal control to regulate (or keep) the state around zero.
- In this chapter, we discuss linear quadratic tracking (LQT) systems, and some related topics in linear quadratic regulator theory.
- It is suggested that the student reviews the material in Appendices A (Vector and Matrices) and B (State Space Analysis) given at the end of the book.

Trajectory Following Systems

- In tracking (trajectory following) systems, we require that the output of a system *track* or *follow* a desired trajectory in some optimal sense.
- Thus, we see that this is a generalization of *regulator* system in the sense that the *desired* trajectory for the regulator is simply the zero state.

4.1 Linear Quadratic Tracking System: Finite Time Case

- Linear quadratic tracking (LQT) system: to maintain the output as close as possible to the desired output with minimum control energy
- A linear, observable system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) \qquad (4.1.1)\end{aligned}$$

where, $x(t)$ is the nth order state vector, $u(t)$ is the rth order control vector, and $y(t)$ is the mth order output vector.

- Let $z(t)$ be the mth order *desired output* and define an error as

$$\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t) \qquad (4.1.2)$$

- Choose a performance index as

$$J = \frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(t_f)\mathbf{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [\underline{\mathbf{e}}'(t)\mathbf{Q}(t)\underline{\mathbf{e}}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt \qquad (4.1.3)$$

with t_f specified and $x(t_f)$ not specified.

- We are dealing with *free-final state system*.
- Also, we assume that $\mathbf{F}(t_f)$ and $\mathbf{Q}(t)$ are $m \times m$ symmetric, positive *semidefinite* matrices, and $\mathbf{R}(t)$ is $r \times r$ symmetric, positive *definite* matrix.

- We now use the Pontryagin Minimum Principle in the following order.
  - Step 1 : *Hamiltonian*
  - Step 2: *Open-Loop Optimal Control*
  - Step 3: *State and Costate System*
  - Step 4: *Riccati and Vector Equations*
  - Step 5: *Closed-Loop Optimal Control*
  - Step 6: *Optimal State*
  - Step **7:** *Optimal Cost*

$$\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t) \tag{4.1.2}$$

$$\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{C}(t)\mathbf{x}(t). \tag{4.1.4}$$

- **Step 1:** *Hamiltonian:*

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)) = \frac{1}{2}\left[\mathbf{z}(t) - \mathbf{C}(t)\mathbf{x}(t)\right]' \mathbf{Q}(t)\left[\mathbf{z}(t) - \mathbf{C}(t)\mathbf{x}(t)\right]$$
$$+\frac{1}{2}\mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t) + \boldsymbol{\lambda}'(t)\left[\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\right]. \tag{4.1.5}$$

- **Step 2:** *Open-Loop Optimal Control:*

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0 \longrightarrow \mathbf{R}(t)\mathbf{u}(t) + \mathbf{B}'(t)\boldsymbol{\lambda}(t) = 0 \tag{4.1.6}$$

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\boldsymbol{\lambda}^*(t).} \tag{4.1.7}$$

Since the second partial of $\mathcal{H}$ in (4.1.5) w.r.t. $\mathbf{u}^*(t)$ is just $\mathbf{R}(t)$, and we chose $\mathbf{R}(t)$ to be positive *definite*, we are dealing with a control which *minimizes* the cost functional (4.1.3).

- **Step 3:** *State and Costate System:*

$$\dot{\mathbf{x}}(t) = \frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\underline{\mathbf{u}(t)} \qquad (4.1.8)$$

$$\Rightarrow \quad \dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) - \mathbf{B}(t)\underline{\mathbf{R}^{-1}(t)\mathbf{B}'(t)\boldsymbol{\lambda}^*(t)}. \qquad (4.1.9)$$

$$\dot{\boldsymbol{\lambda}}^*(t) = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}$$

$$= -\mathbf{C}'(t)\mathbf{Q}(t)\mathbf{C}(t)\mathbf{x}^*(t) - \mathbf{A}'(t)\boldsymbol{\lambda}^*(t) + \mathbf{C}'(t)\mathbf{Q}(t)\mathbf{z}(t) \qquad (4.1.10)$$

$$\Rightarrow \quad \begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ \dot{\boldsymbol{\lambda}}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & -\mathbf{E}(t) \\ -\mathbf{V}(t) & -\mathbf{A}'(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ \boldsymbol{\lambda}^*(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{W}(t) \end{bmatrix} \mathbf{z}(t). \qquad (4.1.12)$$

where

$$\mathbf{E}(t) = \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t), \quad \mathbf{V}(t) = \mathbf{C}'(t)\mathbf{Q}(t)\mathbf{C}(t), \quad \mathbf{W}(t) = \mathbf{C}'(t)\mathbf{Q}(t) \qquad (4.1.11)$$

- This canonical system of *2n* differential equations is **linear, time varying**, but ***nonhomogeneous*** with $W(t)z(t)$ as forcing function.

Boundary conditions:

$$\mathbf{x}(t=t_0)=\mathbf{x}(t_0) \qquad (4.1.13)$$

$$\boldsymbol{\lambda}(t_f)=\frac{\partial}{\partial \mathbf{x}(t_f)}\left[\frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(t_f)\mathbf{e}(t_f)\right]$$

(Fixed final time, free final state
$\Rightarrow$ case (c) in Table 2.1)

$$=\frac{\partial}{\partial \mathbf{x}(t_f)}\left[\frac{1}{2}[\mathbf{z}(t_f)-\mathbf{C}(t_f)\mathbf{x}(t_f)]'\mathbf{F}(t_f)[\mathbf{z}(t_f)-\mathbf{C}(t_f)\mathbf{x}(t_f)]\right]$$

$$=\underline{\mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f)}\mathbf{x}(t_f)-\underline{\mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f)}. \qquad (4.1.14)$$

B.C. $\left\{\begin{array}{l}\boxed{\mathbf{P}(t_f)=\mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f),}\\ \boxed{\mathbf{g}(t_f)=\mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f).}\end{array}\right.$

- **Step 4:** *Riccati and Vector Equations:*

Let $\boldsymbol{\lambda}^*(t)=\underline{\mathbf{P}(t)}\mathbf{x}^*(t)-\underline{\mathbf{g}(t)} \qquad (4.1.15)$

where, the *n×n matrix* $\mathbf{P}(t)$ and *n vector* $\mathbf{g}(t)$ are yet to be determine so as to satisfy the canonical system (4.1.12).

$\Rightarrow$ $$\dot{\boldsymbol{\lambda}}^*(t)=\dot{\mathbf{P}}(t)\mathbf{x}^*(t)+\mathbf{P}(t)\dot{\mathbf{x}}^*(t)-\dot{\mathbf{g}}(t). \qquad (4.1.16)$$

(4.1.12) (4.1.12)

$$-\mathbf{V}(t)\mathbf{x}^*(t) - \mathbf{A}'(t)\left[\mathbf{P}(t)\mathbf{x}^*(t) - \mathbf{g}(t)\right] + \mathbf{W}(t)\mathbf{z}(t) = \dot{\mathbf{P}}(t)\mathbf{x}^*(t)$$
$$+\mathbf{P}(t)[\mathbf{A}(t)\mathbf{x}(t) - \mathbf{E}(t)\{\mathbf{P}(t)\mathbf{x}^*(t) - \mathbf{g}(t)\}] - \dot{\mathbf{g}}(t). \qquad (4.1.17)$$

$$\left[\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}'(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) + \mathbf{V}(t)\right]\mathbf{x}^*(t) -$$
$$[\dot{\mathbf{g}}(t) + \mathbf{A}'(t)\mathbf{g}(t) - \mathbf{P}(t)\mathbf{E}(t)\mathbf{g}(t) + \mathbf{W}(t)\mathbf{z}(t)] = 0. \qquad (4.1.18)$$

$\neq 0$

$$\boxed{\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) - \mathbf{V}(t)} \qquad (4.1.19)$$

$$\boxed{\dot{\mathbf{g}}(t) = [\mathbf{P}(t)\mathbf{E}(t) - \mathbf{A}'(t)]\,\mathbf{g}(\mathbf{t}) - \mathbf{W}(\mathbf{t})\mathbf{z}(\mathbf{t}).} \qquad (4.1.20)$$

Boundary conditions:

$$\boldsymbol{\lambda}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f)\mathbf{x}(t_f) - \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f). \qquad (4.1.14)$$

$$\boldsymbol{\lambda}(t_f) = \mathbf{P}(t_f)\mathbf{x}(t_f) - \mathbf{g}(t_f), \qquad (4.1.21)$$

$$\boxed{\mathbf{P}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f),} \qquad (4.1.22)$$

$$\boxed{\mathbf{g}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f).} \qquad (4.1.23)$$

Thus, the matrix DRE (4.1.19) and the vector equation (4.1.20) are to be solved *backward* using the boundary conditions (4.1.22) and (4.1.23).

- **Step 5:** *Closed-Loop Optimal Control:*

$$\begin{aligned}\mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\left[\mathbf{P}(t)\mathbf{x}^*(t) - \mathbf{g}(t)\right] \\ &= -\mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t) \qquad (4.1.24)\end{aligned}$$

where, $\mathbf{K}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$, is the *Kalman gain.*

- **Step 6:** *Optimal State:*

$$\begin{aligned}\dot{\mathbf{x}}^*(t) &= \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)\right]\mathbf{x}^*(t) \\ &\quad + \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t) \\ &= [\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)]\mathbf{x}^*(t) + \mathbf{E}(t)\mathbf{g}(t). \qquad (4.1.25)\end{aligned}$$

- **Step 7:** *Optimal Cost:* (see [6])

$$J^*(t) = \frac{1}{2}\mathbf{x}^{*\prime}(t)\mathbf{P}(t)\mathbf{x}^*(t) - \mathbf{x}^{*\prime}(t)\mathbf{g}(t) + \mathbf{h}(t) \qquad (4.1.26)$$

where
$$\begin{aligned}\dot{\mathbf{h}}(t) &= -\frac{1}{2}\mathbf{g}'(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t) - \frac{1}{2}\mathbf{z}'(t)\mathbf{Q}(t)\mathbf{z}(t) \\ &= -\frac{1}{2}\mathbf{g}'(t)\mathbf{E}(t)\mathbf{g}(t) - \frac{1}{2}\mathbf{z}'(t)\mathbf{Q}(t)\mathbf{z}(t) \qquad (4.1.27)\end{aligned}$$

with $\mathbf{h}(t_f) = -\mathbf{z}'(t_f)\mathbf{P}(t_f)\mathbf{z}(t_f). \qquad (4.1.28)$

4.1.1 Linear Quadratic Tracking System: Summary

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad \mathbf{x}(t = t_0) = \mathbf{x}(t_0) \qquad t_f \ \textit{specified}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \qquad (4.1.29) \qquad \mathbf{x}(t_f) \ \textit{not specified.}$$

$$\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t)$$

$$J = \frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(t_f)\mathbf{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{e}'(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \qquad (4.1.30)$$

$$\begin{aligned}\mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\left[\underline{\mathbf{P}(t)}\mathbf{x}^*(t) - \underline{\mathbf{g}(t)}\right] \\ &= -\mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t) \qquad (4.1.31)\end{aligned}$$

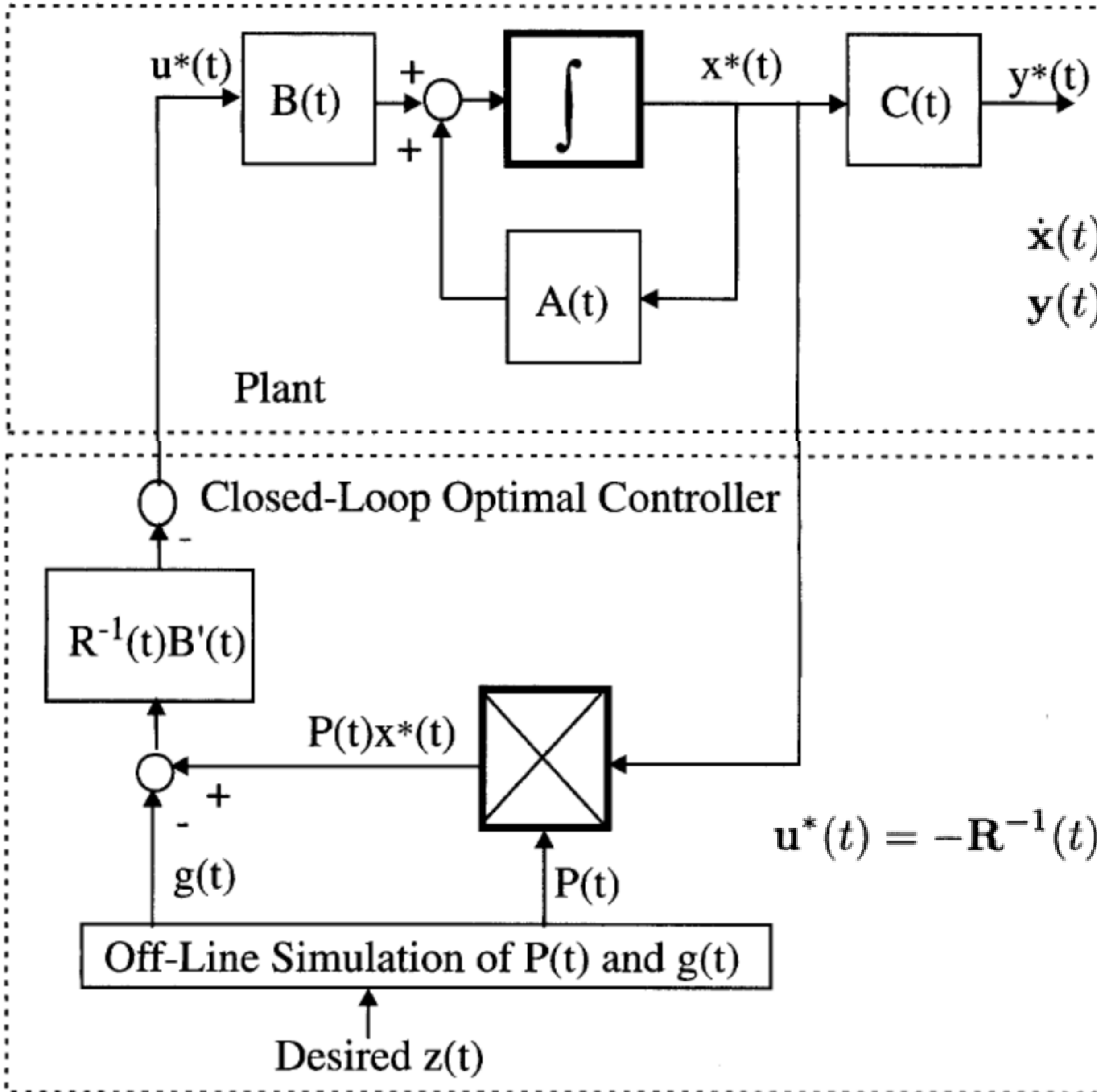
$$\left\{\begin{aligned}\dot{\mathbf{P}}(t) &= -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) - \mathbf{V}(t) \quad (4.1.32) \\ \mathbf{P}(t_f) &= \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f), \qquad (4.1.33)\end{aligned}\right.$$

$$\left\{\begin{aligned}\dot{\mathbf{g}}(t) &= -\left[\mathbf{A}(t) - \mathbf{E}(t)\underline{\mathbf{P}(t)}\right]'\mathbf{g}(t) - \mathbf{W}(t)\mathbf{z}(t) \qquad (4.1.34) \\ \mathbf{g}(t_f) &= \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f) \qquad (4.1.35)\end{aligned}\right.$$

$$\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{E}'(t)\mathbf{P}(t)\right]\mathbf{x}^*(t) + \mathbf{E}(t)\mathbf{g}(t), \qquad (4.1.36)$$

$$J^*(t_0) = \frac{1}{2}\mathbf{x}^{*\prime}(t_0)\mathbf{P}(t_0)\mathbf{x}^*(t_0) - \mathbf{x}^*(t_0)\mathbf{g}(t_0) + \underline{\mathbf{h}(t_0)}. \qquad (4.1.37)$$

Obtained by (4.1.27) and (4.1.28)



$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\left[\mathbf{P}(t)\mathbf{x}^*(t) - \mathbf{g}(t)\right]$$

Figure 4.1 Implementation of the Optimal Tracking System

Table 4.1 Procedure Summary of Linear Quadratic Tracking System

A. Statement of the Problem
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t), \quad \mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t),$ the performance index as $J = \frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(t_f)\mathbf{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[\mathbf{e}'(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)]\,dt,$ and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f)$ is free, find the optimal control, state and performance index.

B. Solution of the Problem	
Step 1	Solve the matrix differential Riccati equation $\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) - \mathbf{V}(t)$, with final condition $\mathbf{P}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f)$, and the non-homogeneous vector differential equation $\dot{\mathbf{g}}(t) = -\left[\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)\right]'\mathbf{g}(t) - \mathbf{W}(t)\mathbf{z}(t)$, with final condition $\mathbf{g}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f)$ where $\mathbf{E}(t) = \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t), \quad \mathbf{V}(t) = \mathbf{C}'(t)\mathbf{Q}(t)\mathbf{C}(t)$, $\mathbf{W}(t) = \mathbf{C}'(t)\mathbf{Q}(t)$.
Step 2	Solve the optimal state $\mathbf{x}^*(t)$ from $\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)\right]\mathbf{x}^*(t) + \mathbf{E}(t)\mathbf{g}(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.
Step 3	Obtain optimal control $\mathbf{u}^*(t)$ from $\mathbf{u}^*(t) = -\mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t)$, where, $\mathbf{K}(t) = \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)$.
Step 4	The optimal cost $J^*(t_0)$ is $J^*(t_0) = \frac{1}{2}\mathbf{x}^{*\prime}(t_0)\mathbf{P}(t_0)\mathbf{x}^*(t_0) - \mathbf{x}^*(t_0)\mathbf{g}(t_0) + \mathbf{h}(t_0)$ where $\mathbf{h}(t)$ is the solution of $\dot{\mathbf{h}}(t) = -\frac{1}{2}\mathbf{g}'(t)\mathbf{E}(t)\mathbf{g}(t) - \frac{1}{2}\mathbf{z}'(t)\mathbf{Q}(t)\mathbf{z}(t)$ with final condition $\mathbf{h}(t_f) = -\mathbf{z}'(t_f)\mathbf{P}(t_f)\mathbf{z}(t_f)$.

4.1.2 Salient Features of Tracking System

1. *Riccati Coefficient Matrix* $\mathbf{P}(t)$*:* We note that the *desired output* $\mathbf{z}(t)$ has no influence on the matrix differential Riccati equation (4.1.32) and its boundary condition (4.1.33). This means that once the problem is specified in terms of the final time t_f, the plant matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, and $\mathbf{C}(t)$, and the cost functional matrices $\mathbf{F}(t_f)$, $\mathbf{Q}(t)$, and $\mathbf{R}(t)$, the matrix function $\mathbf{P}(t)$ is completely determined.

$$\mathbf{V}(t) = \mathbf{C}'(t)\mathbf{Q}(t)\mathbf{C}(t).$$

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) - \mathbf{V}(t),$$

with final condition $\mathbf{P}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{C}(t_f)$,

2. *Closed Loop Eigenvalues:* From the costate relation (4.1.36), we see the closed-loop system matrix $[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{P}(t)]$ is again independent of the desired output $\mathbf{z}(t)$. This means the eigenvalues of the closed-loop, optimal tracking system are *independent* of the desired output $\mathbf{z}(t)$.

$$\dot{\mathbf{x}}^*(t) = [\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)]\,\mathbf{x}^*(t) + \mathbf{E}(t)\mathbf{g}(t)$$

with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

3. *Tracking and Regulator Systems:* The main difference between the optimal *output tracking* system and the optimal *state regulator* system is in the vector $\mathbf{g}(t)$. As shown in Figure 4.1, one can think of the desired output $\mathbf{z}(t)$ as the forcing function of the closed-loop optimal system which generates the signal $\mathbf{g}(t)$.

$\dot{\mathbf{g}}(t) = -\left[\mathbf{A}(t) - \mathbf{E}(t)\mathbf{P}(t)\right]' \mathbf{g}(t) - \mathbf{W}(t)\mathbf{z}(t),$
with final condition $\mathbf{g}(t_f) = \mathbf{C}'(t_f)\mathbf{F}(t_f)\mathbf{z}(t_f)$

4. Also, note that if we make $\mathbf{C}(t) = \mathbf{I}(t)$, then in (4.1.11), $\mathbf{V}(t) = \mathbf{Q}(t)$. Thus, the matrix DRE (4.1.19) becomes the same matrix DRE (3.2.34) that was obtained in LQR system in Chapter 3.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t), \quad \mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t)$$

$$J = \tfrac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(t_f)\mathbf{e}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{e}'(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right]dt$$

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) - \mathbf{V}(t)$$

$$\mathbf{V}(t) = \mathbf{C}'(t)\mathbf{Q}(t)\mathbf{C}(t)$$

Example 4.1

A second order plant

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -2x_1(t) - 3x_2(t) + u(t) \\ \mathbf{y}(t) &= \mathbf{x}(t)\end{aligned} \tag{4.1.38}$$

is to be controlled to minimize the performance index

$$\begin{aligned}J &= [1 - x_1(t_f)]^2 \\ &+ \int_{t_0}^{t_f} \left[[1 - x_1(t)]^2 + 0.002u^2(t) \right] dt.\end{aligned} \tag{4.1.39}$$

The final time t_f is specified at 20, the final state $\mathbf{x}(t_f)$ is free and the admissible controls and states are unbounded. It is required to keep the state $x_1(t)$ close to 1. Obtain the feedback control law and plot all the time histories of Riccati coefficients, $\mathbf{g}$ vector components, optimal states and control.

Solution:

$$z_1(t) = 1 \qquad z_2(t) = 0$$

$$\left.\begin{aligned} &\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{C}\mathbf{x}(t) \\ &\mathbf{C} = \mathbf{I}. \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} e_1(t) &= z_1(t) - x_1(t) \\ e_2(t) &= z_2(t) - x_2(t) \end{aligned}\right.$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{C} = \mathbf{I}; \quad \mathbf{z}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{F}(t_f); \quad \mathbf{R} = r = 0.004. \tag{4.1.40}$$

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}; \quad \mathbf{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}. \tag{4.1.41}$$

$$u^*(t) = -250\left[p_{12}x_1^*(t) + p_{22}x_2^*(t) - g_2(t)\right] \tag{4.1.42}$$

Riccati equation:

$$
\begin{bmatrix} \dot{p}_{11}(t) & \dot{p}_{12}(t) \\ \dot{p}_{12}(t) & \dot{p}_{22}(t) \end{bmatrix} = - \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}
$$

$$
+ \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{0.004} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.1.43}
$$

$$
\Rightarrow \left\{ \begin{aligned} \dot{p}_{11}(t) &= 250 p_{12}^2(t) + 4 p_{12}(t) - 2 \\ \dot{p}_{12}(t) &= 250 p_{12}(t) p_{22}(t) - p_{11}(t) + 3 p_{12}(t) + 2 p_{22}(t) \\ \dot{p}_{22}(t) &= 250 p_{22}^2(t) - 2 p_{12}(t) + 6 p_{22}(t) \end{aligned} \right. \tag{4.1.45}
$$

$$
\begin{bmatrix} p_{11}(t_f) & p_{12}(t_f) \\ p_{12}(t_f) & p_{22}(t_f) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.1.46}
$$

$$\begin{bmatrix}\dot g_1(t)\\ \dot g_2(t)\end{bmatrix} = -\left\{\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\frac{1}{0.004}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}p_{11}(t) & p_{12}(t)\\ p_{12}(t) & p_{22}(t)\end{bmatrix}\right\}'\begin{bmatrix}g_1(t)\\ g_2(t)\end{bmatrix}$$
$$-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}'\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \qquad (4.1.44)$$

$$\left\{\begin{aligned}\dot g_1(t) &= [250p_{12}(t) + 2]\, g_2(t) - 2\\ \dot g_2(t) &= -g_1(t) + [3 + 250p_{22}(t)]\, g_2(t)\end{aligned}\right. \qquad (4.1.47)$$

$$\begin{bmatrix}g_1(t_f)\\ g_2(t_f)\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}. \qquad (4.1.48)$$

Note: One has to try various values of the matrix $\mathbf{R}$ in order to get a better tracking of the states. Solutions for the functions $p_{11}(t), p_{12}(t)$, and $p_{22}(t)$ (Figure 4.2), functions $g_1(t)$ and $g_2(t)$ (Figure 4.3), optimal states (Figures 4.4) and control (Figure 4.5) for initial conditions $\mathbf{x}(0) = [-0.5\ 0]$ and the final time $t_f = 20$ are obtained using MATLAB© routines given in Appendix C under continuous-time tracking system.

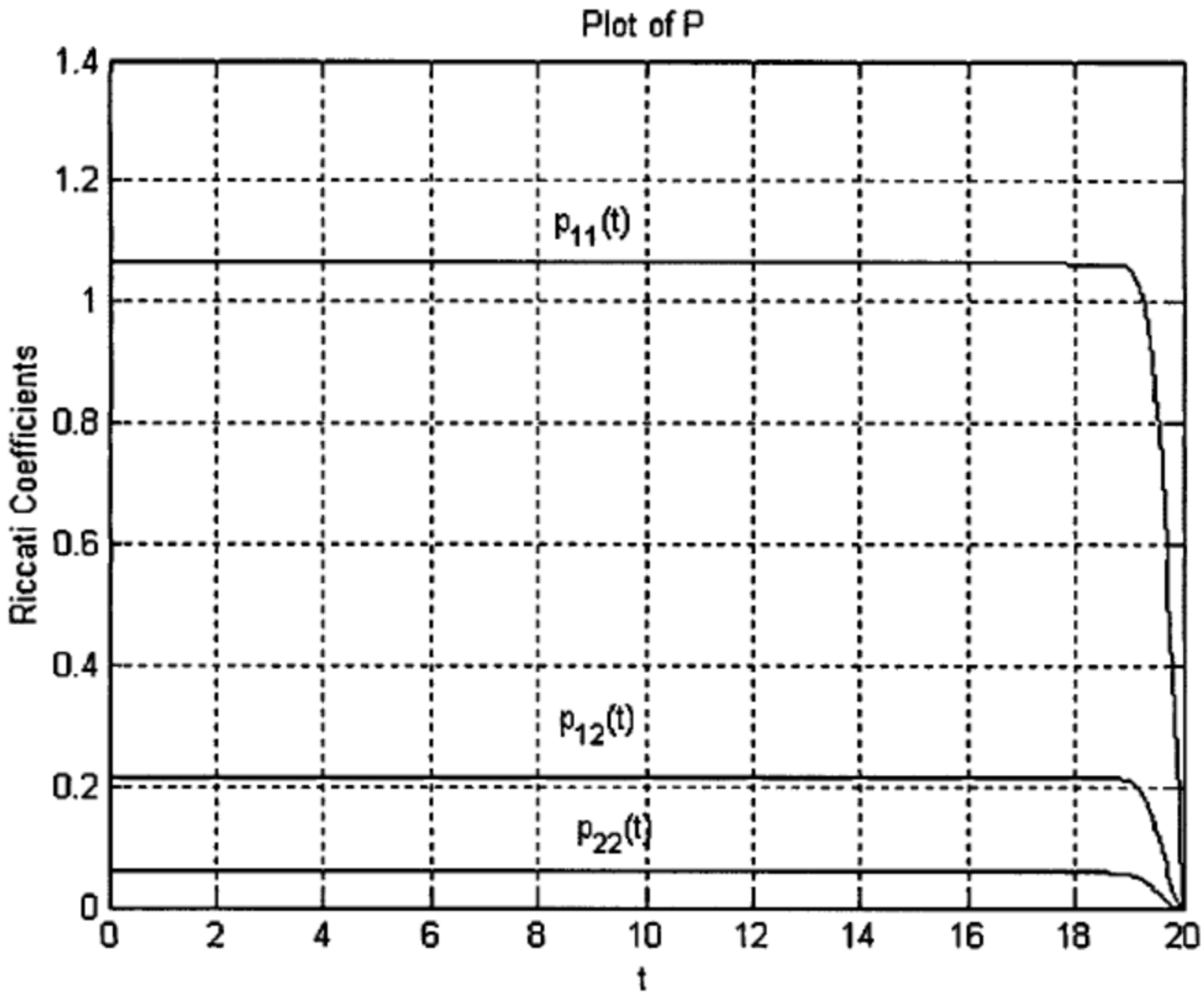


Figure 4.2 Riccati Coefficients for Example 4.1

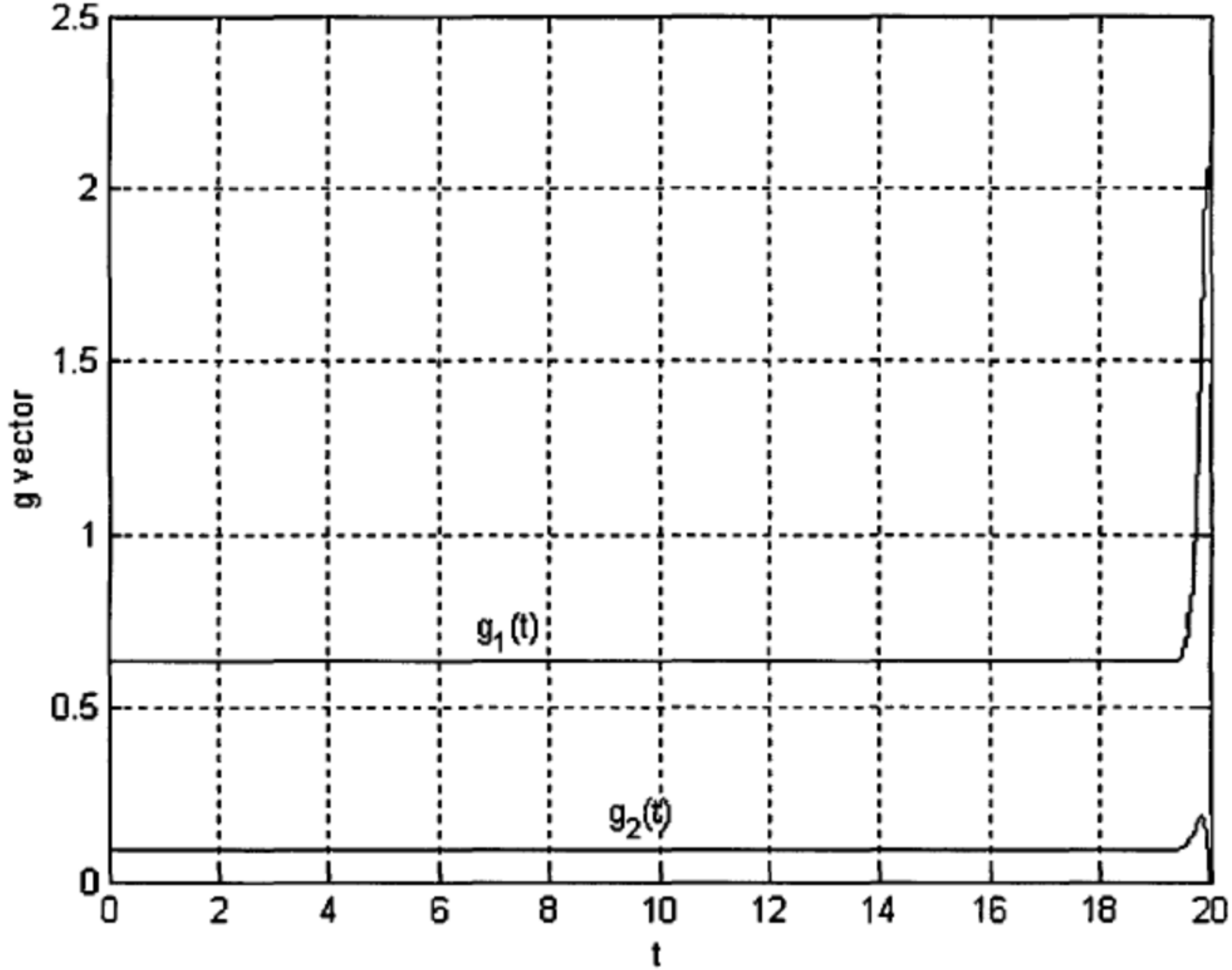


Figure 4.3 Coefficients $g_1(t)$ and $g_2(t)$ for Example 4.1

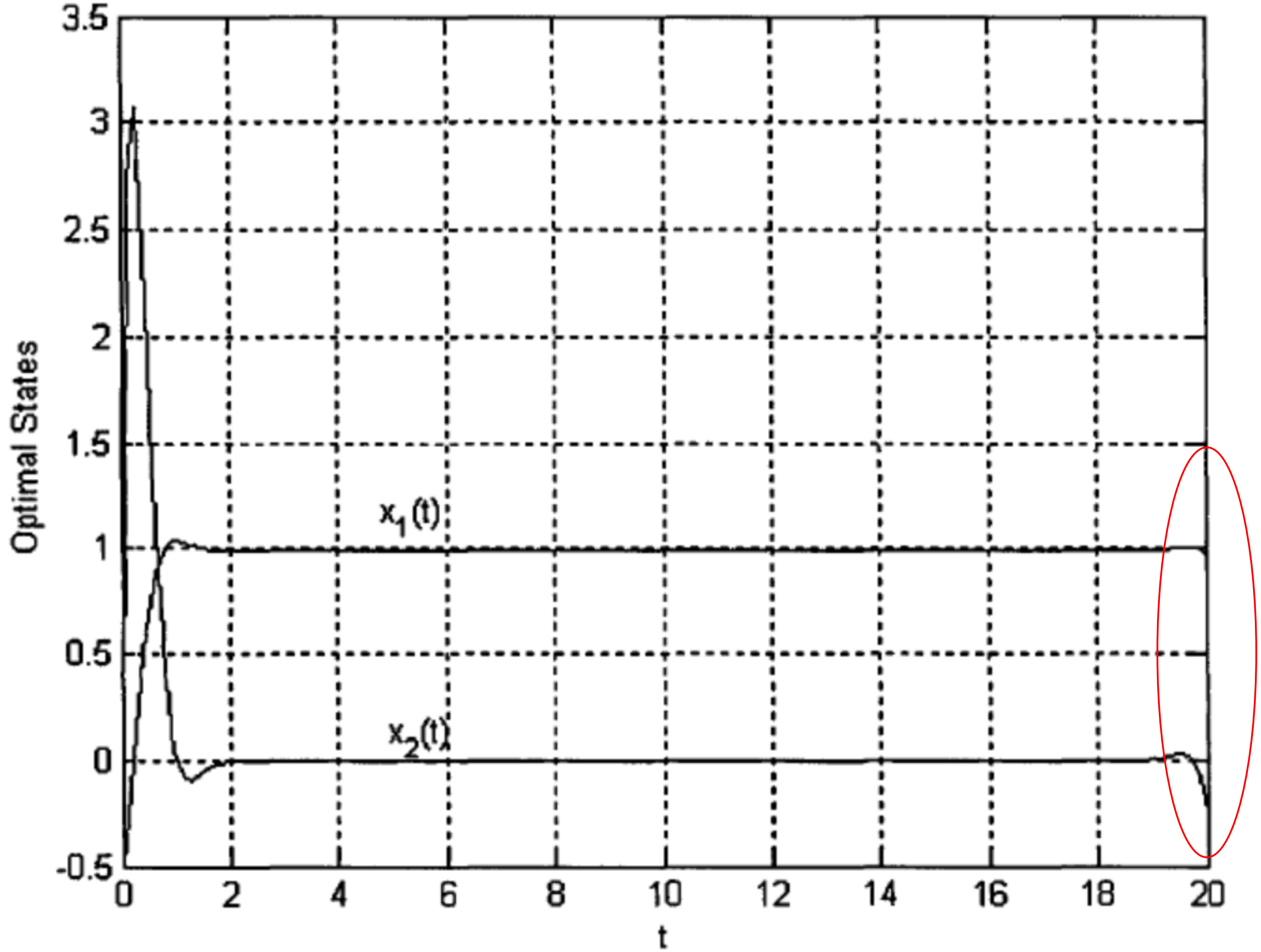


Figure 4.4 Optimal States for Example 4.1

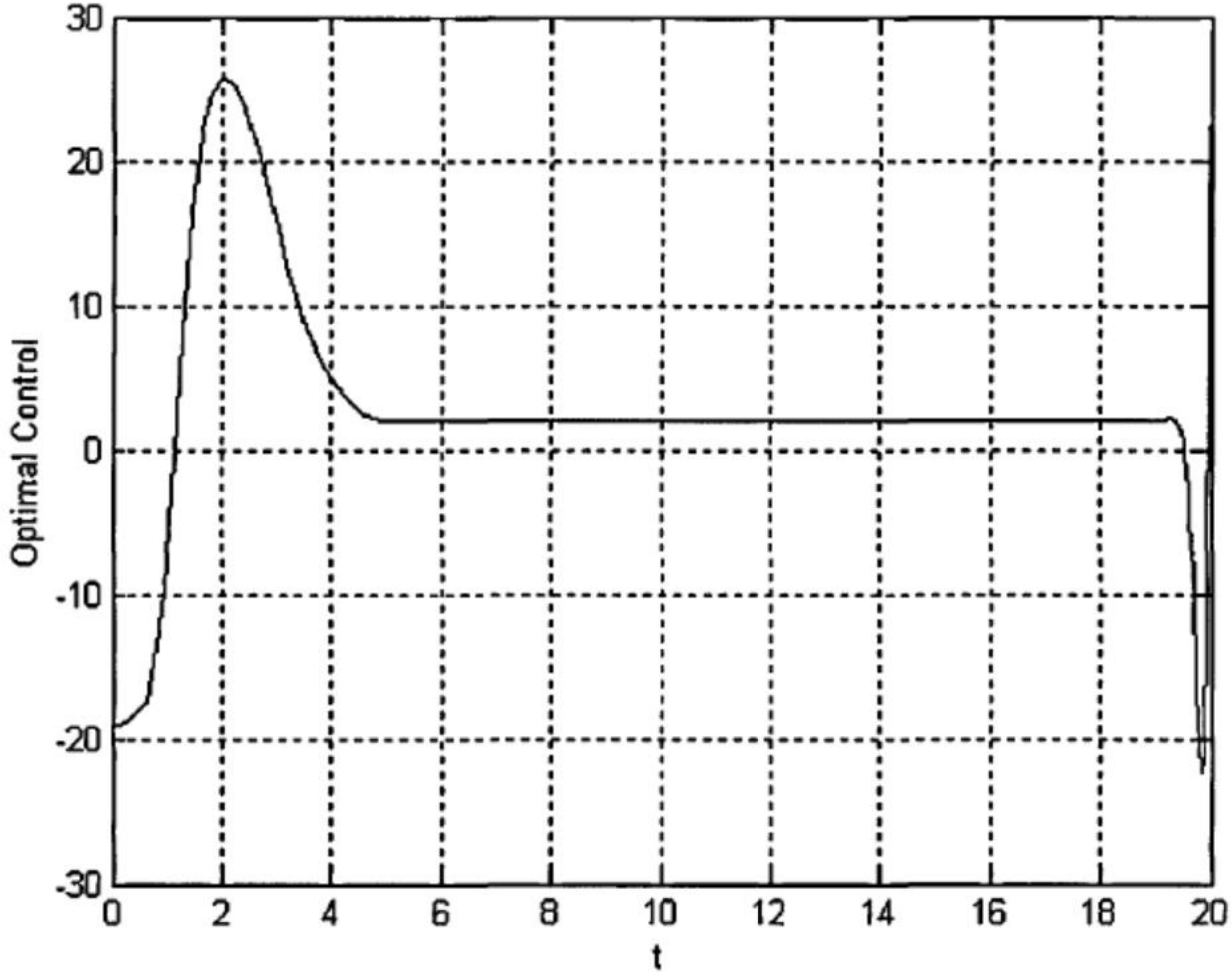


Figure 4.5 Optimal Control for Example 4.1

Example 4.2

Consider the same Example 4.1 with a different PI as

$$J = \int_{t_0}^{t_f} \left[[2t - x_1(t)]^2 + 0.02u^2(t) \right] dt \qquad (4.1.49)$$

where, t_f is specified and $\mathbf{x}(t_f)$ is free. Find the optimal control in order that the state $x_1(t)$ *track* a ramp function $z_1(t) = 2t$ and without much expenditure of control energy. Plot all the variables (Riccati coefficients, optimal states and control) for initial conditions $\mathbf{x}(0) = [-1 \quad 0]'$.

Solution:

$$\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{C}\mathbf{x}(t)$$
$$\mathbf{C} = \mathbf{I}$$

$$\Rightarrow$$

$$e_1(t) = z_1(t) - x_1(t)$$
$$e_2(t) = z_2(t) - x_2(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{C} = \mathbf{I}; \quad \mathbf{z}(t) = \begin{bmatrix} 2t \\ 0 \end{bmatrix};$$

$$\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{R} = r = 0.04. \qquad (4.1.50)$$

Let $\quad \mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}; \quad \mathbf{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}. \qquad (4.1.51)$

$$u^*(t) = -250\left[p_{12}x_1^*(t) + p_{22}x_2^*(t) - g_2(t)\right] \qquad (4.1.52)$$

Riccati equation

$$\begin{bmatrix} \dot{p}_{11}(t) & \dot{p}_{12}(t) \\ \dot{p}_{12}(t) & \dot{p}_{22}(t) \end{bmatrix} = -\begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{0.04}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.1.53)$$

$$\begin{cases} \dot{p}_{11}(t) = 25p_{12}^2(t) + 4p_{12}(t) - 2 \\ \dot{p}_{12}(t) = 25p_{12}(t)p_{22}(t) - p_{11}(t) + 3p_{12}(t) + 2p_{22}(t) \\ \dot{p}_{22}(t) = 25p_{22}^2(t) - 2p_{12}(t) + 6p_{22}(t) \end{cases} \qquad (4.1.55)$$

$$\begin{bmatrix} p_{11}(t_f) & p_{12}(t_f) \\ p_{12}(t_f) & p_{22}(t_f) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.1.56)$$

$$\begin{bmatrix} \dot{g}_1(t) \\ \dot{g}_2(t) \end{bmatrix} = -\left\{ \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{0.04} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \right\}' \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}' \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2t \\ 0 \end{bmatrix} \tag{4.1.54}$$

$$\begin{cases} \dot{g}_1(t) = [25p_{12}(t) + 2]\, g_2(t) - 4t \\ \dot{g}_2(t) = -g_1(t) + [3 + 25p_{22}(t)]\, g_2(t) \end{cases} \tag{4.1.57}$$

$$\begin{bmatrix} g_1(t_f) \\ g_2(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.1.58}$$

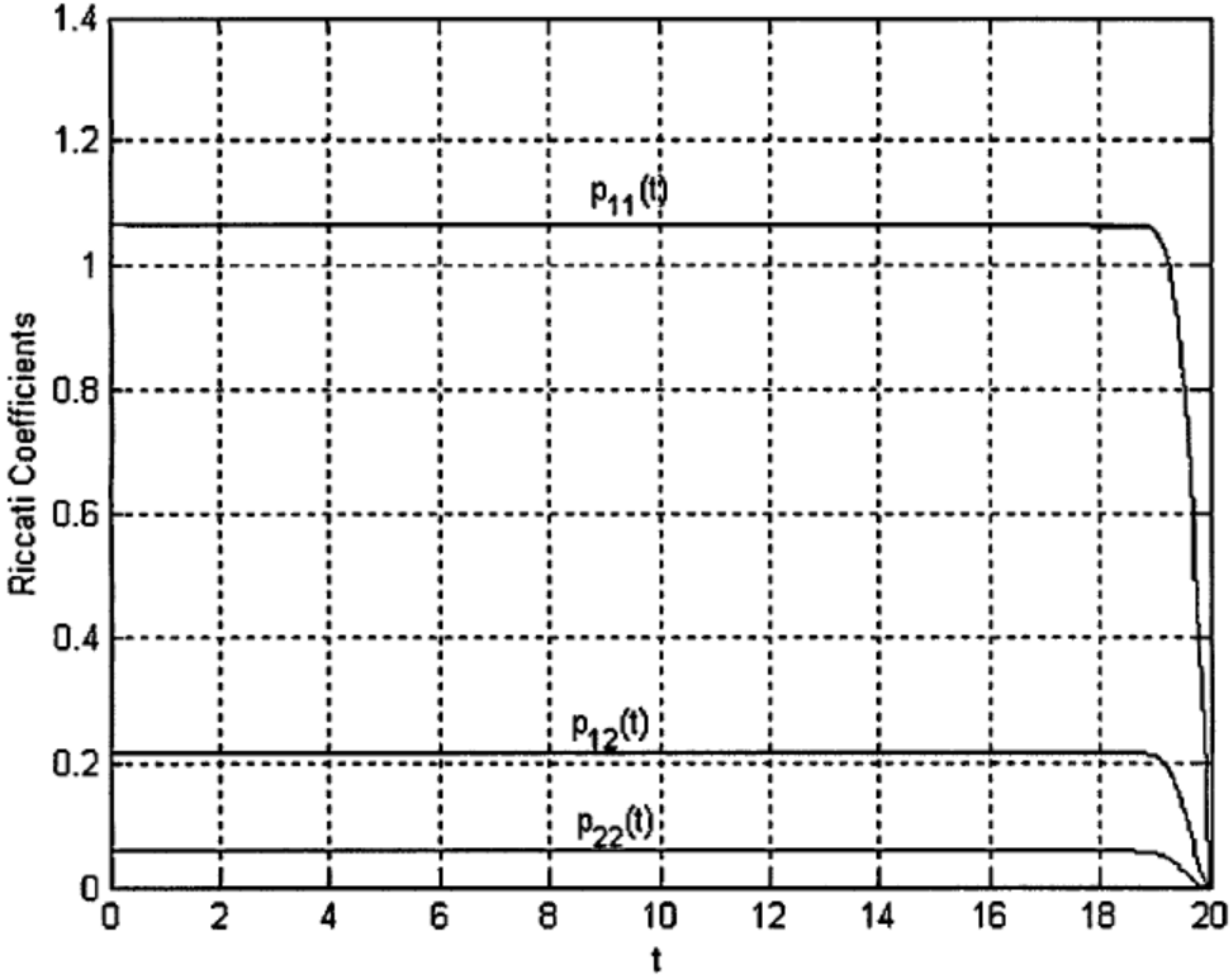


Figure 4.6 Riccati Coefficients for Example 4.2

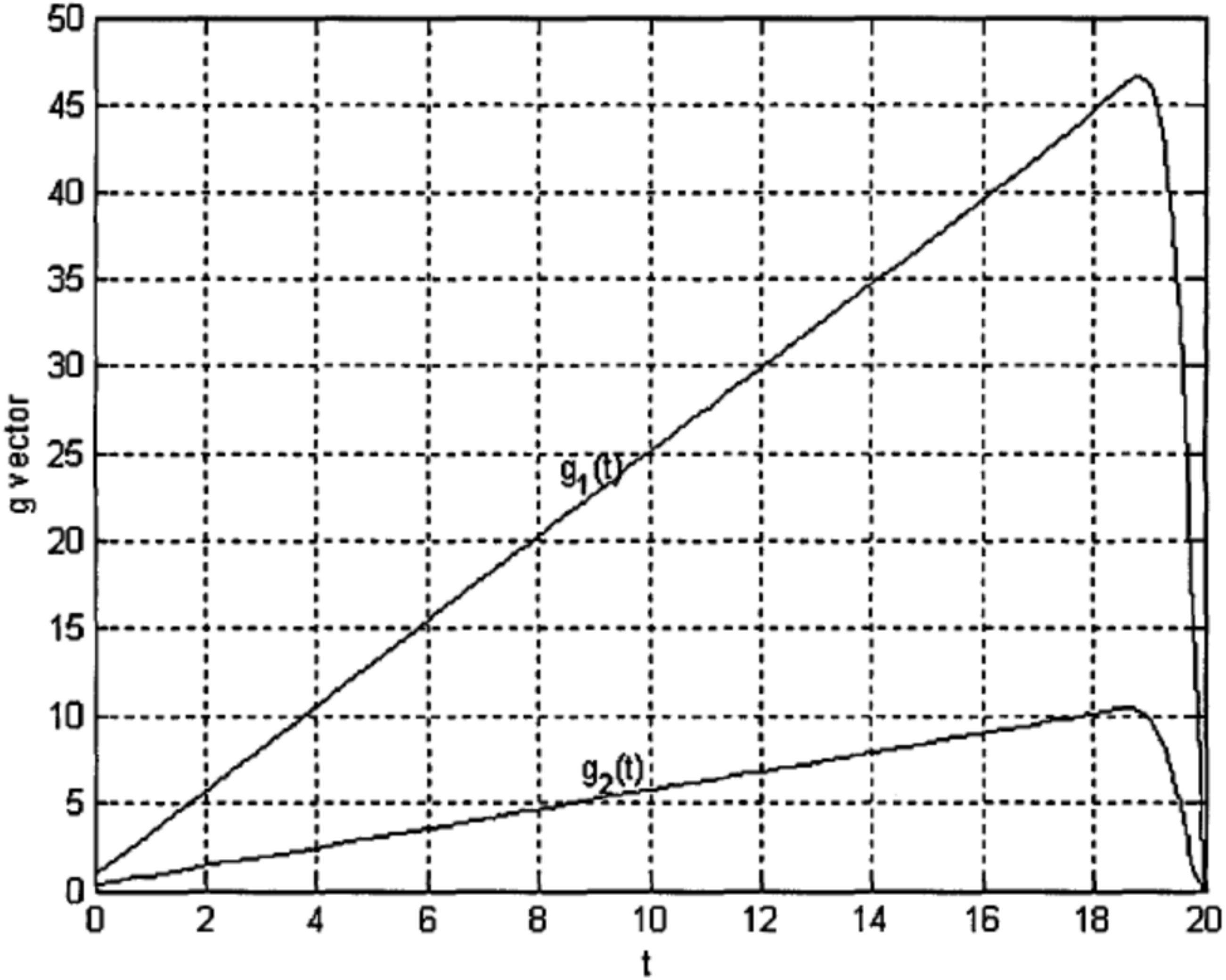


Figure 4.7 Coefficients $g_1(t)$ and $g_2(t)$ for Example 4.2

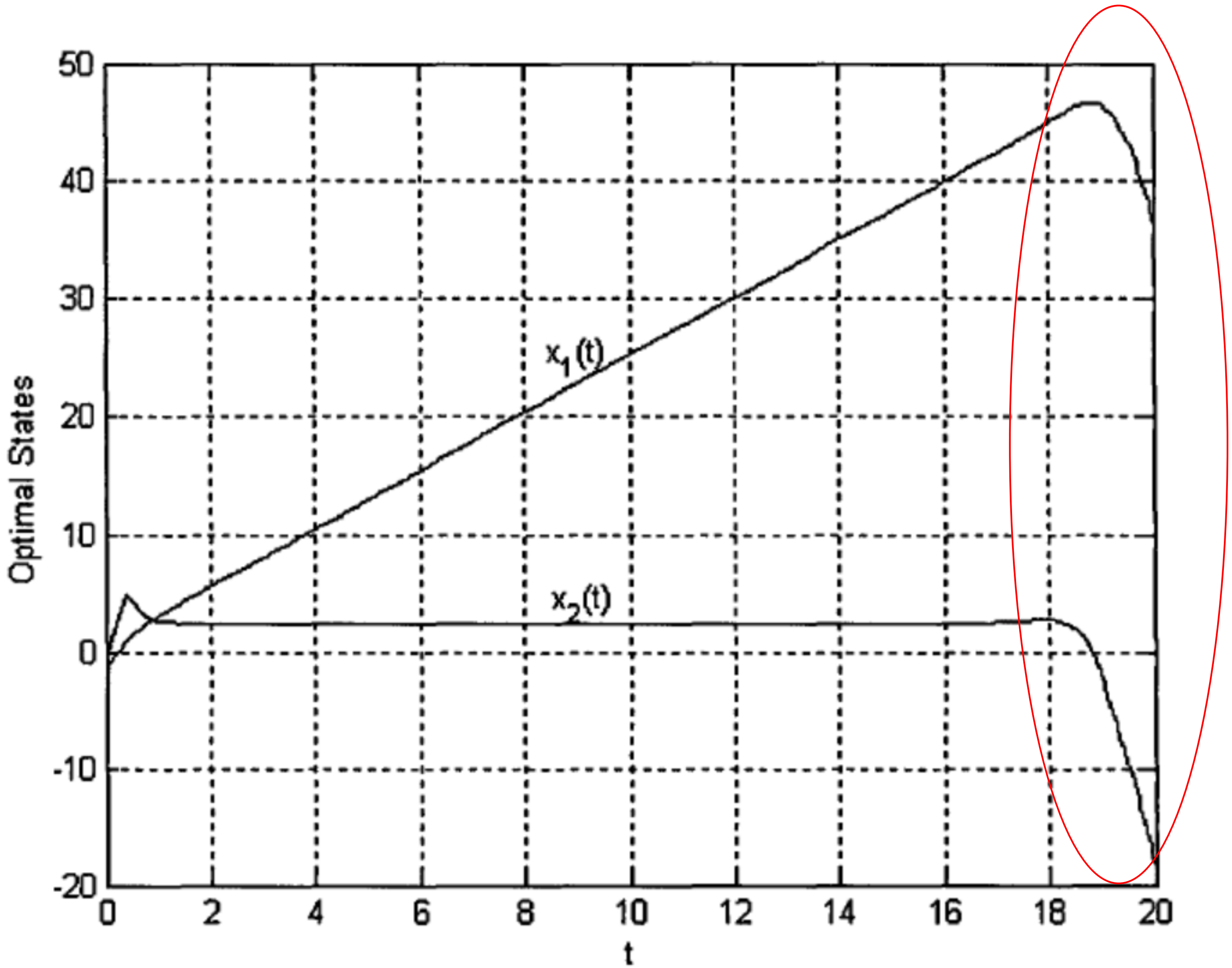


Figure 4.8 Optimal Control and States for Example 4.2

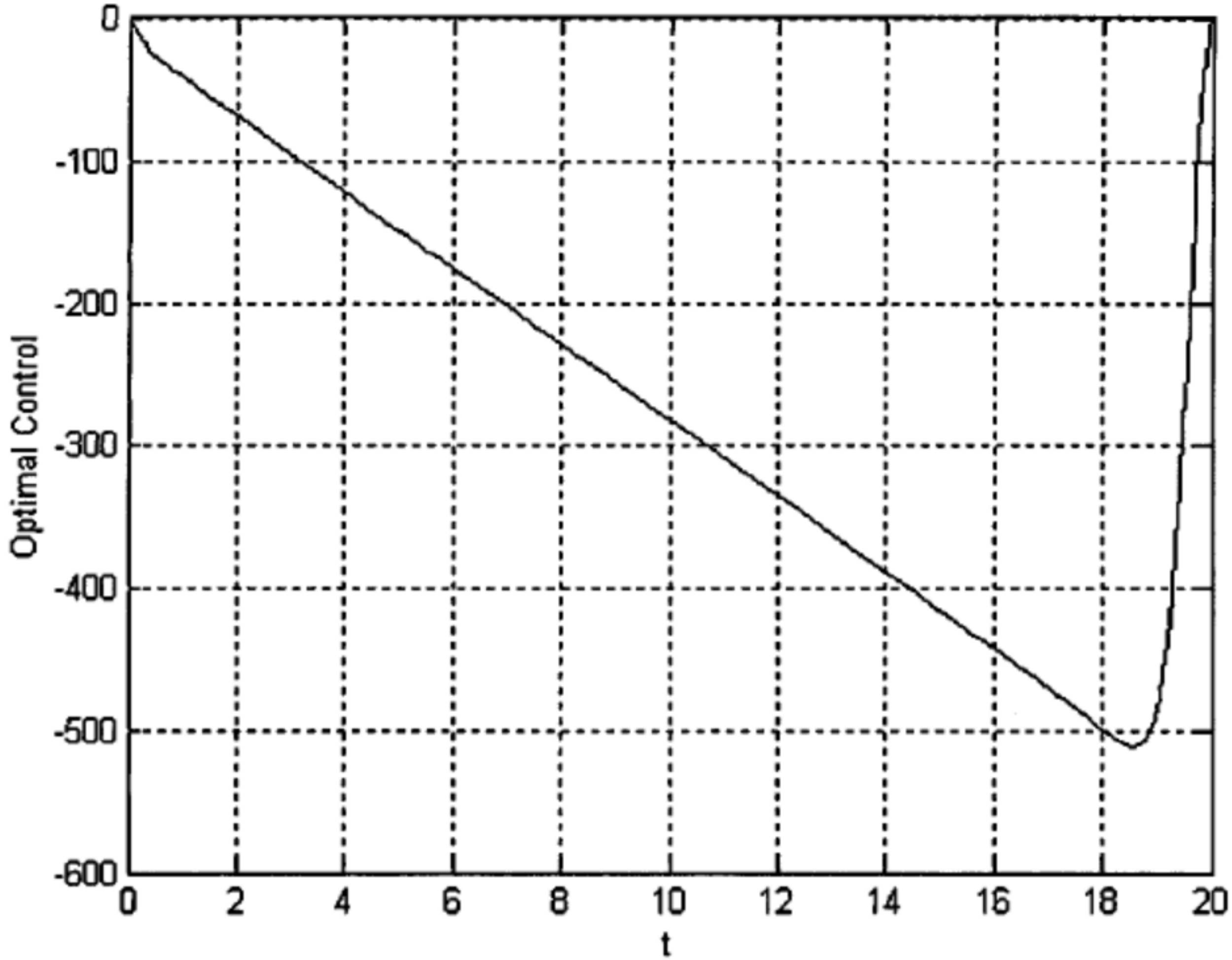


Figure 4.9 Optimal Control and States for Example 4.2

4.2 LQT System: Infinite-Time Case

We now extend the results of *finite-time* case of the linear quadratic tracking system to the case of *infinite time* [3].

Consider a linear *time-invariant* plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (4.2.1)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t). \qquad (4.2.2)$$

Error:

$$\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t), \qquad (4.2.3)$$

Performance index:

$$\lim_{t_f\to\infty} J = \lim_{t_f\to\infty} \frac{1}{2}\int_0^\infty \left[\mathbf{e}'(t)\mathbf{Q}\mathbf{e}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right] dt \qquad (4.2.4)$$

We assume that

Q is an $n\times n$ symmetric, positive *semidefinite matrix*, and

R is a $r\times r$ symmetric, positive *definite* matrix.

Note that there is no terminal cost function in the PI (4.2.4) and hence **F** = 0.

as $t_f \to \infty$,

$$\boxed{\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}'(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{E}(t)\mathbf{P}(t) - \mathbf{V}(t)} \qquad (4.1.19)$$

$$\boxed{-\bar{\mathbf{P}}\mathbf{A} - \mathbf{A}'\bar{\mathbf{P}} + \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} - \mathbf{C}'\mathbf{Q}\mathbf{C} = 0.} \qquad (4.2.5)$$

$$\boxed{\dot{\mathbf{g}}(t) = [\mathbf{P}(t)\mathbf{E}(t) - \mathbf{A}'(t)]\,\mathbf{g}(\mathbf{t}) - \mathbf{W}(\mathbf{t})\mathbf{z}(\mathbf{t}).} \qquad (4.1.20)$$

$$\dot{\bar{\mathbf{g}}}(t) = \left[\bar{\mathbf{P}}\mathbf{E} - \mathbf{A}'\right]\bar{\mathbf{g}}(t) - \mathbf{W}\mathbf{z}(t) \qquad (4.2.6)$$

where, $\mathbf{E} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'$ and $\mathbf{W} = \mathbf{C}'\mathbf{Q}$.

optimal control

$$\begin{aligned}\mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\,[\mathbf{P}(t)\mathbf{x}^*(t) - \mathbf{g}(t)] \\ &= -\mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{g}(t) \qquad (4.1.31)\end{aligned}$$

$$\boxed{\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}'\left[\bar{\mathbf{P}}\mathbf{x}(t) - \bar{\mathbf{g}}(t)\right].} \qquad (4.2.7)$$

4.3 Fixed-End-Point Regulator System

- In this section, we discuss the *fixed-end-point* state regulator system, where the final state $\mathbf{x}(t_f)$ is zero and the final time t_f is *fixed* [5].
- This is different from the conventional *free-end-point* state regulator system with the final time t_f being *free*, leading to the matrix Riccati differential equation that was discussed in Chapter 3.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{4.3.3}$$

$$J(\mathbf{u}) = \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \tag{4.3.4}$$

$$\mathbf{x}(t = t_0) = \mathbf{x}_0; \qquad \mathbf{x}(t = t_f) = \mathbf{x}_f = 0 \tag{4.3.5}$$

where, t_f is ~~*fixed* or given *a priori*.~~
free

$$\boldsymbol{\lambda}(t_f) = \mathbf{F}(t_f)\mathbf{x}(t_f) = \mathbf{P}(t_f)\mathbf{x}(t_f) \qquad (4.3.1)$$

$$\left.\begin{array}{l}\text{for the fixed final condition } \mathbf{x}(t_f) = 0 \\ \text{for arbitrary } \boldsymbol{\lambda}(t_f)\end{array}\right\} \Rightarrow \mathbf{P}(t_f) = \infty \qquad (4.3.2)$$

Alternatively, from Table 2.1 in Chapter 2, this belongs to type (b):

(b). Free-final time and fixed-final state system, Fig. 2.9(b)		
(b)	$\delta t_f \neq 0, \delta \mathbf{x}_f = 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \mathbf{x}_f, \left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0$

$\Rightarrow$ $H(t_f) = 0$

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)) = \frac{1}{2}\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \frac{1}{2}\mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t) + \boldsymbol{\lambda}'(t)\left[\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\right] \qquad (3.2.3)$$

$\Rightarrow$ $\lambda(t_f)$ is arbitrary.

$$\mathbf{P}(t_f) = \infty$$

- This means that for the fixed-end-point regulator system, we solve the matrix DRE (3.2.18) using the final condition (4.3.2).
- In practice, we may start with a very large value of $\mathbf{P}(t_f)$ instead of ∞.

- Alternatively, we present a different procedure to find *closed-loop optimal control* for the fixed-end-point system [5].
- In fact, we will use what is known as *inverse* Riccati transformation between the state and costate variables and arrive at matrix *inverse Riccati* equation.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad (4.3.3)$$

$$J(\mathbf{u}) = \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \qquad (4.3.4)$$

$$\mathbf{x}(t = t_0) = \mathbf{x}_0; \qquad \mathbf{x}(t = t_f) = \mathbf{x}_f = 0 \qquad (4.3.5)$$

where, t_f is *fixed* or given *a priori*.

fixed

We develop the procedure for ~~free~~-end-point regulator system under the following steps (see Table 2.1).

- **Step 1:** *Hamiltonian*
- **Step 2:** *Optimal Control*
- **Step 3:** *State and Costate System*
- **Step 4:** *Closed-Loop Optimal Control*
- **Step 5:** *Boundary Conditions*

- **Step 1:** *Hamiltonian:*

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)) = \frac{1}{2}\mathbf{x}'(t)\mathbf{Q}(t)\mathbf{x}(t) + \frac{1}{2}\mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t) + \boldsymbol{\lambda}'(t)\left[\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\right] \qquad (4.3.6)$$

- **Step 2:** *Optimal Control:*

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0 \longrightarrow \mathbf{R}(t)\mathbf{u}(t) + \mathbf{B}'(t)\boldsymbol{\lambda}(t) = 0 \qquad (4.3.7)$$

$$\Rightarrow \quad \mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\boldsymbol{\lambda}^*(t). \qquad (4.3.8)$$

- **Step 3:** *State and Costate System:*

$$\left\{\begin{array}{ll} \dot{\mathbf{x}}^*(t) = +\dfrac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}} \longrightarrow \dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t), & (4.3.9) \\ \dot{\boldsymbol{\lambda}}^*(t) = -\dfrac{\partial \mathcal{H}}{\partial \mathbf{x}} \longrightarrow \dot{\boldsymbol{\lambda}}^*(t) = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}'(t)\boldsymbol{\lambda}^*(t). & (4.3.10) \end{array}\right.$$

$$\Rightarrow \quad \begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ \dot{\boldsymbol{\lambda}}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & -\mathbf{E}(t) \\ -\mathbf{Q}(t) & -\mathbf{A}'(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ \boldsymbol{\lambda}^*(t) \end{bmatrix} \qquad (4.3.11)$$

where, $\mathbf{E}(t) = \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)$

- **Step 4:** *Closed-Loop Optimal Control:*

$$\boldsymbol{\lambda}^*(t) = \mathbf{P}(t)\mathbf{x}^*(t). \tag{4.3.12}$$

$$\Rightarrow \quad \mathbf{x}^*(t) = \mathbf{M}(t)\boldsymbol{\lambda}^*(t) \tag{4.3.13}$$

$$\Rightarrow \quad \dot{\mathbf{x}}^*(t) = \dot{\mathbf{M}}(t)\boldsymbol{\lambda}^*(t) + \mathbf{M}(t)\dot{\boldsymbol{\lambda}}^*(t)$$

$$\begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ \dot{\boldsymbol{\lambda}}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & -\mathbf{E}(t) \\ -\mathbf{Q}(t) & -\mathbf{A}'(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}^*(t) \\ \boldsymbol{\lambda}^*(t) \end{bmatrix} \tag{4.3.11}$$

$$\Rightarrow \quad \left[\dot{\mathbf{M}}(t) - \mathbf{A}(t)\mathbf{M}(t) - \mathbf{M}(t)\mathbf{A}'(t) - \mathbf{M}(t)\mathbf{Q}(t)\mathbf{M}(t) + \mathbf{B}(t)\mathbf{R}^{-1}\mathbf{B}'(t)\right]\boldsymbol{\lambda}^*(t) = 0 \tag{4.3.14}$$

$$\Rightarrow \quad \boxed{\dot{\mathbf{M}}(t) = \mathbf{A}(t)\mathbf{M}(t) + \mathbf{M}(t)\mathbf{A}'(t) + \mathbf{M}(t)\mathbf{Q}(t)\mathbf{M}(t) - \mathbf{B}(t)\mathbf{R}^{-1}\mathbf{B}'(t).} \tag{4.3.15}$$

Inverse matrix differential Riccati equation (DRE)

- **Step 5:** *Boundary Conditions*:

1. $\mathbf{x}(t_f) = 0$ *and* $\mathbf{x}(t_0) \neq 0$:

$$\mathbf{x}(t_f) = 0 = \mathbf{M}(t_f)\boldsymbol{\lambda}(t_f). \qquad (4.3.16)$$

$$\boxed{\mathbf{M}(t_f) = 0.} \qquad (4.3.17)$$

Solving the inverse matrix DRE (4.3.15) *backward*

2. $\mathbf{x}(t_f) \neq 0$ *and* $\mathbf{x}(t_0) = 0$:

$$\mathbf{x}(t_0) = 0 = \mathbf{M}(t_0)\boldsymbol{\lambda}(t_0) \qquad (4.3.18)$$

$$\boxed{\mathbf{M}(t_0) = 0.} \qquad (4.3.19)$$

Solving the inverse matrix DRE (4.3.15) *forward*

The optimal control (4.3.8) with the transformation (4.3.13) becomes

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\mathbf{M}^{-1}(t)\mathbf{x}^*(t).} \qquad (4.3.20)$$

3. *General Boundary Conditions:* $\mathbf{x}(t_0) \neq 0$ and $\mathbf{x}(t_f) \neq 0$

Assume
$$\mathbf{x}^*(t) = \mathbf{M}(t)\boldsymbol{\lambda}^*(t) + \mathbf{v}(t). \qquad (4.3.21)$$

$$\Rightarrow \quad \dot{\mathbf{x}}^*(t) = \dot{\mathbf{M}}(t)\boldsymbol{\lambda}^*(t) + \mathbf{M}(t)\dot{\boldsymbol{\lambda}}^*(t) + \dot{\mathbf{v}}(t) \qquad (4.3.22)$$

$$\Rightarrow \quad \begin{aligned} &\mathbf{A}(t)\left[\mathbf{M}(t)\boldsymbol{\lambda}^*(t) + \mathbf{v}(t)\right] - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\boldsymbol{\lambda}^*(t) \\ &= \dot{\mathbf{M}}(t)\boldsymbol{\lambda}^*(t) + \mathbf{M}(t)\left[-\mathbf{Q}(t)\left[\mathbf{M}(t)\boldsymbol{\lambda}^*(t) + \mathbf{v}(t)\right]\right. \\ &\qquad \left. - \mathbf{A}'(t)\boldsymbol{\lambda}^*(t)\right] + \dot{\mathbf{v}}(t) \end{aligned} \qquad (4.3.23)$$

$$\Rightarrow \quad \begin{aligned} &\left[\dot{\mathbf{M}}(t) - \mathbf{A}(t)\mathbf{M}(t) - \mathbf{M}(t)\mathbf{A}'(t) - \mathbf{M}(t)\mathbf{Q}(t)\mathbf{M}(t) + \right. \\ &\qquad \left. \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t)\right]\boldsymbol{\lambda}^*(t) + \\ &\qquad \left[\dot{\mathbf{v}}(t) - \mathbf{M}(t)\mathbf{Q}(t)\mathbf{v}(t) - \mathbf{A}(t)\mathbf{v}(t)\right] = 0. \end{aligned} \qquad (4.3.24)$$

$$\Rightarrow \quad \left\{ \begin{aligned} \dot{\mathbf{M}}(t) &= \mathbf{A}(t)\mathbf{M}(t) + \mathbf{M}(t)\mathbf{A}'(t) + \mathbf{M}(t)\mathbf{Q}(t)\mathbf{M}(t) \\ &\quad - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t) \qquad (4.3.25) \\ \dot{\mathbf{v}}(t) &= \mathbf{M}(t)\mathbf{Q}(t)\mathbf{v}(t) + \mathbf{A}(t)\mathbf{v}(t). \qquad (4.3.26) \end{aligned} \right.$$

$$\text{At } t = t_0, \quad \mathbf{x}^*(t_0) = \mathbf{M}(t_0)\boldsymbol{\lambda}^*(t_0) + \mathbf{v}(t_0). \tag{4.3.27}$$

$$\Rightarrow \quad \mathbf{M}(t_0) = 0; \qquad \mathbf{v}(t_0) = \mathbf{x}(t_0). \tag{4.3.28}$$

$$\text{At } t = t_f, \quad \mathbf{x}^*(t_f) = \mathbf{M}(t_f)\boldsymbol{\lambda}^*(t_f) + \mathbf{v}(t_f). \tag{4.3.29}$$

$$\Rightarrow \quad \mathbf{M}(t_f) = 0; \qquad \mathbf{v}(t_f) = \mathbf{x}(t_f). \tag{4.3.30}$$

Thus, the set of the equations (4.3.25) and (4.3.26) are solved using either the *initial* conditions (4.3.28) or *final* conditions (4.3.30).

Thus,

$$\dot{\mathbf{M}}(t) = \mathbf{A}(t)\mathbf{M}(t) + \mathbf{M}(t)\mathbf{A}'(t) + \mathbf{M}(t)\mathbf{Q}(t)\mathbf{M}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}'(t) \qquad (4.3.25)$$

$$\dot{\mathbf{v}}(t) = \mathbf{M}(t)\mathbf{Q}(t)\mathbf{v}(t) + \mathbf{A}(t)\mathbf{v}(t). \qquad (4.3.26)$$

B.C.: $\mathbf{M}(t_0) = 0; \quad \mathbf{v}(t_0) = \mathbf{x}(t_0). \qquad (4.3.28)$

or $\mathbf{M}(t_f) = 0; \quad \mathbf{v}(t_f) = \mathbf{x}(t_f). \qquad (4.3.30)$

Finally, using the transformation (4.3.21) in the optimal control (4.3.8), the closed-loop optimal control is given by

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}'(t)\underbrace{\mathbf{M}^{-1}(t)[\mathbf{x}^*(t) - \mathbf{v}(t)]}_{\lambda(t)} \qquad (4.3.31)$$

where, it is assumed that $\mathbf{M}(t)$ is invertible.

Example 4.3

$$\dot{x}(t) = ax(t) + bu(t), \qquad (4.3.32)$$

$$J = \frac{1}{2}\int_{t_0}^{t_f} [qx^2(t) + ru^2(t)]dt, \qquad (4.3.33)$$

$$x(t=0) = x_0; \qquad x(t=t_f) = 0,$$ (Step 5, Case 1)

Solution:

$$u^*(t) = -r^{-1}bm^{-1}(t)x^*(t) \qquad (4.3.35)$$

$$\dot{m}(t) = 2am(t) + m^2(t)q - \frac{b^2}{r} \qquad (4.3.36)$$ (Inverse DRE)

B.C. $m(t_f) = 0.$

$$\Rightarrow \quad m(t) = \frac{b^2}{r}\left[\frac{e^{-\beta(t-t_f)} - e^{\beta(t-t_f)}}{(a+\beta)e^{-\beta(t-t_f)} - (a-\beta)e^{\beta(t-t_f)}}\right] \qquad (4.3.37)$$

where, $\beta = \sqrt{a^2 + q\frac{b^2}{r}}$

$$\Rightarrow \quad u^*(t) = \frac{1}{b}\left[\frac{(a+\beta)e^{-\beta(t-t_f)} - (a-\beta)e^{\beta(t-t_f)}}{e^{-\beta(t-t_f)} - e^{\beta(t-t_f)}}\right]x^*(t). \qquad (4.3.38)$$

4.4 LQR with a Specified Degree of Stability

- In this section, we examine the state regulator system with infinite time interval and with a specified degree of stability for a time-invariant system [3, 2].

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t); \qquad \mathbf{x}(t = t_0) = \mathbf{x}(0), \tag{4.4.1}$$

$$J = \frac{1}{2}\int_{t_0}^{\infty} e^{2\alpha t}\left[\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right] dt \tag{4.4.2}$$

where, α is a positive parameter.

- Here, we first assume that the pair [$A+\alpha I$, B] is completely *stabilizable* and **R** and **Q** are constant, symmetric, positive *definite* and positive *semidefinite* matrices, respectively.
- The problem is to find the optimal control which minimizes the performance index (4.4.2) under the dynamical constraint (4.4.1).

Let $$\hat{\mathbf{x}}(t) = e^{\alpha t}\mathbf{x}(t); \qquad \hat{\mathbf{u}}(t) = e^{\alpha t}\mathbf{u}(t). \tag{4.4.3}$$

⇨ $$\dot{\hat{\mathbf{x}}}(t) = \frac{d}{dt}\{e^{\alpha t}\mathbf{x}(t)\} = \alpha e^{\alpha t}\mathbf{x}(t) + e^{\alpha t}\dot{\mathbf{x}}(t)$$

$$= \alpha\hat{\mathbf{x}}(t) + e^{\alpha t}[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]$$

$$\underline{\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} + \alpha\mathbf{I})\hat{\mathbf{x}}(t) + \mathbf{B}\hat{\mathbf{u}}(t).} \tag{4.4.4}$$

$$\hat{\mathbf{x}}(t_0) = e^{\alpha t_0}\mathbf{x}(t_0)$$ (I.C.)

⇨ $$\hat{J} = \frac{1}{2}\int_{t_0}^{\infty}[\hat{\mathbf{x}}'(t)\mathbf{Q}\hat{\mathbf{x}}(t) + \hat{\mathbf{u}}'(t)\mathbf{R}\hat{\mathbf{u}}(t)]\,dt. \tag{4.4.5}$$

Optimal control: $$\hat{\mathbf{u}}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\hat{\mathbf{x}}^*(t) = -\bar{\mathbf{K}}\hat{\mathbf{x}}^*(t) \tag{4.4.6}$$

where, $\bar{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}$ and the matrix $\bar{\mathbf{P}}$ is the positive definite, symmetric solution of the algebraic Riccati equation

$$\bar{\mathbf{P}}\underline{(\mathbf{A} + \alpha\mathbf{I})} + \underline{(\mathbf{A}' + \alpha\mathbf{I})}\bar{\mathbf{P}} - \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} + \mathbf{Q} = 0. \tag{4.4.7}$$

Closed-loop system: $$\dot{\hat{\mathbf{x}}}^*(t) = (\mathbf{A} + \alpha\mathbf{I} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}})\hat{\mathbf{x}}^*(t). \tag{4.4.8}$$

$$\mathbf{u}^*(t) = e^{-\alpha t}\hat{\mathbf{u}}^*(t) = -e^{-\alpha t}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}e^{\alpha t}\mathbf{x}^*(t)$$
$$= -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t) = -\bar{\mathbf{K}}\mathbf{x}^*(t). \qquad (4.4.9)$$

This desired (original) optimal control (4.4.9) has the same structure as the optimal control (4.4.6) of the modified system.

$$\hat{J}^* = \frac{1}{2}\hat{\mathbf{x}}^{*\prime}(t_0)\bar{\mathbf{P}}\hat{\mathbf{x}}^*(t_0)$$
$$J^* = \frac{1}{2}e^{2\alpha t_0}\mathbf{x}^{*\prime}(t_0)\bar{\mathbf{P}}\mathbf{x}^*(t_0). \qquad (4.4.10)$$

- We see that the closed-loop optimal control system (4.4.8) has eigenvalues with real parts less than -*a*.
- In other words, the state $\mathbf{x}^*(t)$ approaches zero at least as fast as $e^{-\alpha t}$.
- Then, we say that the closed-loop optimal system (4.4.8) has a degree of stability of at least *a*.

4.4.1 Regulator System with Prescribed Degree of Stability: Summary

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{4.4.11}$$

$$J = \frac{1}{2}\int_{t_0}^{\infty} e^{2\alpha t}\left[\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right] dt, \tag{4.4.12}$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t) = -\bar{\mathbf{K}}\mathbf{x}^*(t) \tag{4.4.13}$$

where, $\bar{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}$

ARE: $$\bar{\mathbf{P}}(\mathbf{A} + \alpha\mathbf{I}) + (\mathbf{A}' + \alpha\mathbf{I})\bar{\mathbf{P}} - \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} + \mathbf{Q} = 0, \tag{4.4.14}$$

$$\dot{\mathbf{x}}^*(t) = \left(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\right)\mathbf{x}^*(t), \tag{4.4.15}$$

$$J^* = \frac{1}{2}e^{2\alpha t_0}\mathbf{x}^{*\prime}(t_0)\bar{\mathbf{P}}\mathbf{x}^*(t_0). \tag{4.4.16}$$

Table 4.2 Procedure Summary of Regulator System with Prescribed Degree of Stability

A. Statement of the Problem	
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, the performance index as $J = \frac{1}{2}\int_{t_0}^{\infty} e^{2\alpha t}\left[\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right]dt$, and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0; \quad \mathbf{x}(\infty) = 0$, find the optimal control, state and index.	
B. Solution of the Problem	
Step 1	Solve the matrix algebraic Riccati equation $\bar{\mathbf{P}}(\mathbf{A} + \alpha\mathbf{I}) + (\mathbf{A}' + \alpha\mathbf{I})\bar{\mathbf{P}} + \mathbf{Q} - \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} = 0$.
Step 2	Solve the optimal state $\mathbf{x}^*(t)$ from $\dot{\mathbf{x}}^*(t) = \left(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\right)\mathbf{x}^*(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.
Step 3	Obtain the optimal control $\mathbf{u}^*(t)$ from $\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t)$.
Step 4	Obtain the optimal performance index from $J^* = \frac{1}{2}e^{2\alpha t_0}\mathbf{x}^{*\prime}(t_0)\bar{\mathbf{P}}\mathbf{x}^*(t_0)$.

Example 4.4

Consider a first-order system

$$\dot{x}(t) = -x(t) + u(t), \qquad x(0) = 1 \tag{4.4.17}$$

and a performance measure

$$J = \frac{1}{2}\int_0^\infty e^{2\alpha t}[x^2(t) + u^2(t)]dt. \tag{4.4.18}$$

Find the optimal control law and show that the closed-loop optimal system has a degree of stability of at least α.

Solution: Essentially, we show that the eigenvalue of this closed-loop optimal system is less than or equal to $-\alpha$. First of all, in the above, we note that $A = a = -1, B = b = 1, Q = q = 1$ and $R = r = 1$. Then, the algebraic Riccati equation (4.4.14) becomes

$$2\bar{p}(\alpha - 1) - \bar{p}^2 + 1 = 0 \quad \longrightarrow \bar{p}^2 - 2\bar{p}(\alpha - 1) - 1. \tag{4.4.19}$$

Solving the previous for positive value of $\bar{p}$, we have

$$\bar{p} = -1 + \alpha + \sqrt{(\alpha - 1)^2 + 1}. \qquad > 0 \tag{4.4.20}$$

The optimal control (4.4.15) becomes

$$u^*(t) = -\bar{p}x^*(t). \tag{4.4.21}$$

The optimal system (4.4.22) becomes

$$\dot{x}^*(t) = \left(-\alpha - \sqrt{(\alpha - 1)^2 + 1}\right) x^*(t). \tag{4.4.22}$$

It is easy to see that the eigenvalue for the system (4.4.22) is related as

$$-\alpha - \sqrt{(\alpha - 1)^2 + 1} < -\alpha. \qquad (4.4.23)$$

This shows the desired result that the optimal system has the eigenvalue less than α.

- Eigenvalue:

  Open loop: -1

  Closed loop: $-\sqrt{2}$

  With α: $-\alpha - \sqrt{(\alpha - 1)^2 + 1}$

Ex. $\alpha = 1 \Rightarrow$ eigenvalue = $-2 < -\sqrt{2} < -1$

4.5 Frequency-Domain Interpretation

- In this section, we use frequency domain to derive some results from the classical control point of view for a linear, time-invariant, continuous-time, optimal control system with infinite-time horizon case.

Consider a controllable, linear, time-invariant plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{4.5.1}$$

the infinite-time interval cost functional

$$J = \frac{1}{2}\int_0^\infty \left[\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right] dt. \tag{4.5.2}$$

optimal control

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t) = -\bar{\mathbf{K}}\mathbf{x}^*(t), \tag{4.5.3}$$

where, $\bar{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}$

matrix ARE

$$-\bar{\mathbf{P}}\mathbf{A} - \mathbf{A}'\bar{\mathbf{P}} + \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} - \mathbf{Q} = 0. \tag{4.5.4}$$

optimal trajectory (state)

$$\dot{\mathbf{x}}^*(t) = \left(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\right)\mathbf{x}^*(t) = (\mathbf{A} - \mathbf{B}\bar{\mathbf{K}})\,\mathbf{x}^*(t), \tag{4.5.5}$$

which is asymptotically stable.

we assume that $[\mathbf{A}, \mathbf{B}]$ is stabilizable and $[\mathbf{A}, \sqrt{\mathbf{Q}}]$ is observable.

the open-loop characteristic polynomial of the system

$$\mathbf{\Delta}_o(s) = |s\mathbf{I} - \mathbf{A}|, \tag{4.5.6}$$

the optimal closed-loop characteristic polynomial

$$\begin{aligned} \mathbf{\Delta}_c(s) &= |s\mathbf{I} - \mathbf{A} + \mathbf{B}\bar{\mathbf{K}}| \\ &= |\mathbf{I} + \mathbf{B}\bar{\mathbf{K}}[s\mathbf{I} - \mathbf{A}]^{-1}|.|s\mathbf{I} - \mathbf{A}|, \\ &= |\mathbf{I} + \bar{\mathbf{K}}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}|\mathbf{\Delta}_o(s). \end{aligned} \tag{4.5.7}$$

This is a relation between the open-loop $\mathbf{\Delta}_o(s)$ and closed-loop $\mathbf{\Delta}_c(s)$ characteristic polynomials.

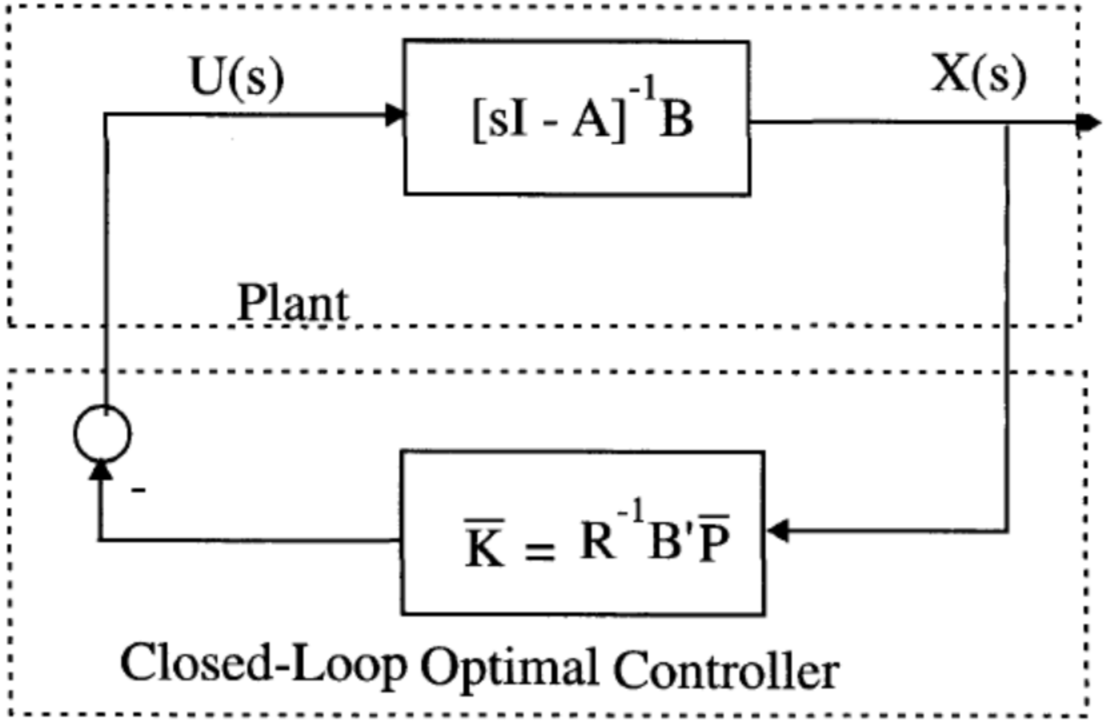


Figure 4.10 Optimal Closed-Loop Control in Frequency Domain

1. $-\bar{\mathbf{K}}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}$ is called the *loop gain matrix*, and
2. $\mathbf{I} + \bar{\mathbf{K}}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}$ is termed *return difference matrix*.

$$\boldsymbol{\Delta}_c(s) = |\mathbf{I} + \bar{\mathbf{K}}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}|\boldsymbol{\Delta}_o(s) \qquad (4.5.7)$$

ARE $$-\bar{\mathbf{P}}\mathbf{A}-\mathbf{A}'\bar{\mathbf{P}}+\bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}=\mathbf{Q}. \tag{4.5.8}$$

First adding and subtracting $s\bar{\mathbf{P}}, s=j\omega$ to the previous ARE, we get

$$\bar{\mathbf{P}}[s\mathbf{I}-\mathbf{A}]+[-s\mathbf{I}-\mathbf{A}']\bar{\mathbf{P}}+\bar{\mathbf{K}}'\mathbf{R}\bar{\mathbf{K}}=\mathbf{Q}. \tag{4.5.9}$$

Next, premultiplying by $\mathbf{B}'\mathbf{\Phi}'(-s)$ and post multiplying by $\mathbf{\Phi}(s)\mathbf{B}$, the previous equation becomes

$$\begin{aligned}\mathbf{B}'\mathbf{\Phi}'(-s)\bar{\mathbf{P}}\mathbf{B}+\mathbf{B}'\bar{\mathbf{P}}\mathbf{\Phi}(s)\mathbf{B}+\mathbf{B}'\mathbf{\Phi}'(-s)\bar{\mathbf{K}}'\mathbf{R}\bar{\mathbf{K}}\mathbf{\Phi}(s)\mathbf{B}&\\ =\mathbf{B}'\mathbf{\Phi}'(-s)\mathbf{Q}\mathbf{\Phi}(s)\mathbf{B}&\end{aligned} \tag{4.5.10}$$

where, we used

$$\mathbf{\Phi}(s)=[s\mathbf{I}-\mathbf{A}]^{-1};\quad \mathbf{\Phi}'(-s)=[-s\mathbf{I}-\mathbf{A}']^{-1}. \tag{4.5.11}$$

Finally, using $\bar{\mathbf{K}}=\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\longrightarrow\bar{\mathbf{K}}'=\bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\longrightarrow\bar{\mathbf{P}}\mathbf{B}=\bar{\mathbf{K}}'\mathbf{R}$ and adding $\mathbf{R}$ to both sides of (4.5.10), we have the desired factorization result as

$$\boxed{\mathbf{B}'\mathbf{\Phi}'(-s)\mathbf{Q}\mathbf{\Phi}(s)\mathbf{B}+\mathbf{R}=\left[\mathbf{I}+\bar{\mathbf{K}}\mathbf{\Phi}(-s)\mathbf{B}\right]'\mathbf{R}\left[\mathbf{I}+\bar{\mathbf{K}}\mathbf{\Phi}(s)\mathbf{B}\right]} \tag{4.5.12}$$

The Kalman gain **K** can be obtained by solving this equation.

or equivalently,

$$\begin{aligned}\mathbf{B}'[-s\mathbf{I}-\mathbf{A}']^{-1}\mathbf{Q}[s\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}+\mathbf{R} \\ = \left[\mathbf{I}+\bar{\mathbf{K}}[-s\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}\right]'\mathbf{R}\left[\mathbf{I}+\bar{\mathbf{K}}[s\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}\right].\end{aligned} \tag{4.5.13}$$

The previous relation is also called the *Kalman equation in frequency domain*.

4.5.1 Gain Margin and Phase Margin

- We know that in classical control theory, the features of gain and phase margins are important in evaluating the system performance with respect to robustness to plant parameter variations and uncertainties.
- The engineering specifications often place lower bounds on the phase and gain margins.

$$(4.5.13) \xrightarrow{s = j\omega} \mathbf{B}'[-j\omega\mathbf{I} - \mathbf{A}']^{-1}\mathbf{Q}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{R}$$

$$= \left[\mathbf{I} + \bar{\mathbf{K}}[-j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\right]' \mathbf{R}[\mathbf{I} + \bar{\mathbf{K}}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}]. \qquad (4.5.14)$$

$$\Rightarrow \quad \mathbf{M}(j\omega) = \mathbf{W}'(-j\omega)\mathbf{W}(j\omega) \qquad (4.5.15)$$

where $\mathbf{W}(j\omega) = \mathbf{R}^{1/2}\left[\mathbf{I} + \bar{\mathbf{K}}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\right]$

$$\mathbf{M}(j\omega) = \mathbf{R} + \mathbf{B}'[-j\omega\mathbf{I} - \mathbf{A}']^{-1}\mathbf{Q}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}. \qquad (4.5.16)$$

Note that $\mathbf{M}(j\omega) \geq \mathbf{R} > 0$. Using $\mathbf{Q} = \mathbf{C}\mathbf{C}'$, $\mathbf{R} = \mathbf{D}\mathbf{D}' = \mathbf{I}$

and the notation $\mathbf{W}'(-j\omega)\mathbf{W}(j\omega) = ||\mathbf{W}(j\omega)||^2$, $\qquad (4.5.17)$

$$(4.5.14) \Rightarrow \boxed{||\mathbf{I} + \bar{\mathbf{K}}'[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}||^2 = \mathbf{I} + ||\mathbf{C}'[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}||^2.} \qquad (4.5.18)$$

This result can be used to find the optimal feedback matrix $\bar{\mathbf{K}}$ given the other quantities $\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R} = \mathbf{I}$. Note that in (4.5.18), we need not solve for the Riccati coefficient matrix $\bar{\mathbf{P}}$, instead we directly obtain the feedback matrix $\bar{\mathbf{K}}$.

In the single-input case, the various matrices become scalars or vectors as $\mathbf{B} = \mathbf{b}, \mathbf{R} = r, \bar{\mathbf{K}} = \bar{\mathbf{k}}$. Then, the factorization result (4.5.14) boils down to

$$
\begin{aligned}
r + \mathbf{b}'[-j\omega\mathbf{I} - \mathbf{A}']^{-1}\mathbf{Q}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b} \\
= r|1 + \bar{\mathbf{k}}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}|^2.
\end{aligned}
\tag{4.5.19}
$$

In case $\mathbf{Q} = \mathbf{c}\mathbf{c}'$, we can write (4.5.19) as

$$
r + |\mathbf{c}'[j\omega\mathbf{I} - \mathbf{A}']^{-1}\mathbf{b}|^2 = r|(1 + \bar{\mathbf{k}}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b})|^2. \tag{4.5.20}
$$

The previous result may be called another version of the *Kalman equation in frequency domain.* The previous relation (also from (4.5.18) for a scalar case) implies that

$$
\boxed{|(1 + \bar{\mathbf{k}}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b})|^2 \geq 1.} \tag{4.5.21}
$$

Thus, the return difference is lower bounded by 1 for all ω.

$$\mathbf{I} + \bar{\mathbf{K}}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}$$

Example 4.5

Consider a simple example where we can verify the analytical solutions by another known method. Find the optimal feedback coefficients for the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) \end{aligned} \tag{4.5.22}$$

and the performance measure

$$J = \frac{1}{2}\int_0^{\infty} \left[x_1^2(t) + x_2^2(t) + u^2(t)\right] dt. \tag{4.5.23}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{R} = r = 1. \qquad (4.5.24)$$

$$\mathbf{C} = \mathbf{D} = \mathbf{I}$$

Kalman equation

$$||\mathbf{I} + \bar{\mathbf{K}}'[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}||^2 = \mathbf{I} + ||\mathbf{C}'[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}||^2. \qquad (4.5.25)$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \frac{1}{s} & 0 \\ \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}; \quad \bar{\mathbf{K}} = \bar{\mathbf{k}} = [k_{11} \quad k_{12}]. \qquad (4.5.26)$$

$$\left[1 + \begin{bmatrix} k_{11} & k_{12} \end{bmatrix} \begin{bmatrix} \frac{1}{-s} & \frac{1}{(-s)^2} \\ 0 & \frac{1}{-s} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] \left[1 + \begin{bmatrix} k_{11} & k_{12} \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] =$$

$$1 + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{-s} & \frac{1}{(-s)^2} \\ 0 & \frac{1}{-s} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad (4.5.27)$$

$$1 + (2k_{11} - k_{12}^2)\frac{1}{s^2} + k_{11}^2\frac{1}{s^4} = 1 - \frac{1}{s^2} + \frac{1}{s^4} \qquad (4.5.28)$$

$$k_{11} = 1, \qquad k_{12} = \sqrt{3} \qquad (4.5.29)$$

$$u^*(t) = -\bar{\mathbf{K}}\mathbf{x}^*(t) = -\left[1 \ \sqrt{3}\right]\mathbf{x}^*(t). \qquad (4.5.30)$$

Note: This example can be easily verified by using the algebraic Riccati equation (3.5.15) (of Chapter 3)

$$\bar{\mathbf{P}}\mathbf{A} + \mathbf{A}'\bar{\mathbf{P}} - \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} + \mathbf{Q} = 0 \qquad (4.5.31)$$

$$\bar{\mathbf{P}} = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} \qquad (4.5.32)$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}^*(t) = -\left[1 \ \sqrt{3}\right]\mathbf{x}^*(t) \qquad (4.5.33)$$

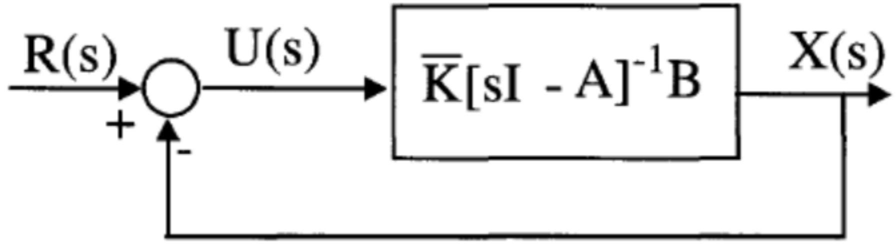


Figure 4.11 Closed-Loop Optimal Control System with Unity Feedback

Here, we can easily recognize that for a single-input, single-output case, the optimal feedback control system is exactly like a classical feedback control system with unity negative feedback and transfer function as $G_o(s) = \bar{\mathbf{k}}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}$. Thus, the frequency domain interpretation in terms of gain margin, phase margin can be easily done using Nyquist, Bode, or some other plot of the transfer function $G_o(s)$.

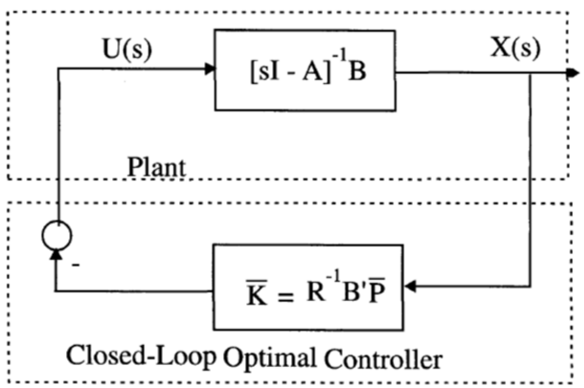
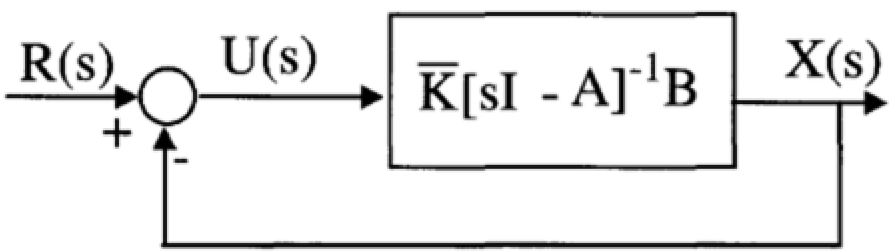


Figure 4.10

Nyquist stability criterion

- According to Nyquist stability criterion, for closed-loop stability, the *Nyquist plot* (or diagram) makes CCW encirclements as many times as there are poles of the transfer function $G_o(s)$ lying in the *right half* of the *s*-plane.
- Z = N + P

  1. P = 0, N = 0 => Z = 0
  2. P = -N => Z = 0

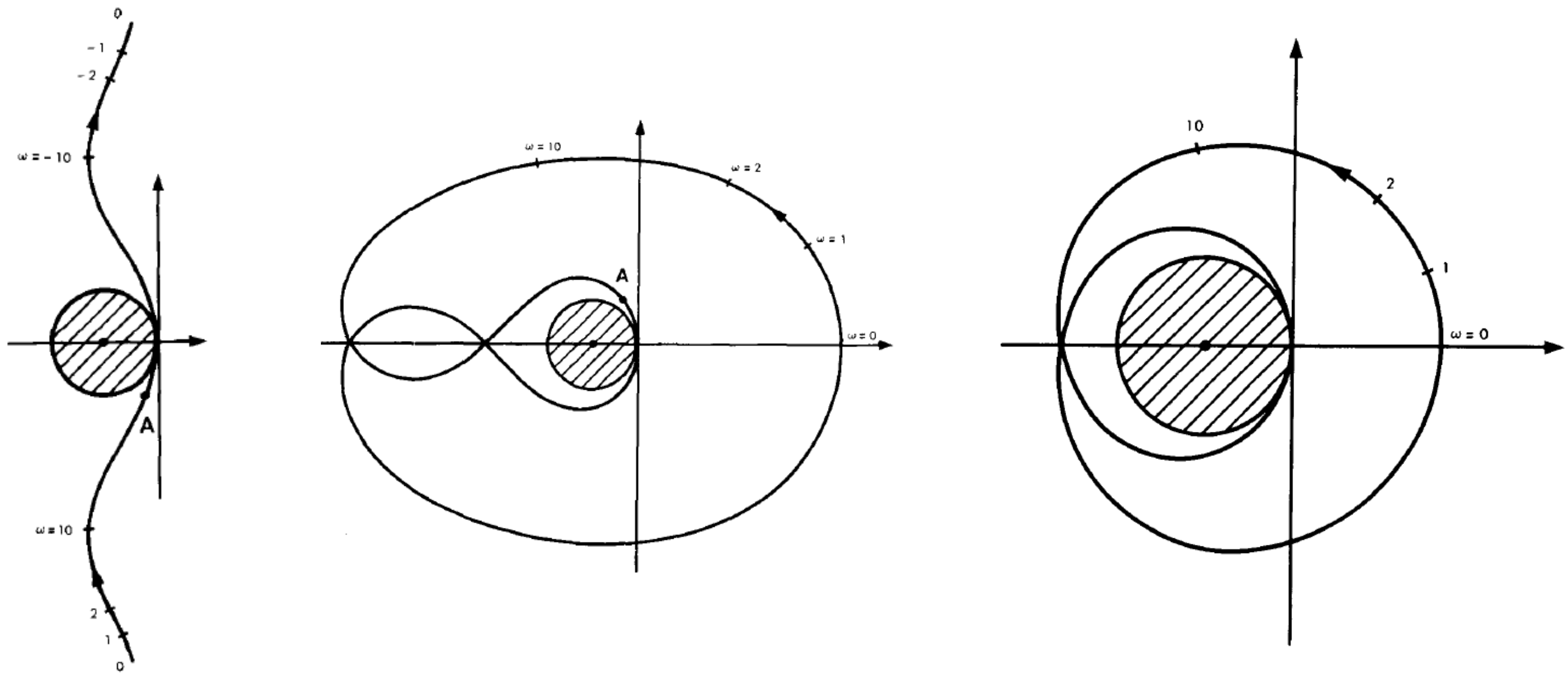


- According to the Nyquist stability theorem,

  Closed-loop stability => Z = 0 => N = -P, the encirclement around the point (-1, j0)

  - If the open loop is stable, P = 0 => N = 0

  - If the open loop is unstable, N = - P $\neq$ 0



Gain Margin

From Figure 4.11 and the return difference relation (4.5.21),

$$\left|(1 + \bar{\mathbf{k}}[j\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b})\right| \geq 1 \tag{4.5.34}$$

implies that the distance between the critical point -1 + *j*0 and any point on the Nyquist plot is at least 1 and the resulting Nyquist plot is shown in Figure 4.12 for all positive values of ω (i.e., 0 to ∞).

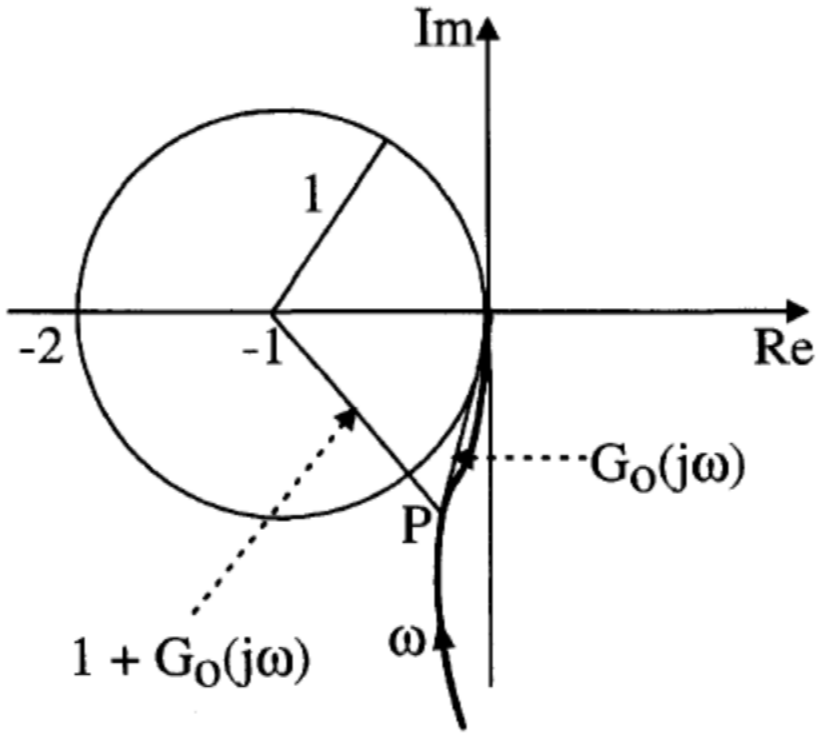


Figure 4.12 Nyquist Plot of $G_o(j\omega)$

Thus, it is clear that the closed-loop optimal system has *infinite gain margin*.

- The number of encirclements of the point -1+j0 is the same as the number of encirclements of any other point inside the circle of unit radius and center -1+j0.

Consider $1+\beta G_0(s) \Rightarrow -\dfrac{1}{\beta}$

- If $\beta < 1/2$, $-1/\beta$ lies outside the unit circle and contradicts the Nyquist criterion.
- Thus we have an infinite gain margin on the upper side and lower gain margin of $\beta = 1/2$.

Phase Margin

- Let us first recall that the phase margin is the amount of phase shift in CW direction (without affecting the gain) through which the Nyquist plot can be rotated about the origin so that the gain crossover (unit distance from the origin) passes through the -1+ j0 point. Simply, it is the amount by which Nyquist plot can be rotated CW to make the system unstable.

- Consider a point *P* at unit distance from the origin on the Nyquist plot (see Figure 4.12). Since we know that the Nyquist plot of an optimal regulator must avoid the unit circle centered at -1+ *j*0, the set of points which are at unit distance *from the origin* and lying on Nyquist diagram of an optimal regulator are constrained to lie on the portion marked X on the circumference of the circle with unit radius and centered at the origin as shown in Figure 4.13.

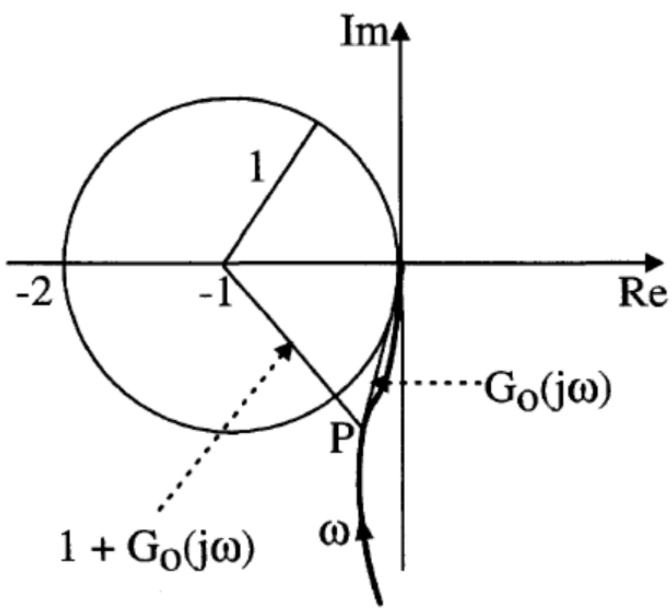


Figure 4.12 Nyquist Plot of $G_o(j\omega)$

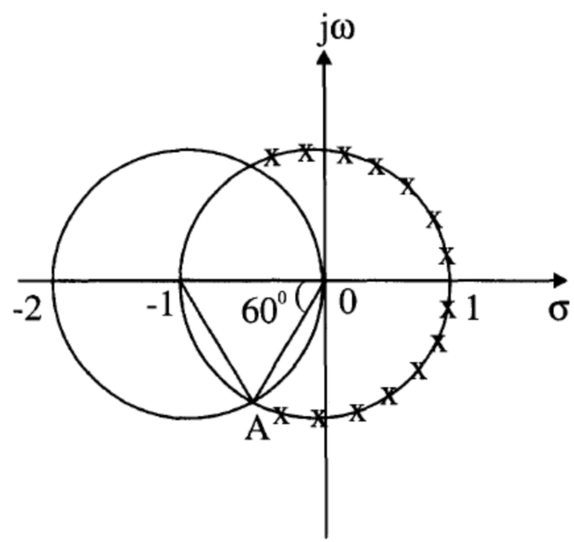


Figure 4.13 Intersection of Unit Circles Centered at Origin and $-1 + j0$

- Here, we notice that the *smallest angle* through which one of the admissible points *A* on (the circumference of the circle centered at origin) the Nyquist plot could be shifted in a CW direction to reach $-1+j0$ point is 60 degrees. Thus, the closed-loop optimal system or LQR system has a phase margin of at least 60 degrees.

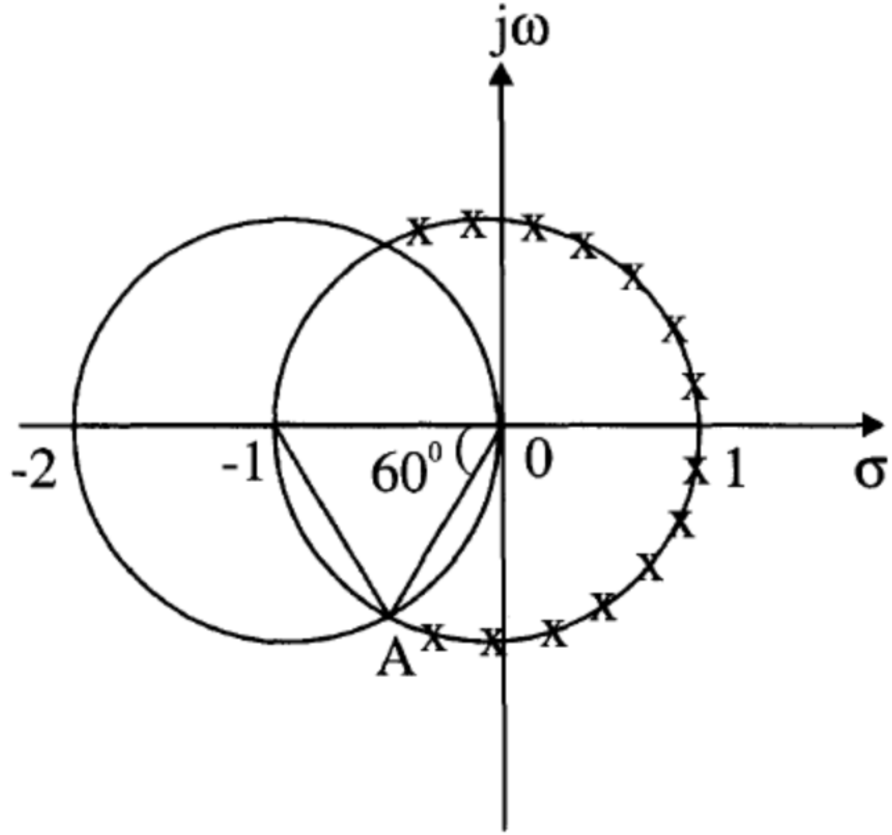


Figure 4.13 Intersection of Unit Circles Centered at Origin and $-1 + j0$

Summary:
Linear Quadratic Regulation/Tracking

Linear:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \qquad (4.1.1)$$

Quadratic:

$$J = \frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(t_f)\mathbf{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{e}'(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}'(t)\mathbf{R}(t)\mathbf{u}(t)\right] dt \qquad (4.1.3)$$

where $\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t) \qquad (4.1.2)$

Tracking $\rightarrow$ regulation:

$z(t) = 0$
$C(t) = \mathbf{I}$

t_f

- Finite → Differential Riccati eq.
- Infinite → $\mathbf{F}(t_f) = 0$ → Controllable/stablizable
  - Time varying → Differential Riccati eq.
  - Time invariant → Algebraic Riccati eq.

Tracking

- Regulation ($z(t) = 0$, $C(t) = \mathbf{I}$)
- Observable/detectable