Homework 3

Alexander Gould

January 28, 2015

91

4. Show that the negative of an even number is even. Use a direct proof.

Statement	Reason
n is even.	Given
$\exists k\,(n = 2k)$	Definition of an even number
$-n = 2 * -k$	Multiply both sides by -1.
$-n = 2a$	Assign $a = -k$.
$-n$ is even.	Definition of an even number

6. Use a direct proof to show that the product of two odd numbers is odd.

Statement	Reason
a and b are odd.	Given
$\exists c\,(a = 2c + 1)$,$\exists d\,(b = 2d + 1)$	Definition of an odd number
$ab = (2c + 1)\,(2d + 1)$	Multiply a and b
$ab = 4cd + 2c + 2d + 1$	FOIL
$ab = 2\,(2cd + c + d) + 1$	Distributive Property
$(2cd + c + d) \in \mathbb{Z}$	The sums and products of integers are integers
ab is odd.	Definition of an odd number

11. Prove or disprove that the product of two irrational numbers is irrational.

We can disprove this by counter-example. $\sqrt{3}$ is irrational, but the product of $\sqrt{3}$ and $\sqrt{3}$ is either 3 or -3, both rational.

16. Prove that if m and n are integers and mn is even, then m is even or n is even.

Using our answer from Question 6, we know that if both m and n are odd, mn will always be odd. Therefore, at least one of them *has* to be even.

17. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using:

A. A proof by contraposition (If n is odd, $n^3 + 5$ is even.)

Statement	Reason
n is odd.	Given
$\exists k\,(n = 2k + 1)$	Definition of an odd number
$n^3 + 5 = 2\left(4k^3 + 6k^2 + 3k + 3\right)$	FOIL
$\left(4k^3 + 6k^2 + 3k + 3\right) \in \mathbb{Z}$	The sums and products of integers are integers
$n^3 + 5$ is even.	Definition of an even number

B. A proof by contradiction

We start by assuming both n and $n^3 + 5$ are odd. We can use the reasoning from Part A up until the end of the proof, where we've proved that $n^3 + 5$ has to be even. Since there's a contradiction, we know that either n or $n^3 + 5$ is actually even. Since $n^3 + 5$ is odd is a given, we know n is even.

24. Show that at least three of any 25 days chosen must fall in the same month of the year.

If we assume that this is false, that means at most 2 days can be in the same month, meaning we'd need 24 days max. Since we know there are 25 days, there is a contradiction and the statement is true.

26. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

(Not sure how to put this one in a table.) Using our answer from Question 16, we know that we need at least one even factor to get an even product. Since 7 is odd, $7n$ will have the same parity as n. We can then use the definition of an even number to say that an even number stays even only when another even number is added to it. (Othereise the "+1" would turn it into an odd number.) Likewise, an even number added to an odd number yields an odd number. (Otherwise, the 2 "+1"'s would cancel each other out.) Therefore, we know that $7n + 4$ will always have the same parity as n. Since the 2 are equivelant, they're dependent on each other, proving the "if and only if" true.

28. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

Since we know the absolute values remain the same (two identical positive integers squared will still be equal), we just need to prove that a negative number times a negative number is positive. You can do this by factoring out the two impied "-1"'s, squaring both now-identical positive numbers, and then reapplying 1 "-1" on each side of the equals sign. The numbers will now be negative, but still be equal. Since we now know that both a positive and a negative squared will yeild a positive, and we know that the absolute value is the same, we know that the statement is true. If m is any value other than n or $-n$, the absolute value part won't hold up. Since the condition goes both ways, the "if and only if" is true.

30. Show that these three statements are equivalent, where a and b are real numbers: (i) $a < b$, (ii) $\frac{a+b}{2} > a$, and (iii) $\frac{a+b}{2} < b$.

If the statements are equivelant, we should be able to get any statement from any other. So $a + b > 2a$, $b > 2a - a$, $a < b$ gets us from ii to i. Likewise, $a + b < 2b$, $a < 2b - b$, $a < b$ gts us from iii to i. And we can get from i to iii via $a + b < b + b$, $\frac{a+b}{2} < \frac{2b}{2}$, $\frac{a+b}{2} < b$, and from i to ii via $a + a < b + a$, $\frac{2a}{2} < \frac{a+b}{2}$, $\frac{a+b}{2} > a$.

38. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

15. It's greater than the squares of the first 3 squares, so those are the only ones we can use. But no combination of those gets us 15.