Automata Homework 1

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1 Logic, Reasoning, Induction

1.1

1.1.1

$$\begin{aligned}
p \to (q \vee r) &\equiv \neg p \vee (q \vee r) \\
&\equiv (\neg p \vee q) \vee r \\
&\equiv \neg (p \wedge \neg q) \vee r \\
&\equiv (p \wedge \neg q) \to r
\end{aligned}$$

1.1.2

$$\begin{aligned}
\neg(p \vee q) \vee (\neg p \wedge q) \vee &\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q) && \text{(De Morgan)} \\
&\equiv \Big[\neg p \wedge (\neg q \vee q) \Big] \vee (p \wedge q) && \text{(Distributive)} \\
&\equiv \big[\neg p \wedge T \big] \vee (p \wedge q) \\
&\equiv \neg p \vee (p \wedge q) \\
&\equiv (\neg p \vee p) \wedge (\neg p \vee q) && \text{(Distributive)} \\
&\equiv T \wedge (\neg p \vee q) \\
&\equiv \neg p \vee q \\
&\equiv \neg (p \wedge \neg q) && \text{(De Morgan)}
\end{aligned}$$

1.2

We are going to prove these statements both with contradiction and inference rules:

1.2.1

$$(p \to q) \implies (\neg q \to \neg p) \tag{1}$$

Proof with contradiction:

$$\begin{aligned}
(p \to q) \wedge \neg(\neg q \to \neg p) &\equiv (\neg p \vee q) \wedge \neg(q \vee \neg p) \\
&\equiv (\neg p \vee q) \wedge (\neg q \wedge p) \\
&\equiv \neg q \wedge p \wedge (\neg p \vee q) && \text{(reorder)} \\
&\equiv \neg q \wedge \Big[(p \wedge \neg p) \vee (p \wedge q)\Big] && \text{(Distributive)} \\
&\equiv \neg q \wedge \Big[F \vee (p \wedge q)\Big] \\
&\equiv \neg q \wedge (p \wedge q) \\
&\equiv p \wedge (\neg q \wedge q) && \text{(Associative)} \\
&\equiv p \wedge F \\
&\equiv F
\end{aligned}$$

Thus we have a contradiction.

Proof with inference rules:

(1)	$\neg q$	Assumption
(2)	$\mid p \to q$	Premise
(3)	$\mid \neg p$	Modus Tollens: 1, 2
(4)	$\neg q \to \neg p$	$\to$ introduction: 1-3

1.2.2

Porof by contradiction:

$$\Big[(p \to q) \wedge (p \to \neg q)\Big] \implies \neg p \tag{3}$$

$$\begin{aligned}
\Big[(p \to q) \wedge (p \to \neg q)\Big] \wedge \neg(\neg p) &\equiv \Big[(\neg p \vee q) \wedge (\neg p \vee \neg q)\Big] \wedge p \\
&\equiv \Big[\neg p \vee (q \wedge \neg q)\Big] \wedge p \\
&\equiv \Big[\neg p \vee F\Big] \wedge p \\
&\equiv \neg p \wedge p \\
&\equiv F && \therefore \text{Contradiction}
\end{aligned}$$

Proof with inference rules:

(1)	p	Assumption
(2)	$\mid p \to q$	Premise
(3)	$\mid q$	Modus Ponens: 1, 2
(4)	$\mid p \to \neg q$	Premise
(5)	$\mid \neg p$	Modus Tollens: 3, 4
(6)	$\mid \perp$	$\neg$ elimination: 1, 5
(7)	$\neg p$	$\neg$ introduction: 1-6

($\perp$ means contradiction).
Note that we've used $p \to q$ and $p \to \neg q$ as premises for simplicity. If we don't consider them our direct premises, then we can use Simplification rule on $(p \to q) \wedge (p \to \neg q)$ to get to them .

1.2.3

Proof by contradiction:

$$\Big[(p \vee q) \wedge (\neg p \vee r)\Big] \implies (q \vee r) \tag{5}$$

$$\begin{aligned}
\Big[(p \vee q) \wedge (\neg p \vee r)\Big] \wedge \neg(q \vee r) &\equiv \Big[(p \vee q) \wedge (\neg p \vee r)\Big] \wedge (\neg q \wedge \neg r) \\
&\equiv \Big[(p \wedge \neg q) \vee (q \wedge \neg q)\Big] \wedge \Big[(\neg p \wedge \neg r) \vee (r \wedge \neg r)\Big] \\
&\qquad\qquad \text{(Distributive law } \times 2) \\
&\equiv \Big[(p \wedge \neg q) \vee F\Big] \wedge \Big[(\neg p \wedge \neg r) \vee F\Big] \\
&\equiv (p \wedge \neg q) \wedge (\neg p \wedge \neg r) \\
&\equiv \neg q \wedge \neg r \wedge (p \wedge \neg p) \\
&\equiv \neg q \wedge \neg r \wedge F \\
&\equiv F \qquad \therefore \text{ Contradiction}
\end{aligned}$$

Proof with inference rules:

(1)	$\neg(q \vee r)$	Assumption
(2)	$\mid \neg q \wedge \neg r$	De Morgan: 1
(3)	$\mid \neg q$	Simplification: 2
(4)	$\mid p \vee q$	Premise
(5)	$\mid p$	Disjunctive Syllogism: 3, 4
(6)	$\mid \neg p \vee r$	Premise
(7)	$\mid \neg r$	Simplification: 2
(8)	$\mid \neg p$	Disjunctive Syllogism: 6, 7
(9)	$\mid \perp$	$\neg$ elimination: 5, 8
(10)	$\neg\neg(q \vee r)$	$\neg$ introduction:1-9
(11)	$q \vee r$	$\neg\neg$ elimination

1.3

1.3.1

If you write a truth table for both sides, you'll see that this equivalence does not hold. But rather we can only derive the following statement and we are going to prove that instead.

$$(p \to r_1) \wedge (r_1 \to r_2) \wedge \ldots \wedge (r_n \to q) \implies p \to q \tag{7}$$

It is easily proved using induction.
Base Case:

$$(p \to r_1) \wedge (r_1 \to q) \implies p \to q \tag{8}$$

(1)	p	Assumption
(2)	$\mid p \to r_1$	Premise
(3)	$\mid r_1$	Modus Ponens: 1, 2
(4)	$\mid r_1 \to q$	Premise
(5)	$\mid q$	Modus Ponens: 3, 4
(6)	$p \to q$	$\to$ introduction: 1-5

Induction Hypothesis: Let's assume for $n = k$ this equivalence holds. Then:

$$(p \to r_1) \wedge (r_1 \to r_2) \wedge \ldots \wedge (r_k \to z) \implies p \to z \tag{10}$$

Induction Step: We can show that it also holds for $n = k + 1$.

(1)	$(p \to r_1) \wedge (r_1 \to r_2) \wedge \ldots \wedge (r_{k+1} \to q)$	Premise
(2)	$(p \to r_1) \wedge (r_1 \to r_2) \wedge \ldots \wedge (r_k \to r_{k+1})$	Simplification: 1
(3)	$p \to r_{k+1}$	Induction Hypothesis: 2
(4)	$r_{k+1} \to q$	Simplification: 1
(5)	p	Assumption
(6)	$\quad \mid r_{k+1}$	Modus Ponens: 3, 5
(7)	$\quad \mid q$	Modus Ponens: 4, 6
(8)	$p \to q$	$\to$ introduction:

1.3.2

We prove this equivalence with induction.
Base Case: for $n = 1$ it's obvious:

$$p = r_1$$
$$p \to q \equiv r_1 \to q$$

Induction Hypothesis: Let's assume for $n = k$ this equivalence holds:

$$p_1 = r_1 \vee \ldots \vee r_k$$
$$p_1 \to q \equiv (r_1 \to q) \wedge \ldots \wedge (r_k \to q)$$

Induction Step: Now we now show that it holds for $n = k + 1$ as well:

$$p = \underbrace{r_1 \vee \ldots \vee r_k}_{p_1} \vee \underbrace{r_{k+1}}_{p_2} = p_1 \vee p_2$$

$$\begin{aligned}
(r_1 \to q) \wedge \ldots \wedge (r_k \to q) \wedge (r_{k+1} \to q) &\equiv (p_1 \to q) \wedge (r_{k+1} \to q) && \text{(Induction Hypothesis)} \\
&\equiv (p_1 \to q) \wedge (p_2 \to q) \\
&\equiv (\neg p_1 \vee q) \wedge (\neg p_2 \vee q) \\
&\equiv (\neg p_1 \wedge \neg p_2) \vee q && \text{(Distributive law)} \\
&\equiv \neg(p_1 \vee p_2) \vee q && \text{(De Morgan)} \\
&\equiv \neg p \vee q \\
&\equiv p \to q \quad \therefore
\end{aligned}$$

2 Properties of sets, recursive definitions, countability and uncountability

2.1

Yes it is a equivalence relation:

- **Reflexive:** $\forall x \in L$ we define u and v as: $u = x, v = \epsilon$. Therefore $x = uv = vu \implies xRx$.
- **Symmetric:** It's obvious! (you only need to swap u and v values).
- **Transitive:** Assume $|x| = n = i + j$ and x is split into u and v at index i (i.e., $|u| = i, |v| = j$) and $y = vu$. And we know yRz; i.e., if we split y at some index and swap the resulting substrings, we get z. There is three possibilities for index at which y is split. If y is split at index $k = j$, then $x = z$ and xRz by reflexiveness. If y is split at $0 \leq k < j$, then one can rewrite x, y, and z as follow: $y = abu, x = \underbrace{ua}_{u\prime}\underbrace{b}_{v\prime}, z = \underbrace{b}_{v\prime}\underbrace{ua}_{u\prime}$ (where $|a| = k$) and as you can see, you can convert x to z and vice versa, by swapping $u\prime$ and $v\prime$. Thus xRz. You can show same thing in the case where split is done at index $j \leq k < n$ with a similar procedure. Hence R is transitive.

2.2

2.2.1

We name the mentioned set S and define S recursively:

(1) $a \in S$

(2) $\forall x, y \in S: \ x + y \in S$

(3) $\forall x, y \in S: \ x \times y \in S$

(4) $\forall x \in S: \ (x) \in S$

Now we prove string $(a + a \times (a + a))$ belongs to S by constructing it step by step with the above rules:

1. $\xrightarrow{(1)} a \in S$
2. $\xrightarrow{1,(2)} a + a \in S$
3. $\xrightarrow{2,(4)} (a + a) \in S$

4. $\xrightarrow{1,3,(3)} a \times (a+a) \in S$

5. $\xrightarrow{1,4,(2)} a + a \times (a+a) \in S$

6. $\xrightarrow{5,(4)} (a + a \times (a+a)) \in S$

(Numbers in parenthesis refer to rules defined above).

2.2.2

We name the mentioned set S and define S recursively:

(1) $\epsilon \in S$

(2) $\forall x \in S: \ (x) \in S$

(3) $\forall x, y \in S: \ xy \in S$

Now we prove $()(()) \in S$ by constructing it step by step:

1. $\xrightarrow{(1)} \epsilon \in S$
2. $\xrightarrow{1,(2)} () \in S$
3. $\xrightarrow{2,(2)} (()) \in S$
4. $\xrightarrow{3,2,(3)} (())() \in S$

2.3

We assume disks in A do not have any overlap with each other.
You can find a point (x, y) at every disk such that $x, y \in \mathbb{Q}$ (because every disk covers a range of real number in both axes, and you can find a rational number in every real range). We map every disk to one of its points with rational coordinates and since disks are disjoint, this mapping would be a one-to-one function. Thus we have mapped the set A to a subset of $\mathbb{Q}^2$ and as we know, $\mathbb{Q}$ and $\mathbb{Q}^2$ are both countable sets. Thus $|A| \leq |\mathbb{Q}^2|$ and A is countable.

2.4

In this case, A can be uncountable. Consider only set of circles with their center at origin and a radius of $r \in (0, 1]$. There can be as many (non-intersecting) circles in this set as cardinality of $(0, 1]$ which we know is higher than $|J|$ hence it is not countable.

3 Basics of Automata

3.1

We add a new state to the automata (s) and want to determine new transition function's output for state q and letter t (i.e., $\delta'(q,t)$). There is three cases:

1. If q is an accepting state in A and there is some state which goes to q with t, then in the new automata, $\delta'(q,t) = s$. For example, state 4 is an accepting state and state 2 goes to 4 by receiving letter b. So in the new automata, $\delta'(4,b) = s$.

2. If q is not in accepting states and if all states that have an edge to q, go to q with all receiving letters (except t), then $\delta'(q,t) = s$. For example, $\delta'(2,a) = s$. Because all of in-neighbors (neighbors with an incoming edge) of 2 (only 1) are connected to 2 with $\Sigma - \{a\} = b$.

3. If it's non of above rules applied, δ' acts similar to δ.

By applying these rules, the new state machine would be as follow:

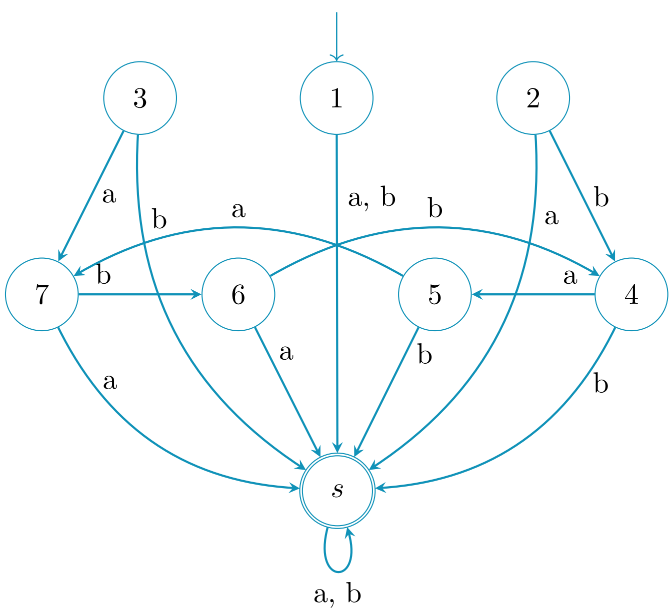

Figure 1: New automata (A')

Note that rule number 2 is vacuously true for states 1 and s; because they have no in-neighbors in transition function δ. So resulting state machine would be the same as a state machine with only states 1 and s with $\delta'(1,a) = s$, $\delta'(1,b) = s$, $\delta'(s,a) = s$, $\delta'(s,b) = s$ as its only edges.