11. Let U be uniform on (0, 1). Use simulation to approximate the following:

(a) $\mathrm{Corr}\left(U, \sqrt{1-U^2}\right)$.

(b) $\mathrm{Corr}\left(U^2, \sqrt{1-U^2}\right)$.

12. For uniform (0, 1) random variables $U_1, U_2, \ldots$ define

$$N = \text{Minimum}\left\{n: \sum_{i=1}^{n} U_i > 1\right\}$$

That is, N is equal to the number of random numbers that must be summed to exceed 1.

(a) Estimate $E[N]$ by generating 100 values of N.

(b) Estimate $E[N]$ by generating 1000 values of N.

(c) Estimate $E[N]$ by generating 10,000 values of N.

(d) What do you think is the value of $E[N]$?

Ans:

11(a): -0.9217748029375138

11(b): -0.9802089544580987

12(a): 2.68 ;12(b): 2.716 ;12(c): 2.784

12(d):proof

$$\mathrm{E(N)} = \sum_{n=0}^{\infty} P(N > n) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

Note:

$$\mathrm{P(N=n)} = \mathrm{P}(U_1 + U_2 + \cdots + U_{n-1} < 1) + \mathrm{P}(U_1 + U_2 + \cdots + U_n > 1)$$

$$= \mathrm{P}(U_1 + U_2 + \cdots + U_{n-1} < 1) - \mathrm{P}(U_1 + U_2 + \cdots + U_n < 1)$$

And $\mathrm{P}(U_1 + U_2 + \cdots + U_n < 1) = \int_0^1 \int_0^{1-u_1} \ldots \int_0^{1-u_1-u_2-\cdots-u_{n-1}} 1 \, du_n \, du_{n-1} \ldots du_1 = \frac{1}{n!}$

$$\therefore \mathrm{P(N=n)} = \frac{1}{(n-1)!} - \frac{1}{n!}$$

Then $\mathrm{P(N>n)} = P(N \geq n+1) = \sum_{k=n+1}^{\infty} \left\{\frac{1}{(n-1)!} - \frac{1}{n!}\right\} = \frac{1}{n!}$

Code:

```
import numpy as np

U = np.random.uniform(0, 1, 100)
print("11(a) = ", np.corrcoef(U, np.sqrt(1-pow(U,2)))[0,
1])
print("11(b) = ", np.corrcoef(pow(U,2), np.sqrt(1-pow(U,
2)))[0,1])

11(a) =  -0.9217748029375138
11(b) =  -0.9802089544580987
import numpy as np

num = [100, 1000, 1000]
value = []
for k in num:
    N = []
    n = k
    for i in range(n):
        count = 1
        U = np.random.uniform(0, 1)
        for j in range(10000):
            if U < 1:
                U = U + np.random.uniform(0, 1)
                count = count + 1
            else:
                N.append(count)
                count = 0
                break
    value.append(sum(N)/n)
value

 [2.68, 2.716, 2.784]
```