11. Suppose the joint density of X, Y, Z is given by

$$f(x, y, z) = Ce^{-(x+y+z+axy+bxz+cyz)}, \quad x > 0,\ y > 0,\ z > 0$$

where a, b, c are specified nonnegative constants, and C does not depend on x, y, z. Explain how we can simulate the vector X, Y, Z, and run a simulation to estimate $E[XYZ]$ when $a = b = c = 1$.

$\mathrm{f(x|y,z)} = \mathrm{C}e^{-(y+z+cyz)}e^{-(ay+bz+1)x} = C_1 e^{-(ay+bz+1)x} \sim \mathrm{Exp}(\lambda = ay + bz + 1)$

$\mathrm{f(y|x,z)} = \mathrm{C}e^{-(x+z+bxz)}e^{-(ax+cz+1)y} = C_2 e^{-(ax+cz+1)y} \sim \mathrm{Exp}(\lambda = ax + cz + 1)$

$\mathrm{f(z|x,y)} = \mathrm{C}e^{-(x+y+axy)}e^{-(bx+cy+1)z} = C_3 e^{-(bx+cy+1)z} \sim \mathrm{Exp}(\lambda = bx + cy + 1)$

$Initial\ set\ y = z = 0.5$

Generate x from $\mathrm{Exp}(\lambda = \mathrm{y + z + 1})$

$Then\ generate\ y\ from$ $\mathrm{Exp}(\lambda = \mathrm{x + z + 1})$

$Then\ generate\ z\ from$ $\mathrm{Exp}(\lambda = \mathrm{x + y + 1})$

$Repeat\ k\ times, get\ x_k, y_k, z_k, compute\ x_k * y_k * z_k$

$$Then\ repeat\ n\ times, E(XYZ) = \frac{\sum x * y * z}{n}$$

By simulation:

```
E(XYZ) =  0.08868526247511184
```

12. Suppose that for random variables X, Y, N

$$P\{X = i, y \le Y \le y + dy, N = n\} \approx C\binom{n}{i} y^{i+\alpha-1}(1-y)^{ni+\beta-1}e^{-\lambda}\frac{\lambda^n}{n!}dy$$

where $i = 0, \ldots, n, n = 0, 1, \ldots, y \ge 0$, and where α, β, λ are specified constants. Run a simulation to estimate $E[X]$, $E[Y]$, and $E[N]$ when $\alpha = 2$, $\beta = 3$, $\lambda = 4$.

$$\text{Correct} \approx \mathrm{C}\binom{n}{i} y^{i+\alpha-1}(1-y)^{n-i+\beta-1}e^{-\lambda}\frac{\lambda^n}{n!}\mathrm{dy}$$

$\mathrm{P(X = i \mid y \le Y \le y + dy, N = n)} \approx C_1\binom{n}{i} y^i(1-y)^{n-i} \sim Bin(n, y)$

$\mathrm{P(y \le Y \le y + dy \mid X = i, N = n)}$
$\approx C_2 y^{i+\alpha-1}(1-y)^{n-i+\beta-1} \sim Beta(i + \alpha, n - i + \beta)$

$$\mathrm{P}(\mathrm{N}=\mathrm{n} \mid \mathrm{y} \leq \mathrm{Y} \leq \mathrm{y}+\mathrm{dy}, \mathrm{X}=\mathrm{i}) \approx C_3 \frac{(1-y+\lambda)^{n-i}}{(n-i)!} \mathrm{e}^{-(1-y+\lambda)}, \text{let } \mathrm{j}=(\mathrm{n}-\mathrm{i})$$

$$=\mathrm{P}(\mathrm{J}=\mathrm{j} \mid \mathrm{y} \leq \mathrm{Y} \leq \mathrm{y}+\mathrm{dy}, \mathrm{X}=\mathrm{i}) \sim \mathrm{Poi}(1-\mathrm{y}+\lambda)$$

$Initial\ set\ y = 0.5, N = 5$

Generate x from $Bin(n, y)$

$Then\ generate\ y\ from\ Beta(i+\alpha, n-i+\beta)$

$Then\ generate\ j\ from\ \mathrm{Poi}(1-\mathrm{y}+\lambda)$

$Repeat\ k\ times, get\ x_k, y_k, j_k, n_k = j_k + x_k$

$Then\ repeat\ t\ times, E(X) = \frac{\sum x}{t}, E(Y) = \frac{\sum y}{t}, E(N) = \frac{\sum n}{t}$

By simulation:

```
E(X) =  1.589
E(Y) =  0.39917226868347067
E(N) =  3.979
```