1. (60%)

Show how to generate a random variable whose distribution function is

$$F(x) = \frac{1}{2}(x + x^2), \quad 0 \le x \le 1$$

using

(a) the inverse transform method;
(b) the rejection method;
(c) the composition method.

Which method do you think is best for this example? Briefly explain your answer.

$(a)\ Let\ u = F(x) = \frac{1}{2}(x + x^2)\ , u \sim U(0,1)$

$$\Rightarrow x + x^2 = 2u$$

$$\Rightarrow x + x^2 + \frac{1}{4} = 2u + \frac{1}{4}$$

$$\Rightarrow \left(x + \frac{1}{2}\right)^2 = 2u + \frac{1}{4}$$

$$\Rightarrow x + \frac{1}{2} = \pm\sqrt{2u + \frac{1}{4}}(\text{負不合})$$

$$\Rightarrow x = \sqrt{2u + \frac{1}{4}} - \frac{1}{2}$$

$Algorithm:$
$Step1: U \sim U(0,1)$

$$Step2: X = \sqrt{2U + \frac{1}{4}} - \frac{1}{2}$$

$(b)\ \ f_X(x) = \frac{1}{2}(1 + 2x), 0 \le x \le 1$

$Choose\ f_Y(y) = 1, 0 \le y \le 1.$

$$So\ c = \max_x \frac{f_X(x)}{f_Y(x)} = \max_x \frac{1}{2}(1 + 2x) \Rightarrow x = 1, c = \frac{3}{2}.$$

$$Then\ g(y) = \frac{1}{3}(1 + 2y), 0 \le y \le 1.$$

$Algorithm:$
$Step1: U_1 \sim U(0,1), Y = U_1$
$Step2: U \sim U(0,1)$
$Step3: If\ U \leq \frac{1}{3}(1+2Y), set\ X = Y; otherwise\ return\ step1$

$(c)\ Since\ F(x) = \frac{1}{2}(x+x^2), f(x) = \frac{1}{2}(1+2x), 0 \leq x \leq 1.$

$Let\ g_1(x) = c_1 * 1, g_2(x) = c_2 * 2x\ , be\ density\ of\ two\ components.$

$So\ c_1 = c_2 = 1, f(x) = \frac{1}{2}(1+2x) = \frac{1}{2}g_1(x) + \frac{1}{2}g_2(x).$

$Thus\ G_1^{-1}(x) = x,\ G_2^{-1}(x) = \sqrt{x}$

$Algorithm:$
$Step1: Generate\ U_1, U_2 \sim U(0,1)$
$Step2: If\ U_1 \leq \frac{1}{2}\ , set\ X = U_2; otherwise\ X = \sqrt{U_2}$

如果 F(x)很容易就能算出 inverse function 如此例題，則直接選擇 inverse method 會比較方便。

2. (20%)

Give two algorithms for generating a random variable having distribution function

$$F(x) = 1 - e^{-x} - e^{-2x} + e^{-3x}, \quad x > 0$$

$(1)\ AR - method$

$f_X(x) = e^{-x} + 2e^{-2x} - 3e^{-3x}, x > 0.$

$Choose\ f_Y(y) = e^{-y}, y > 0.$

$So\ c = \max_x \frac{f_X(x)}{f_Y(x)} = 1 + 2e^{-x} - 3e^{-2x} \Longrightarrow x = \ln 3, c = \frac{4}{3}$

$Then\ g(y) = \frac{3}{4}(1 + 2e^{-y} - 3e^{-2y}), y > 0.$

$Algorithm:$
$Step1: U_1 \sim U(0,1), Y = -lnU_1$
$Step2: U \sim U(0,1)$
$Step3: If\ U \leq \frac{3}{4}(1 + 2e^{-Y} - 3e^{-2Y}), set\ X = Y; otherwise\ return\ step1$

(2)

$F(x) = 1 - e^{-x} - e^{-2x} + 3e^{-3x} = (1 - e^{-x})(1 - e^{-2x}), x > 0.$

$Let\ Y \sim Exp(1), Z \sim Exp(2),$ and Y 和 Z 獨立

$So\ F(x) = F(y)F(z) = P(max(Y, Z) \leq x)$

$Algorithm:$

$Step1: U_1, U_2 \sim U(0,1), Y = -lnU_1, Z = -\frac{1}{2}lnU_2$

$Step2: X = max(Y, Z)$

3. (20%)

Give two algorithms for generating a random variable having density function

$$f(x) = \frac{1}{4} + 2x^3 + \frac{5}{4}x^4, \quad 0 < x < 1$$

$(1)\ AR - method$

$f_X(x) = \frac{1}{4} + 2x^3 + \frac{5}{4}x^4, 0 < x < 1$

$Choose\ f_Y(y) = 1, 0 \leq y \leq 1$

$So\ c = \max_x \frac{f_X(x)}{f_Y(x)} = \frac{1}{4} + 2x^3 + \frac{5}{4}x^4 \Longrightarrow x = 1, \text{c} = 3.5$

$Then\ g(y) = \dfrac{\frac{1}{4} + 2y^3 + \frac{5}{4}y^4}{3.5}, 0 \leq y \leq 1$

$Algorithm:$

$Step1: U_1 \sim U(0,1), Y = U_1$

$Step2: U \sim U(0,1)$

$Step3: If\ U \leq \dfrac{\frac{1}{4} + 2y^3 + \frac{5}{4}y^4}{3.5}, set\ X = Y; otherwise\ return\ step1$

$(2)\ Composition\ method$

$f_X(x) = \frac{1}{4} + 2x^3 + \frac{5}{4}x^4 = \frac{1}{4}(1 + 8x^3 + 5x^4), 0 < x < 1.$

$Let\ g_1(x) = c_1 * 1, g_2(x) = c_2 * 8x^3, g_3(x) = c_3 * 5x^4$

$be\ density\ of\ three\ components.$

$So\ c_1 = 1, c_2 = \frac{1}{2}, c_3 = 1 \Longrightarrow f_X(x) = \frac{1}{4}(g_1(x) + 2 * g_2(x) + g_3(x))$

$Thus\ G_1^{-1}(x) = x,\ G_2^{-1}(x) = \sqrt[4]{x},\ G_3^{-1}(x) = \sqrt[5]{x}$

$Algorithm$:

$Step1$: $Generate\ U_1, U_2 \sim U(0,1)$

$Step2$: $If\ U_1 \leq \dfrac{1}{2}\ , set\ X = \sqrt[4]{U_2}\ ;$

$else\ if\ \dfrac{1}{2} < U_1 \leq \dfrac{3}{4}\ , set\ X = \sqrt[5]{U_2}\ ;$

$else\ set\ X = U_2$