統計計算 HW2 0752617 張家綸

1. (20%) Estimate $\int_0^\infty \int_0^x e^{-(x+y)}\,dy\,dx$ **via simulation.**

<sol>

$$\int_0^\infty \int_0^x e^{-(x+y)}\,dydx\ ,0 < y < x < \infty$$

$$= \int_0^\infty \int_0^1 xe^{-x(z+1)}dzdx\ \left(令z = \frac{y-0}{x-0}\right), 0 < zx < x$$

$$< \infty\ (必成立,因z \sim U(0,1))$$

$$= \int_0^1 \int_0^1 \frac{\left(\frac{1}{w}-1\right)e^{-\left(\frac{1}{w}-1\right)(y+1)}}{w^2}dzdw(令w = \frac{1}{(1+x)})$$

Via simulation =

```
print("第一題 = ", sum(Total)/n)
```

```
第一題 =  0.510147375938053
```

2. (30%)

A deck of 100 cards—numbered 1, 2, . . . , 100—is shuffled and then turned over one card at a time. Say that a "hit" occurs whenever card i is the ith card to be turned over, $i = 1, \ldots, 100$. Write a simulation program to estimate the expectation and variance of the total number of hits. Run the program. Find the exact answers and compare them with your estimates.

<sol>

Simulation:

```
print("E(N) = ", sum(X)/sim)
print("Var(N) = ", sum(np.power(X,2))/sim - np.power(sum(X)/sim,2))
```

```
E(N) =  1.04
Var(N) =  1.0104
```

類似配對問題: 第 k 個信封是否在第 k 個

$$P(\text{恰有}k\text{個配對}) = \sum_{n=k}^{100}(-1)^{n-k} * \binom{n}{k} * S_k = \frac{1}{k!}(1-\frac{1}{1!}+\frac{1}{2!}-\cdots+(-1)^{100-k} *)$$

$$Where S_k = \sum_{i1} \dots \sum_{ik} P(A_{i1}\cap \dots \cap A_{ik}) = \frac{1}{k!}, (Ai: \text{第}i\text{個配對})$$

$$\text{E(N)} = \sum_{N=0}^{100} N * P(\text{恰有}N\text{個配對})$$

$$= \sum_{N=1}^{100} N * \frac{1}{N!}\left(1-\frac{1}{1!}+\frac{1}{2!}-\cdots+(-1)^{100-N} * \frac{1}{(100-N)!}\right)$$

$$\text{Var(N)} = \text{E}(N^2) - (E(N))^2$$

```
print("Exactly E(N)", sum(N))
print("Exactly Var(N)",sum(N_2) - np.power(sum(N),2))
```

```
Exactly E(N) 0.9999999999999997
Exactly Var(N) 1.0000000000000002
```

誤差:E(N) = 0.04

Var(N) = 0.0104

3. (50%)

The negative binomial probability mass function with parameters (r, p), where r is a positive integer and $0 < p < 1$, is given by

$$p_j = \frac{(j-1)!}{(j-r)!(r-1)!} p^r (1-p)^{j-r}, \quad j = r, r+1, \ldots$$

(a) Use the relationship between negative binomial and geometric random variables and the results of Example 4d to obtain an algorithm for simulating from this distribution.

(b) Verify the relation

$$p_{j+1} = \frac{j(1-p)}{j+1-r} p_j$$

(c) Use the relation in part (b) to give a second algorithm for generating negative binomial random variables.

(d) Use the interpretation of the negative binomial distribution as the number of trials it takes to amass a total of r successes when each trial independently results in a success with probability p, to obtain still another approach for generating such a random variable.

<sol>

(a) 因為$X_i \sim Gep(p)$, 參考投影片$P.26\sim28$

$Algorithm$:

Step1: Generate $U \sim U(0,1)$

Step2: $X = \mathrm{int}\left(\frac{\log(U)}{\log(1-p)}\right) + 1$

Step3: Repeat step2 r times; $\sum_{i=1}^{r} X_i \sim NB(r,p)$

(b) $P_{j+1} = \frac{j!}{(j+1-r)!(r-1)!} p^r (1-p)^{j+1-r}$

$$= \frac{j(1-p)(j-1)!}{(j+1-r)(j-r)!\,(r-1)!} p^r (1-p)^{j-r}$$

$$= \frac{j(1-p)}{(j+1-r)} P_j$$

(c)

Algorithm: Start with $p_0 = p^r$

Step1: Generate $U\sim U(0,1)$

$$Step2: X = \begin{cases} r\ , 0 \le U < p_0 \\ r+1\ , p_0 \le U < p_1 \\ \vdots \end{cases} ,where\ P_{j+1} = \frac{j(1-p)}{(j+1-r)} P_j$$

(d)

Algorithm:

Step1: Set $n = 0$

Step2: Set $n = n + 1$

Step3: Generate $U\sim U(0,1)$

Step4: If $U < p$, set $X_i = n$, stop; else return step2

Step5: Repeat step1~4 r times, $\sum_{i=1}^{r} X_i \sim NB(r,p)$