Random Numbers

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Notation: $\mathbf{X}$, random variable associated with certain distribution.

$X = x,$ a **realization** of X.

Note: Random numbers ($U(0,1)$) or random variates (transformation of $U(0,1)$) simulate *realizations* of random variables.

- The building block of a simulation study is the ability to generate random numbers, where a random number represents the value of a random variable uniformly distributed on $(0, 1)$.
- In this chapter we explain how such numbers are computer generated and also begin to illustrate their uses.

Pseudo-random Number Generation

- Mechanical approach:
  - **Manual methods**: Spinning wheels/ rolling dice/ shuffling cards, . . .
    – **Handy** but slow and **not** *reproducible*!
  - **Random number table**: **Reproducible**.
    – Slow and easy to **run out of table**.
  - **Mid-square method**: Middle part of the square of the preceeding numbers.
    – Random? **Terminated at zero**!
- Modern approach: **Computer**.
  – VonNeumann and Ulam during World War II.

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Pseudo-random Number Generators

$$x_i = f(x_{i-1}, \ldots, x_{i-k}), \quad \text{multiple recursion.}$$

Note: 1. The number of previous numbers used, k, is called the *order* of the generator.

2. The value at the **start** of the resursion is called the *seed*.

3. Each time the recursion is begun with the **same seed**, the **same sequence** is generated. It's reproducible!

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Congruential Generator

(Lehmer (1951).)

Notation: a, b, c, k are integers, and $c \neq 0$.

$a \equiv b \mod c$ if and only if $b = ck + a$, for some k and $0 \leq a < c$.

Congruential Method:

$$\mathbf{X_i \equiv (aX_{i-1} + c) \mod m, \quad i = 1, 2, 3, \ldots,}$$

where a, c, m and X_0 are integers.

Take $U_i = X_i/m \in [\mathbf{0}, \mathbf{1}), i = 1, 2, 3, \ldots$.

Note: 1. The *initial value* X_0 is the **seed**. Here, order=1.

2. $c = 0$ – *multiplicative congruential method.*

3. $c \neq 0$ – *mixed congruential method.*

4. $\{U_1, U_2, \ldots\}$ contains **at most** m distinct numbers

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<u>Ex</u>. 1. $a = 2, c = 1, m = 8$, i.e. $X_i \equiv 2X_{i-1} + 1 \mod 8$.

Choose $X_0 = 2$.

i	0	1	2	3	4	$\cdots$
X_i	2	5	3	7	7	$\cdots$
U_i	.25	.625	.375	.875	.875	$\cdots$

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Ex. 1. $a = 5, c = 1, m = 8$, i.e. $X_i \equiv 5X_{i-1} + 1 \mod 8$.

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i	0	1	2	3	4	5	6	7	8
X_i	2	3	0	1	6	7	4	5	2

"**Full**" period! □

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Note: a and m should be chosen such that

1. For any initial seed, the resultant sequence has the "appearance" of being a sequence of **independent uniform** $(0, 1)$ random variables.
2. For any initial seed, the number of variables that can be generated before repetition begins is large.
3. The values can be computed efficiently on a digital computer.

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Suggestions: 1. For a 32-bit word machine,

$$m = 2^{31} - 1, a = 7^5 = 16807, c = 0.$$

2. For a 36-bit word machine,

$$m = 2^{35} - 31, a = 5^5, c = 0.$$

3. In IMSL, $m = 2^{31} - 1$, $a = 16807$, 397204094, 950706376 with **CALL RNOPT(OPT)**, OPT=1, 3, 5, respectively.

We assume that we have a "black box" that gives a random number on request.

Splus:

- Continuous uniform(a, b): `runif(n, min=a, max=b)`, `runif(10, 2, 4)`, `runif(10, max=4)`, `runif(10)`
- Discrete uniform(m, n): `m + floor((n-m+1)*r)` where $r \sim U(0, 1)$.

Applications

One of the earliest applications of random numbers was in the computation of integral.

Case 1. Integration.

Ex. 1. $\int_0^1 g(x)dx = ?$

Recall: $(SLLN)$ If $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with pdf $f(\cdot)$, then

$$\mathbf{\frac{1}{n}\sum_{i=1}^{n} g(X_i) \longrightarrow E[g(X)],}\ \text{a.s.,} \quad \text{as } n \to \infty,$$

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provided that $E[g(X)] = \int g(x)f(x)dx$ exists.

Now consider $X \sim U(0,1)$, then $f(x) = \mathbf{1}_{(0,1)}(x)$. Thus,

$$\theta = \int_0^1 g(x)dx = \int_{\mathbf{0}}^{\mathbf{1}} g(x) \cdot \mathbf{1} dx = \int g(x)\mathbf{f}(\mathbf{x})dx = \mathbf{E}[\mathbf{g}(\mathbf{X})],$$

with $X \sim U(0,1)$. $\square$

Hence, the integral can be found via **statistical approach**.

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Algorithm: 1. **Generate** $U_1, U_2, \ldots, U_n \sim U(0,1)$.

2. Compute $g(U_i), i = 1, 2, \ldots, n$.

3. $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} g(U_i)$. □

Note: This approach to approximating integrals is called the **Monte-Carlo** approach. □

Splus code

to approximate $\theta = \int_0^1 e^x \, dx$:

```
a_runif(100000);
v_exp(a);
print("Approximate value: ");
mean(v);
print("True value: ");
print(exp(1)-1)
```

Ex. 2. $\int_a^b g(x)dx = ?$

Method I: Here

$$\theta = \int_a^b g(x)dx = (b-a)\int_a^b g(x)\frac{1}{b-a}dx = (b-a)E[g(X)],$$

where $X \sim U(a,b)$.

So $\hat{\theta} = \frac{b-a}{n}\sum_{i=1}^n g(X_i), \quad X_i \sim U(a,b)$???

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Recall: $U \sim U(0,1)$ then $\mathbf{X} = a + (b-a)U \sim \mathbf{U(a,b)}$.

Algorithm: 1. Generate $U_1, U_2, \ldots, U_n \sim U(0,1)$.

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Method II: Make a *change of variables* such that

$$\int_a^b g(x)dx = \int_0^1 h(y)dy = E[h(Y)],$$

with $Y \sim U(0,1)$ for some h.

Consider $y = (x-a)/(b-a)$ so $dx = (b-a)dy$ and

$h(y) = g(a+(b-a)y)(b-a)$. Thus

$$\theta = \int_0^1 g(a+(b-a)y)(b-a)dy = E[(b-a)g(a+(b-a)Y)],$$

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Ex. 3. $\int_0^\infty g(x)dx = ?$

Consider $y = 1/(1+x)$ s.t. $\theta = \int_0^1 h(y)dy$, where

$$h(y) = \frac{g(\frac{1}{y} - 1)}{y^2}.$$

□

Ex. 4. $\int_{-\infty}^0 g(x)dx = ?$

Consider $y = 1/(1-x)$ s.t. $\theta = \int_0^1 h(y)dy$, for some h. □

Ex. 5. $\int_{-\infty}^\infty g(x)dx = ?$

Note that $\int_{-\infty}^\infty g(x)dx = \int_{-\infty}^0 g(x)dx + \int_0^\infty g(x)dx$. □

<u>Ex</u>. 3. $\int_0^{\infty} g(x)dx = ?$

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<u>Ex</u>. 4. $\int_{-\infty}^{0} g(x)dx = ?$

Consider $y = 1/(1-x)$ s.t. $\theta = \int_0^1 h(y)dy$, for some h. $\square$

<u>Ex</u>. 5. $\int_{-\infty}^{\infty} g(x)dx = ?$

Note that $\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{0} g(x)dx + \int_0^{\infty} g(x)dx$. $\square$

Ex. 3. $\int_0^{\infty} g(x)dx = ?$

Consider $y = 1/(1+x)$ s.t. $\theta = \int_0^{1} h(y)dy$, where

$$h(y) = \frac{g(\frac{1}{y} - 1)}{y^2}.$$

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Similarly, for $\theta = \int_0^1 \ldots \int_0^1 g(x_1, \ldots, x_k) dx_1 \ldots dx_k = \mathbf{E}[\mathbf{g}(\boldsymbol{X})]$, where $\boldsymbol{X} = (X_1, \ldots, X_k) \sim U[0,1]^k$, or $X_1, \ldots, X_k \overset{\text{i.i.d.}}{\sim} U(0,1)$,

we can use

$$\hat{\theta} = \frac{\mathbf{1}}{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{g}(\boldsymbol{X}_{\mathbf{j}}) = \frac{\mathbf{1}}{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{g}(\mathbf{X}_{\mathbf{1}}^{\mathbf{j}}, \ldots, \mathbf{X}_{\mathbf{k}}^{\mathbf{j}}),$$

where $X_i^j \sim U(0,1)$, for $i = 1, \ldots, k, \;\; j = 1, \ldots, n.$ $\square$

Case 2. Estimating Probabilities.

Ex. The estimation of π.

$$\pi = \text{Area of a circle of radius 1.}$$

i.e. Area of $A = \{(x, y) : x^2 + y^2 \leq 1\}$.

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Let $\Omega = [-1,1] \times [-1,1]$ and (X, Y) be a point **randomly** selected within Ω, then

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\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} 1/4 & \text{if } -1 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases} = f_X(x) \cdot f_Y(y) \\
&= \frac{1}{2}\mathbf{1}_{(-1,1)}(x)\,\frac{1}{2}\mathbf{1}_{(-1,1)}(y).
\end{aligned}
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That is, $X, Y \overset{\text{i.i.d.}}{\sim} U(-1,1)$, or (X, Y) is **uniformly distributed over Ω**.

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Consider $A = \{(x, y) : x^2 + y^2 \le 1, \} \subset \Omega$. Then $|A| = \pi$ and $|\Omega| = 4$.

Thus,

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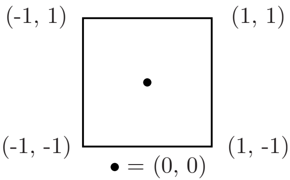
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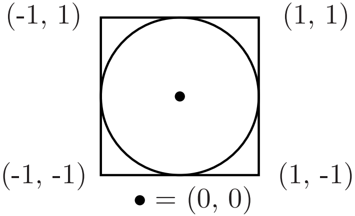
(1, 1)
(-1, -1)
(1, -1)
• = (0, 0)


(-1, 1)
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Q: How to estimate $\theta = P((X, Y) \in A)$?

Answer: Let $Z = 1_A(X, Y)$, an indicator random variable. Then

$$\mathbf{E(Z) = P((X, Y) \in A)} = \theta.$$

Hence, if $Z_1, \ldots, Z_n \overset{\text{i.i.d.}}{\sim} Z$, we have

$$\hat{\theta} = \mathbf{\frac{1}{n}\sum_{i=1}^{n} Z_i} \longrightarrow \theta, \text{ a.s., as } n \to \infty.$$

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Algorithm: 1. Generate $(X_1, Y_1), \ldots, (X_n, Y_n) \overset{\text{i.i.d.}}{\sim} U(-1, 1)^2$.

$(X_i = -1 + 2U_i,\ Y_i = -1 + 2U_i',\ i = 1, \ldots, n.)$

2. **If** $X_i^2 + Y_i^2 \leq 1$, set $Z_i = 1$; **otherwise**, set $Z_i = 0$, for $i = 1, \ldots, n$.

3. Set $\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} Z_i$.

4. Set $\hat{\pi} = 4 \cdot \hat{\theta}$.

Splus code

:

```
estimatePI_function(n){
u1_runif(n,-1,1);
u2_runif(n,-1,1);
s2_u1∧2+u2∧2;
cat("Esimate of PI: ", length(s2[s2<1])/n,"\n");
}
n_100000;
estimatePI(n);
```