Equivalence of Semidomain and Integral Domain Property of Grothendieck Gr

We aim to prove that for a semiring S, the following conditions are equivalent:

(a) S is a semidomain.

(b) The multiplication of S extends to the Grothendieck group $\mathcal{G}(S)$ of $(S,+)$ turning $\mathcal{G}(S)$ into an integral domain.

(a) implies (b):

Suppose S is a semidomain. We want to show that the multiplication on S extends to the Grothendieck group $\mathcal{G}(S)$, making it an integral domain.

- Consider the Grothendieck group $\mathcal{G}(S)$. The elements of $\mathcal{G}(S)$ can be written as formal differences $a - b$ with $a, b \in S$, and the operation in $\mathcal{G}(S)$ is induced by the addition in S.
- We define multiplication in $\mathcal{G}(S)$ as follows: for $a - b$ and $c - d$ in $\mathcal{G}(S)$, set

$$(a - b)(c - d) = ac + bd - (ad + bc).$$

- Since S is a semidomain, $ac = 0$ implies $a = 0$ or $c = 0$. Thus, $\mathcal{G}(S)$, with this multiplication, has no zero divisors, making it an integral domain.

(b) implies (a):

Now, suppose the multiplication on S extends to the Grothendieck group $\mathcal{G}(S)$, turning it into an integral domain. We aim to show that S is a semidomain.

- If S were not a semidomain, there would exist non-zero elements $a, b \in S$ such that $ab = 0$. However, in $\mathcal{G}(S)$, the elements of S naturally embed, so $a - 0$ and $b - 0$ are non-zero in $\mathcal{G}(S)$.

- The product $(a - 0)(b - 0) = ab - 0 = 0$ would contradict the fact that $\mathcal{G}(S)$ is an integral domain with no zero divisors.

- Hence, S must be a semidomain, since $ab = 0$ cannot occur for non-zero a and b.

Conclusion: Both implications hold, thus S is a semidomain if and only if the multiplication on S extends to the Grothendieck group $\mathcal{G}(S)$, turning it into an integral domain.