Max-flow and Circulation Comparison

Max-flow

- $G = (V, E)$ = directed graph,
- Two distinguished nodes:
  - s = source, t = sink.
- c(e) = capacity of edge e.
- $$0 \le f(e) \le c(e)$$
  $$\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$$
- max flow = min cut
- Algorithms:
- Generic augmenting path:
  - O(m val(f*)).
- Capacity scaling:
  - $O(m^2 \log C)$
- *Shortest augmenting path:
  - $O(m^2 n)$.
- * Preflow-Push:
  - $O(m\, n^2)$ or $O(n^3)$.

Circulation with demands

G

- Node supply and demands d(v), $v \in V$.
  - demand if $d(v) > 0$;
  - supply if $d(v) < 0$;
  - transshipment if $d(v) = 0$
- Conservation
  $$\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$$
- $\Rightarrow$ Necessary condition to have a circulation
  $$\sum_{v : d(v) > 0} d(v) = \sum_{v : d(v) < 0} - d(v) =: D$$
- Convert to network flow:

G'

  - Add new source s and sink t.
  - For each v with $d(v) < 0$, add edge (s, v) with capacity $-d(v)$.
  - For each v with $d(v) > 0$, add edge (v, t) with capacity $d(v)$.
  - Claim: G has circulation iff G' has max flow of value D
    (saturates all edges leaving s and entering t)

$A \Leftrightarrow B$

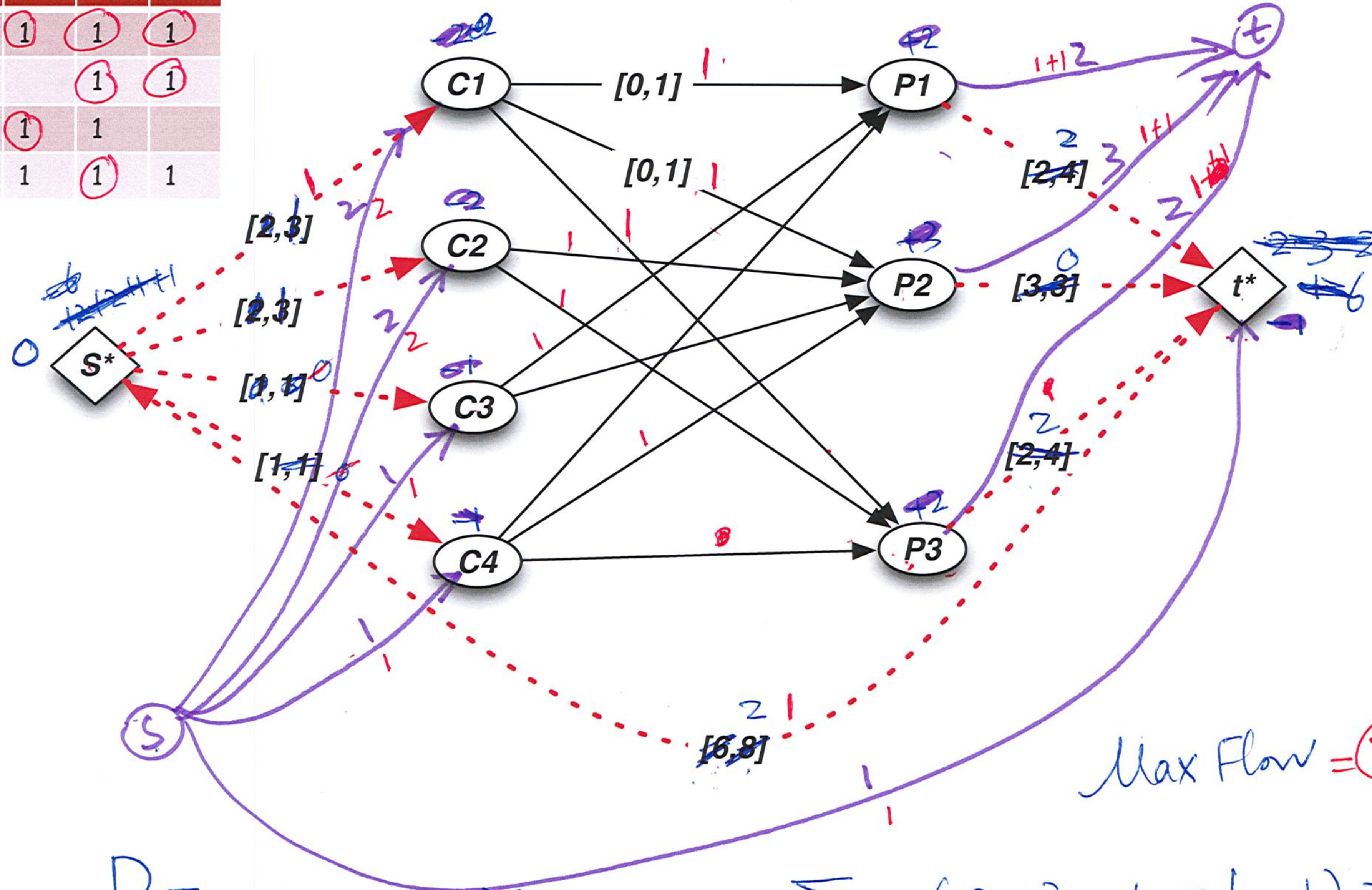
- with Demands and Lower Bound:
  $\ell(e) \le f(e) \le c(e)$
- Transfer each edge e: (v, w):

$$v \xrightarrow{l(e)} w$$
$d(v)+l(e)$ $\qquad$ $d(w)-l(e)$

- $d(v)=d(v)+l(e)$; $d(w)=d(w)-l(e)$; $c(e)=c(e)-l(e)$

	P1	P2	P3
	3	3	2
3 C1	1	1	1
2 C2		1	1
C3	1	1	
C4	1	1	1

Convert to Network Flow Problem



Max Flow = 7

$$D = \sum_{d(v)>0} 2+3+2 = 7 = -\sum_{d(v)<0} (-2,-2,-1,-1,-1) = 7$$

demand supply

Survey Design Example

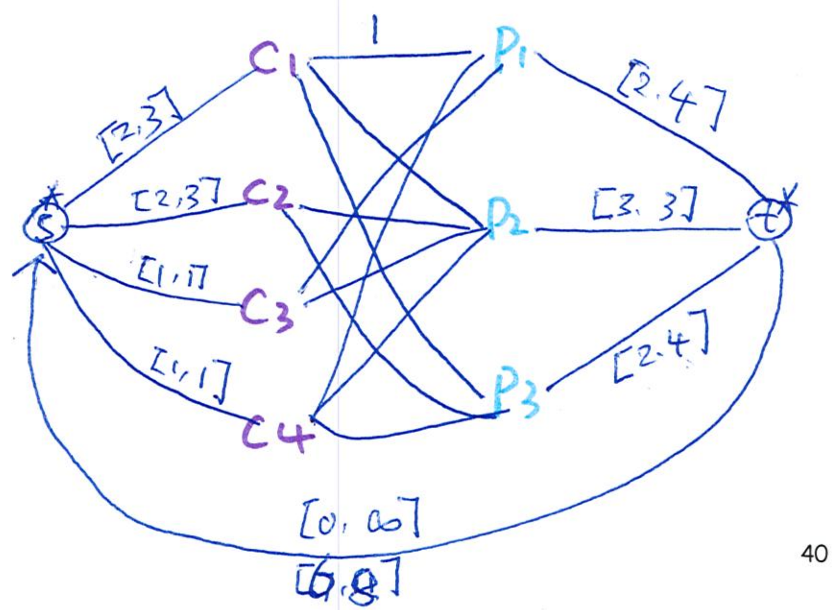
Survey design.

- Design survey asking 4 consumers about 3 products.
- Can only survey consumer i about product j if they own it, see table below.
- Ask consumer 1, consumer 2 each between 2 and 3 questions.
- Ask consumer 3, consumer 4 each 1 question only.
- Ask between 2 and 4 consumers about product 1.
- Ask 3 consumers about product 2.
- Ask between 2 and 4 consumers about product 3.

Link: possible asking
flow: actual question

Goal. Design a survey that meets these specs, if possible.

	P1	P2	P3
C1	1	1	1
C2		1	1
C3	1	1	
C4	1	1	1



Project Selection

can be positive or negative

Projects with prerequisites.

- Set P of possible projects. Project v has associated revenue p_v.
  - some projects generate money: create interactive e-commerce interface, redesign web page
  - others cost money: upgrade computers, get site license
- Set of prerequisites E. If $(v, w) \in E$, can't do project v and unless also do project w.
- A subset of projects $A \subseteq P$ is feasible if the prerequisite of every project in A also belongs to A.

-10 UC $\longrightarrow$ BW $+100$

X	A.	BW	X
−10	B.	UC	✓
90	C.	UC. BW	✓
0	D.	∅	

Project selection. Choose a feasible subset of projects to maximize revenue.

C

Project	Prerequisites	Profit
P1		-10
P2	P1	15
P3	P1, P2	-5
P4	P2	10
P5	P3	20

$p_v > 0$ 45

$p_v < 0$ −15