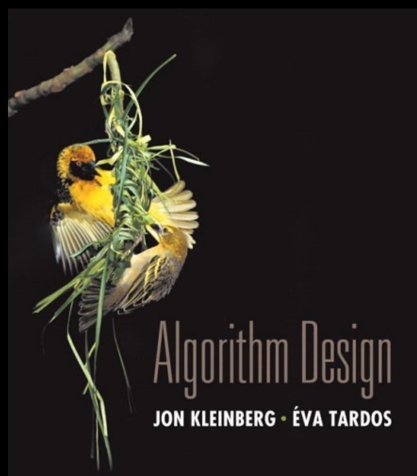

Chapter 4

Greedy Algorithms

1

4.0 Greedy Algorithm

- ✓ Build up a solution in small steps
- ✓ Choosing a decision myopically to optimize some criterion.
- ✓ Not always find the global optimal solution
- ✓ Challenge 1: how to choose the criterion used at each step
- ✓ Challenge 2: how to prove it works when it does find the optimal solution

2

Coin Changing

> Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.
>
> \- *Gordon Gecko (Michael Douglas)*

3

Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest number of coins.

Ex: 34¢.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex: $2.89.

4

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Coin-Changing: Greedy Algorithm

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

```
Sort coins denominations by value: c1 < c2 < … < cn.

  coins selected
S ← φ
while (x ≠ 0) {
   let k be largest integer such that ck ≤ x
   if (k = 0)
      return "no solution found"
   x ← x - ck
   S ← S ∪ {k}
}
return S
```

Q. Is cashier's algorithm optimal?

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Coin-Changing: Analysis of Greedy Algorithm

Theorem. Greed is optimal for U.S. coinage: 1, 5, 10, 25, 100.

Pf. (by induction on x)

- Consider optimal way to change $c_k \le x < c_{k+1}$: greedy takes coin k.
- We claim that any optimal solution must also take coin k.
  - if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x
  - table below indicates no optimal solution can do this
- Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm. ▪

k	c_k	All optimal solutions must satisfy	Max value of coins 1, 2, …, k-1 in any OPT
1	1	$P \le 4$	-
2	5	$N \le 1$	4
3	10	$N + D \le 2$	4 + 5 = 9
4	25	$Q \le 3$	20 + 4 = 24
5	100	no limit	75 + 24 = 99

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Coin-Changing: Analysis of Greedy Algorithm

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.
- Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
- Optimal: 70, 70.

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4.1 Interval Scheduling

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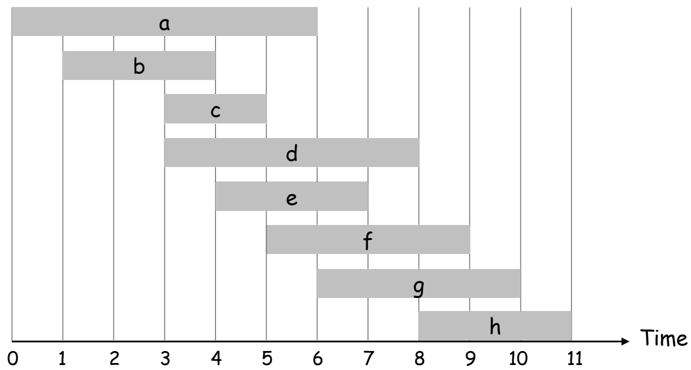
Interval Scheduling

Interval scheduling.
- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



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Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of start time s_j.
- [Earliest finish time] Consider jobs in ascending order of finish time f_j.
- [Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.
- [Fewest conflicts] For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

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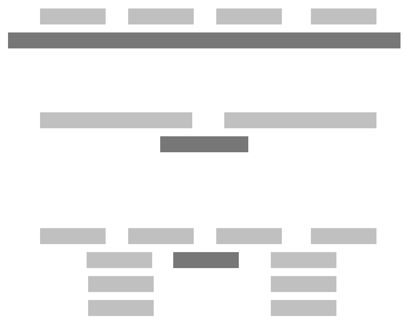
Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.



breaks earliest start time

breaks shortest interval

breaks fewest conflicts

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Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

  jobs selected
A ← ϕ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. – How to check "`job j compatible with A`"?
- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.

Q: Time complexity?

O(n log n).

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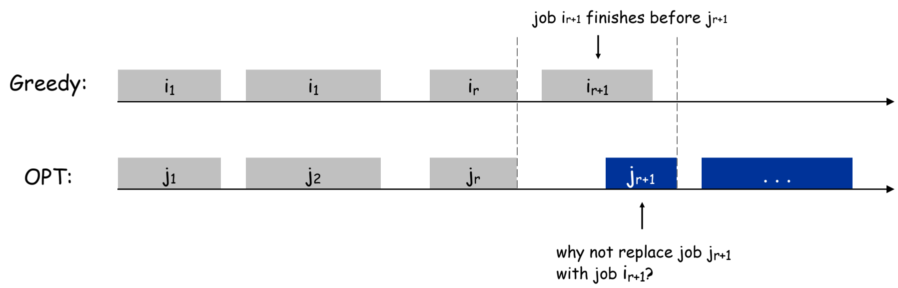
Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



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Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

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- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
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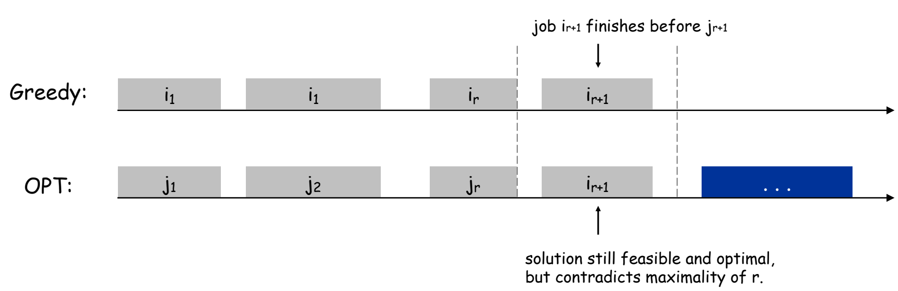

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4.1b Interval Partitioning

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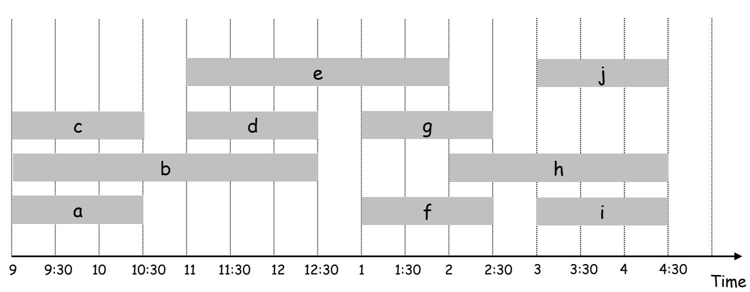
Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

Q: can you do better?



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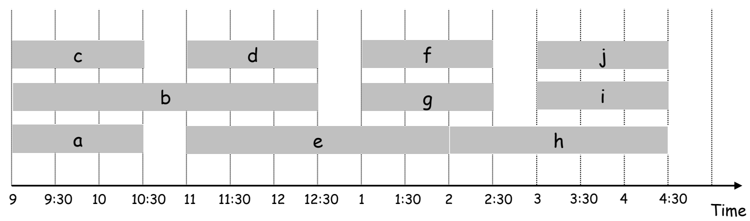
Interval Partitioning

Interval partitioning.
- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



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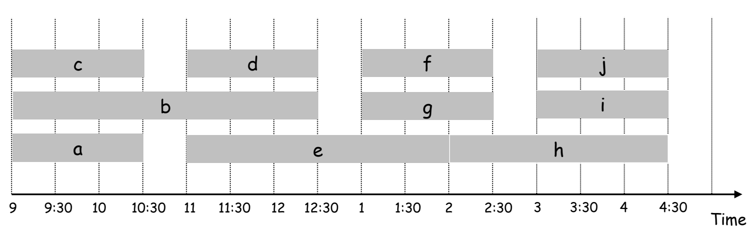
Interval Partitioning: Lower Bound on Optimal Solution

Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = $\underset{\text{a, b, c all contain 9:30}}{\underset{\uparrow}{3}}$ $\Rightarrow$ schedule below is optimal.

Q. Does there always exist a schedule equal to depth of intervals?



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Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0   ← number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. Q: How to check if "lecture j is compatible with some classroom k"

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Q: Time complexity?

O(n log n).

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Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation $\Rightarrow$ all schedules use $\geq$ d classrooms. ▪

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4.2 Scheduling to Minimize Lateness

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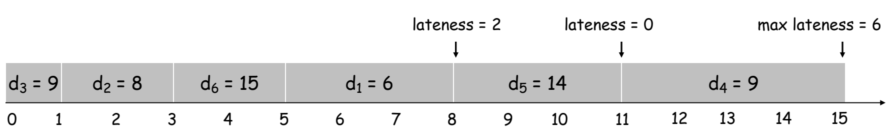
Scheduling to Minimizing Lateness

Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.
- Goal: schedule all jobs to minimize **maximum** lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



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Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j.
- [Earliest deadline first] Consider jobs in ascending order of deadline d_j.
- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

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Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

counterexample

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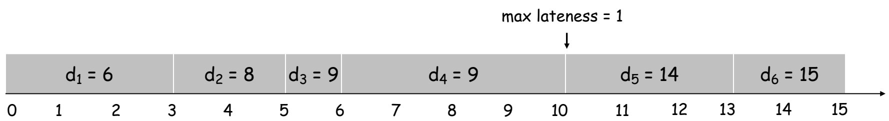
Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

```
Sort n jobs by deadline so that d1 ≤ d2 ≤ … ≤ dn

t ← 0
for j = 1 to n
   Assign job j to interval [t, t + tj]
   sj ← t, fj ← t + tj
   t ← t + tj
output intervals [sj, fj]
```



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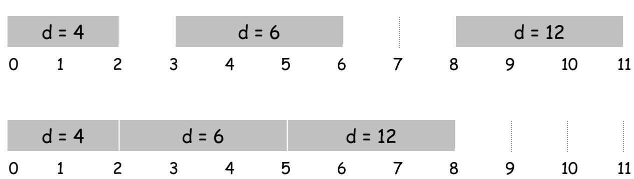
Minimizing Lateness: No Idle Time

Observation. There exists an optimal schedule with no idle time.



Observation. The greedy schedule has no idle time.

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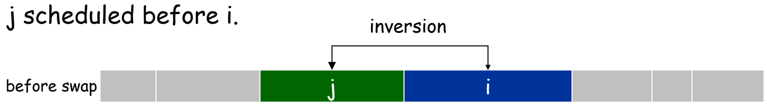
Minimizing Lateness: Inversions

Def. An inversion in schedule S is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i.



Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Consider the four jobs: i, i+1,...j-1, j with deadline $d_i, d_{i+1}, \ldots, d_{j-1}, d_j$
If job i and j is an inversion, ➔ $d_i > d_j$, then one of the following must be true:
1. job i, i+1 is another inversion
2. job j-1, j is another inversion
3. job i+1, j-1 is another inversion (if they are not consecutive, repeat the reasoning process)

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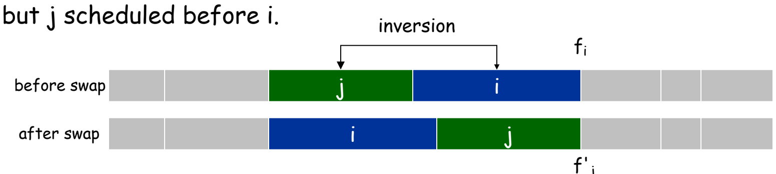
Minimizing Lateness: Inversions

Def. An inversion in schedule S is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i.



Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.
Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

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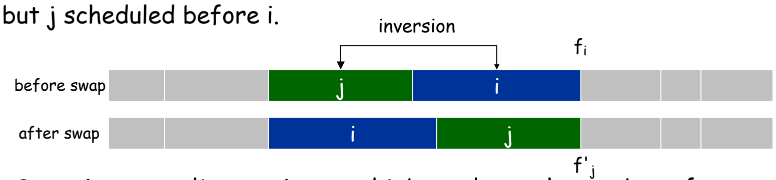
Minimizing Lateness: Inversions

Def. An inversion in schedule S is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i.



Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell_j \leq \ell_i$ (old max)
- $\ell = \max(\ell_j, \ell_i) = \ell_i$
- $\ell'_i \leq \ell_i$ (old max)
- If job j is late:
- $\ell'_j \leq \ell_i$
- $\max(\ell'_i, \ell'_j) \leq \ell_i$

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && d_i < d_j \\ &\leq \ell_i && \text{(definition)} \end{aligned}$$

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Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S* has no idle time.
- If S* has no inversions, then S = S*.
- If S* has an inversion, let i-j be an adjacent inversion.
  - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of S* ▪

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Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

- Example: interval scheduling

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

- Example: scheduling to minimize lateness

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

- Example: interval partition

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