Marriage Theorem

Marriage Theorem. [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

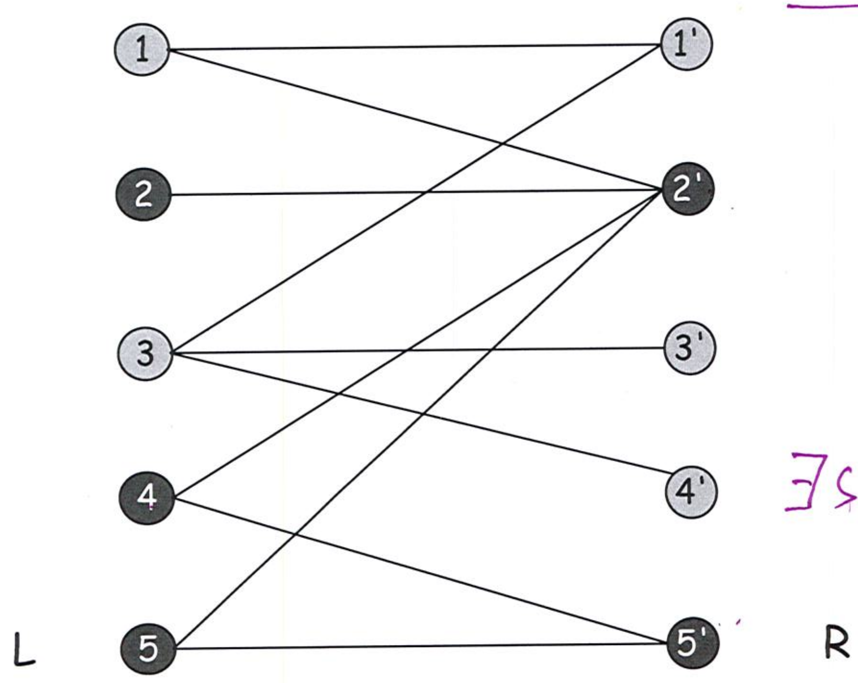
$|N(S)| = |S|$

Pf. $\Rightarrow$ This was the previous observation.

$S = \{1, 2\}$ $\quad$ $S = \{2, 4, 5\}$

$N(S) = \{1', 2'\}$ $\quad$ $N(S) = \{2', 5'\}$

$|N(S)| = 2 < |S| = 3$



No perfect matching:

S = { 2, 4, 5 }

N(S) = { 2', 5' }.

$\exists S,\ |N(S)| < |S|$

No Perfect Matching.

Edge Disjoint Paths

Disjoint path problem. Given a digraph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s-t paths.

Def. Two paths are edge-disjoint if they have no edge in common.

Ex: communication networks.

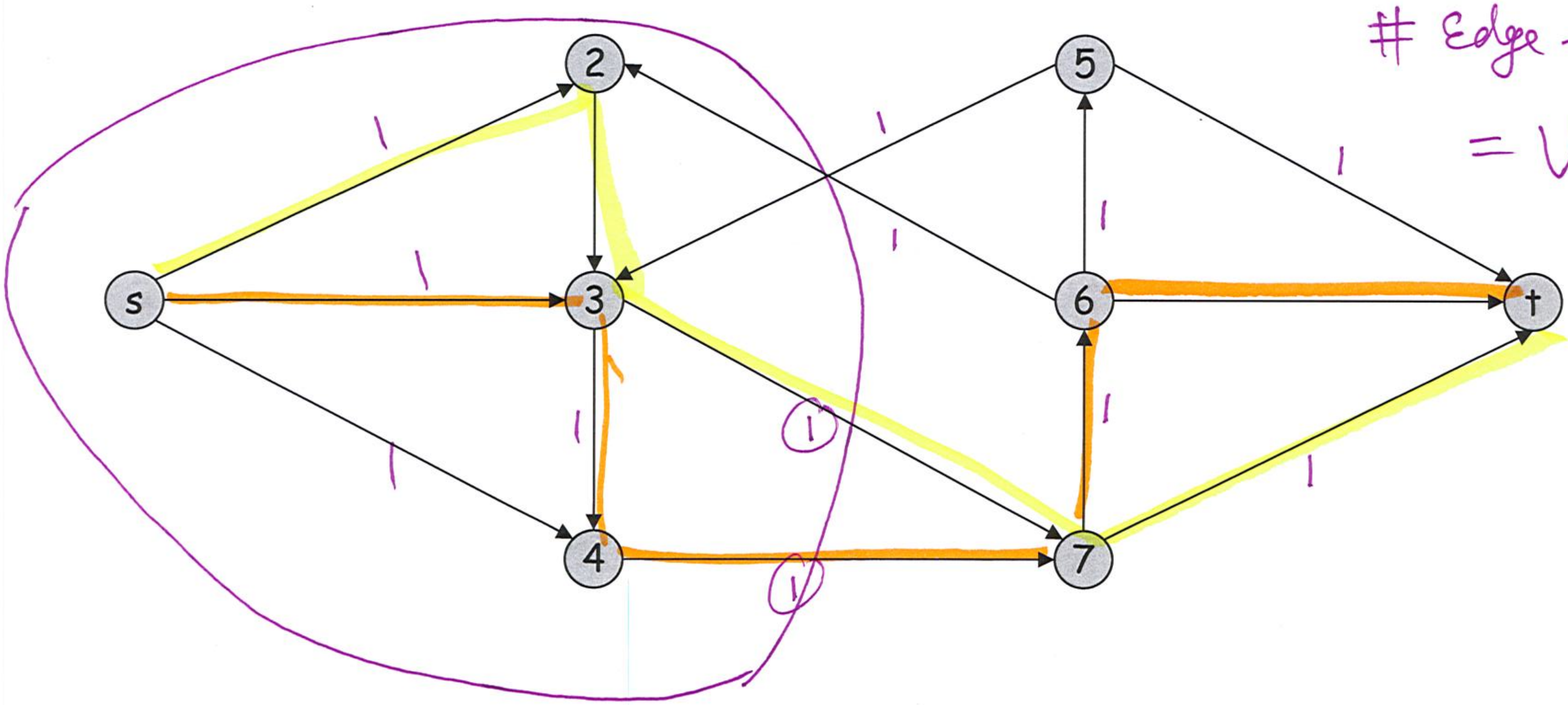
max flow = 2

min Cut = 2

$$\frac{\text{Flow-Network}}{c(e)}$$

Class Exercise: Find the max number of edge-disjoint s-t paths!

Edge-disjoint Path = Val(max-flow)



Circulation with Demands

Circulation with demands.

- Directed graph $G = (V, E)$.
- Edge capacities $c(e)$, $e \in E$.
- Node supply and demands $d(v)$, $v \in V$.

demand if $d(v) > 0$; supply if $d(v) < 0$; transshipment if $d(v) = 0$

• No s. t.

Flow Network $d(v)=0$

$d(a)=1$ demand

$\xrightarrow[3]{5\ \cancel{3}} a \xrightarrow[5]{\cancel{2}\ 6}$

$d(a)=-2$ supply

Def. A circulation is a function that satisfies:

- For each $e \in E$: $\quad 0 \le f(e) \le c(e)$ (capacity)
- For each $v \in V$: $\quad \sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem: given (V, E, c, d), does there exist a circulation?

Flow Network S source supply. $d(s)$ ~~>0~~ $\boxed{<0}$

Sink t. $\boxed{d(t) > 0}$ demand.

other nodes. v. $d(v) = 0$ $\quad \sum_{e \text{ in } v} f(e) = \sum_{e \text{ out } v} f(e)$

Circulation with Demands

Necessary condition: sum of supplies = sum of demands.

$$\sum_{v : d(v) > 0} d(v) = \sum_{v : d(v) < 0} -d(v) =: D$$

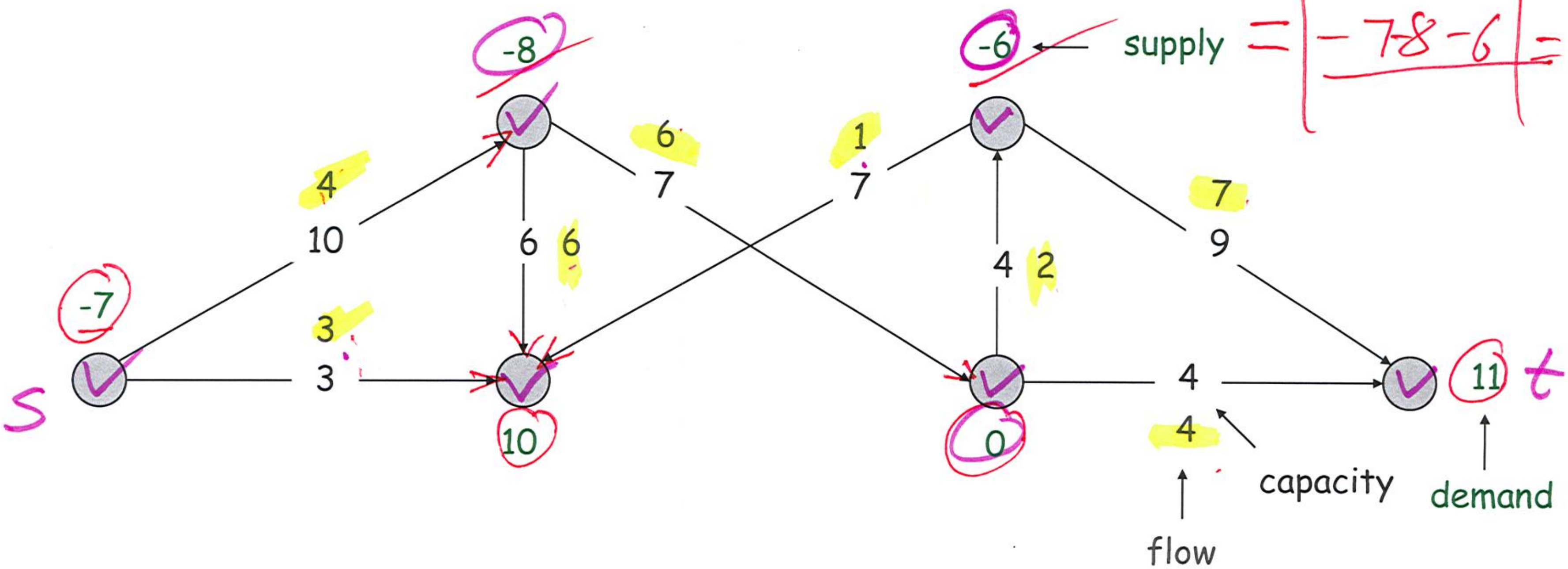
amount of total demand

Pf. Sum conservation constraints for every demand node v.

= |total supply|.

$D = 10 + 11 = 21$

$= |-7 - 8 - 6| = |-21|$



Circulation with Demands

$$\sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v) =: D$$

Max flow formulation.

- Add new source s and sink t.
- For each v with d(v) < 0, add edge (s, v) with capacity -d(v). >0
- For each v with d(v) > 0, add edge (v, t) with capacity d(v).
- Claim: G has circulation iff G' has max flow of value D.

YES.

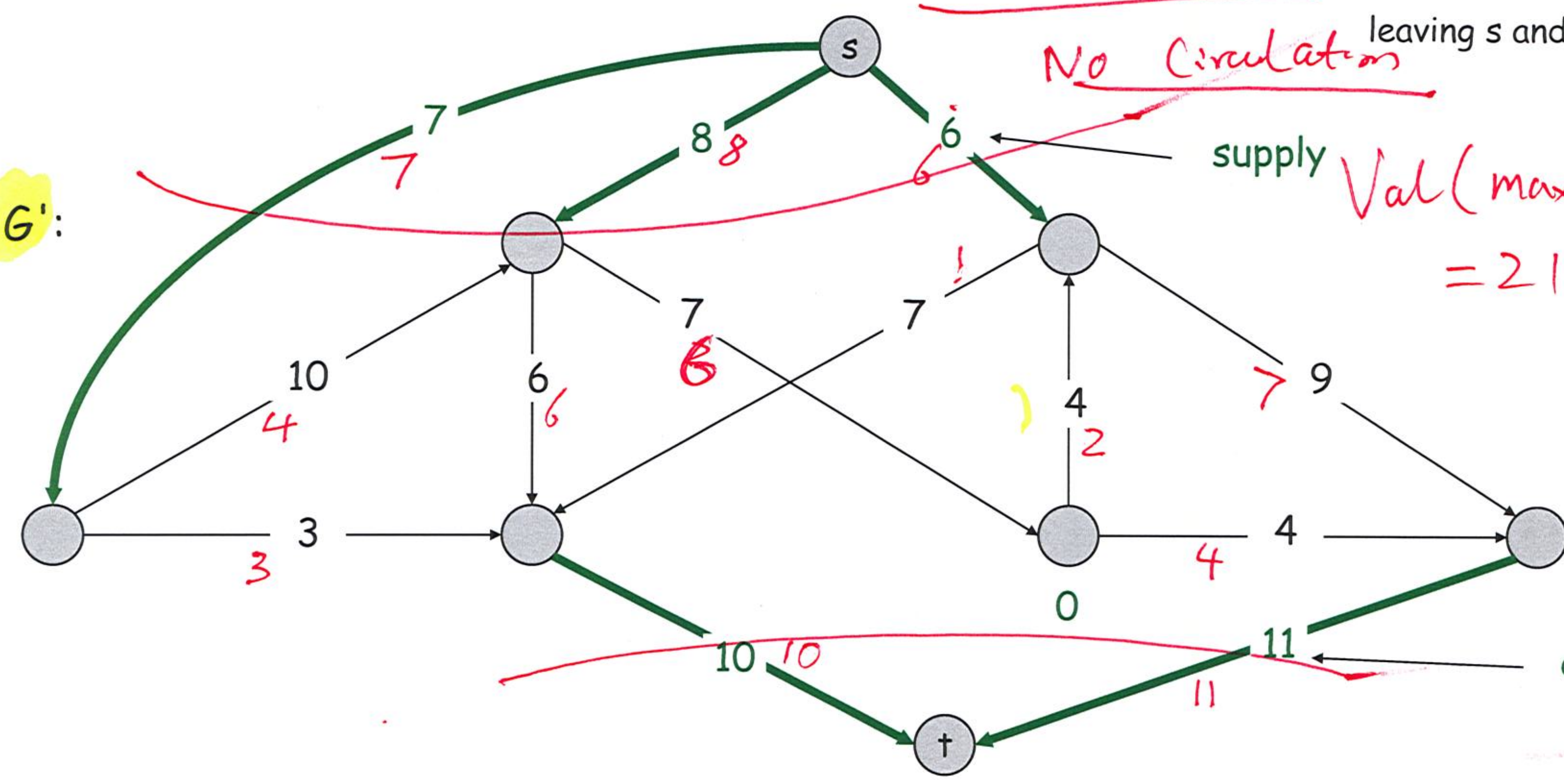
V(max flow) < D

No Circulation

saturates all edges leaving s and entering t

Val(max flow) = 21 = D.

G':

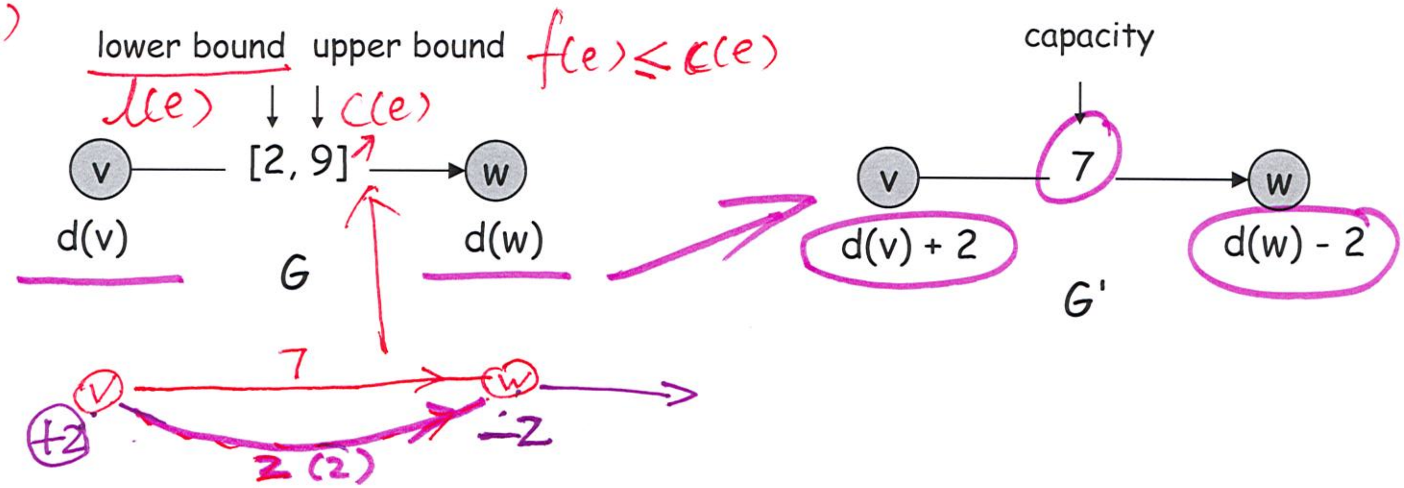


Circulation with Demands and Lower Bounds

Idea. Model lower bounds with demands.

- Send $\ell(e)$ units of flow along edge e.
- Update demands of both endpoints.

$\ell(e) \leq f(e)$
min speed



Theorem. There exists a circulation in G iff there exists a circulation in G'. If all demands, capacities, and lower bounds in G are integers, then there is a circulation in G that is integer-valued.

Pf sketch. f(e) is a circulation in G iff $f'(e) = f(e) - \ell(e)$ is a circulation in G'.