Curves on torus layers and coding for continuous alphabet sources

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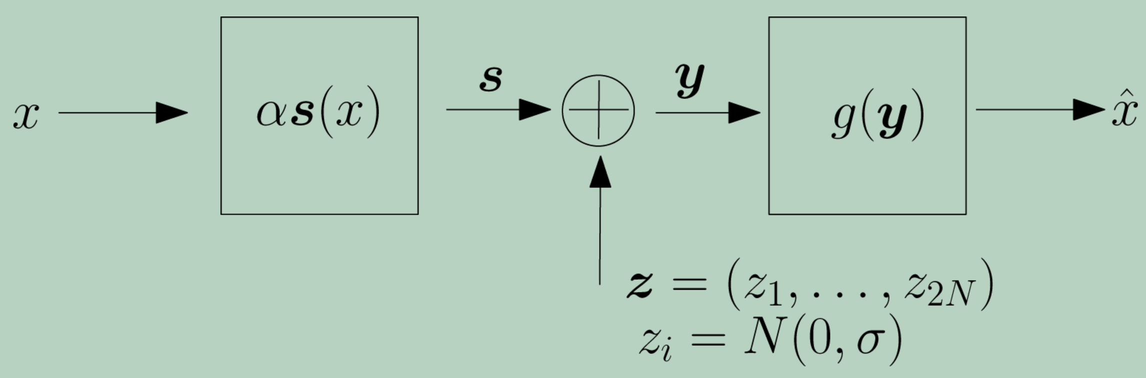
Introduction

The problem of designing good codes for continuous alphabet sources to be transmitted over a channel with power constraint can be viewed as the one of constructing curves in the Euclidean space of maximal length and such that its folds (or laps) are a good distance apart. In this work we consider the problem of transmitting a continuous alphabet discrete-time source over an AWGN channel and propose a scheme based on curves on layers of torus lying on the surface of a sphere in $\mathbb{R}^{2N}$. We show that the perfomance of this scheme is related to geometrical properties of spherical codes and projections of N-dimensional rectangular lattices. An approach to these problems, as well as comparisons with some previous constructions are provided.

Communication Model

The communication system we consider here is illustrated below. Given an input real value x, within the unit interval $[0, 1]$, the encoder maps x into a point $\boldsymbol{s}(x)$ of a curve $\boldsymbol{s}$ on S^{2N-1}, which will be sent over an AWGN channel with power constraint and dimension $2N$. The objective of the decoder is to compute an estimate for the sent value attempting to minimize the mean squared error (mse) $E[(X - \hat{X})^2]$ of the process.



Given $\rho > 0$, let E_ρ denote the event $\|\boldsymbol{z}\| > \rho$. Then the mse can be evaluated as:

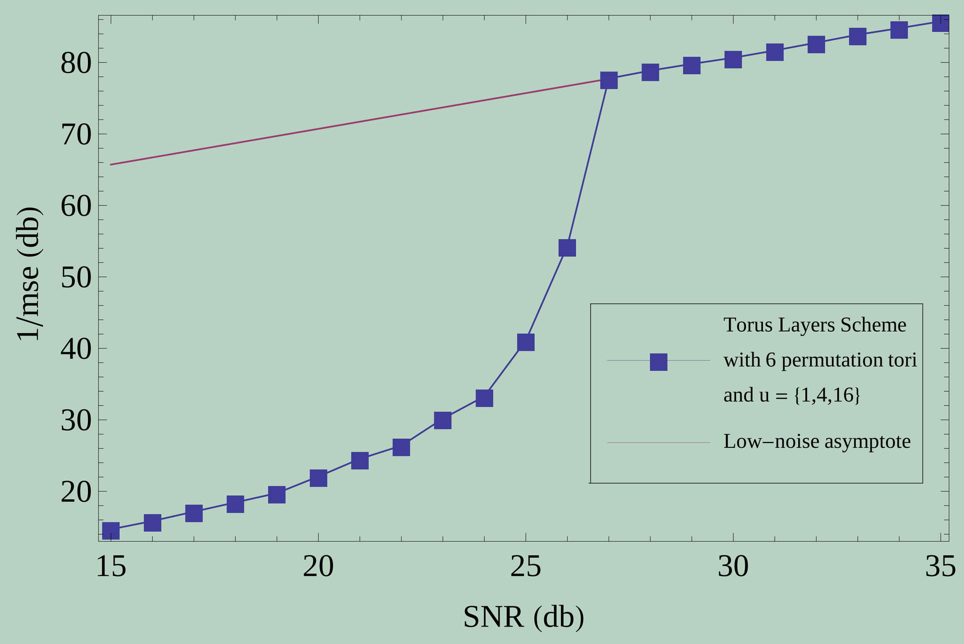
$$E[(X - \hat{X})^2] = E[(X - \hat{X})^2 | E_\rho] P(E_\rho) + E[(X - \hat{X})^2 | E_\rho^c] P(E_\rho^c) \quad (1)$$

The terms E_ρ and E_ρ^c are called small-error and large-error events, respectively.

Coding schemes for this communication model generally behave as follows:

- When the SNR of the channel is low, the system performs poorly due to large errors on the decoding process.
- When the SNR reaches a certain threshold, the term $E[(X - \hat{X})^2 | E_\rho]$ will dominate Equation (1) and the scheme will reach a low noise asymptote.

This behavior is displayed in the figure below for some choice of parameters of our proposed scheme.

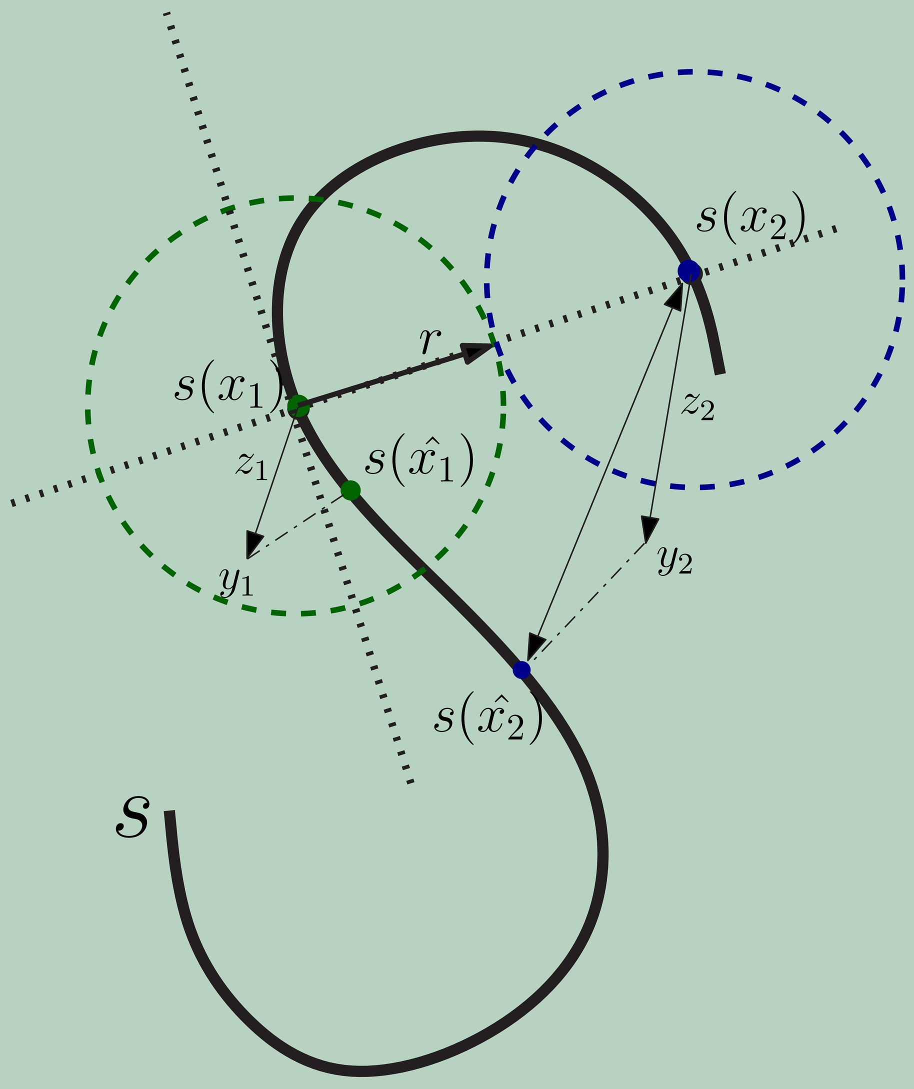


Curves on a sphere

When the transmission power is constant, the signal locus is a curve on the surface of a sphere. From an information theoretical point of view, two important properties of this curve are its stretch and small-ball radius defined as

- The *stretch* $\mathcal{S}(x) \triangleq \|\dot{\boldsymbol{s}}(x)\|$ gives the resolution when estimating the sent value (i.e., the low noise asymptote).
- The *small-ball radius* is the largest $\delta > 0$ such that $B_\delta(\boldsymbol{s}(x)) \cap H(x) \subset V(x)$ for all $x \in [0, 1]$. It defines how "large" is the threshold to the large-error event. See figure on the right.

We are interested in curves with both stretch *and* small-ball radius as large as possible (which are contrary objectives!).



Our scheme

In our scheme, the interval $[0, 1)$ will be partitioned into M sub-intervals and each of them mapped into a curve on a flat torus lying on the sphere S^{2N-1}. Given $x \in [0, 1]$ the encoding function is:

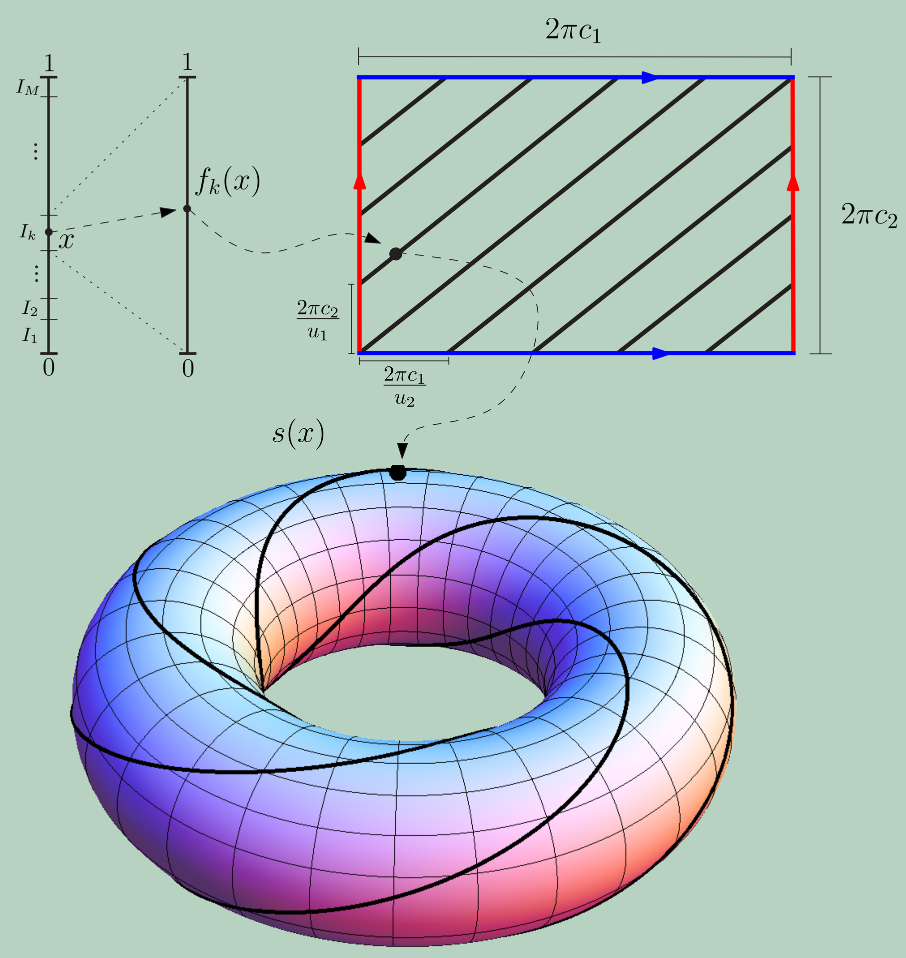
$$\boldsymbol{s}(x) := \boldsymbol{s}_{T_k}(f_k(x)), \text{ if } x \in I_k, \text{where} \quad (2)$$

$$f_k(x) = \frac{x - \sum_{j=1}^{k-1} l_j / L}{l_k / L}$$

and each curve $\boldsymbol{s}_{T_k}(f_k(x))$ is such that:

$$(\boldsymbol{s}_{T_k}(x)_{2i-1}, \boldsymbol{s}_{T_k}(x)_{2i}) = c_i(\cos(2\pi u_i x), \sin(2\pi u_i x))$$

Each curve is defined by a vector $\boldsymbol{c} = (c_1, \dots, c_N)$, s.t. $\|\boldsymbol{c}\| = 1$ and an integer vector $\boldsymbol{u} = (u_1, \dots, u_N)$. The vector $\boldsymbol{c}$ defines a flat torus on the sphere surface and $\boldsymbol{u}$ defines the curve on each torus. The objective is to choose a set of vectors $\boldsymbol{c}_1, \dots, \boldsymbol{c}_M$ and $\boldsymbol{u}_1, \dots, \boldsymbol{u}_M$ such that each curve has small-ball radius at least δ and each torus is at a distance at least 2δ from each other, assuring a total of at least δ for the small-ball radius of the whole scheme. This scheme is a generalization of the one proposed in [3].



We can show [1] that:

(i) Choosing a good set of tori is related to finding a good N-dimensional spherical code.

(ii) Finding good curves in each torus is equivalent to finding a vector $\boldsymbol{u}$ of a given length, such that the projection of the rectangular lattice $c_1\mathbb{Z} \oplus \cdots \oplus c_N\mathbb{Z}$ onto $\boldsymbol{u}^\perp$ is a dense lattice.

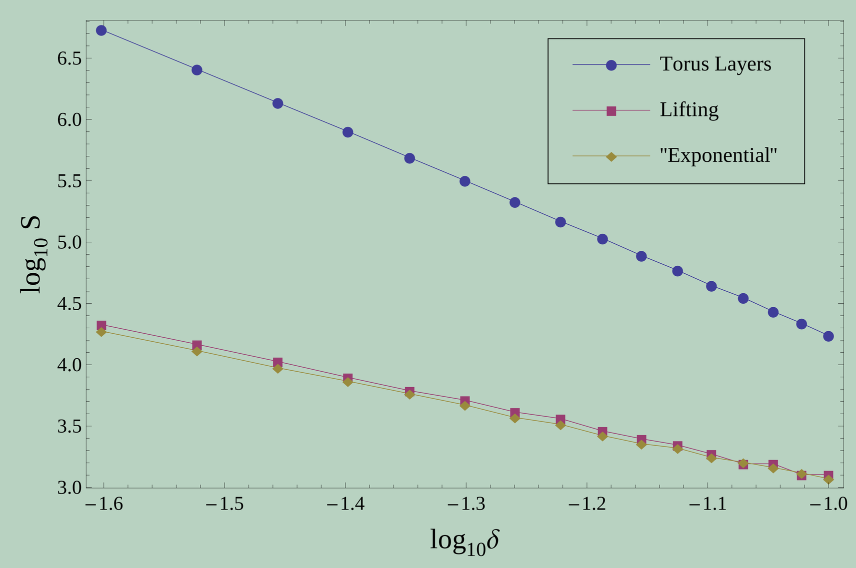
Results

The next theorem gives an answer to the problem of finding good lattices which are projection of $c_1\mathbb{Z} \oplus \cdots \oplus c_N\mathbb{Z}$ by generalizing the main construction of Sloane et al [2].

Theorem 1 *Let Λ be any $(N-1)$-dimensional lattice and let $\Lambda_{\boldsymbol{c}} = c_1\mathbb{Z} \oplus \cdots \oplus c_N\mathbb{Z}$. There exists a sequence of vectors $\boldsymbol{u}_w$ such that Λ_w, the projection of $\Lambda_{\boldsymbol{c}}$ onto $\boldsymbol{u}_w^\perp$, satisfies:*

$$(1/w)\Lambda_w \to \Lambda, \text{ as } w \to \infty. \quad (3)$$

From this result and a good strategy for producing a spherical code, we show that our scheme outperforms previous constructions in terms of small-ball radius and curve length. Some comparisons with [2] and [3] are shown next. Comparisons were made for an $1 : 6$ expansion map (i.e., sources transmitted over a Gaussian channel of dimension 6).



References

[1] A. Campello, C. Torezzan, and S. I. R. Costa. Curves on torus layers and coding for continuous alphabet sources. *IEEE International Symposium of Information Theory (to appear)*, 2012.

[2] N. J. A. Sloane, V. A. Vaishampayan, and S. I. R. Costa. The lifting construction: A general solution for the fat strut problem. In *IEEE International Symposium on Information Theory Proceedings*, 2010.

[3] V. A. Vaishampayan and S. I. R. Costa. Curves on a sphere, shift-map dynamics, and error control for continuous alphabet sources. *IEEE Transactions on Information Theory*, 2003.