# Chain Rule

John Purcell

December 9, 2021

## 1 Functions of One Variable

Suppose we have some variable $z$ which is a function of a variable $y$. $y$ in turn is a function of a variable $x$.

$$y = f(x)$$

$$z = g(y)$$

The rate of change of $y$ with respect to $x$ (or, "how fast y changes when x changes") is:

$$\frac{dy}{dx}$$

And the rate of change of $z$ with respect to $y$ is:

$$\frac{dz}{dy}$$

The chain rule gives us a formula for the rate of change of $z$ with respect to $x$:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

It's easy to remember this formula, because if you pretend that the term on the right of the equals sign is a fraction, it looks as though the $dy$ terms cancel out; it's as if we are both dividing by $dy$ and multiplying by it.

## 2 Functions of More Than One Variable

Now suppose that $z$ is a function of multiple variables, $y_1$, $y_2$, $y_3$ ...

$$z = g(y_1, y_2, y_3, ...)$$

And each of these variables is a function of $x$.

$$y_1 = f_1(x)$$

$$y_2 = f_2(x)$$

$$y_3 = f_3(x)$$

...

To find the rate of change of $z$ with respect to $x$, given the rates of change of each of the $y_i$ with respect to $x$, we must add the contributions to the overall rate from each of these $y_i$.

In the following formula I've replaced the $d$ symbols with $\partial$ symbols, as a reminder that we are considering partial rates of change: for example the rate at which $z$ changes with respect to $y_1$, if $y_2$, $y_3$, etc., remain constant.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y_1}\frac{\partial y_1}{\partial x} + \frac{\partial z}{\partial y_2}\frac{\partial y_2}{\partial x} + \frac{\partial z}{\partial y_3}\frac{\partial y_3}{\partial x} + ...$$

We can write this formulate more compactly using the summation symbol.

$$\frac{\partial z}{\partial x} = \sum_i \frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x}$$