Introducing the Chain Rule

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February 26, 2022

1 Exchange Rates

Suppose we have three currencies: UK pounds (£), US dollars ($) and European Union euros (€).

We'll denote these with the symbols p, s and e respectively for simplicity.

Suppose a *bureau de change* cites the following exchange rates:

GBP/USD	(dollars per pound)	1.33
USD/EUR	(euros per dollar)	0.90

Given this information, it's easy to work out the exchange rate for GBP/EUR (euros per pound).

$$GBP/EUR = GBP/USD \times USD/EUR = 1.33 \times 0.90 = 1.20$$

2 Exchange Rates as Functions

We can think of these as functions that accept amounts in one currency and return the equivalent amount in another currency.

$$s = f_{ps}(p) = 1.33p$$

$$e = f_{se}(s) = 0.9s$$

$$e = f_{pe}(p) = 1.20p$$

It's then easy to figure out the rate at which dollars change with respect to pounds, the rate at which euros change with respect to dollars, and the rate at which euros change with respect to pounds.

$$\frac{ds}{dp} = f'_{ps}(p) = 1.33$$

$$\frac{de}{ds} = f'_{se}(s) = 0.90$$

$$\frac{de}{dp} = f'_{pe}(p) = 1.20$$

So if for example you're changing pounds into dollars and you get a certain number of dollars, then you increase the number of pounds you're exchanging by a certain small amount, the number of dollars you get will increase by 1.33 times that amount.

Since these functions are linear, it's also true that the number of dollars you get for a certain number of pounds is simply always just 1.33 times the number of pounds.

We can see that the following is true:

$$\frac{de}{dp} = \frac{de}{ds}\frac{ds}{dp}$$

Notice that if we pretend these are fractions (which they aren't), it looks as though the ds cancel out.

This is known as the *chain rule* (for functions of one variable) and it also works for non-linear functions.