



# Further Maths

This book is currently a *preliminary draft*.

It is probably full of errors, lies, paradoxes and communist propaganda.

Send corrections to <https://github.com/aDotInTheVoid/a-level-notes>.

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# 1 Proof by induction

## Example 1.0.1

Prove that every term the sequence given by  $a_1 = 3$  and  $a_{n+1} = a_n + 2$  is odd.

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When  $n = 1$ ,  $a_n = a_1 = 3$  which is odd. Therefore the theorem is true for  $n = 1$

Assume the theorem is true for  $n = k$ . Therefore  $a_k = 2p + 1$  where  $p \in \mathbb{Z}$

Now we need to show the theorem is true for  $n = k + 1$ . Target statement:  $a_{k+1} = 2q + 1$  where  $q \in \mathbb{Z}$ .

$$\begin{aligned} a_{k+1} &= a_k + 2 \\ &= 2p + 1 + 2 && \text{where } p \in \mathbb{Z} \\ &= 2(p + 1) + 1 \\ &= 2q + 1 && \text{where } q \in \mathbb{Z} \end{aligned}$$

Because it is true for  $n = 1$  and  $n = k$  being true implies the theorem is true for  $n = k + 1$ , so it is true for all  $n \in \mathbb{N}^+$

## Example 1.0.2

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# A Formulae and Identities

These will not be given. You are expected to know them.

## §A.1 Pure Mathematics

### §A.1.1 Quadratic Equations

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### §A.1.2 Laws of Indices

$$\begin{aligned}a^x a^y &\equiv a^{x+y} \\ a^x \div a^y &\equiv a^{x-y} \\ (a^x)^y &\equiv a^{xy}\end{aligned}$$

### §A.1.3 Laws of Logarithms

$$x = a^n \iff n = \log_a x \text{ where } a > 0 \text{ and } x > 0$$

$$\begin{aligned}\log_a x + \log_a y &\equiv \log_a (xy) \\ \log_a x - \log_a y &\equiv \log_a \left(\frac{x}{y}\right) \\ k \log_a x &\equiv \log_a (x^k)\end{aligned}$$

### §A.1.4 Coordinate Geometry

A straight line graph, gradient  $m$  passing through  $(x_1, y_1)$  has equation

$$y - y_1 = m(x - x_1)$$

Straight lines with gradients  $m_1$  and  $m_2$  are perpendicular when  $m_1 m_2 = -1$

### §A.1.5 Sequences

General term of an arithmetic progression:

$$u_n = a + (n - 1)d$$

General term of a geometric progression:

$$u_n = ar^{n-1}$$

### §A.1.6 Trigonometry

In the triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{Area} = \frac{ab \sin C}{2}$$

$$1 \equiv \cos^2 A + \sin^2 A$$

$$\sec^2 A \equiv 1 + \tan^2 A$$

$$\csc^2 A \equiv 1 + \cot^2 A$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

### §A.1.7 Mensuration

Circumference and Area of circle, radius  $r$  and diameter  $d$

$$C = 2\pi r = \pi d$$

$$A = \pi r^2$$

Pythagoras' Theorem: In any right-angled triangle where  $a$ ,  $b$  and  $c$  are the lengths of the sides and  $c$  is the hypotenuse:

$$c^2 = a^2 + b^2$$

Area of a trapezium where where  $a$  and  $b$  are the lengths of the parallel sides and  $h$  is their perpendicular separation:

$$\text{Area} = \frac{(a+b)h}{2}$$

Volume of a prism = area of cross section  $\times$  length

For a circle of radius  $r$ , where an angle at the centre of  $\theta$  radians subtends an arc of length  $l$  and encloses an associated sector of area  $a$ :

$$l = r\theta$$

$$a = \frac{r^2\theta}{2}$$

### §A.1.8 Complex Numbers

For two complex numbers  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

**Loci in the Argand diagram**

$|z - a| = r$  is a circle radius  $r$  centred at  $a$

$\arg(z - a) = \theta$  is a half line drawn from  $a$  at angle  $\theta$  to a line parallel to the positive real axis

**Exponential Form**

$$e^{i\theta} = \cos \theta + i \sin \theta$$

**§A.1.9 Matrices**

For a 2 by 2 matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  the determinant  $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ . The inverse is  $\frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The transformation represented by matrix **AB** is the transformation represented by matrix **B** followed by the transformation represented by matrix **A**.

For matrices **A**, **B**:  $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

**§A.1.10 Algebra**

$$\sum_{r=1}^n r = \frac{n(n+1)}{2}$$

For  $ax^2 + bx + c = 0$  with roots  $\alpha$  and  $\beta$ :

$$\alpha + \beta = \frac{-b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

For  $ax^3 + bx^2 + cx + d = 0$  with roots  $\alpha$ ,  $\beta$  and  $\gamma$ :

$$\sum \alpha = \frac{-b}{a}$$

$$\sum \alpha\beta = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

**§A.1.11 Hyperbolic Functions**

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}$$

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}$$

$$\tanh x \equiv \frac{\sinh x}{\cosh x}$$

## §A.1.12 Calculus and Differential Equations

### Differentiation

$$\frac{d}{dx}x^n \equiv nx^{n-1}$$

$$\frac{d}{dx}\sin kx \equiv k \cos kx$$

$$\frac{d}{dx}\cos kx \equiv -k \sin kx$$

$$\frac{d}{dx}\sinh kx \equiv k \cosh kx$$

$$\frac{d}{dx}\cosh kx \equiv k \sinh kx$$

$$\frac{d}{dx}e^{kx} \equiv ke^{kx}$$

$$\frac{d}{dx}\ln x \equiv \frac{1}{x}$$

$$\frac{d}{dx}f(x) + g(x) \equiv f'(x) + g'(x)$$

$$\frac{d}{dx}f(x)g(x) \equiv f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}f(g(x)) \equiv f'(g(x))g'(x)$$

### Integration

$$\int x^n dx \equiv \frac{x^{n+1}}{n+1} + c \text{ where } n \neq -1$$

$$\int \cos kx dx \equiv \frac{\sin kx}{k} + c$$

$$\int \sin kx dx \equiv -\frac{\cos kx}{k} + c$$

$$\int \cosh kx dx \equiv \frac{\sinh kx}{k} + c$$

$$\int \sinh kx dx \equiv \frac{\cosh kx}{k} + c$$

$$\int e^{kx} dx \equiv \frac{e^{kx}}{k} + c$$

$$\int \frac{1}{x} dx \equiv \ln|x| + c \text{ where } x \neq 0$$

$$\int f'(x) + g'(x) dx \equiv f(x) + g(x) + c$$

$$\int f'(g(x))g'(x) dx \equiv f(g(x)) + c$$

### Area under a curve

$$\int_a^b y dx \text{ where } y \geq 0$$

**Volumes of revolution about the x and y axes**

$$V_x = \pi \int_a^b y^2 dx$$

$$V_y = \pi \int_c^d x^2 dy$$

**Simple Harmonic Motion**

$$\ddot{x} = -\omega^2 x$$



# B Number Sets

## §B.1 Integers

- $\mathbb{Z}^+$ : Positive integers without 0.  $\{1, 2, 3 \dots\}$
- $\mathbb{Z}_0^+$ : Positive integers with 0.  $\{0, 1, 2, 3 \dots\}$
- $\mathbb{Z}$ : Integers.  $\{\dots - 3, -2, -1, 0, 1, 2, 3 \dots\}$

## §B.2 Unused Integers

I will try to avoid these as they are ambiguous.

- $\mathbb{N}$ : Natural numbers. May or may not include 0. I will use  $\mathbb{Z}^+$  or *natzero* to be explicit.

## §B.3 Non-Integers

- $\mathbb{Q}$ : Numbers that can be expressed as a fraction:  $\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}^+\}$
- $\mathbb{R}$ : Real numbers. The definition gets axiomatic fast, so I will just say that they are just 1 number (not a matrix/vector) and are not complex / imaginary.
- $\mathbb{C}$ : A number with a real part and a complex part.  $\mathbb{C} = \{a + bi \mid a \in \mathbb{R} b \in \mathbb{R}\}$