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OCR

H245

Further Maths

This book is curently a *preliminary draft*.

It is probably full of errors, lies, paradoxes and communist propaganda.

Send corections to `https://github.com/aDotInTheVoid/a-level-notes`.

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Pure Core

1 Proof by induction

Example 1.0.1

Prove that every term the sequence given by $a_1 = 3$ and $a_{n+1} = a_n + 2$ is odd.

When $n = 1$, $a_n = a_1 = 3$ which is odd. Therfor the theorem is true for $n = 1$

Assume the theorem is true for $n = k$. Therefor $a_k = 2p + 1$ where $p \in \mathbb{Z}$

Now we need to show the theorem is true for $n = k + 1$. Target statement: $a_{k+1} = 2q + 1$ where $q \in \mathbb{Z}$.

$$\begin{aligned}
a_{k+1} &= a_k + 2 \\
&= 2p + 1 + 2 && \text{where } p \in \mathbb{Z} \\
&= 2(p + 1) + 1 \\
&= 2q + 1 && \text{where } q \in \mathbb{Z}
\end{aligned}$$

Because it is true for $n = 1$ and $n = k$ being true implys the theorem is true for $n = k + 1$, so it is true for all $n \in \mathbb{N}^+$

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Appendices

A Formulae and Identities

These will not be given. You are expected to know them.

§A.1 Pure Mathematics

§A.1.1 Quadratic Equations

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

§A.1.2 Laws of Indices

$$\begin{aligned} a^x a^y &\equiv a^{x+y} \\ a^x \div a^y &\equiv a^{x-y} \\ (a^x)^y &\equiv a^{xy} \end{aligned}$$

§A.1.3 Laws of Logarithms

$x = a^n \iff n = \log_a x$ where $a > 0$ and $x > 0$

$$\begin{aligned} \log_a x + \log_a y &\equiv \log_a(xy) \\ \log_a x - \log_a y &\equiv \log_a(\frac{x}{y}) \\ k \log_a x &\equiv \log_a(x^k) \end{aligned}$$

§A.1.4 Coordinate Geometry

A straight line graph, gradient m passing through (x_1, y_1) has equation

$$y - y_1 = m(x - x_1)$$

Straight lines with gradients m_1 and m_2 are perpendicular when $m_1 m_2 = -1$

§A.1.5 Sequences

General term of an arithmetic progression:

$$u_n = a + (n-1)d$$

General term of a geometric progression:

$$u_n = ar^{n-1}$$

§A.1.6 Trigonometry

In the triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\text{Area} = \frac{ab\sin C}{2}$$

$$1 \equiv \cos^2 A + \sin^2 A$$
$$\sec^2 A \equiv 1 + \tan^2 A$$
$$\csc^2 A \equiv 1 + \cot^2 A$$
$$\sin 2A \equiv 2\sin A\cos A$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A$$
$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

§A.1.7 Mensuration

Circumference and Area of circle, radius r and diameter d

$$C = 2\pi r = \pi d$$
$$A = \pi r^2$$

Pythagoras' Theorem: In any right-angled triangle where a, b and c are the lengths of the sides and c is the hypotenuse:

$$c^2 = a^2 + b^2$$

Area of a trapezium where where a and b are the lengths of the parallel sides and h is their perpendicular separation:

$$\text{Area} = \frac{(a+b)h}{2}$$

$$\text{Volume of a prism} = \text{area of cross section} \times \text{length}$$

For a circle of radius r, where an angle at the centre of θ radians subtends an arc of length l and encloses an associated sector of area a:

$$l = r\theta$$
$$a = \frac{r^2\theta}{2}$$

§A.1.8 Complex Numbers

For two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Loci in the Argand diagram

$|z-a|=r$ is a circle radius r centred at a

$\arg(z-a)=\theta$ is a half line drawn from a at angle θ to a line parallel to the positive real axis

Exponential Form

$$e^{i\theta}=\cos\theta+i\sin\theta$$

§A.1.9 Matrices

For a 2 by 2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the determinant $\Delta=\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right|=ad-bc$. The inverse is $\frac{1}{\Delta}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$

The transformation represented by matrix $\mathbf{AB}$ is the transformation represented by matrix $\mathbf{B}$ followed by the transformation represented by matrix $\mathbf{A}$.

For matrices $\mathbf{A}$, $\mathbf{B}$: $(\mathbf{AB})^{-1}=\mathbf{A}^{-1}\mathbf{B}^{-1}$

§A.1.10 Algebra

$$\sum_{r=1}^{n} r=\frac{n(n+1)}{2}$$

For $ax^2+bx+c=0$ with roots α and β:

$$\alpha+\beta=\frac{-b}{a}$$
$$\alpha\beta=\frac{c}{a}$$

For $ax^3+bx^2+cx+d=0$ with roots α, β and γ:

$$\sum\alpha=\frac{-b}{a}$$
$$\sum\alpha\beta=\frac{c}{a}$$
$$\alpha\beta\gamma=\frac{-d}{a}$$

§A.1.11 Hyperbolic Functions

$$\cosh x\equiv\frac{e^x+e^{-x}}{2}$$
$$\sinh x\equiv\frac{e^x-e^{-x}}{2}$$
$$\tanh x\equiv\frac{\sinh x}{\cosh x}$$

§A.1.12 Calculus and Differential Equations

Differentiation

$$\frac{d}{dx}x^n \equiv nx^{n-1}$$
$$\frac{d}{dx}\sin kx \equiv k\cos kx$$
$$\frac{d}{dx}\cos kx \equiv -k\sin kx$$
$$\frac{d}{dx}\sinh kx \equiv k\cosh kx$$
$$\frac{d}{dx}\cosh kx \equiv k\sinh kx$$
$$\frac{d}{dx}e^{kx} \equiv ke^{kx}$$
$$\frac{d}{dx}\ln x \equiv \frac{1}{x}$$
$$\frac{d}{dx}f(x)+g(x) \equiv f'(x)+g'(x)$$
$$\frac{d}{dx}f(x)g(x) \equiv f'(x)g(x)+f(x)g'(x)$$
$$\frac{d}{dx}f(g(x)) \equiv f'(g(x))g'(x)$$

Integration

$$\int x^n\,dx \equiv \frac{x^{n+1}}{n+1}+c \text{ where } n \neq -1$$
$$\int \cos kx\,dx \equiv \frac{\sin kx}{k}+c$$
$$\int \sin kx\,dx \equiv \frac{-\cos kx}{k}+c$$
$$\int \cosh kx\,dx \equiv \frac{\sinh kx}{k}+c$$
$$\int \sinh kx\,dx \equiv \frac{\cosh kx}{k}+c$$
$$\int e^{kx}\,dx \equiv \frac{e^{kx}}{k}+c$$
$$\int \frac{1}{x}\,dx \equiv \ln|x|+c \,\text{where}\, x \neq 0$$
$$\int f'(x)+g'(x)\,dx \equiv f(x)+g(x)+c$$
$$\int f'(g(x))g'(x)\,dx \equiv f(g(x))+c$$

Area under a curve

$$\int_a^b y\,dx \text{ where } y \geq 0$$

Volumes of revolution about the x and y axes

$$V_x = \pi \int_a^b y^2 \, dx$$
$$V_y = \pi \int_c^d x^2 \, dy$$

Simple Harmonic Motion

$$\ddot{x} = -\omega^2 x$$

B Number Sets

§B.1 Integers

- $\mathbb{Z}^+$: Positive integers without 0. $\{1, 2, 3 \ldots\}$
- $\mathbb{Z}_0^+$: Positive integers with 0. $\{0, 1, 2, 3 \ldots\}$
- $\mathbb{Z}$: Integers. $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$

§B.2 Unused Integers

I will try to avoid these as they are ambiguous.

- $\mathbb{N}$: Natural numbers. May or may not include 0. I will use $\mathbb{Z}^+$ or $natzero$ to be explicit.

§B.3 Non-Integers

- $\mathbb{Q}$: Numbers that can be expressed as a fraction: $\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}^+\}$
- $\mathbb{R}$: Real numbers. The definition gets axiomatic fast, so I will just say that they are just 1 number (not a matrix/vector) and are not complex / imaginary.
- $\mathbb{C}$: A number with a real part and a complex part. $\mathbb{C} = \{a + bi \mid a \in \mathbb{R} b \in \mathbb{R}\}$