Non-Binary Information Propagation Theory and Applications

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This paper studies the phase transitions of a non-linear dynamical system on a discrete-time, multi-agent network. We examine a variety of network models with the goal of examining the effect of network structure on the phase transition points. Using a modified logistic map, we show both analytically and empirically that the structure of the network, quantified by the first eigenvalue of it's adjacency matrix (λ), affects the phase transition points. Despite the simplicity of our system, we show that these results include a variety of dynamical systems and apply to diverse scientific fields, from biology to Internet routing.

I. PRELIMINARIES

A. Problem Formulation

Consider a graph $G = (V, E)$, where V is a set of vertices (nodes) and E is a set of edges. Each node $v_i \in V$ influences its neighbors according to a logistic map function, where $0 \leq i \leq N-1$. Furthermore, G is represented as $\mathbf{A} = [a_{ij}]$, where $a_{ij} = 1$ when node v_i connects to v_j and 0 otherwise. Let $x_i(t)$ denote the state of node v_i at time t and $\vec{x}(t) = [x_0(t), x_1(t), \cdots, x_{N-1}(t)]^T$ denote the vector state of the nodes in the system. At time $t+1$, the amount of influence spread $x_i(t+1)$ by node v_i is

$$x_i(t+1) = r_0 \times \sum_{j=0}^{N-1} a_{ji}x_j(t) \times \left[1 - \sum_{j=1}^{N-1} a_{ji}x_j(t)\right] \quad (1)$$

where the node v_j is a neighbor of v_i (i.e., edge $(v_j, v_i) \in E$ or $a_{ji} = 1$). For convenience, we refer to this model as the **Networked Logistic Map (NLM)**.

B. Contributions

In this paper, we examine the effect of graph topology on the phase transitions of the system described above. Using a variety of graph models, we find the following:

1. **Existence Threshold,** r_e. There exists an existence threshold r_e, such that, for all $0 \leq r_0 \leq r_e$, the system will converge to $\mathbf{X}^* = \{\vec{0}\}$ as $t \to \infty$. Furthermore, I show that $r_e = \frac{1}{\lambda}$, where λ is the largest eigenvalue of the adjacency matrix for all graph models. These results are analyzed in §II 1 and numerical simulation results are given in §III D.

2. **Dampening Threshold,** r_d. There exists a dampening threshold r_d, such that, for any $r_e < r_0 \leq r_d$, the system will converge to a fixed point $\mathbf{X}^* = \{\vec{x}_i\}$, where $\sum_i x_i > 0$. If $r_d < r_0 \leq r_p$, the system will converge to a fixed point and exhibit a dampening behavior. Furthermore, we show empirically that $r_d \approx \frac{2}{\lambda}$ (§II 2 and §III E).

3. **Periodic Threshold,** r_p. There exists a periodic threshold r_p such that, for any $r_p < r_0 \leq r_c$, where r_c is the *Chaotic Threshold*, the system will exhibit a periodic steady state behavior, i.e., there will exist a period p such that $\vec{x}(t) = \vec{x}(t-p)$. Thus, the steady state behavior is a set $\mathbf{X}^* = \{\vec{x}(t-p+1), \vec{x}(t-p+2), \cdots, \vec{x}(t)\}$. The periodic threshold will occur in the range $\frac{2}{\lambda} < r_p \leq \frac{3}{\lambda}$. We show this results analytically for complete and regular graphs (§II 3) and empirically for random graphs (§III F)

4. **Initial Condition Insensitivity**. Any set of initial conditions $\vec{x}(0) = [x_0(0), x_1(0), \cdots, x_{N-1}(0)]^T$ within appropriate bounds will cause the system to converge to a steady-state $\mathbf{X}^*$.

II. ANALYSIS

Figure 1 provides the intuition behind the networked logistic map system. The system is represented as a vector of node states $\vec{x}(t) = [x_0(t), x_1(t), \cdots, x_{N-1}(t)]^T$ at time t. Our system-level propagation function is encoded as the iterate map $F_{\mathbf{A}} : \mathbb{R}^N \to \mathbb{R}^N$. That is, $F_{\mathbf{A}}$ maps the vector of node states $\vec{x}(t)$ to $\vec{x}(t+1)$.

In this section, we examine the aggregate behavior of system states, that is $s(t)$ as calculated by the aggregation function $H : \mathbb{R}^N \to \mathbb{R}$. Recall from Section I B, we define the propagation function $F_{\mathbf{A}}$ as the networked logistic map and the aggregation function H as $\sum_i x_i$.

1. *Existence Threshold Condition*

As exemplified in Figure 2, we observe a phase transition from 0 to non-0 values at transition #1. We refer to

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Symbol	Definition	Symbol	Definition
N	Number of agents in the system (vertices)	M	Number of connections in the system (edges)
$V = \{v_0, v_1, \cdots, v_{N-1}\}$	Set of vertices v_i in G, where $\|V\| = N$	$E = \{e_0, e_1, \cdots, e_{M-1}\}$	Set of edges in G, where $e_i = (v_k, v_j)$ and $\|E\| = M$
$G = (G, E)$	Graph topology	$\mathbf{A} = [a_{ij}]$	The adjacency matrix representation of G, where element $a_{ij} = 1$ iff $(v_i, v_j) \in E$
$\vec{x}(t) = [x_0(t), x_1(t), \cdots, x_{N-1}(t)]^T$	The vector of states for each agent v_i, where $\vec{x} \in \mathbb{R}^N$.	$\vec{s} = [s_0, s_1, \cdots, s_{N-1}]^T$	The vector of aggregate states.
$x_i(t+1) = f(x_i(t))$	the node-level dynamical function	$\vec{x}(t+1) = F_{\mathbf{A}}(\vec{x}(t))$	the system-level dynamical function
$s(t) = H(\vec{x}(t))$	the aggregate value of $\vec{x}$ at time t	$N_i(G)$	the neighbors of node i in graph G

TABLE I. Terminology

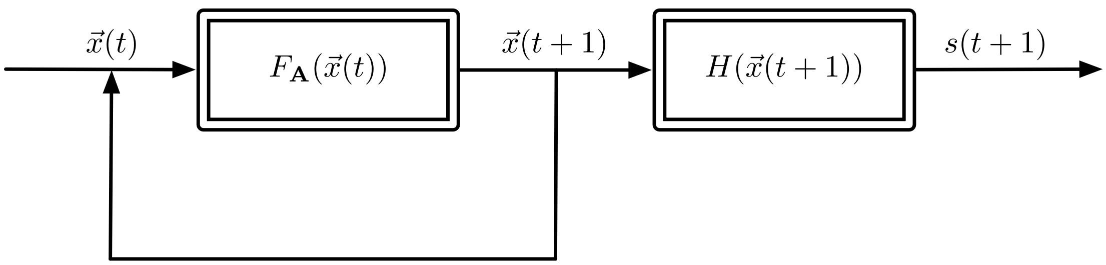


FIG. 1. General block diagram for our dynamic process.

this transition as the 'Existence Threshold'.

Theorem 1 *Our system, defined as* $s(t+1) = H(F_{\mathbf{A}}(\vec{x}(t)))$*, is asymptotically stable at the equilibrium point* $H(F_{\mathbf{A}}(\vec{x}^*)) = s^*$*, if the eigenvalues of the Jacobian* $\mathbf{J} = \nabla H(F_{\mathbf{A}}(\vec{x}(t)))$ *are less than 1 in absolute value, where* $\mathbf{J} = [J_{ij}]$ *and* $J_{ij} = [\nabla H(x)]_{ij} = \frac{\partial H(F_{\mathbf{A}}(\vec{x})))_i}{\partial s_i}|_{H(F_{\mathbf{A}}(\vec{x}))=0}$.

Proof. *Outline of Proof.* The proof consists of two sequential steps. We begin by finding the fixed point corresponding to $s(t+1) = s(t)$, which we will refer to as $s^\#$. More precisely, we find the fixed point corresponding to $\vec{x}(t+1) = \vec{x}(t)$. Then, once we have the fixed point $s^\#$, we must determine its stability criteria.

1. Fixed Point. The fixed point $s^\#$ corresponds to all agents in the system maintaining no state information, i.e., $x_i = 0$ for all i. Consider the traditional, non-networked logistic map $x(t+1) = f(x(t)) = rx(t)(1-x(t))$, where initially $0 \le x_0 \le 1$. The fixed points are solutions of $x^* = rx^*(1-x^*)$ and these solutions are $x^\# = 0$ or $x^* = 1 - 1/r$. In order to determine the stability of these points, we need $f'(x) = r(1-2x)$. So, for $|f'(x^\#)| = r$. Since a fixed point is stable if $f'(x^\#) < 1$, this fixed point is stable if $r < 1$.

We apply the same logic to the NLM implementation. Essentially, our system has a fixed point, $s^\# = 0$, as indicated by line #1 in Figure 2.

2. Jacobian of the NLDS. Now, we know we have a fixed point at $s^\# = 0$, yet we do not know the stability criteria of $s^\#$. For convenience, I will consider the entire system state $\vec{x}(t)$, rather than the aggregated state $s(t)$, knowing that $s(t)$ is stable if and only if $\vec{x}(t)$ is stable. Essentially, we are looking for a r value where the Jacobian $\nabla F(\vec{x}^\#) = \vec{1}$. It is well known (TODO: find this theorem) that $\nabla F(\vec{x}^\#) = \vec{1}$ if and only if the eigenvalues of the system are all less than 1 in absolute value. By applying the Perron-Frobenius theorem, this statement is equivalent to stating $F(\vec{x}^\#) = \vec{1}$ if and only if the largest eigenvalue of the system is less than 1.

The Jacobian of our NLM is the partial derivatives of all first-order partial derivatives of our vector-function. Again, focusing on $\vec{x}^\#$, this

$$J = \nabla F(x(t+1)) = \frac{\partial(x_0(t+1), x_1(t+1), \cdots, x_{N-1}(t+1))}{\partial(x_0(t), x_1(t), \cdots, x_{n-1}(t))}$$

Now, the shape of the Jacobian is similar to the shape of the adjacency matrix A, i.e., $J_{ij} = (\neq)0$ iff $A_{ij} = (\neq)0$. This, we can state now that the fixed point $\vec{x}^\#$ is stable if and only if the largest eigenvalue of the adjacency matrix is less than 1 in absolute value. This equates to a stable region of range $0 < r_e < 1$, where $r_e = \lambda_1 * r$.

2. Dampened Behavior Threshold Conditions

Here, we introduce another fixed point behavior of our system, which we will refer to as the Dampened Behavior Threshold, r_d. Again, we use the derivative of the logistic map, $f'(x) = r(1-2x)$ to determine the stability criteria of this point. Using the same logic detailed above, we can show that our system $\vec{x}(t+1) = F_{\mathbf{A}}(\vec{x}(t))$ is stable if and only if $\nabla F_{\mathbf{A}}(x(t+1)) < 1$. Further, we have already shown that the $F_{\mathbf{A}}$ implicitly accounts for the adjacency matrix $\mathbf{A}$ and the logistic map, ultimately, $J_{F_{\mathbf{A}}} = \nabla F_{\mathbf{A}} = \frac{\partial A_i}{\partial A_j} * f'(\vec{x}(t))$. Again, this

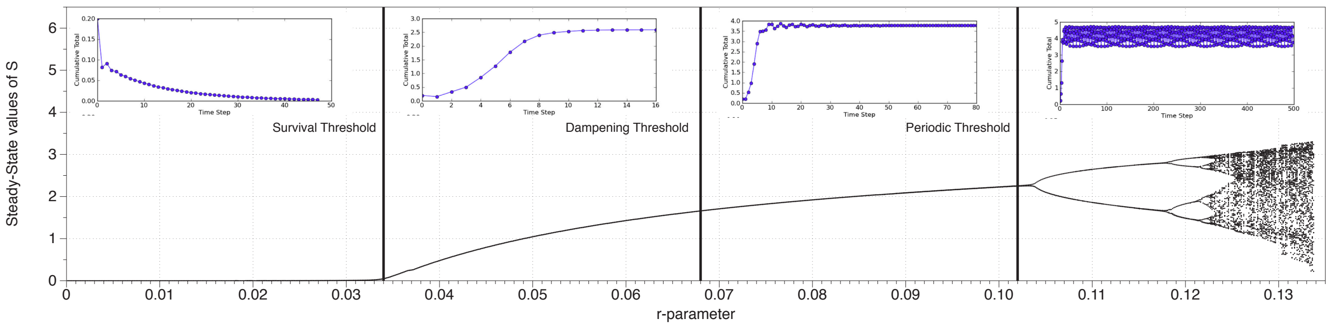


FIG. 2. Example of networked logistic map thresholds simulated on an Erdős-Rényi random graphs with $N = 100$, $p = 0.3$. Each vertical line indicates a particular threshold condition: 1. Existence Threshold, 2. Dampened Behavior Threshold, 3. Periodicity Threshold.

problem is separated into finding where $\nabla \mathbf{A} < 1$ and $\nabla f(x) < 1$. For the second fixed point, the each element of $f'(x_i^*) = 1$. Recall the solution to the logistic maps second fixed point is $x^* = 1 - 1/r$. Thus, each $x \in \vec{x}^*$ must satisfy $|f'(x)| = |2 - r| < 1$. So, for the logistic map portion only, this fixed point is stable in the range $1 < r < 3$ for each $x_i \in \vec{x}^*$. The same stability criteria applies to the adjacency matrix, so the global behavior of this section is stable for the range $2 < r_d < 3$, where $r_d = \lambda_1 * r$.

3. *Periodic Threshold Conditions*

From basic dynamical systems theory, we define a period-k point x^{*_k} as a point for which $x_{t+k} = x_t$. For our system, $\vec{x}^* = F_{\mathbf{A}}^{(k)}(\vec{x}^*)$ is a periodic fixed point with period-k. Also note that, as previously defined, $F_{\mathbf{A}}^{(k)}$ is a k-fold. We will begin by assuming that a period-2 bifurcation exists in our system. Such a bifurcation will satisfy $\vec{x}^* = F_{\mathbf{A}}^{(2)}(\vec{x}^*)$, i.e., the 2-fold of our system.

We now ask, what range of r satisfy a period-2 bifurcation? Using similar methods to those discussed above, we find that the period-2 behavior is stable in the range $3 < \lambda_1 * r < 1 + \sqrt{6} \approx 3.449$. If we continue increasing period-k values, we find that period behavior is stable in the range $3 < r_p < 4$, where $r_p = \lambda_1 * r$.

III. APPLICATIONS AND IMPLICATIONS

A. Simulation and Numerical Results

In this section, we present the results from numerous simulations and numerical evaluations. Unless otherwise state, each experiment was conducted on at least 4 graph models: 1. Erdős-Rényi, 2. Barabási-Albert , 3. Chain, and 4. Grid.

B. Graph Models

Throughout this paper, we refer to a number of graph models. This section will detail their specifics.

1. *Complete Graphs*

A complete graph, denoted K_N, when represented by the adjacency matrix has eigenvalues $\lambda_1 = N - 1$ and $\lambda_2 = \cdots = \lambda_N = -1$.

2. *Erdős-Rényi Random Graphs*

The Erdős-Rényi random graph models is defined as follows. Given N nodes and edge probability p, an edge exists between nodes i and j as follows:

$$e_{ij} = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } (1-p) \end{cases} \tag{2}$$

3. *Barabási-Albert Random Graphs*

This model is often referred to as the *Preferential Attachment Model.* Barabási-Albert grows a power-law network (with $\gamma = 3$) as follows: Begin with an initial network of $m_0 \geq 2$ nodes with degree of each node at least 1. New nodes are added until the desired graph size of N is reached. Each new node is connected to m existing nodes with a probability that is proportional to the number of edges that the existing nodes already have, more formally:

$$p_i = \frac{k_i}{\sum_j k_j}, \tag{3}$$

where k_i is the degree of node i.

4. *Grid Models*

A grid graph is an $m \times n$ graph $G_{m,n}$ has $N = m + n$ nodes.

C. Methodology

For each trial of the following simulations, we instantiated a specific graph based on one of the four models mentioned above. Once instantiated, we selected a node seeding methodology, either *individual, uniform (iu)*, *subset, uniform (su)*, or *all, uniform (au)*. These designations refer to the quantity of nodes that receive a uniform random seed value drawn from the range $[0, 1]$. Each node implements the logistic map function described in Section § I B.

D. Evaluation of Existence Threshold, r_e

Recall that the existence threshold (r_e) is the point below which the system converges to 0. In Figures 3 and 4, we show an example of the existence threshold–identified at $1/\lambda$–for Barabási-Albert and Erdős-Rényi networks, respectively. Specifically, in Fig. 3, the point at which the steady-state values "take-off" is at $r_e = 1/\lambda = 0.109$, and for Fig. 4, the same point is observed at $r_e = 1/\lambda = 0.034$.

The existence threshold is important in a variety of disciplines, particularly epidemiology. In that field, the existence threshold is often referred to as r_0 (Pronounced "*r-not*," not to be confused with our terminology denoting the homogeneous initial value of r_0 in Eq. 1).

E. Evaluation of Dampening Threshold, r_d

The dampening threshold is indicated at the Figures 3 and 4 at the point $2/\lambda$. Again, this result corresponds to the theoretical results of section § II 3.

The dampening threshold is also important in a variety of fields, most notably computer network routing. Consider a routing protocol attempting to converge on a particular system-wide state. If the protocol operates below the dampening threshold, one is guaranteed that a fixed state is reached utilizing the least "energy" required to converge.

F. Evaluation of Periodic Threshold, r_p

Recall in Section § II 3, we predict the periodic threshold to occur within the range $r_d < r_p <= 3/\lambda$. This result is shown nicely in both figures 3 and 4. Note that for the Barabási-Albert graph model, the bifurcation point occurs around $r_p = 0.26$, yet $3/\lambda = 0.33$. This extreme discrepancy does not exist in the other graph family examined (Erdős-Rényi, Chain, Grid).

The periodic threshold indicates a final state that remain in a constant flux between various values.

G. Evaluation of Initial Condition Sensitivity

Finally, each of the plots presented in Figures 3 and 4 were seeded at random from a different seed node. Yes, as shown in these figures, the final steady-state values expected for each value of r are equivalent, despite the starting position. Like the logistic map, we expect the outcome of our NLM model to be insensitive to initial starting conditions. That is, we expect that no matter how the graph is seeded, the characteristic curve of the steady-state value of S versus the r-parameter for each instantiated graph will be identical.

IV. RELATED WORKS

In this section, we briefly cover works related to our own.

A. Information Propagation

At its heart, the networked logistic map is a means of information propagation, analogous to disease (or epidemic) spreading processes. For a gentle introduction to this subject, refer to the survey by Hethcote [**?**]. Early propagation models assumed *homogeneous* populations; that is, a population with no social or spatial structure. Newer models apply underlying structure to the population, thus creating a *heterogeneous* population. We too apply a network structure to our model through the use of generative graph models, such as Erdős-Rényi or Barabási-Albert . The key difference between our work and such disease spreading processes is our introduction of a continuous valued measure of node state.

B. Applications

The study of dynamical processes on networks applies to a wide variety of diverse scientific fields, ranging from computer science and engineering to neuroscience to finance. For a comprehensive survey, refer to [1] or [2].

1. *Computer Science Applications*

The area of distributed systems benefits from deeper understanding of the behavior of dynamical systems operating on networks. Consensus algorithms often require the distributed system to achieve a global state (signifying a consensus among the population). Understanding

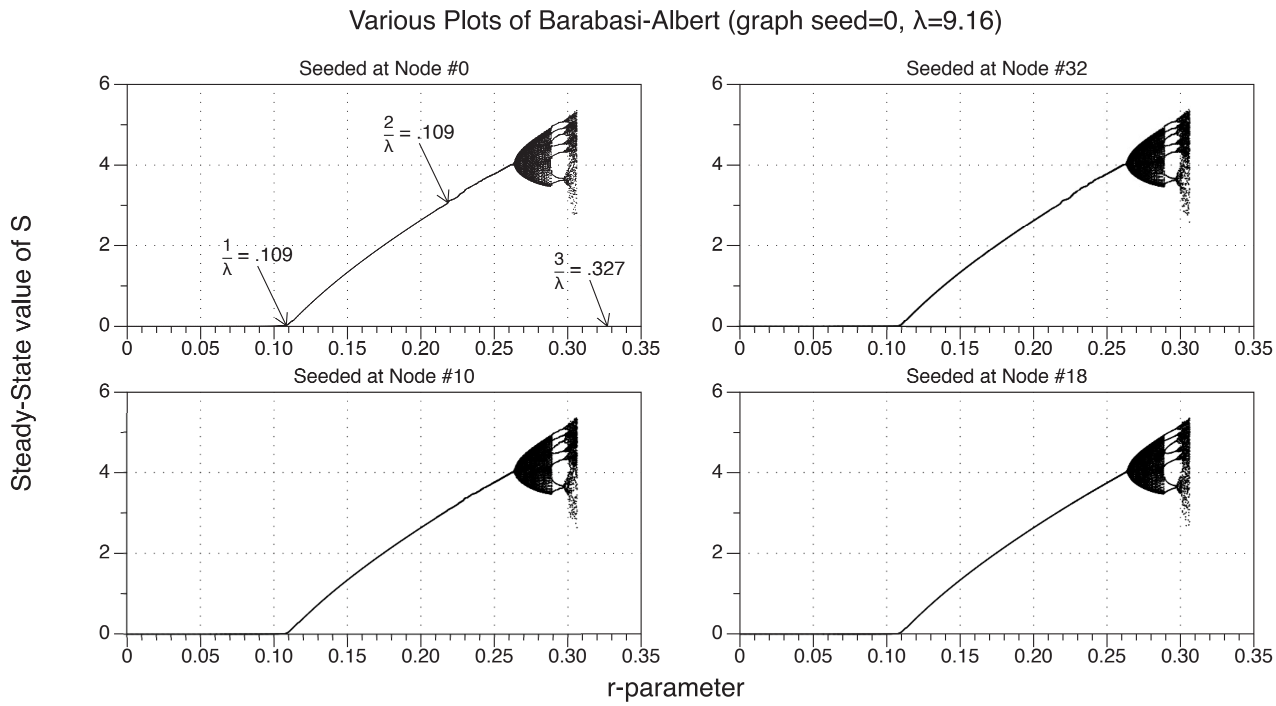


FIG. 3. Bifurcation point of a Network Logistic Map applied to a Barabási-Albert Graph.

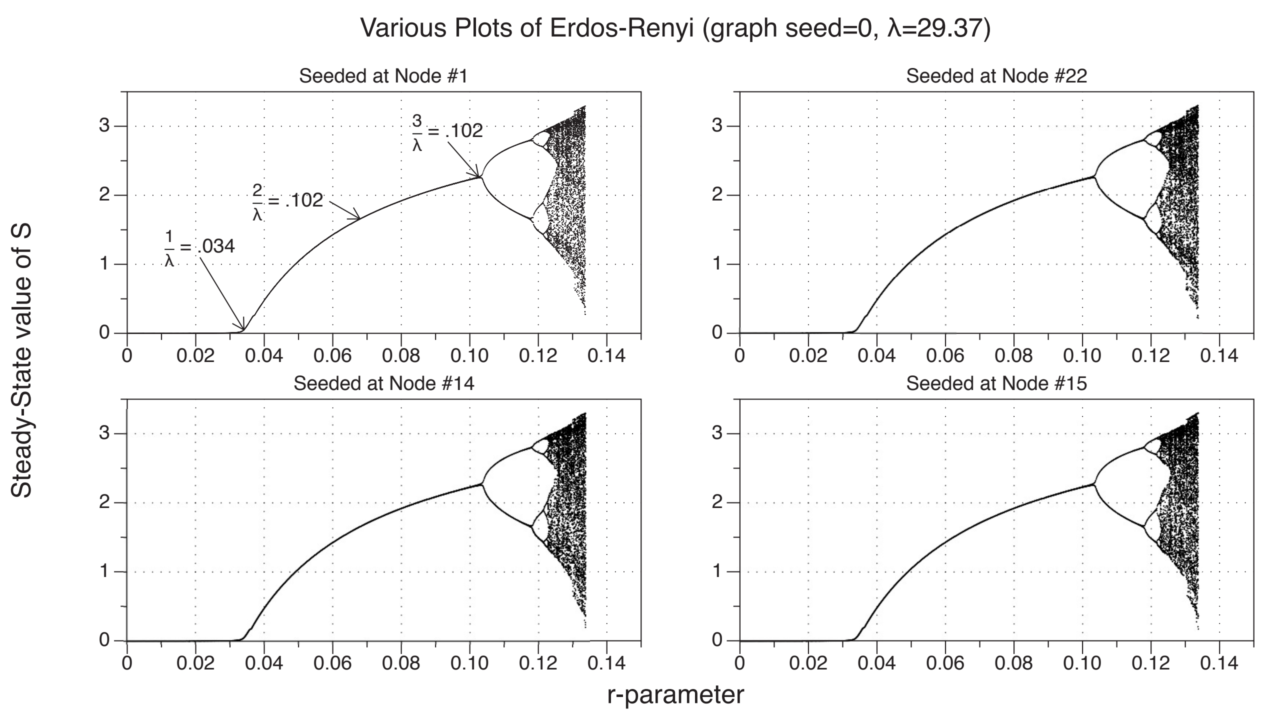


FIG. 4. Bifurcation point of a Network Logistic Map applied to a Erdős-Rényi Graph.

the effect of networks on such dynamical processes will result in better performance of such systems. Global coordination algorithms (such as routing algorithms) will also benefit from a deeper understanding of dynamical processes on networks. TODO: Add notes on wireless communication networks.

The efficiency gains promised by a deeper understanding will fall into one of two general categories: 1) Modifying dynamical system to exploit a fixed structure, or 2) Modifying structure to benefit a dynamical system.

The area of data mining often exploits relations between data to effect efficient extraction of knowledge from large data sets. Thus, it is another natural area that will benefit from the understanding of dynamical systems and their relation to graph structures.

V. CONCLUSIONS

In this study, we have effectively de-coupled the propagation model from the connectivity structure for a deterministic, non-binary propagation function. Further, we showed that, specific to the logistic map, that the effective macro-level parameter, $r_M = \lambda_1 * r_0$.

TODO: Strengthen what types of functions F satisfy this construct.

TODO: Change aggregation function to an evaluation of family of aggregation functions?

Appendix A: Introduction

Numerous real-world systems are modeled as networks consisting of a set of nodes and edges connecting node pairs. Such networked systems can further be categorized into dynamical processes *of* networks themselves, or dynamical processes acting *on* networks. This paper focuses on the latter.

Specifically, we extend the classic binary–or categorical–epidemic model of information propagations. Our key insight is that certain phenomena are not adequately described by a single binary digit (i.e., infected/not infected), but rather as a spectrum. For example, market penetration, product adoption, gossip/rumors and public opinion are phenomena that spread across networked populations in a non-binary fashion. In this paper, we propose modeling this spectrum as a single continuous value.

Now, due to the our change to the nature of the propagating information, we must modify the traditional models of information propagation. For ease of analysis and simulation implementation, we elect to use the traditional logistic map to model the propagation of our continuous valued information.

The key difference between previous models and our effort herein is, in our model, node state is described by a continuous, real value and is akin to "strength." For example, consider adoption of an Android smartphone. One's excitement about the smartphone is not strictly "excited" or "not excited," rather one experiences a range of excited states, and one's strength of excitement influences family, friends and others proportionally. The strength of one's excitement can be measured as a continuous real value that spreads across one's social network of acquaintances.

In this study, we focus on developing the simplest possible model that can characterize the propagation of real value properties across a networked system. To the best of our knowledge, this is the first work that examines such a network wide propagation problem and sets the stage for further theoretical analysis of these systems. From a theoretical point of view, our work may be seen as a first attempt to explore the outcome of two competing phenomena: *attrition* at every propagation hop and *amplification* by propagation of properties from adjacent nodes (neighbors).

In this paper, we show the following:

1. **Existence Threshold,** r_e. There exists an existence threshold r_e, such that, for all $r_0 < r_e$, the system will converge to $\vec{x} = \vec{0}$. Furthermore, I show that $r_e = \frac{1}{\lambda}$, where λ is the largest eigenvalue of the adjacency matrix.

2. **Initial Condition Insensitivity**. Any set of initial conditions $\vec{x}_0 = [x_{0,0}, x_{1,0}, \cdots, x_{N-1,0}]$ within appropriate bounds will cause $\vec{x}$ to converge to a steady-state of $\vec{x}^* = [x_0^*, x_1^*, \cdots, x_{N-1}^*]$.

3. **Periodic Threshold,** r_p. There exists a periodic threshold r_p such that, for any $r_0 > r_p$, the system will exhibit a periodic steady state behavior, i.e., there will exist a period p such that $\vec{x}_t = \vec{x}_{t-p}$. Thus, the steady state behavior is a set $\mathbf{X}^* = \{\vec{x}_{t-p+1}, \vec{x}_{t-p+2}, \cdots, \vec{x}_t\}$. The periodic threshold will occur in the range $\frac{2}{\lambda} < r_p <= \frac{3}{\lambda}$.

4. **Dampening Threshold,** r_d. There exists a dampening threshold r_d, such that, for any $r_e < r_0 < r_d$, the system will converge to a fixed point $\sum_i x_i > 0$. If $r_d < r_0 < r_p$, the system will converge to a fixed point and exhibit a dampening behavior. Furthermore, we show empirically that $r_d \approx \frac{2}{\lambda}$.

TODO: The above contributions are a duplicate of those below. I'm leaving in place to determine which position reads better (also, due to the cutting I've done below, section 2 seems light).

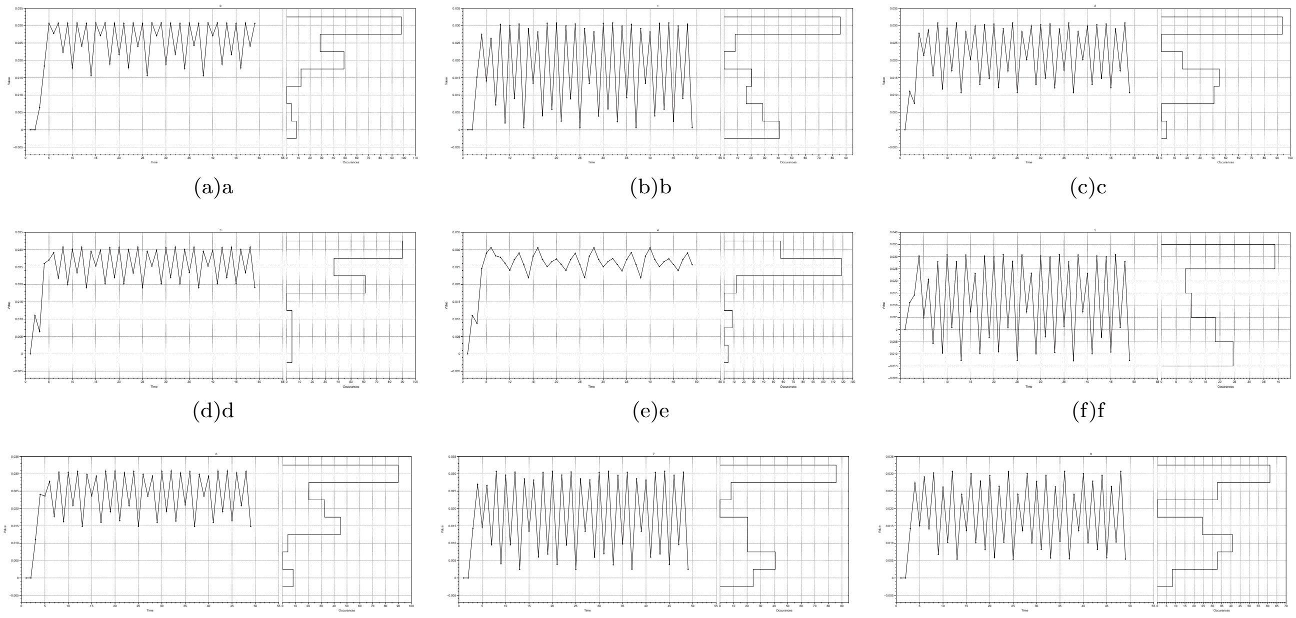
Appendix B: Additional Plots, Future/Ongoing Work

1. Accuracy of Predicted Threshold Values

Refer to Figure B 1.

2. Predicted vs. Numerical Results

Refer to Figure B 2.

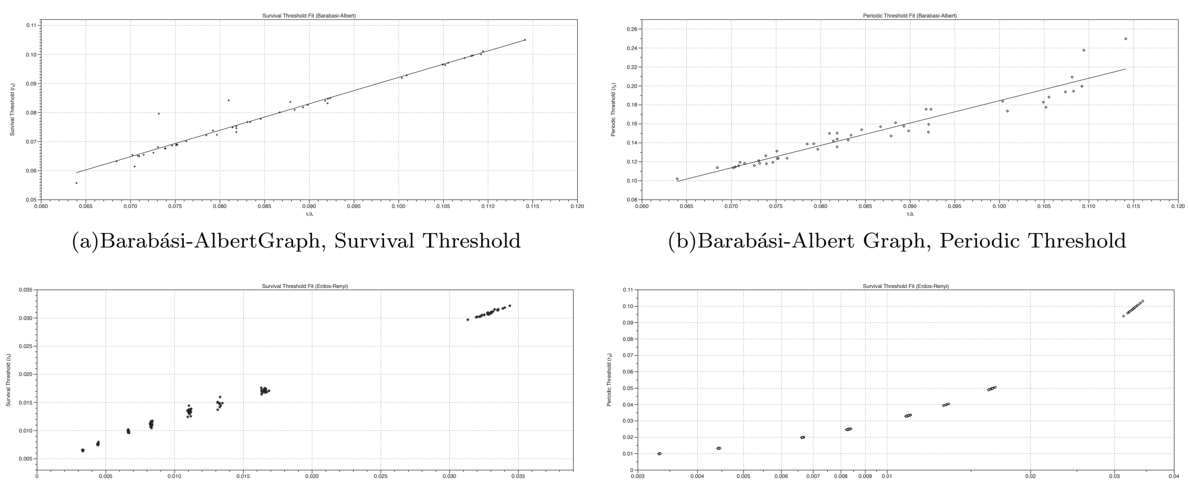


(a)a (b)b (c)c

(d)d (e)e (f)f

(g)g (h)h (i)i

FIG. 5. a



(a)Barabási-AlbertGraph, Survival Threshold (b)Barabási-Albert Graph, Periodic Threshold

(c)Erdős-Rényi Graph, Survival Threshold (d)Erdős-Rényi Graph, Periodic Threshold

FIG. 6. a

Appendix C: The Implicit Function Theorem

Copied verbatim from Wolfram online: Given

$$F_1(x, y, z, u, v, w) = 0 \tag{C1}$$

$$F_2(x, y, z, u, v, w) = 0 \tag{C2}$$

$$F_3(x, y, z, u, v, w) = 0, \tag{C3}$$

if the determinant of the Jacobian

$$|JF(u, v, w)| = |\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}| \neq 0, \tag{C4}$$

then u, v, w can be solved for in terms of x, y, x and partial derivatives of u, v, w with respect to x, y, z can be found by differentiating implicitly.

More generally, let A be an open set in $\mathbb{R}^{n+k}$ and let $f : A \mapsto \mathbb{R}^n$ be of class C^k (i.e., *smooth*). Write f in the form $f(x, y)$, where x and y are elements of $\mathbb{R}^k$ and $\mathbb{R}^n$. Suppose that (a, b) is a point in A such that $f(a, b) = 0$ and the determinant of the $n \times n$ matrix whose elements are the derivatives of the n component functions of f with respect to the n variables, written as y, evaluated at (a, b), is not equal to zero. The latter may be rewritten as

$$rank(Df(a,b)) = n. \tag{C5}$$

Then there exists a neighborhood B of a in $\mathbb{R}^k$ and a unique C^k function $g : B \mapsto \mathbb{R}^n$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in B$.

Appendix D: Notes on Spectral Graph Metrics

1. Eigenvector Centrality

The per-component eigenvalue of the k-th eigenvector is called the *Eigenvector Centrality* of node v_j:

$$(x_k)_j = \frac{(Ax_k)_j}{\lambda_k} = \frac{1}{\lambda_k}\sum_{l=0}^{N-1} a_{jl}(x_k)_l = \frac{1}{\lambda_k}\sum_{v_l \in Neig(v_j)} (x_k)_l$$

This represents the measure of importance node v_j has in the network with respect to eigenvector x_k. The largest eigenvector is most often considered. The best known example of this metric is the Google Page Rank algorithm.

2. Graph Energy

The graph energy E_G is

$$E_G = \sum_{j=0}^{N-1} |\lambda_j(A)|$$

3. Effective Graph Resistance

Consider a network where a flow with magnitude I is injected at node v_a and leaves at v_b. The in-flows and out-flows at node v_i follow the conservation law:

$$\sum_{v_j \in Neigv_i} y_{ij} = \sum_{j=0}^{N-1} a_{ij} y_{ij} = I(1_{i-a} - 1_{i=b})$$

The effective graph resistance R_G is defined as:

$$R_G = \frac{1}{2}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} \omega_{ab} = \frac{1}{2} u^T \Omega u$$

R_G measures the difficulty of transport in a graph.

[1] L. J. Allen and P. van den Driessche, Journal of Difference Equations and Applications (2008).

[2] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Physics Reports **469**, 93 (2008).