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Second Order Linear DE.

form: $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)$$

Ex:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = e^x$$

$$\rightarrow \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + 0.y = \frac{e^x}{y^2} \Rightarrow 2^{nd} \text{ Order Linear DE.}$$

Case: 1

If $P(x)$ = constant, $Q(x)$ = constant

then the equation is known as 2^{nd} Order Linear Differencial equation with constant coefficients.

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = f(x)$$

case: 2

If $P(x)$ and $Q(x)$ are function of x, then the equation is known as 2^{nd} Order Linear DE with variable coefficients.

[Note: In the form if $f(x)$ is 0, the DE is 'homogeneous' otherwise its 'non-homegeneous'.]

Case 1

Second order homogeneous Linear Differencial equation with constant coefficient :-

$\because P(x), Q(x)$ are constant and $f(x) = 0$

$$\therefore a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \quad \text{--- (1)}$$

Solution Procedure :-

Let $y = e^{mx}$ be a solution of the equation.

Then, $\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2 e^{mx}$

$\because y = e^{mx}$ is a solution of (1), we must have:

$$a_2 m^2 e^{mx} + a_1 m e^{mx} + a_0 e^{mx} = 0$$

or, $(a_2 m^2 + a_1 m + a_0) e^{mx} = 0$

$\therefore a_2 m^2 + a_1 m + a_0 = 0 \quad [\because e^{mx} \neq 0]$

Which is known as auxilary equation.

Solve this equation for the values of m and accordingly, we will have solution of (1).

Depending on the values of m, we have 3 different case :-

case (i) Two values of m are real and distinct.

Let, the roots be m_1 and m_2 $\therefore m_1 \neq m_2$

$m_1, m_2 \in \mathbb{R}$

$\therefore$ Solution is given by :-

$$\boxed{y = Ae^{m_1 x} + Be^{m_2 x}}$$

case (ii)

Two values of m are real and equal.

$\therefore m_1 = m_2$

$(m_1, m_2 \in \mathbb{R})$.

$\therefore$ Solution is given by :-

$$\boxed{y = e^{m_1 x}(A + Bx)}$$

$m_1 = m_2$

$$\boxed{xe^{m_1 x} = e^{m_2 x}}$$

case (iii) Two values of m are complex.

$\therefore$ Let $m_1 = \alpha + i\beta$

$\therefore m_2 = \alpha - i\beta \rightarrow$ Conjugate of m_1

$y = e^{m_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x} \cdot e^{i\beta x}$

$y = e^{m_2 x} = e^{(\alpha - i\beta)x} = e^{\alpha x} \cdot e^{-i\beta x}$

We know,

$$e^{ix} = \cos x + i \sin x$$

$$\therefore e^{i\beta x} = \cos\beta x + i\sin\beta x$$

and, $$e^{-i\beta x} = e^{i(-\beta x)} = \cos\beta x - i\sin\beta x$$

$\therefore$ Solution is given by:-

$$y = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$

In all of the above cases, A & B are arbitrary constant

Question Find Solution $\rightarrow$ $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$

Sol^n Let $y = e^{mx}$ be a solution ~~of the solution~~ of the equation

$\therefore$ The auxilary equation is given by:-

$$m^2 - 6m + 9 = 0$$

$$\Rightarrow m^2 - 3m - 3m + 9 = 0$$

or, $m(m-3) - 3(m-3) = 0$

$$(m-3)^2 = 0 \qquad \therefore m = 3, 3.$$

$\therefore$ Solution $\Rightarrow$ $y = e^{3x}(A + Bx)$

Question Solve : $y'' - 2y' + 5y = 0$, $y(0) = -3$, $y'(0) = 1$

Sol Let $y = e^{mx}$ be a solution of the equation

$y(0) = -3$

For, $x = 0$, $y = -3$

Then the auxillary equation is given by:-

$y'(0) = 1$

for $x = 0$, $\frac{dy}{dx} = 1$

$$m^2 - 2m + 5 = 0$$

$$m_1 = \frac{-(-2) + \sqrt{4-20}}{2} = 1 + 2i$$

$$m_2 = \frac{-(-2) - \sqrt{4-20}}{2} = 1 - 2i$$

∴ The solution is :-

$$y = e^x (A\cos 2x + B\sin 2x)$$

↳ where A, B are arbitrary constants.

As, $y(0) = -3$

∴ We have, $-3 = e^0 (A\cos 0 + B\sin 0)$

$\Rightarrow -3 = A$

∴ $A = -3$.

now, $\frac{dy}{dx} = e^x (A\cos 2x + B\sin 2x) + e^x (-2A\sin 2x + 2B\cos 2x)$

As, $y'(0)=1$

$x=0,\ \frac{dy}{dx}=1$

now,

$\Rightarrow 1 = e^0(A\cos 2\times 0 + B\sin 2\times 0) + e^0(-2A\sin 2\times 0 + 2B\cos 2\times 0)$

$\Rightarrow 1 = A+2B$

$\longrightarrow$ but $A=-3$

or, $1=-3+2B$

or, $B=2$.

$\therefore A=-3,\ B=2$

$\therefore$ The solution of the problem is

$y=e^x(-3\cos 2x + 2\sin 2x)$ for $y(0)=-3$ and $y'(0)=1$.

HW

- $9y''+6y'+5y=0$

  $y(0)=6,\ y'(0)=0$

- $y''-4y'+4y=0$

  $y(0)=3,\ y'(0)=1$

- $y''-6y'+8y=0$

  $y(0)=1,\ y'(0)=6$

Wronskian

Two functions y_1 and y_2 are independant if their wronskian W is non-zero

where, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

ex: Let $y_1 = x$, $y_2 = e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = (x-1)e^x \neq 0$$

if $x \neq 1$

$\therefore$ y_1 and y_2 are 2 two independant functions $\forall x \neq 1$.

ex: let ~~cos~~ $y_1 = \cos x$ $y_2 = \sin x$

$$\therefore W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0 \ \forall x$$

$\therefore$ y_1 and y_2 are independant functions. $\forall x$.

ex $y_1 = e^{mx}$, $y_2 = xe^{mx}$

$$W = \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & (mxe^{mx} + e^{mx}) \end{vmatrix} = (1+mx)e^{2mx} - x m e^{2mx}$$

$$= e^{2mx} \neq 0 \ \forall x.$$

$\therefore$ y_1 and y_2 are independant functions $\forall x$.

D-operator

$D = \frac{d}{dx}$ | $\frac{dy}{dx} = Dy$

$\frac{d^2y}{dx^2} = D^2y$

Second Order non-homogeneous linear differential equation with constant coefficients:-

form: $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = f(x) \Rightarrow (D^2 + a_1 D + a_2)y = f(x)$

$\therefore y = \frac{1}{D^2 + a_1 D + a_2} f(x)$

If $f(x) \neq 0$, then the equation is non-homogeneous

Solution of the equation =

= complementary function (C.F) + Particular integral (P.I)

CF = Solution of the corresponding homogeneous equation taking $f(x) = 0$

PI = Particular solution for non-homogeneous pair.

Particular Integrals (P.I)

case I: $f(x) = e^{ax}$

case I: $PI = \frac{1}{(D^2 + a_1 D + a_2)} e^{ax}$ → replace D with a

Where m_1 or $m_2 \neq a$

$= \frac{1}{a^2 + a_1 a + a_2} e^{ax}$ → if $a^2 + a_1 a + a_2 \neq 0$. and $a \neq m$.

→ Solve : $(D^2 + 2D + 1)y = 2e^{3x}$

The auxilary equation of the given problem is

$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0.$

$m_1 = m_2 = -1$

$\therefore CF = e^{-x}(A + Bx)$

$\therefore$ P.I $= \frac{1}{(D+1)^2} \cdot 2e^{3x}$

$= \frac{2e^{3x}}{(3+1)^2} = \frac{e^{3x}}{8}$

$\therefore$ Solution is $y = e^{-x}(A+Bx) + \frac{e^{3x}}{8}$

where A & B are arbitrary constant.

→ Solve: $(D^2 - 5D + 6) = e^{4x}$

Auxilary equation :- $m^2 - 5m + 6 = 0$

$\Rightarrow m^2 - 3m - 2m + 6 = 0$

$\Rightarrow m(m-3) - 2(m-3) = 0$

$\therefore m = 2, 3$

$CF = Ae^{3x} + Be^{2x}$.

now,

$PI = \frac{1}{(D-2)(D-3)} \cdot e^{4x}$

or, $PI = \frac{1}{(4-2)(4-3)} e^{4x} = \frac{e^{4x}}{2}$.

$\therefore$ Solution is $y = Ae^{3x} + Be^{2x} + \frac{e^{4x}}{2}$

where A and B are arbitrary constants.

classmate

Date

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Case II

Suppose the auxillary equation has roots.

$m = m_1$ & m_2

and $a = m_1$ or m_2

$\hookrightarrow e^{ax}$

then, $P.I = \frac{1}{(D-m_1)(D-m_2)}\{e^{ax}\}$

$$= \begin{cases} \frac{xe^{ax}}{(a-m_2)} & \text{if } a = m_1 \\ \frac{xe^{ax}}{(a-m_1)} & \text{if } a = m_2 \end{cases}$$

→ Solve: $(D^2 - 5D + 6) = e^{3x}$

Auxilary equation :- $m^2 - 5m + 6 = 0$

$\therefore m = 3, 2.$

$CF = Ae^{3x} + Be^{2x}$

now, $m_1 = a = 3.$

$\therefore PI = \frac{xe^{3x}}{(3-2)}$ $\left[\frac{xe^{ax}}{(a-m_2)} \text{ where, } a = 3, m_2 = 2\right]$

$= xe^{3x}$

$\therefore$ Solution: $Ae^{3x} + Be^{2x} + xe^{3x}$ where A & B are arbitrary constants.

Case III

If $a = m_1 = m_2$

then, $PI = \frac{x^2}{2!} e^{ax}$

→ Solve: $(D^2 + D - 2)y = 2e^x + 7e^{-2x} + 4e^{2x}$.

Let $f(x)$ be 0.

$\therefore (m^2 + m - 2)y = 0$

$\Rightarrow m^2 + m - 2 = 0$

$\Rightarrow m^2 + 2m - m - 2 = 0$

$\Rightarrow \cancel{D-1)} (m-1)(m+2) = 0$

$\therefore m_1 = 1 \qquad m_2 = -2$

$C.F = Ae^x + \frac{B}{e^{2x}}$

$PI_1 = \frac{2xe^x}{(1-(-2))} = 2\frac{xe^x}{3}$

$PI_2 = \frac{7 \times xe^{-2x}}{(-2-1)} = -\frac{7}{3}xe^{-2x}$

$PI_3 = 4\frac{e^{2x}}{(D-1)(D+2)} = \frac{4e^{2x}}{4} = e^{2x}$

$\therefore$ Solution: $Ae^x + \frac{B}{e^{2x}} + \frac{2}{3}xe^x - \frac{7}{3}xe^{-2x} + e^{2x}$

Case III :

$f(x) = x^n$

$$\frac{1}{1-D} = 1 + D + D^2 + D^3 + \dots$$

$$\frac{1}{1+D} = 1 - D + D^2 - D^3 + \dots$$

$$\frac{1}{(1-D)^2} = 1 + 2D + 3D^2 + 4D^3 + \dots$$

$$\frac{1}{(1+D)^2} = \cancel{1 + 2D + 3D^2} \dots \quad 1 - 2D + 3D^2 - \dots$$

Question $(D^2 - 2D + 1)y = 3x^3$

Sol^n Auxilary eq: $m^2 - 2m + 1 = 0$

$\Rightarrow m^2 - m - m + 1 = 0$

$\Rightarrow m(m-1) - (m-1) = 0$

$\Rightarrow (m-1)^2 = 0$

$\therefore m = 1, 1$

$CF = e^x (A + Bx)$

$$PI = \frac{1}{D^2 - 2D + 1}\{3x^3\} = \frac{1}{(D-1)^2}\{3x^3\} = \frac{1}{(1-D)^2}\{3x^3\}$$

$$= \{1 + 2D + 3D^2 + 4D^3 + \dots\}\{3x^3\}$$

$$= \{1 + 2D + 3D^2 + 4D^3 + \dots\}\{3x^3\}$$

$$= 3x^3 + 2\frac{d}{dx}(3x^3) + 3\frac{d^2}{dx^2}(3x^3) + 4\frac{d^3}{dx^3}(3x^3)$$

$$= 3x^3 + 18x^2 + 54x + 72$$

$$\therefore y = Ae^x + Bxe^x + 3x^3 + 18x^2 + 54x + 72.$$

Question $(D^2 - 4D + 4)y = 2x + 3$

Auxilary equation is given by:

$\Rightarrow m^2 - 4m + 4 = 0$

or, $m^2 - 2m - 2m + 4 = 0$

or, $m(m-2) - 2(m-2) = 0$

or, $(m-2)^2 = 0$

or, $m = 2, 2.$

$Cf = e^{2x}(A + Bx)$

$$PI = \frac{1}{(D-2)^2}\{2x + 3\}$$

$$= \frac{1}{\left[\left(2\left(\frac{D}{2} - 1\right)\right)^2\right]}\{2x + 3\}$$

$$= \frac{1}{4\left(1 - \frac{D}{2}\right)^2}\{2x + 3\}$$

$$= \frac{1}{4}\left[1 + 2\cdot\frac{D}{2} + 3\cdot\left(\frac{D}{2}\right)^2 + 4\cdot\left(\frac{D}{2}\right)^3 + \cdots\right]\left\{2x+3\right\}$$

$$= \frac{1}{4}\left[1 + D\right]\left\{2x+3\right\}$$

$$= \frac{1}{4}\left[(2x+3) + \frac{d}{dx}(2x+3)\right] = \frac{1}{4}\left[2x+3+2\right] = \frac{2x+5}{4}.$$

Ex: Let $PI = \frac{1}{D^2+D+1}\left\{5x^3\right\}$

$$= \frac{1}{1 + \underline{\underline{(D^2+D)}}}\left\{5x^3\right\}$$
$$\mathcal{D}$$

$$= \frac{1}{1+\mathcal{D}}\left\{5x^3\right\} \quad \text{where, } \underline{\underline{\mathcal{D} = D^2+D}}.$$

$$= \left\{1 - (D+D^2) + (D+D^2)^2 - (D+D^2)^3 + \cdots\right\}\left\{5x^3\right\}.$$

Let $PI = \frac{1}{D^2+2D-3}\left\{2x+7\right\}$

$$= \frac{1}{-3\left(1 - \left(\frac{2D+D^2}{3}\right)\right)}\left\{2x+7\right\}$$

$$= -\frac{1}{3}\left[1 + \frac{2D+D^2}{3} + \left(\frac{2D+D^2}{3}\right)^2 + \cdots\right]\left\{2x+7\right\}$$

Question: $(D^2-1)y = 2x^4 - 3x + 1$

$$PI = \frac{1}{D^2-1}\cdot\left\{2x^4 - 3x + 1\right\}$$

$$= -\frac{1}{1-D^2}\left\{2x^4 - 3x + 1\right\}$$

$$= -\left[1 + D^2 + D^4\right]\left\{2x^4 - 3x + 1\right\}$$

$$= -\left[2x^4 - 3x + 1 + \frac{d^2}{dx^2}\left(2x^4 - 3x + 1\right) + \frac{d^4}{dx^4}\left(2x^4 - 3x + 1\right)\right]$$

$$= -\left[2x^4 - 3x + 1 + 24x^2 + 48\right]$$