21.02.23

Comparision Test (Limit Form) :-

Let $\sum u_n$ and $\sum v_n$ be two infinite series where $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite), then if $\sum u_n$ ~~is convergent~~ and ~~then~~ $\sum v_n$ ~~is also convergent and if $\sum v_n$ is divergent then $\sum u_n$ is also divergent.~~ converge and diverge together.

Ex: Test the convergence of: $1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \dots$

ans: Here: $u_n = \frac{2^{n-1}}{(n-1)!}$

$$\therefore u_n = \frac{2}{(n-1)} \cdot \frac{2}{(n-2)} \cdots\cdots \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{2}{1} \ (n-1) \text{ times}$$

$$< \frac{2}{3} \cdot \frac{2}{3} \cdots\cdots \frac{2}{3} \ (n-3) \text{ times} \times 2$$

$$= \left(\frac{2}{3}\right)^{n-3} \cdot 2$$

now, let $v_n = 2\left(\frac{2}{3}\right)^{n-3}$

look like $\rightarrow 2\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 \cdots\cdots \left(\frac{2}{3}\right)^{n-3}\right)$

where $\sum v_n$ is a geometric series
where $x = \frac{2}{3} < 1$
and is thus convergent.

$\therefore$ By comparision test, $\sum u_n$ is also convergent
Since, $\sum u_n < \sum v_n$

Question: Test the convergence of : $\frac{6}{1.3.5}+\frac{8}{3.5.7}+\frac{10}{5.7.9}+\dots$

ans : Here -

$$u_n = \frac{2n+4}{(2n-1)(2n+1)(2n+3)}$$

Let $v_n = \frac{1}{n^2}$

$$\therefore \frac{u_n}{v_n} = \frac{n^2(2n+4)}{(2n-1)(2n+1)(2n+3)}$$

now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2+\frac{4}{n})}{(2-\frac{1}{n})(2+\frac{1}{n})(2+\frac{3}{n})} = \frac{2}{8} = \frac{1}{4}$ (finite)

Here, $\sum v_n = \sum 1/n^2$ is a p series where $p=2>1$ and is thus convergent.

$\therefore$ By comparision ~~th~~ test, u_n is also convergent

$\because$ ~~$v_n = u_n$ is~~ $\lim_{n\to\infty} \frac{u_n}{v_n} =$ finite.

Series.

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Question Show that, the series $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots\dots$ is convergent.

Sol. $u_n = \frac{1}{n!}$

$$\sum u_n = 1+\frac{1}{2!}+\frac{1}{3!}-\dots-\dots+\frac{1}{n!}$$

now, $\frac{1}{2!} = \frac{1}{2}$

$$\frac{1}{3!} < \frac{1}{2^2}$$

$$\frac{1}{4!} < \frac{1}{2^3}$$

$\therefore$ let $\sum v_n = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots$

so, $u_n \leq v_n$

and since $\sum v_n$ is a geometric series with $x = \frac{1}{2} < 1$ and is thus convergent.

Thus, by comparision test, $\sum u_n$ is also convergent since, $\sum u_n < \sum v_n$.

Cauchy's Root Test :

If $\sum u_n$ is a series of positive terms, such that -

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = l \text{ (finite) then -}$$

i) the series is convergent if $l<1$.
ii) the series is divergent if $l>1$.
iii) if $l=1$, then the test fails and we need to check by some another test.

Ex: $\sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Sol. Here, $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$-n^{3/2} \times \frac{1}{n}$$
$$= -n\sqrt{n} \times \frac{1}{n}$$
$$= -\sqrt{n}.$$

$$\therefore \lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$= \frac{1}{\lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1.$$

$\therefore$ By Cauchy's Root Test, $\sum u_n$ is convergent.

01/03/23

D'Alembert's ratio test :-

If $\sum u_n$ is a series of positive terms, such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$ (finite) $[l \in \mathbb{R}]$; then

$\frac{u_n}{u_{n+1}}$ $l>1$ conv, $l<1$ div

i) the series is convergent if $l<1$
ii) the series is divergent if $l>1$
iii) there is no ~~contd~~ conclusion if $l=1$.

Question Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$.

Here, $u_n = \frac{n!\,2^n}{n^n}$

$\therefore$ ~~$u_{n+1} =$~~ $u_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}} \Big/ \frac{n!\,2^n}{n^n} = \frac{2(n+1)}{(n+1)^{n+1}} \times n^n$$

$$= \frac{2n^n}{(n+1)^n}$$

$$= \frac{2}{\left(\frac{n+1}{n}\right)^n} = \frac{2}{(1+1/n)^n}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2}{(1+1/n)^n} = \frac{2}{\lim_{n\to\infty}(1+1/n)^n} = \frac{2}{e} < 1$$

$\therefore$ By D'Alembert's ratio test, $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$ is convergent

Raabe's Test :-

If $\sum u_n$ be a series of positive terms such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l \text{ (finite) } [l \in \mathbb{R}], \text{ then}$$

i) the series is convergent if $l > 1$
ii) the series is divergent if $l < 1$
iii) the test fails if $l = 1$.

Question Test the convergence of $\rightarrow \sum u_n$ where -

$$u_n = \frac{3.6.9 \dots 3n}{7.10.13 \dots (3n+4)}$$

$$\therefore u_{n+1} = \frac{3.6.9 \dots 3n.(3n+3)}{7.10.13 \dots (3n+4)(3n+7)}$$

now, $\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n+3}{3n+7} = \lim_{n\to\infty} \frac{3+3/n}{3+7/n} = \frac{3}{3} = 1.$$

Thus, D'Alembert's test fails.

$\therefore$ now, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+3} - 1\right) = \lim_{n\to\infty} \frac{4n}{3n+3}$$

$$= \lim_{n\to\infty} \frac{4}{3+3/n} = \frac{4}{3} > 1.$$

Thus, Raabe's Test $\sum u_n$ is convergent.

Question: Examine the convergence of $\frac{1.2}{3} + \frac{2.3}{5} + \frac{3.4}{7} + \dots$

ans: Here :- $u_n = \frac{n(n+1)}{2n+1}$

now, let us consider $v_n = n$

then, $\frac{u_n}{v_n} = \frac{n(n+1)}{2n+1} \times \frac{1}{n} = \frac{n+1}{2n+1} = \frac{1+1/n}{2+1/n}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$ which is finite.

So, u_n & v_n converge or diverge together according to convergence test (limit form)

now, $\sum v_n$ is a p-series of $p=-1$ and is thus divergent.

$\therefore \sum u_n$ is also divergent.

Question Test the convergence of the series -

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots$$

Hint : Apply comparision test limit form.

$$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{\cancel{n}\left(2-\frac{1}{n}\right)}{\cancel{n}.n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let. $v_n = \frac{1}{n^2}$

Now, $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)\times n^2}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\times n^2} = 2 = \text{finite}$

Σv_n is a p. series and $P = 2 > 1$. So v_n is convergent. $\therefore$

By comparision test, Σu_n is convergent.

(limit form)

Question Test the convergence of the series-

$$\frac{6}{1.3.5} + \frac{8}{3.5.7} + \frac{10}{5.7.9} + \cdots$$

Hint : Apply comparision test limit form

Now, $u_n = \dfrac{2n+4}{(2n-1)(2n+1)(2n+3)}$

~~Let~~.
$$= \frac{\cancel{n}\left(2+\frac{4}{n}\right)}{\cancel{n}.n^2\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)}$$

Let, $v_n = \dfrac{1}{n^2}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\left(2+\frac{4}{n}\right)}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)} = \frac{2}{2\times 2\times 2} = \frac{1}{4} = \text{finite}$$

Σv_n is a p-series and $p = 2 > 1$ which is convergent.

So, by comparision test Σu_n is also convergent.

Question Test the convergence of the series :

$$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \ldots n$$

Hint : Apply comparision test limit form.

ans

$$u_n = \frac{1+2+3+\ldots(n+1)}{(n+1)^3}$$

$$u_n = \frac{1+2+3\ldots+(n+1)}{(n+1)^3}$$

$$= \frac{\frac{(n+1)(n+2)}{2}}{(n+1)^3}$$

$$= \frac{n+2}{2(n+1)^2} \quad \cancel{= \frac{n}{2(n+1)^2} + \frac{1}{(n+1)^2}}$$

$$\frac{n+1}{2(n+1)^2} = \frac{1}{2} \cdot \frac{1}{n+1}$$

$$\cancel{u_{n+1} = \frac{n+3}{2(n+2)^2}}$$

$$\cancel{\frac{u_{n+1}}{u_n} = \frac{n+3}{2(n+2)^2} \times \frac{2}{n}}$$

$$u_n = \frac{(n+2)}{2(n+1)^2} \qquad v_n = \frac{1}{n+1} = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} \ldots$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$$

is convergent

$$\frac{u_n}{v_n} = \frac{n+2}{2(n+1)} = \frac{1+2/n}{2(1+\frac{1}{n})}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left| \frac{1+2/n}{2(1+1/n)} \right| = \frac{1}{2} \text{ (finite)}$$

$\therefore \Sigma u_n$ is also convergent

Question Test the convergence of the series whose n^{th} term is : $u_n = \sqrt{n^2+1} - n$.

ans. Here : $u_n = \sqrt{n^2+1} - n$

$$u_n = \frac{\left(\sqrt{n^2+1} - n\right)\left(\sqrt{n^2+1} + n\right)}{\left(\sqrt{n^2+1} + n\right)}$$

$$= \frac{1}{\sqrt{n^2+1} + n}$$

Let us assume, $v_n = \frac{1}{n}$

$$\therefore \frac{u_n}{v_n} = \frac{1}{\sqrt{n^2+1}+n} \Big/ \frac{1}{n}$$

$$= \frac{n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{1+\frac{1}{n^2}} + 1}$$

$\therefore \lim\limits_{n \to \infty} = \frac{1}{2}$ which is finite.

So, u_n and v_n converge together according to Convergence test limit form.

now, $\sum v_n$ is a p-series of $p=1$ and is thus convergent

$\therefore \sum v_n$ is also convergent.

Question: Test the convergence of the series

$$\left(\frac{1}{3}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \dots + \infty$$

Hint: Apply Cauchy's Root Test

ans: Here: $u_n = \left(\frac{n}{2n+1}\right)^n$

Now, $\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\frac{n}{2n+1}\right)^{n\times\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n\left(2+\frac{1}{n}\right)} = \frac{1}{2}$

< 1

$\therefore$ By Cauchy's Root test the series is convergent

10.03.23

SERIES

Question: Test the convergence of the series:

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots$$

→ Let, $\sum u_n = 1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots$

$$\therefore u_n = \frac{n^2}{n!}, \quad u_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

now, $$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2}{n!} \times \frac{(n+1)!}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{n^2}{n!} \times \frac{(n+1)n!}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n}{1+1/n} = \infty > 1.$$

$\therefore$ By D'Alembert's ratio test, $\sum u_n$ is convergent

Question: Test the convergence of the series:

$$\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \dots$$

→ Let, $\sum u_n = \frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \dots$

$$\therefore u_n = \frac{n}{1+2^n}, \quad u_{n+1} = \frac{(n+1)}{1+2^{(n+1)}}$$

now, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n}{1+2^n} \times \frac{1+2^{(n+1)}}{(n+1)}$

$= \lim_{n\to\infty} \frac{n}{(n+1)} \times \frac{1+2^{n+1}}{1+2^n} = \lim_{n\to\infty} \frac{1}{1+1/n} \cdot \frac{1+2\cdot 2^n}{1+2^n}$

$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \times \frac{\frac{1}{2^n}+2}{\frac{1}{2^n}+1}$

$= 1 \times 2 = 2 > 1$

$\therefore$ By D'Alembert's ratio test, $\sum u_n$ is convergent.

Question Test the convergence of the series:

$$1 + \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1.3.5}{2.4.6} + \cdots$$

$\rightarrow$ let, $\sum u_n = 1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \cdots$

$\therefore u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$, $u_{n+1} = \frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots 2n(2n+2)}$

now, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2n+2}{2n+1} = \lim_{n\to\infty} \frac{1+1/n}{1+1/2n} = 1$

$\therefore$ D'Alembert's ratio test fails and have no conclusion

now, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - 1\right)$

$= \lim_{n\to\infty} n\left(\frac{1}{2n+1}\right) = \lim_{n\to\infty} \frac{1}{2 + 1/n} = \frac{1}{2} < 1.$

$\therefore$ By Raabe's Test, $\sum u_n$ is divergent.

14.03.23

Question Test the convergence of the series:

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\dots\dots$$

→ Here, $u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$

$$\therefore u_n^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1}$$

$$= \frac{n}{n+1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$\therefore \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$$

$$= 1\left[e-1\right]^{-1} < 1.$$

$\therefore$ By Cauchy's Root Test, $\sum u_n$ is convergent.

Question Examine the convergence of:

$$\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\cdot\frac{2}{5}\right)^2+\left(\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\right)^2+\dots$$

→ Here, $u_n = \left[\frac{1.2.3\dots n}{3.5.7\dots(2n+1)}\right]^2$

$\therefore u_{n+1} = \left[\frac{1.2 \ldots \ldots n(n+1)}{3.5.7 \ldots \ldots (2n+1)(2n+3)}\right]^2$

now, $\frac{u_{n+1}}{u_n} = \left(\frac{n+1}{2n+3}\right)^2$

$\therefore \lim_{n\to\infty} = \left(\frac{1+1/n}{2+3/n}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} < 1$

Thus, by D'A lemberts ratio test; $\sum u_n$ is convergent.

Question Test the convergence of the series:

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots$$

Hint : Apply D'A lemberts ratio test.

Now, $u_n = \frac{n}{2^n}$ $\quad \therefore u_{n+1} = \frac{n+1}{2^{n+1}}$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \times \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})}{2}$

$= \frac{1}{2} < 1$

$\therefore$ By D'A lembert's ratio test the series is convergent.

Question Discuss the convergence of $\sum_{n=1}^{\infty} n^4 e^{-n^2}$

→ Here, $\frac{u_{n+1}}{u_n} = \frac{(n+1)^4 e^{-(n+1)^2}}{n^4 e^{-n^2}}$

$$= \left(1+\frac{1}{n}\right)^4 e^{-(n+1)^2+n^2}$$

$$= \left(1+\frac{1}{n}\right)^4 e^{-2n-1}$$

$$= \frac{\left(1+\frac{1}{n}\right)^4 \cdot e^{-1}}{e^{2n}}$$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1.$

Thus, by D'Alembert's ratio test, $\sum u_n$ is convergent

Question Test the convergence of the series : (by Raabe's Test)

$$1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7} + \cdots$$

→ $u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} \times \frac{1}{2n+1}$

$u_{n+1} = \frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots 2n(2n+2)} \times \frac{1}{\cancel{(2n+1)}(2n+3)}$

Incomplete