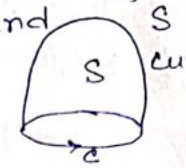
EM-II

10.05.2024

• **Stoke's Theorem:** Let $\vec{F}$ be a continuous differentiable vector point function and S be a surface bounded by a closed curve then,



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds.$$

NOTE: If $\vec{F}$ is conservative, then find $\int_C \vec{F}\cdot d\vec{r}$.

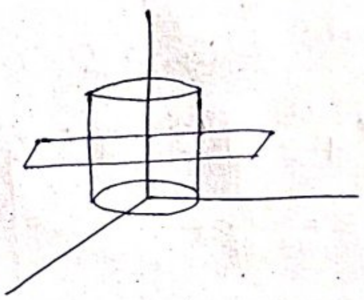
$$= \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds \text{ [using Stoke's Theorem]}$$

$$= \iint_S \vec{0}\cdot\hat{n}\,ds = 0.$$

Q. Use Stoke's Theorem to evaluate $\int_C \cos x\,dx + 2y^2dy + z\,dz$ where C is the curve $x^2+y^2=1$, $z=1$.

→ using Stoke's theorem,

$$\int_C \cos x\,dx + 2y^2dy + z\,dz$$



[Here $\vec{F} = \cos x\,\hat{i} + 2y^2\hat{j} + z\hat{k}$]

$$= \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

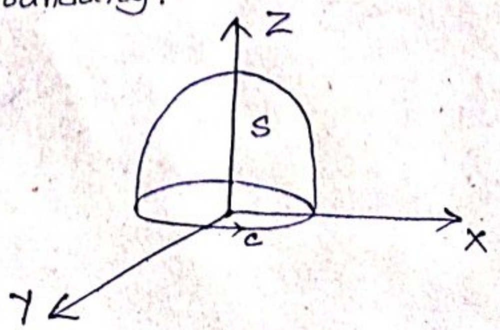
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & 2y^2 & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial}{\partial z}2y^2\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial}{\partial z}\cos x\right) + \hat{k}\left(\frac{\partial}{\partial x}2y^2 - \frac{\partial}{\partial y}\cos x\right)$$

$$= \vec{0}$$

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S \vec{0}\cdot\hat{n}\,ds = 0.$$

Q. Verify Stoke's Theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is the boundary.



→ On the boundary c we have, $x^2+y^2=1$ and $z=0$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (2x-y)\,dx - yz^2\,dy \cdot y^2z\,dz$$

$$= \int_C (2x-y)\,dx$$

Let, $x=\cos\theta$ and $y=\sin\theta$ → Parametric form

$$= \int_{\theta=0}^{2\pi} (2\cos\theta-\sin\theta)(-\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} (\sin^2\theta - 2\sin\theta\cos\theta)\,d\theta$$

$\boxed{\begin{array}{l} \cos n\pi = (-1)^n \\ \sin n\pi = 0 \end{array}}$

$$= \frac{1}{2}\int_0^{2\pi} 2\sin^2\theta\,d\theta - \int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1-\cos 2\theta)\,d\theta + \left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[(\theta)_0^{2\pi} - \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}\right] + \frac{1}{2}\left[\cos 4\pi - \cos 0\right]$$

$$= \frac{1}{2}\left[2\pi - \frac{1}{2}\times 0\right] + 0$$

$$= \underline{\pi}$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & (-y^2z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x-y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right]$$

$$= \hat{i}\left[-z\cdot 2y + y\cdot 2z\right] - 0 + 1(\hat{k})$$

$$= \hat{k}.$$

$$\hat{n} = \text{grad}(x^2+y^2+z^2-1)$$

$$= \vec{\nabla}(x^2+y^2+z^2-1) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-1)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}.$$

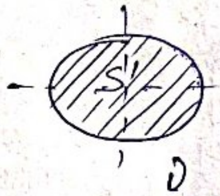
$$\therefore \hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}} = x\hat{i} + y\hat{j} + z\hat{k} \left[\because x^2+y^2+z^2 = 1 \text{ on } S\right]$$

$$\therefore \text{curl}\,\vec{F}\cdot\hat{n} = \hat{k}\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = z.$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S z\,ds$$

$$= \iint_{S'} z \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$ [S' is the orthogonal projection of S on xy plane

$$= \iint_{S'} \cancel{z} \frac{dx\,dy}{\cancel{z}}$$



$x^2+y^2 \leq 1$

= Area of S'

$$= \pi\cdot(1)^2 = \pi$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

Hence Stoke's Theorem is verified.