First Order Linear Differencial Equation :

Form: $\frac{dy}{dx} + P(x)y = Q(x)$

Ex: $h(x)\frac{dy}{dx} + f(x)y = g(x)$

$$\frac{dy}{dx} + \frac{f(x)}{h(x)}y = \frac{g(x)}{h(x)}$$

I.F of the given form $\rightarrow e^{\int P(x)\,dx}$

Solution: $y \times IF = \int (Q(x) \times IF)\,dx + C$

Question $\frac{dy}{dx} - y = 0$

Sol^n: Comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

We have, $P(x) = -1$
$Q(x) = 0$

$\therefore$ For the linear DE :

$$IF = e^{\int P(x)\,dx} = e^{\int -1\,dx} = e^{-x}$$

$\therefore$ Solution $\rightarrow y \times IF = \int (Q(x) \times IF)\,dx + C$

or, $y \times e^{-x} = \int (0 \times e^{-x})\,dx + C$

or, $ye^{-x} = C$

or, $y = e^{x}C$.

Question $\frac{dy}{dx} + xy = e^{x}$

ans: Comparing with $y' + P(x)y = Q(x)$

We have, $P(x) = x$

$Q(x) = e^{x}$

$\therefore$ for linear DE:

$$IF = e^{\int P(x)\,dx} = e^{\int x\,dx} = e^{x^2/2}$$

Solution: $y \times IF = \int [Q(x) \times IF]\,dx + C$

or, $y e^{x^2/2} = \int [e^{x} \cdot e^{x^2/2}]\,dx + C$

$y e^{x^2/2} = \int \left(e^{\frac{2x + x^2}{2}}\right) dx + C$

Question $x\frac{dy}{dx} - y = x^3$

ans $x\frac{dy}{dx} - y = x^3$

or, $\frac{dy}{dx} - \frac{1}{x}y = x^2$

Comparing to $\frac{dy}{dx} + P(x)y = Q(x)$

$P(x) = -\frac{1}{x}$ $\quad Q(x) = x^2$

$\therefore IF = e^{\int P(x)\,dx} = e^{-\int \frac{1}{x}\,dx} = \cancel{e^{-\int x^{-1}\,dx}}\ e^{-\log x} = \frac{1}{x}$

Solution: $y \times IF = \int (Q(x) \times IF)\,dx + C$

or, $y \cdot \frac{1}{x} = \int \left(x^2 \cdot \frac{1}{x}\right) dx + C$ $\quad$ or, $\frac{y}{x} = \frac{x^2}{2} + C$

~~or $y = x^2 + xC$~~

Question $x\dfrac{dy}{dx} + y = x\sin x$

ans $x\dfrac{dy}{dx} + y = x\sin x$

or, $\dfrac{dy}{dx} + \dfrac{y}{x} = \sin x$

~~or,~~ $\therefore \; P(x) = \dfrac{1}{x} \qquad Q(x) = \sin x$

$\text{IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Solution: $y \times \text{IF} = \int \left(Q(x) \times \text{IF} \right) dx + C$

$\Rightarrow xy = \int (\sin x \cdot x)\, dx + C$

$\Rightarrow xy = \cancel{x \frac{d}{dx} x \sin} - x\cos x + \sin x + C$