EM-II

• Gauss Divergence Theorem

Let $\vec{F}$ be a vector point function defined over the volume V with surface S, then $\iiint_V \operatorname{div}\vec{F}\, dv = \underbrace{\iint_S \vec{F}\cdot\hat{n}\, ds}_{\text{flux}}$

NOTE: If $\vec{F}$ is solinoidal, i.e. $\operatorname{div}\vec{F} = 0$

Then, $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv = \iiint_V 0\, dv = 0.$

Q. Using Gauss Divergence Theorem evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$, where $\vec{F} = 3xz\hat{i} + y^2\hat{j} - 3yz\hat{k}$ and S is the surface of the cube bounded by $x=0,\ y=0,\ z=0,\ x=1,\ y=1,\ z=1$.

→ By Divergence Theorem,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv \quad \text{—— (i)}$$

Now, $\operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(3xz\hat{i} + y^2\hat{j} - 3yz\hat{k})$

$$= \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial z}(3yz)$$

$$= 3z + 2y - 3y = 3z - y$$

From (i),

$$\iiint \operatorname{div}\vec{F}\, dv = \iiint (3z - y)\, dv$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (3z - y)\, dz\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[\frac{3z^2}{2} - yz\right]_0^1 dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[\frac{3}{2}\cdot 1 - y\right] dy\, dx$$

$$= \int_{x=0}^{1} \left[\frac{3}{2}y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_{x=0}^{1} \cancel{\left[y^2/2\right]_0^1 dx} \left[\frac{3}{2} - \frac{1}{2}\right] dx$$

$$= \int_{x=0}^{1} \left[\frac{2}{2}\right] dx$$

$$= [x]_0^1 = 1.\ \text{(Ans.)}$$

Q. Using Divergence then evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 2xz\hat{i} + y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube ~~ob~~ bounded by $x=0$, $y=0$, $z=0$, $x=1$, $y=1$, $z=1$.

→ By Divergence Theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\vec{F}\,dv \text{ —— (i)}$$

Now, $\mathrm{div}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2xz\hat{i} + y^2\hat{j} + yz\hat{k})$

$$= \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 2z + 2y + y = 3y + 2z$$

~~For (i~~ From (i),

$$\iiint_V \mathrm{div}\vec{F}\cdot dv$$

$$= \iiint_V (2z + 3y)\,dv$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2z+3y)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[2z^2/2 + 3yz\right]_0^1 dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[z^2 + 3yz\right]_0^1 dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} [1 + 3y]\,dy\,dx$$

$$= \int_{x=0}^{1} \left[y + \frac{3}{2}y^2\right]_0^1 dx$$

$$= \int_{x=0}^{1} \left[1 + \frac{3}{2}\right] dx = \int_{x=0}^{1} [5/2]\,dx$$

$$= \frac{5}{2}\int_{x=0}^{1} dx = \frac{5}{2}[x]_0^1 = \frac{5}{2}. \text{ (Ans.)}$$

Q. Use Divergence Theorem to evaluate

$\iint_S (x^3 dy\,dz + x^2 y\,dz\,dx + x^2 z\,dy\,dx)$ where S is ~~cattet~~ colsed surface consisting of the cylinder $x^2+y^2=4$, $0 \leq z \leq 3$ and the circular discs $z=0$, $z=3$.

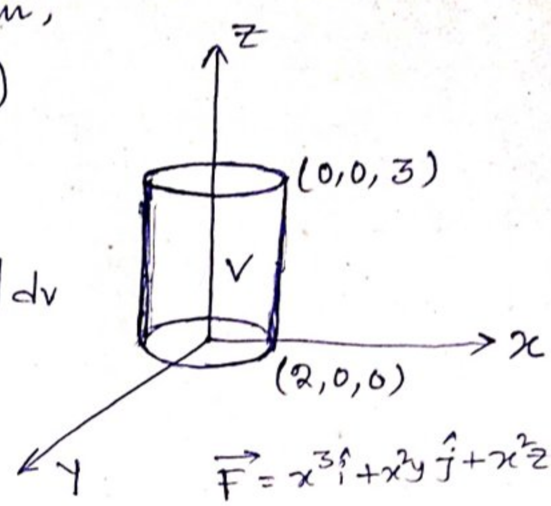
→ By Gauss Divergence Theorem,

$$\iint_S (x^3 dy\,dz + x^2 y\,dz\,dx + x^2 z\,dy\,dx)$$

$$= \iiint_V \operatorname{div}\vec{F}\,dv \quad \text{——— (i)}$$

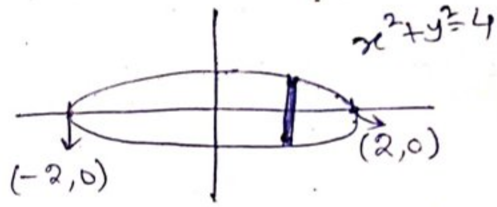
$$= \iiint_V \left[\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)\right] dv$$

$$= \iiint_V 5x^2\,dv$$



$\vec{F} = x^3\hat{i} + x^2 y\,\hat{j} + x^2 z\,\hat{k}$

$$= \int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 5x^2\,dy\,dx\,dz$$



$$= 5\cdot 2\cdot 2\int_{z=0}^{3}\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} x^2\,dy\,dx\,dz$$

vertical strip for finding y limit
$y = \sqrt{4-x^2}$

$$= 20\int_{z=0}^{3}\int_{x=0}^{2} x^2\,[y]_0^{\sqrt{4-x^2}}\,dx\,dz$$

$$= 20\int_{z=0}^{3}\int_{x=0}^{2} x^2\sqrt{4-x^2}\,dx\,dz$$

$$\left[\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx\right.$$

$$= 20\int_{z=0}^{3} \pi\,dz = 20\pi \times 3 = 60\pi.$$

$4-x^2 = z^2$

$\Rightarrow 4 - z^2 = x^2$

$\Rightarrow \cancel{2}x\,dx = -\cancel{2}z\,dz$

$\Rightarrow x\,dx = -z\,dz$