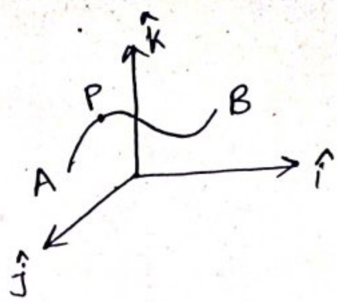
Line Integral:

$$\int_a^b f(x)\,dx$$



Let, Γ be a space curve whose initial position ~~B~~ is A and final position is B.

Let, P be any point on the curve.

$s = \overset{\frown}{AP}$.

Then the tangent at P of the curve Γ is give by $\vec{t} = \frac{d\vec{r}}{ds}$,

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of ~~P~~ P.

Then the line integral of a vector point function $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$ along the curve Γ is defined as

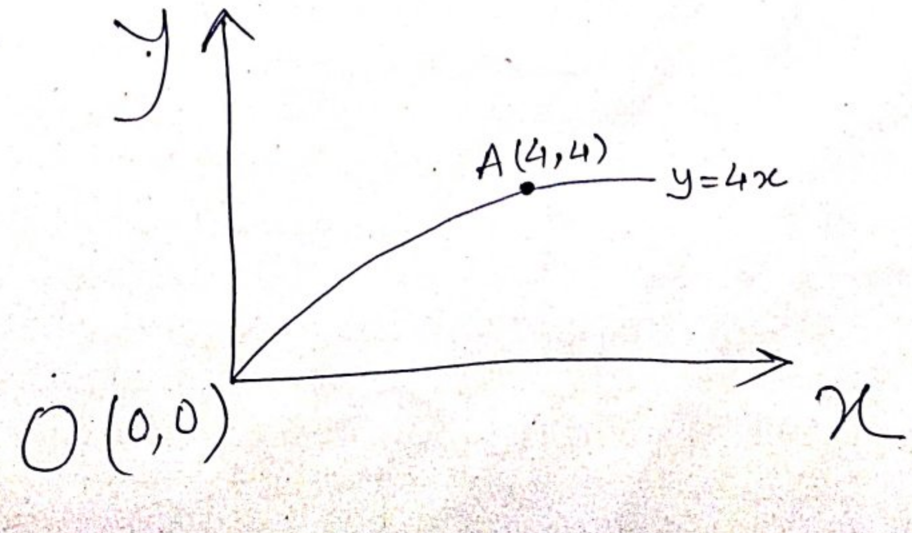
$$\int_{AB} \vec{F}\cdot\vec{t}\,ds = \int_{s_1}^{s_2} \vec{F}\cdot\frac{d\vec{r}}{ds}\,ds$$

$$= \int_{s_1}^{s_2} \vec{F}\cdot d\vec{r} = \int_{s_1}^{s_2} (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_{s_1}^{s_2} F_1\,dx + F_2\,dy + F_3\,dz \quad \left[\begin{aligned} \vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ d\vec{r} &= dx\hat{i} + dy\hat{j} + dz\hat{k} \end{aligned}\right]$$

Q: Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2y^2\hat{i} + y\hat{j}$ and the curve c is $y^2 = 4x$ in the xy-plane from $(0,0)$ to $(4,4)$.

→



$$\int_{\widehat{OA}} \vec{F}\cdot d\vec{r} = \int_{\widehat{OA}} (x^2y^2\hat{i} + y\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_{\widehat{OA}} x^2y^2dx + y\,dy$$

$y^2 = 4x$

$\Rightarrow y = 2\sqrt{x}$

$2y\,dy = 4\,dx$

$\Rightarrow dy = \frac{2}{y}dx$

$= \frac{\cancel{2}}{\cancel{2}\sqrt{x}}dx$

$\Rightarrow dy = \frac{dx}{\sqrt{x}}$.

$$= \int_{x=0}^{4} (x^2 4x)dx + 2\cancel{\sqrt{x}}\cdot\frac{dx}{\cancel{\sqrt{x}}}$$

$$= \int_{x=0}^{4} (4x^3+2)dx$$

$$= 4\cdot\left[\frac{x^4}{4}\right]_0^4 + 2[x]_0^4$$

$$= \cancel{4}\cdot\frac{256}{\cancel{4}} + 2\times 4 = 256+8 = 264.$$

Q. $\vec{A} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$, then find $\int \vec{A}\cdot d\vec{r}$ along the curve c given by $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

$\rightarrow$

$$\int \vec{A}\cdot d\vec{r} = \int (3xy\hat{i} - 5z\hat{j} + 10x\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$dx = 2t\,dt$, $dy = 4t\,dt$, $dz = 3t^2dt$

$$\int \vec{A}\cdot d\vec{r} = \int (3xy\hat{i} - 5z\hat{j} + 10x\hat{k})\cdot(2tdt\hat{i} + 4tdt\hat{j} + 3t^2dt\hat{k})$$

$$= \int \{3(t^2+1)\cdot(2t^2)\hat{i} - 5(t^3)\hat{j} + 10(t^2+1)\hat{k}\}\cdot (2tdt\hat{i} + 4tdt\hat{j} + 3t^2dt\hat{k})$$

$$= \int \{(6t^4 + 6t^2)\hat{i} - 5t^3\hat{j} + (10t^2+10)\hat{k}\}\cdot \{2tdt\hat{i} + 4tdt\hat{j} + 3t^2dt\hat{k}\}$$

$$= \int 12t^5dt + 12t^3dt - 20t^4dt + 30t^4dt + 30t^2dt$$

$$= \int_1^2 12t^5dt + \int_1^2 12t^3dt + \int_1^2 10t^4dt + \int_1^2 30t^2dt$$

$$= 12\cdot\left[t^6/6\right]_1^2 + 12\left[t^4/4\right]_1^2 + 10\left[t^5/5\right]_1^2 + 30\left[t^3/3\right]_1^2$$

$$= 2 \cdot [64 - 1] + 3\,[16 - 1] + 2\,[32 - 1] + 10\,[8 - 1]$$

$$= 126 + 45 + 62 + 70 = 303.$$

EM-II

Q. If $\vec{F} = (5x^2+6y)\hat{i} - (3x+2y^2)\hat{j} + 2xz^2\hat{k}$, then evaluate $\int_C \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve C, given by

(i) $x=t,\ y=t^2,\ z=t^3$.

→

$$\vec{F}\cdot d\vec{r} = \{(5x^2+6y)\hat{i} - (3x+2y^2)\hat{j} + 2xz^2\hat{k}\}\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= (5x^3+6xy - 3xy - 2y^3 + 2xz^3)$$

$$= (5x^3+3xy-2y^3+2xz^3)$$

$x = t \Rightarrow dx = dt$ $\qquad z = t^3 \Rightarrow dz = 3t^2dt$

$y = t^2 \Rightarrow dy = 2t\,dt$

$$\int \vec{F}\cdot d\vec{r} = \int\{(5t^2+6t^2)\hat{i} - (3t+2t^4)\hat{j} + 2t^7\hat{k}\}\cdot\{dt\,\hat{i} + 2t\,dt\,\hat{j} + 3t^2dt\,\hat{k}\}$$

$$= \int (11t^2)dt - \int 6t^2dt - \int 4t^5dt + \int 6t^9dt$$

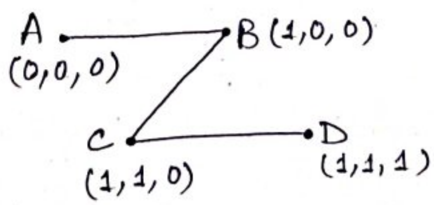
$$= \left[11\cdot\frac{t^3}{3} - 6\cdot\frac{t^3}{3} - 4\cdot\frac{t^6}{6} + 6\cdot\frac{t^{10}}{10}\right]_0^1$$

$$= \left[\frac{t^3}{3}\times 5 - \frac{4t^6}{6} + \frac{6t^{10}}{10}\right]_0^1$$

$$= \left[\frac{100t^3 - 40t^6 + 36t^{10}}{60}\right]_0^1$$

$$= \frac{1}{60}\left[100 - 40 + 36\right] = \frac{96}{60} = \frac{32}{20} = \frac{8}{5}.$$

(ii) the curve C is the st. lines joining the points $(0,0,0)$ to $(1,0,0)$, then $(1,0,0)$ to $(1,1,0)$ and then $(1,1,0)$ to $(1,1,1)$.



In the st. line AB

$y = z = 0$ & $x \in [0,1]$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int (5x^2+6y)dx - (3x+2y^2)dy + 2xz^2dz$$

$$= \int_{x=0}^{1} (5x^2+0)dx + 0 = \frac{5}{3}.$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int (5x^2+6y)\,dx - (3x+2y^2)\,dy + 2xz^2\,dz$$

$$\cancel{= -\int_0^1 2y^2\,dy = -2\left[y^3/3\right]_0^1 = -}$$

$$= -\int_0^1 \left[3(1)+2y^2\right] dy = -3[y]_0^1 - 2\left[y^3/3\right]_0^1$$

$$= -3 - \frac{2}{3} = -\frac{11}{3}$$

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_0^1 2xz^2\,dz = \int_0^1 2\cdot z^2\,dz = 2\cdot\left[\frac{z^3}{3}\right]_0^1 = \frac{2}{3}.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r}$$

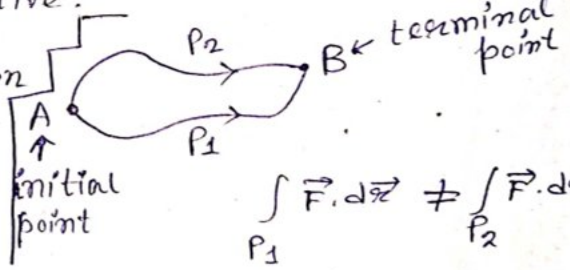
$$= \frac{5}{3} - \frac{11}{3} + \frac{2}{3} = -\frac{4}{3}.$$

EM-II

16.04.2024

• Conservative Vector field:

If $\int_{AB} \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r}$ is path independent then the vector field $\vec{F}$ is called conservative.



$$\int_{P_1} \vec{F}\cdot d\vec{r} \neq \int_{P_2} \vec{F}\cdot d\vec{r}$$

▲ Let v be a scalar point function such that $\vec{F} = \vec{\nabla} v = \text{grad } v$

Then, $\int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} \vec{\nabla} v\cdot d\vec{r}$

$$= \int_{AB} \left(\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial v}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \int_{AB} \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right)$$

$$= \int_{AB} dv \quad \text{[By chain Rule]}$$

$$= [v]_A^B$$

$$= v(B) - v(A).$$

Note: v is called potential of the vector field.

Q. If $\vec{F} = (x^2 - y)\hat{i} + (y^2 + x)\hat{j}$, then evaluate $\int_c \vec{F}\cdot d\vec{r}$ where c is the curve ~~in the~~ $y = x^2 + 1$, on x-y plane, from (0,1) to (1,2).

$\rightarrow \int \vec{F}\cdot d\vec{r} = \int_c \{(x^2 - y)\hat{i} + (y^2 + x)\hat{j}\}\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= \int_c \{(x^2 - y)\,dx + (y^2 + x)\,dy\}$

$= \int_c (x^2 - y)\,dx + \int_c (y^2 + x)\,dy$

$= \left[\frac{x^3}{3}\right]_0^1 - y(x)_0^1 + \left[\frac{y^3}{3}\right]_1^2 + x(y)_1^2$

$= \frac{1}{3}(1 - 0) - y\cdot 1 + \frac{1}{3}(8 - 1) + x\cdot 1$

$= \frac{1}{3} - y + \frac{7}{3} + x = x - y + \frac{8}{3}$.

$= 1 - 1 + \frac{8}{3} = \frac{8}{3}$.

$x(t) = t$

$y(t) = t^2 + 1$

$t = 0 \rightarrow (0, 1)$

$t = 1 \rightarrow (1, 2)$

$\int_c \{(x^2 - y)\,dx + (y^2 + x)\,dy\}$

$= \int_{t=0}^{1} \{(\cancel{t^2} - \cancel{t^2} - 1)\,dt + (t^4 + 2t^2 + 1 + t)\cdot 2t\,dt\}$

$= -[t]_0^1 + 2/6\,[t^6]_0^1 + 4/4\,[t^4]_0^1 + 2/2\,[t^2]_0^1 + \frac{2}{3}[t^3]_0^1$

$= -1 + \frac{1}{3}\times 1 + 1\times 1 + 1\times 1 + \frac{2}{3}\times 1$

$= -\cancel{1} + \frac{1}{3} + \cancel{1} + 1 + \frac{2}{3} = \frac{1+3+2}{3} = \frac{6}{3} = 2$.