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Bernoulli's Equation.

(non-linear ODE)

form $$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\frac{1}{y^n}\frac{dy}{dx} + P(x).\frac{y}{y^n} = Q(x)$$

or, $$\frac{1}{y^n}\frac{dy}{dx} + P(x).\frac{1}{y^{n-1}} = Q(x) \quad \text{— (i)}$$

now, let $y^{1-n} = z$

$$\therefore \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

or, $$\frac{1}{(1-n)}\frac{dz}{dx} = \frac{1}{y^n}\frac{dy}{dx} \quad \text{— (ii)}$$

Plugging (ii) into (i):

$$\frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x)$$

or, $$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

or, $$\frac{dz}{dx} + P'(x)z = Q'(x)$$

where, $P'(x) = (1-n)P(x)$

$Q'(x) = (1-n)Q(x)$

now, $IF = e^{\int P'(x)\, dx}$

$\therefore$ Solution :-

$$\boxed{z \times IF = \int (Q'(x) \times IF)\, dx + C}$$

where $z = y^{1-n}$

Question Solve : $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

Sol^n: $\frac{dy}{dx} - \frac{1}{2x} \cdot y = \frac{1}{2x^2} \cdot y^2$

This is a Bernoulli's Equation.
We can rewrite this equation as :-

$$\frac{1}{y^2} \cdot \frac{dy}{dx} - \frac{1}{2x} \cdot \frac{y}{y^2} = \frac{1}{2x^2}$$

or, $y^{-2} \frac{dy}{dx} - \frac{1}{2x} y^{-1} = \frac{1}{2x^2}$ — (i)

now, let $y^{-1} = z$

$\therefore \frac{dz}{dx} = -y^{2} \frac{dy}{dx}$ — (ii)

Plugging (ii) int (i);

$$\Rightarrow -\frac{dz}{dx} - \frac{1}{2x} z = \frac{1}{2x^2}$$

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or, $\frac{dz}{dx} + \frac{1}{2x} z = -\frac{1}{2x^2}$

Comparing to
$\frac{dz}{dx} + P(x) z = Q(x)$

Here, $P(x) = \frac{1}{2x}$

$Q(x) = -\frac{1}{2x^2}$

$\therefore IF = e^{\int P(x)\,dx} = e^{\int \frac{1}{2x} dx} = \sqrt{x}$

$\therefore$ Solution: $1/y \sqrt{x} = -\int \left(\frac{1}{2x^2} \cdot \sqrt{x}\right) dx + C$

$= -\frac{1}{2} \int \left(\frac{x^{1/2}}{x^2}\right) dx + C$

$= -\frac{1}{2} \int (x^{1/2 - 2}) dx + C$

$= -\frac{1}{2} \int x^{-3/2} dx + C$

$= -\frac{1}{2} \frac{x^{-1/2}}{-1/2} + C$

Solution: $\frac{1}{y} \sqrt{x} = \frac{1}{\sqrt{x}} + C$

or, $\frac{x}{y} = 1 + C\sqrt{x}$

H/W i) $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$

ii) $(x^2y^3 + xy)\,dy = dx$ → Hint: do $\frac{dx}{dy}$

Question $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$

$$\Rightarrow \frac{1}{\sqrt{y}} \frac{dy}{dx} + \frac{x}{1-x^2} \sqrt{y} = x \quad - (1)$$

~~Comparing to~~ y Let $\sqrt{y} = z$

$$\frac{dz}{dx} = \frac{d}{dx}(y^{1/2}) = \frac{1}{2} y^{-1/2} \frac{dy}{dx} = \frac{1}{2\sqrt{y}} \frac{dy}{dx}$$

Multiplying (1) by 1/2

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} + \frac{x}{2(1-x^2)} \sqrt{y} = \frac{x}{2}$$

now,

$$\frac{dz}{dx} + \frac{x}{2(1-x^2)} z = \frac{x}{2}$$

now, $P(x) = \frac{x}{2(1-x^2)}$ $Q(x) = \frac{x}{2}$

$$\text{If} = e^{\int P(x)\,dx} = e^{\int \frac{x}{2(1-x^2)} dx}$$