Theorem: The necessary and sufficient condition for a vector function $\vec{f}(t)$ to be constant is $\frac{df}{dt} = \vec{0}$.

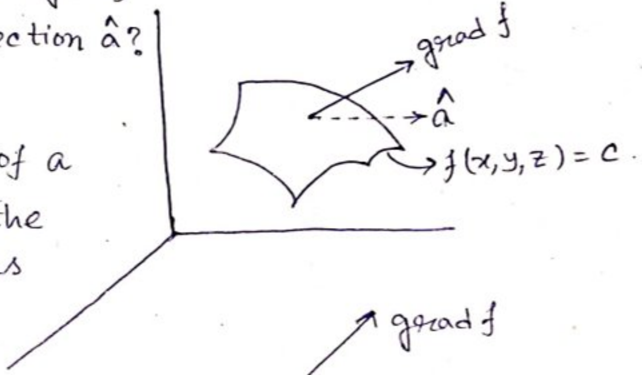
e.g. $\vec{f}(t) = 2\hat{i} + 3\hat{j} - \hat{k}$

Theorem: The necessary and sufficient condition for a vector function $\vec{f}(t)$ have a constant magnitude is $\frac{d\vec{f}}{dt} \cdot \vec{f} = 0$.

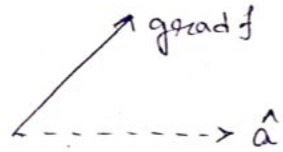
Theorem: The necessary and sufficient condition for a vector function $\vec{f}(t)$ have a constant direction is $\frac{d\vec{f}}{dt} \times \vec{f} = \vec{0}$.

- Directional Derivative

What is the rate of change of $f(x,y,z) = c$ in the unit direction $\hat{a}$?



The directional derivative of a ~~vector~~ scalar function $\vec{f}(t)$ along the unit vector $\hat{a}$ is defined as $\vec{\nabla} f \cdot \hat{a}$.



grad $f \cdot \hat{a} = \vec{\nabla} f \cdot \hat{a}$

(1) Find the directional derivative of $\phi = xy^2z + 4x^2z$ at $(-1,1,2)$ in the direction $2\hat{i} + \hat{j} - 2\hat{k}$.

$\rightarrow$ $$\vec{\nabla}\phi = \frac{\partial}{\partial x}(xy^2z + 4x^2z)\,\hat{i} + \frac{\partial}{\partial y}(xy^2z + 4x^2z)\,\hat{j} + \frac{\partial}{\partial z}(xy^2z + 4x^2z)\,\hat{k}.$$

$$= (y^2z + 8xz)\hat{i} + (2xyz)\hat{j} + (xy^2 + 4x^2)\hat{k}$$

$$= (2 - 16)\hat{i} + (-4)\hat{j} + (-1 + 4)\hat{k}$$

$$= -14\hat{i} - 4\hat{j} + 3\hat{k}.$$

The unit vector along $2\hat{i} + \hat{j} - 2\hat{k}$ is

$$= \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}.$$

$\therefore$ The directional derivative at $(-1,1,2)$ in the direction $2\hat{i} + \hat{j} - 2\hat{k}$ is

$$(\text{grad}\,\phi)\Big|_{(-1,1,2)} \cdot \hat{a} = (-14\hat{i} - 4\hat{j} + 3\hat{k}) \cdot \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= -\frac{38}{3}.$$

(ii) Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at $(1,2,3)$ in the direction $5\hat{i} + 4\hat{k}$.

$$\vec{\nabla}\phi = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$
$$= 2\hat{i} - 4\hat{j} + 12\hat{k}.$$

The unit vector along $5\hat{i} + 4\hat{k}$ is $\dfrac{5\hat{i}+4\hat{k}}{\sqrt{25+16}} = \dfrac{5\hat{i}}{\sqrt{41}} + \dfrac{4\hat{k}}{\sqrt{41}}$

Directional derivative is

$$\vec{\nabla}\phi\Big|_{(1,2,3)} \cdot \hat{a} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \left(\frac{5\hat{i}}{\sqrt{41}} + \frac{4\hat{k}}{\sqrt{41}}\right)$$
$$= \frac{10}{\sqrt{41}} + \frac{48}{\sqrt{41}} = \frac{1}{\sqrt{41}}(10+48)$$
$$= \frac{58}{\sqrt{41}}$$

(iii) Find the maximum value of directional derivative of $\phi = x^2 + z^2 - y^2$ at the point $(1,3,2)$. Also find the direction for the same.

$$\vec{\nabla}\phi = 2x\hat{i} - 2y\hat{j} + 2z\hat{k}.$$
$$= 2\hat{i} - 6\hat{j} + 4\hat{k}$$

unit vector $= \dfrac{2\hat{i} - 6\hat{j} + 4\hat{k}}{\sqrt{4+36+16}} = \dfrac{2\hat{i}}{\sqrt{56}} - \dfrac{6\hat{j}}{\sqrt{56}} + \dfrac{4\hat{k}}{\sqrt{56}}$

Directional derivative is for maximum

$$= |\vec{\nabla}\phi|^2 = \left(\sqrt{4+36+16}\right)^2 = (\sqrt{56})^2 = 56.$$