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First Order homogeneous equation:

A first order differential equation $y' = f(x,y)$ is said to a homogeneous equation if $f(x,y)$ is a homogeneous function of degree of homogenity 0.

Ex: $y' = \frac{x^2+y^2}{xy}$ → Homogeneous diff. eq.

$$f(x,y) = \frac{x^2+y^2}{xy} = \frac{x^2(1+(y/x)^2)}{x^2(y/x)} = f(y/x)$$

$y' = \frac{x^2+xy+y^2}{x^3+y^3}$ → not a homogeneous diff. eq.

Process for Solving homogeneous differential equation :-

- Consider the variable transformation, $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

- Substitute the values of y and $\frac{dy}{dx}$ in terms of x and v in the given equation.

- Use variable seperation method for solving the problem and finally substitute $v = \frac{y}{x}$ in the obtained solution.

Ex: $(x^2 + 4y^2 + xy)\,dx - x^2 dy = 0$

$\Rightarrow (x^2 + 4y^2 + xy)\,dx = x^2 dy$

or, $\frac{dy}{dx} = \frac{x^2 + 4y^2 + xy}{x^2} = 1 + 4(y/x)^2 + (y/x)$

Let us consider, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = 1 + 4v^2 + v$

or, $x\frac{dv}{dx} = 1 + 4v^2$

or, $\frac{dv}{1+4v^2} = \frac{dx}{x}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

or, $\int \frac{dv}{1+4v^2} = \int \frac{dx}{x}$

or, $\frac{1}{4} \int \frac{dv}{1/4 + v^2} = \int \frac{dx}{x}$

or, $\frac{1}{4} \cdot \frac{1}{1/2} \tan^{-1}\left(\frac{v}{1/2}\right) = \ln x + c$

$\Rightarrow \frac{1}{2} \tan^{-1}(2v) = \ln x + c$

$\Rightarrow \frac{1}{2} \tan^{-1}\left(\frac{2y}{x}\right) = \ln x + c.$

Question: $y' = \frac{(4x+y)^2}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{(4x+y)^2}{x^2}$

$\frac{dy}{dx} = \frac{16x^2 + y^2 + 8xy}{x^2} = 16 + y^2/x^2 + 8\frac{y}{x}$

Let, $y = vx \quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = 16 + v^2 + 8v$

or, $x\frac{dv}{dx} = 16 + v^2 + 7v$

or, $\int \frac{dv}{16 + v^2 + 7v} = \int \frac{dx}{x}$

or, $\int \frac{dv}{v^2 + 2 \cdot \frac{7}{2} \cdot v + \left(\frac{7}{2}\right)^2 + 16 - \left(\frac{7}{2}\right)^2} = \ln x + c$

$$\text{or, } \int \frac{dv}{(v+7/2)^2 + \left(\frac{\sqrt{15}}{2}\right)^2} = \ln x + c$$

$$\text{or, } \cancel{\frac{2}{\sqrt{15}} \tan^{-1}\left(\frac{\sqrt{15}}{2v+7}\right) = \ln x + c}$$

$$\text{or, } \frac{2}{\sqrt{15}} \tan^{-1}\left(\frac{2v+7}{\sqrt{15}}\right) = \ln x + c$$

$$\text{or, } \frac{2}{\sqrt{15}} \tan^{-1}\left(\frac{2(y/x)+7}{\sqrt{15}}\right) = \ln x + c \qquad \text{or, } \frac{2}{\sqrt{15}} \tan^{-1}\left(\frac{2y+7x}{\sqrt{15}\,x}\right) = \ln x + c$$

H/W, 1. $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$

2. $xy' = x e^{-y/x} + y$

3. $xy' = y + x \sec(y/x)$

4. $x^2 y' - xy = x^2 + y^2$

5. $3xy' - 3y + \sqrt{x^2+y^2} = 0$

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H/W

Question: $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$

or, $\dfrac{dy}{dx} = -\dfrac{(3xy + y^2)}{x^2 + xy}$

Let, $y = vx$ $\quad\therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore v + x\dfrac{dv}{dx} = -\dfrac{(3x^2v + v^2x^2)}{x^2 + x^2v} = \dfrac{-(3v + v^2)}{1 + v}$

or, $x\dfrac{dv}{dx} = \dfrac{-(3v + v^2)}{1 + v} - v = \dfrac{-[3v + v^2 + v(1+v)]}{1 + v}$

$= -\dfrac{(4v + 2v^2)}{1 + v} = -\dfrac{(2+v)\,2v}{1+v}$

or, $\dfrac{1+v\,(dv)}{2v(2+v)} = -\dfrac{1}{x}\,dx$

or, $\displaystyle\int \frac{1}{2v(2+v)}\,dv + \int \frac{1}{2(2+v)}\,dv = -\ln x + c$

or, $\displaystyle\int \left(\frac{1}{4v} - \frac{1}{4(2+v)}\right) dv + \frac{1}{2}\ln(v+2) = -\ln x + c$

or, $\dfrac{1}{4}\ln v - \dfrac{1}{4}\ln(v+2) + \dfrac{1}{2}\ln(v+2) = -\ln x + c$

or, $\dfrac{1}{4}\left[\ln v - \ln(v+2) + 2\ln(v+2)\right] = -\ln x + c$

or, $\frac{1}{4}\left[\ln v + \ln(v+2)\right] = -\ln x + c$

or, $\frac{\ln v(v+2)}{4} = -\ln x + c$.

$\because v = y/x$

$\therefore \frac{1}{4}\ln\left(\frac{y^2+2xy}{x^2}\right) = -\ln x + c$

Question: $xy' = xe^{-y/x} + y$

or, $y' = e^{-y/x} + \frac{y}{x}$

Let, $y = vx$ $\quad\therefore y' = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = e^{-v} + v$

or, $x\frac{dv}{dx} = \frac{1}{e^v}$

or, $e^v\,dv = \frac{1}{x}\,dx$

or, $\int e^v\,dv = \int \frac{1}{x}\,dx$

or, $e^v = \ln x + c$

or, $e^{y/x} = \ln x + c$

Question: $xy' = y + x\sec(y/x)$

or, $y' = y/x + \sec(y/x)$

Let $v = y/x$ or $y = vx$ $\therefore y' = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = v + \sec v$

or, $x\frac{dv}{dx} = \sec v$

or, $\frac{1}{\sec v}dv = \frac{1}{x}dx$

or, $\int \cos v\, dv = \int \frac{1}{x}dx$

or, $\sin v = \ln x + c$

or, $\sin(y/x) = \ln x + C$

Question: $x^2y' - xy = x^2 + y^2$

or, $y' - y/x = \frac{x^2+y^2}{x^2}$

Let, $y = vx$ $\therefore y' = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} - v = \frac{x^2 + (vx)^2}{x^2}$

or, $x\frac{dv}{dx} = \frac{x^2(1+v^2)}{x^2}$

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or, $\frac{1}{1+v^2}\,dv = \frac{1}{x}\,dx$

or, $\int \frac{1}{1+v^2}\,dv = \int \frac{1}{x}\,dx$

or, $\tan^{-1} v = \ln x + c$ or, $\tan^{-1}(y/x) = \ln x + c$

Question: $3xy' - 3y + \sqrt{x^2+y^2} = 0$

or, $3xy' = 3y - \sqrt{x^2+y^2}$

or, $y' = \frac{y}{x} - \frac{1}{3}\sqrt{1+(y/x)^2}$ Let, $v = y/x \Rightarrow y = vx$

or, $y' = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = v - \frac{1}{3}\sqrt{1+v^2}$

or, $x\frac{dv}{dx} = -\frac{1}{3}\sqrt{1+v^2}$

or, $\frac{dv}{\sqrt{1+v^2}} = -\frac{1}{3}\frac{dx}{x}$

$$\boxed{\begin{aligned}\int \frac{dx}{\sqrt{a^2+x^2}} &= \frac{1}{a}\ln|x+\sqrt{a^2+x^2}| \\ &= \frac{1}{a}\ln|x+\sqrt{a^2+x^2}|\end{aligned}}$$

or, $\ln(v + \sqrt{1+v^2}) = -\frac{1}{3}\ln x + \ln C$

or, $(v + \sqrt{1+v^2})^3 = C^3/x$

Let, $K = C^3$.

or, $\frac{y+\sqrt{x^2+y^2}}{x^3} = K/x$

or, $(y+\sqrt{x^2+y^2})^3 = Kx^2$.

$\log_e a = \ln a$