06.03.23

Exact Differential Equation.

Let us consider a first order linear diff. equation as

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{---} \ (1)$$

i) We will check the exactness of this type of equations

ii) If it is exact, then how to find the solution of the given D.E

The DE (1) is said to be exact diff. equation if and only if it satisfies the following conditions:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex: $(2xy - 3x^2)\,dx + (x^2 - 2y)\,dy = 0$

→ Here, $M = 2xy - 3x^2$

$N = x^2 - 2y$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy - 3x^2) = 2x$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 - 2y) = 2x$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Given, eq. is exact.

Solution of Exact DE.

Step 1 $\int M\,dx$, treating y as a constant

Step 2 $\int N\,dy$, treating x as a constant

Step 3 Compute, $\int M\,dx + \int N\,dy = C$, but common term are only counted once.

Ex: $y\,dx + x\,dy = 0$

→ Here, $M = y$
$N = x$

$\int M\,dx = \int y\,dx = xy$

$\int N\,dy = \int x\,dy = xy$

$\therefore$ Solution → $xy = C$

Ex: $\cos x \cos y\,dx - \sin x \sin y\,dy = 0$

→ Here, $M = \cos x \cos y$
$N = -\sin x \sin y$

$\int M\,dx = \cos y \sin x$

$\int N\,dy = -\sin x(-\cos y)$
$= \cos y \sin x$

$\therefore$ Solution ⇒ $\sin x \cos y = C$

classmate

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Exact DE

15.03.23

Question: Solve the following IVP:

$$e^x(\cos y\,dx - \sin y\,dy) = 0 \quad , y(0) = 0 \rightarrow \text{Initial condition}$$

For $x=0$, $y=0$

Sol- Comparing the given equation with
$M dx + N dy = 0$

We have, $M = e^x \cos y$
$N = -e^x \sin y$

now, $\frac{\partial M}{\partial y} = -e^x \sin y$

$\frac{\partial N}{\partial x} = -e^x \sin y$

$\because$ We observe that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence, the given equation is an exact DE.

Let, $f(x,y) = C$ be the solution of given DE, then -

$f(x,y) = \int M dx$ (partial integration wrt x)

$= \int e^x \cos y\,dx = \cos y \int e^x dx = e^x \cos y$

again, $f(x,y) = \int N dy$ (partial integration wrt y)

$= \int -e^x \sin y\,dy = -e^x \int \sin y\,dy = e^x \cos y$

$\therefore f(x,y) = e^x \cos y = C$ is the general solution of the equation now, for $x=0, y=0$,

$$e^0\cos(0)=c \Rightarrow c=1.$$

$\therefore e^x\cos y = 1$ is the solution of the IVP.

Question Find the value of a & b so that the following equation becomes an exact D.E. Hence solve the equation.

$$(y+x^3)dx + (ax+by^3)dy = 0$$

ans Comparing to $Mdx + Ndy = 0$

$$M = y+x^3 \qquad N = ax+by^3$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y+x^3) = 1+0 = 1.$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(ax+by^3) = a$$

now, for exact DE,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \therefore a=1., \; b\in\mathbb{R}$$

now, let $f(x,y)=C$ be the solution of given DE.

$$\therefore f(x,y) = \int M\,dx = \int (y+x^3)\,dx = xy + \frac{x^4}{4}$$

again, $f(x,y) = \int N\,dy = \int (1.x+by^3)\,dy = xy + b\frac{y^4}{4}$

$$\therefore f(x,y) = \frac{x^4}{4} + xy + b\frac{y^4}{4} = xy + \frac{1}{4}(x^4+by^4)$$

where $b\in\mathbb{R}$

$\therefore f(x,y) = xy + \frac{1}{4}(x^4+by^4) = C$ is the general solution of the DE.