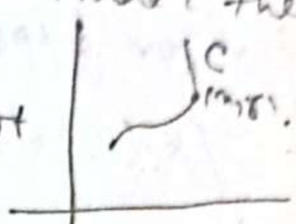
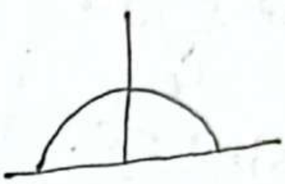
Complex Integration

Let $z = x+iy$ be a complex number. As z represents a pt. (x,y) in Argand plane, so z varies as (x,y) moves on the plane. If $x=\phi(t)$, & $y=\psi(t)$ for real variable t then $z=\phi(t)+i\psi(t)$. (t = parameter).

Simple curve:-

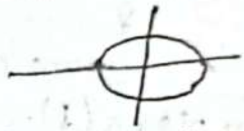


A curve is simple if it does not intersects itself. So a curve $C: z=\phi(t)+i\psi(t)$ is simple if $t_1 \neq t_2 \Rightarrow \cancel{\phi(t} \; z(t_1) \neq z(t_2)$.



Closed curve:-

A simple curve is called closed if two end points of the curve coincide.

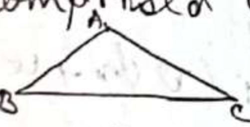


Smooth curve:-

A curve C is called smooth if it posses unique tangent at every pt. Above two figures of curve is of that type.

Contour or piecewise smooth curve:-

A curve is called contour or piecewise smooth if it is comprised of a finite number of smooth curves.



Here AB, BC, CA are smooth

Riemann's Def^n of integration:-

Let $f(z)$ be a function where z varies over a piecewise smooth and simple curve. Let A & B be two end points of the curve & we divide this curve into n arc by means of points $A=z_0, z_1 \dots z_n=B$. Let ξ_r be any pt. on the arc between z_{r-1} & z_r & $|z_{r-1}-z_r|$ is length of the arc. Let $\|P\| = \max\limits_{1\le r\le n} |z_{r-1}-z_r|$

Let $S_P = \sum\limits_{r=1}^{n} (z_{r}-z_{r-1}) f(\xi_r)$ be the sum.

Then P is partition of the curve C (path of integration or contour)

$$\therefore \int_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{r=1}^{n} (z_r - z_{r-1}) f(\xi_r)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1}) f(\xi_r).$$

Properties :-

(i) $\int_C [f(z)+\phi(z)]\,dz = \int_C f(z)\,dz + \int_C \phi(z)\,dz$.

(ii) $\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$.

$-C$ for opposite direction of C.

(iii). $\int_C K f(z)\,dz = K\int_C f(z)\,dz$. [$K$ = const]

(iv) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$.

where $C_1 + C_2 = C$.

Line integral along a piecewise Smooth Curve

Theorem:- (complex integral as a sum of two line integrals)

$f(z) = ~~u~~ u+iv$ be continuous ~~and~~ on a piecewise, smooth simple arc C. Then $f(z)$ is integrable along C & is given by,

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy).$$

As $z = x+iy \Rightarrow dz = dx + i\,dy$.

$\therefore f(z)\,dz = (u\,dx - v\,dy) + i\,(v\,dx + u\,dy)$.

Problems:-

1) Evaluate $\int^{2-i} (3xy + iy^2)\,dz$.

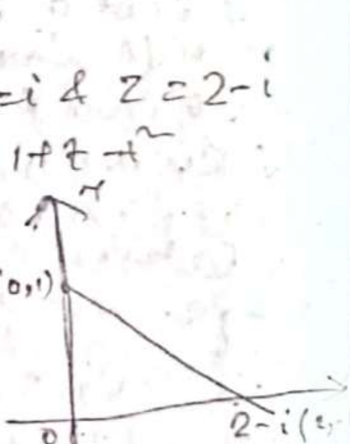
(a) along the st. line joining $z=i$ & $z=2-i$

(b) along the curve $x = 2t-2$, $y = 1+t-t^2$.

$\Rightarrow$ (a). Equ^n of the line joining i & $2-i$ is:

$$\frac{y-1}{-1} = \frac{x-0}{2-0}.$$

$\Rightarrow x = -y+1$.



Parametric form is $y=t$,

$x = -t+1$.

$$\int_i^{2-i}(3xy+iy^2)\,dz$$

$$= \int_i^{2-i}(3xy+iy^2)(dx+idy)$$

for $z=i$, $x=0$, $y=1$ $\therefore t=1$

& $z=2-i$, $t=-1$.

$$= \int_1^{-1}\{3(-t+1)\cdot t+it^2\}(-dt+i\,dt)$$

$$= (-1+i)\int_1^{-1}(-3t^2+3t+it^2)\,dt$$

$$= (-1+i)\left[-t^3+\frac{3t^2}{2}+i\frac{t^3}{3}\right]_1^{-1}$$

$$= (-1+i)\left(2-\frac{2i}{3}\right) = -\frac{4}{3}+i\frac{8}{3}.$$

(b) at $z=i$, $x=0$, $y=1$ $\therefore t=1$.

$z=2-i$, $x=2$, $y=-1$. $\therefore t=2$.

$$\therefore \int_i^{2-i}(3xy+iy^2)\,dz = \int_i^{2-i}(3xy+iy^2)(dx+idy)$$

$$= \int_1^2\{3(2t-2)(1+t-t^2)+i(1+t-t^2)\}\{2dt+i(1-2t)dt\}$$

2) Evaluate $\int \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve C given by a st. line joining $z=0$ & $z=4+2i$.

$\Rightarrow$ Line joining $z=0$ & $z=4+2i$ is $\frac{y-0}{2-0}=\frac{x-0}{4-0} \Rightarrow x=2y$.

$\therefore$ parametric equ^n is $y=t$, $x=2t$.

at $z=0$, $t=0$ & $z=4+2i$, $t=2$.

$$\therefore \int_C \bar{z}\,dz = \int_C (x-iy)(dx+idy)$$

$$= \int_0^2 (2t-it)(2dt+idt)$$

$$= (2+i)\int_0^2(2t-it)\,dt.$$

Cauchy's Theorem :-

Let $f(z)$ be an analytic function and $f'(z)$ is continuous at each pt. of within the domain D bounded by closed contour C. Then

$$\int_C f(z)\,dz = 0.$$

Cauchy-Goursat theorem :-

Let $f(z)$ be an analytic function within and on a simple closed contour C. Then $\oint_C f(z)\,dz = 0$.

Formula :-

(i) $\oint_C \frac{dz}{z-z_0} = 0$, if C is a closed curve and $z = z_0$ is an exterior pt. of C & then $\frac{1}{z-z_0}$ is analytic.

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(ii) $\oint_C \frac{dz}{z-z_0} = 2\pi i$ if C is simple closed curve and $z=z_0$ is interior pt of C then integration is taken in the anticlockwise direction.

(iii). $\oint_C \frac{dz}{(z-z_0)^n} = 0$, $n = 2, 3, \ldots$ if C is any simple closed curve and $z=z_0$ is an interior pt. of C.

Problems:-

1) Evaluate $\int_C \frac{3z^2-2}{z-1} dz$, where C is the circle $|z| = \frac{1}{2}$.

$\Rightarrow$ Here $(0,0)$ is centre & $\frac{1}{2}$ is the radious

$f(z) = \frac{3z^2-2}{z-1}$ $\therefore$ $z=1$ is the singular pt. outside C & $f(z)$ is analytic.

$\therefore$ By Cauchy-Goursat th.

$$\int_C \frac{3z^2-1}{z-1} dz = 0.$$

2) Evaluate $\oint_C \frac{z}{z^2-3z+2} dz$, where C is the circle $|z-2| = \frac{1}{2}$.

$\Rightarrow$ Here C: $|z-2| = \frac{1}{2}$ circle of centre $(2,0)$ & radius $\frac{1}{2}$.

$\therefore z^2-3z+2 = (z-1)(z-2)$

$\therefore \frac{z}{z^2-3z+2} = \frac{z}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}$.

$\therefore z=2$ is interior pt.

$$\therefore \oint \frac{2}{z-2} = 2\pi i \times 2$$

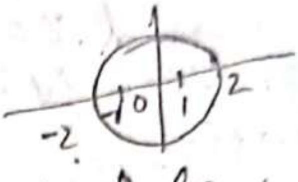
& $z=1$ is exterior pt.

$$\therefore \oint_C \frac{1}{z-1} = 0.$$

$\therefore \oint_C \frac{z}{z^2-3z+2} = 4\pi i$.

3) Evaluate: $\oint \frac{dz}{z^2-1}$, where C is the circle $x^2+y^2=4$..

$\Rightarrow$ Since $z=\pm 1$ lie inside C.



$$\therefore \oint_C \frac{dz}{z^2-1} = \frac{1}{2}\oint_C \left\{\frac{1}{z-1} - \frac{1}{z+1}\right\} dz.$$

$$= \frac{1}{2}\cdot 2\pi i - \frac{1}{2}\cdot 2\pi i$$

$$= 0.$$

4) Find $\oint_C \frac{z^2+2z+1}{(z-1)z^2(z-2)} dz$, $C: |z| = 6$.

$$\Rightarrow \frac{z^2+2z+1}{z^2(z-1)(z-2)} = \frac{A_1}{z-1} + \frac{A_2}{z} + \frac{A_3}{z^2} + \frac{A_4}{z-2}.$$

$$\therefore (z+1)^2 = A_1 z^2(z-2) + A_2(z-1)(z-2)\cdot z + A_3(z-2)(z-1) + A_4(z-1)z^2.$$

$\underline{z=2}$. $9 = A_4 \times 4$ $\quad$ $\underline{z=1}$ $\quad -4 = A_1$

$A_4 = \frac{9}{4}$

$\underline{z=0}$. $\frac{1}{2} = A_3$. $\quad \therefore A_2 = \frac{7}{4}$.

$$\therefore \oint_C \frac{z^2+2z+1}{z^2(z-1)(z-2)} dz = -4\int \frac{dz}{z-1} + \frac{7}{4}\int \frac{1}{z-0} dz + \frac{1}{2}\int \frac{dz}{(z-0)^2} + \frac{9}{4}\int \frac{dz}{z-2}.$$

C: (0,0) is centre & 6 is radius.

$z = 0, 1, 2$ lie within C.

$$\therefore \oint_C \frac{z^2+2z+1}{z^2(z-1)(z-2)} dz = -4\times 2\pi i + \frac{7}{4}\times 2\pi i + \frac{1}{2}\times 0 \text{ (by (III))} + \frac{9}{4}\times 2\pi i$$

$$= 0.$$

5) Evaluate $\oint_C \frac{z+1}{z^2-2z}\,dz$, where C is the Circle $C: |z|=5$.

$\Rightarrow$ Here $(0,0)$ is centre & 5 is radius of C

$\therefore \frac{z+1}{z^2-2z} = \frac{1}{2}\cdot\frac{2z+2}{z(z-2)} = \frac{1}{2}\left\{\frac{3}{z-2} - \frac{1}{z}\right\}$

$\therefore \oint_C \frac{z+1}{z^2-2z}\,dz = \frac{3}{2}\int \frac{dz}{z-2} - \frac{1}{2}\int \frac{dz}{z-0}$

$= \frac{3}{2} \times 2\pi i - \frac{1}{2} \times 2\pi i$

$= 2\pi i$.

6) Evaluate $\oint_C \frac{4-3z}{(z-1)z(z-3)}\,dz$, where C is the Circle $|z| = \frac{5}{2}$.

$\Rightarrow \frac{4-3z}{(z-1)z(z-3)} = \frac{A_1}{z-1} + \frac{A_2}{z} + \frac{A_3}{z-3}$

$\therefore 4-3z = A_1 z(z-3) + A_2(z-1)(z-3) + A_3(z-1)z$.

$z=0,\ 4 = 3A_2 \quad A_2 = \frac{4}{3}$

$z=3,\ -5 = 6A_3 \quad A_3 = -\frac{5}{6}$.

$z=1,\ 1 = -2A_1,\ A_1 = -\frac{1}{2}$.

$\therefore \oint_C \frac{4-3z}{(z-1)z(z-3)}\,dz = -\frac{1}{2}\int\frac{dz}{z-1} + \frac{4}{3}\int\frac{dz}{z-0} - \frac{5}{6}\int\frac{dz}{z-3}$.

$C: |z| = \frac{5}{2}$. Centre is $(0,0)$ & radious $\frac{5}{2}$.

$\therefore z = 0, 1$ lies inside C.

$\therefore \oint_C \frac{4-3z}{(z-1)z(z-3)}\,dz = -\frac{1}{2}\times 2\pi i + \frac{4}{3}\times 2\pi i$

$= -\pi i + \frac{8}{3}\pi i$

$= \frac{5}{3}\pi i$.

Cauchy's integral formula:-

Th. If $f(z)$ is analytic within and on a simple closed curve C and α is any point within C, then.

$$f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}\,dz.$$

Th. : (for derivative of analytic function).

If $f(z)$ is analytic within and on a closed curve C, then the derivative of $f(z)$ at any ~~pt.~~ interior pt. α of C is given by,

$$f'(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^2}\,dz.$$

Th. (Successive derivative)

If $f(z)$ is analytic within and on a closed curve C, then the nth order derivative of $f(z)$ at any point α of C is given by

$$f^n(\alpha) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-\alpha)^{n+1}}$$

Problems:-

1) Evaluate $\int_C \frac{e^z\,dz}{z^2+\pi^2}$, where C is the circle $|z|=4$.

$\Rightarrow$ Here $(0,0)$ is the centre & 4 is radius of C.

$$\frac{1}{z^2+\pi^2} = \frac{1}{(z+\pi i)(z-\pi i)} = \frac{1}{2\pi i}\left\{\frac{1}{z-\pi i} - \frac{1}{z+\pi i}\right\}$$

As e^z is analytic within & on C & $\pm\pi i$ lie within C so by Cauchy integral formula

$$\oint_C \frac{e^z\,dz}{z^2+\pi^2} = \frac{1}{2\pi i}\int \frac{e^z}{z-\pi i}\,dz - \frac{1}{2\pi i}\int \frac{e^z}{z+\pi i}\,dz.$$

$$= e^{\pi i} - e^{-\pi i}$$

$$= (\cos\pi + i\sin\pi) - (\cos\pi - i\sin\pi)$$

$$= 0$$

2) Use Cauchy integral formula to evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$. where C is the circle $|z|=3$.

$\Rightarrow$ Here $(0,0)$ is centre & 3 is radius of C. e^{2z} is analytic within & on C & $z=-1$ lies inside C. By Cauchy integral formula of derivative we have,

$$\frac{3!}{2\pi i}\int_C \frac{e^{2z}}{(z+1)^4}dz = \left[\frac{d^3(e^{2z})}{dz^3}\right]_{z=-1} = 8e^{-2}.$$

$$\Rightarrow \int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i e^{-2}}{3}.$$

) Evaluate : $\int \frac{e^{3z}\,dz}{(4z-\pi i)^3}$.

$$\Rightarrow I = \int_{|z|=1} \frac{e^{3z}\,dz}{(4z-\pi i)^3} = \frac{1}{64}\int_{|z|=1} \frac{e^{3z}\,dz}{(z-\frac{\pi i}{4})^3}.$$

where the path of integration is the circle $|z|=1$ whose centre is $(0,0)$ & radius 1. e^{3z} is analytic within & on $|z|=1$ & $z=\frac{\pi i}{4}$ lie within $|z|=1$.

By Cauchy integral formula:

$$\frac{2!}{2\pi i}\int_{|z|=1} \frac{e^{3z}\,dz}{(z-\frac{\pi i}{4})^{2+1}} = \left[\frac{d^2(e^{3z})}{dz^2}\right]_{z=\frac{\pi i}{4}} = 9e^{3\cdot\frac{\pi i}{4}}.$$

$$\therefore \int_{|z|=1} \frac{e^{3z}\,dz}{(z-\frac{\pi i}{4})^3} = 9\pi i\left\{\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right\}.$$

4). Find $\oint_C \frac{e^{2z}\,dz}{(z-1)(z-2)}$; $C: |z|=3$

$\Rightarrow$ Here $(0,0)$ is centre & 3 is radius of $|z|=3$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$\therefore f(z) = e^{2z}$ is analytic within & on C. & $z=1,2$ lies inside C.

$$\therefore \frac{1}{2\pi i}\oint_C \frac{e^{2z}\,dz}{(z-1)(z-2)} = \frac{1}{2\pi i}\oint_C \frac{e^{2z}}{z-2}dz - \frac{1}{2\pi i}\oint_C \frac{e^{2z}}{z-1}dz.$$

$$= e^4 - e^2.$$

$$\therefore \oint_C \frac{e^{2z}\,dz}{(z-1)(z-2)} = 2\pi i(e^4 - e^2).$$

5) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$. where C is the circle $|z| = 3$.

$\Rightarrow$ Here $(0,0)$ is the centre & 3 is radius of the circle C

$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$.

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within and on the circle C & $z = 1, 2$ lie within C, so by Cauchy's integral formula,

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{f(z)dz}{z-2} - \oint_C \frac{f(z)}{z-1} dz.$$

$$= 2\pi i f(2) - 2\pi i f(1)$$
$$= 2\pi i (\sin 4\pi + \cos 4\pi) - 2\pi i (\sin \pi + \cos \pi)$$
$$= 2\pi i + 2\pi i = 4\pi i$$

6) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$ ~~$= \frac{1}{z-2} - \frac{1}{z-1} - \frac{1}{(z-1)^2}$~~, where C is circle $|z| = 3$.

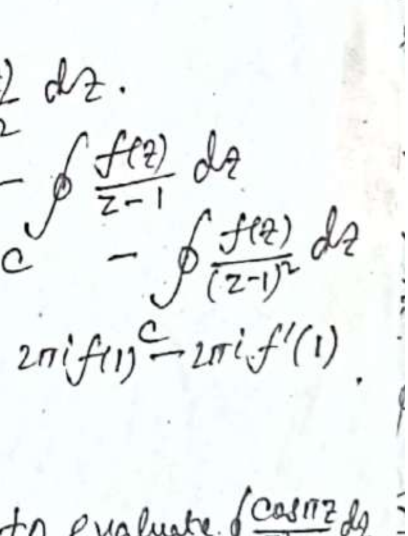
$\Rightarrow \frac{1}{(z-1)^2(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} - \frac{1}{(z-1)^2}$

Since $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within & on C & $z = 1, 2$ lie within C.

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz = \oint_C \frac{f(z)}{z-2} dz - \oint_C \frac{f(z)}{z-1} dz - \oint_C \frac{f(z)}{(z-1)^2} dz$$

$$= 2\pi i f(2) - 2\pi i f(1) - 2\pi i f'(1).$$

7) Use Cauchy's integral formula to evaluate $\oint \frac{\cos \pi z}{z^2-1} dz$, around a rectangle with vertices $2 \pm i$, $-2 \pm i$.



$\Rightarrow f(z) = \cos \pi z$ is analytic within & on the rectangle & $z = 1, -1$ lies within this. By Cauchy integral formula,

$$\int_C \frac{\cos \pi z}{z^2-1} dz = \frac{1}{2} \oint_C \frac{\cos \pi z}{z-1} dz - \frac{1}{2} \oint_C \frac{\cos \pi z}{z+1} dz$$

$$= \frac{1}{2} \times 2\pi i \cos \pi - \frac{1}{2} 2\pi i \cos(-\pi)$$

$$= 0.$$