classmate

Date
Page

Series

17.03.23

Power Series:

A series of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$ is called power series.

It is also denoted as $\sum_{n=0}^{\infty} a_n x^n$, where a_n are coefficients.

Note: If $x=0$, then the power series is always convergent, and if the series is convergent only for $x=0$, then it is called 'nowhere' convergent.

Note: If the series is convergent for all values of x, then it is called 'everywhere' convergent.

1. $(1-x)^{-1}$
$= 1 + x + x^2 + x^3 + \dots$

2. $(1+x)^{-1}$
$= 1 - x + x^2 - x^3 + \dots$

now, express $\to 1 + 2x + 3x^2 + 4x^3 + \dots$
$= (1-x)^{-2}$

Note: If the power series converges for some values of x, then the values of x for which the series converges is called 'region of convergence'.

Note: If for the power series $\sum_{n=0}^{\infty} a_n x^n$, there exists a positive number R, such that the series converges for $|x| < R$ and divergence for $|x| > R$, then R is called radius of convergence.

Radius of Convergence :

The radius of convergence R of a power series $\sum a_n x^n$ is defined to be equal to :

$$R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}} \quad , \text{ where } \lim_{n\to\infty} |a_n|^{\frac{1}{n}} > 0$$

$$R = \infty \quad , \text{ when } \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$$

$$R = 0 \quad , \text{ when } \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$$

The radius of convergence for a power series $\sum a_n x^n$ is also given by:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Radius of Convergence.

21.03.23

Question: Find the Radius of convergence of the series :

i) $1 + 2x + 3x^2 + 4x^3$

→ Here, $a_n = n+1$

now, $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n+1)^{1/n} = A$ (say)

$\therefore \log A = \lim_{n\to\infty} \frac{1}{n} \log(n+1)$

$= \lim_{n\to\infty} \frac{\log(n+1)}{n} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{n\to\infty} \frac{1/n+1}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$

now, $\log A = 0$

$\therefore A = e^0 = \underline{1}.$

$\therefore R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}} = \frac{1}{1} = \underline{1}.$

ii) $1 + x + 2!\, x^2 + 3!\, x^3 + \ldots$

→ Here, $a_n = n!$

~~now,~~ $\cancel{\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n!)^{1/n}}$

now, $a_{n+1} = (n+1)!$

$\therefore R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)\,n!}$

$= \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\therefore R = 0$ → ∴ The series is not convergent for any non-zero values of x.

(no region of convergence)

iii) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Here → $a_0 = 1$

$a_1 = \frac{1}{2!}$

$a_2 = \frac{1}{3!}$

$\therefore a_n = \frac{1}{(n+1)!}$

iii) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Here → $a_1 = \frac{1}{1!}$

$a_2 = \frac{1}{2!}$ $\therefore a_n = \frac{1}{n!}$

$$a_{n+1} = \frac{1}{(n+1)!} = \frac{1}{(n+1)n!}$$

$$\frac{a_n}{a_{n+1}} = n+1$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} n+1 = \infty$$

$\therefore$ ~~for all~~ Series is convergent when $x \neq 0 \ \forall x$

iv) $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \dots$

v) $\frac{1}{3} - x + \frac{x^2}{3^2} + -x^3 + \frac{x^4}{3^4} - x^4 + \dots$