EM – II

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Differential Equation

- Ordinary differential equation (Number of independent variable must be one)
- Partial differential equation (Number of independent variable more than one)

* A differential equation is a relation between independent variable, dependent variable and the different order of derivative terms.

* Linear Differential Equation and Non-linear Differential Equation.

* Non-linearity of any differential equation:

(i) The power of derivative term or dependent variable must not be equal to 1. $\rightarrow \left(\frac{dy}{dx}\right)^{3/2} + xy = 0$.

(ii) dependent variable must be multiplied by derivative term. $\rightarrow \frac{d^2y}{dx^2} - y^2\frac{dy}{dx} + y = \sin x$

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(iii) Having any transedental function.

$\hookrightarrow \frac{dy}{dx} + xe^y = \tan x$

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- **Differential Equation:** Relationship between the dependent variable, independent variable and their derivatives. ~~is called a dif~~

- **Order of a differential equation:** The order of the highest order derivative of a differential equation is called order of that differential equation.

$$\left(\frac{d^2y}{dx^2}\right)^2 + 5\frac{dy}{dx} + 6y = x^3 \rightarrow \text{order} = 2.$$

$$\frac{dy}{dx} + 6y = \left(\frac{d^5y}{dx^5}\right)$$

- **Degree of a differential equation:** The power of the highest order derivative of a d.e. is called degree of that d.e. after the equation is free from fraction or radicals.

Integration Formula

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

2. $\int \frac{dx}{x} = \log|x| + c$

3. $\int \sin x\, dx = -\cos x + c$

4. $\int \cos x\, dx = \sin x + c$

5. $\int \sec x \tan x\, dx = \sec x + c$

6. $\int \sec^2 x \, dx = \tan x + c$

7. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

8. $\int e^x \, dx = e^x + c$

9. $\int e^{mx} \, dx = \frac{e^{mx}}{m} + c$.

10. $\int a^x \, dx = \frac{a^x}{\log a} + c$

11. $\int a^{mx} \, dx = \frac{a^{mx}}{m \log a} + c$.

12. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

13. $\int \frac{dx}{1+x^2} = \tan^{-1} + c$.

1st order Ordinary Differential Equation:

$$\frac{dy}{dx} = f(x, y).$$

$$M dx + N dy = 0.$$

M and N are functions of x and y.

The equation $M dx + N dy = 0$ will be exact if we can write $M dx + N dy$ as dU where U is a function of x and y.

e.g., $e^x \sin y \, dx + (e^x + 1) \cos y \, dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(e^x \sin y)$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\{\cos y (e^x + 1)\}$
$= e^x . \cos y$	$= \cos y . e^x$

As $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are equal, the differential equation is exact.

The necessary and sufficient condition for which the eq^n $Mdx + Ndy = 0$ will be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

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• Formation of Differential equation:

(i) $y = ax + b$, where a and b are arbitrary constants.

→ Given:

$$y = ax + b \quad \text{—— (i)}$$

Differentiate w.r.t. x both sides —

$$\frac{dy}{dx} = a$$

Again diff w.r.t. x —

$$\frac{d^2y}{dx^2} = 0.$$

$$\cancel{y = ae^{2x} + be^{2x}.x}$$

(ii) $y = (a+bx)e^{2x}$, where a and b are arbitrary constant.

$$\Rightarrow \frac{dy}{dx} = (a+bx).e^{2x}.2 + e^{2x}.b$$

$$\Rightarrow \frac{dy}{dx} = 2y + be^{2x} \rightarrow be^{2x} = \frac{dy}{dx} - 2y$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + b.e^{2x}.2 + 0 = 2be^{2x} + 2\frac{dy}{dx}.$$

$$\cancel{y = ae^{2x} + be^{2x}.x}$$

$$\cancel{\Rightarrow \frac{dy}{dx} = ae^{2x}.2 + 0 + bx.e^{2x}.2 + e^{2x}.b}$$

$$\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - 2y\right) + 2\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 2\frac{dy}{dx} - 4y$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0}$$

(iii) $y = kx + k - k^3$, where k is arbitrary constant

$$\cancel{\Rightarrow \frac{dy}{dx} = k + 0 - 0} \quad \Rightarrow \frac{y}{k} = x + 1 - k^2$$

$$\cancel{\Rightarrow \frac{dy}{dx} = k} \quad \Rightarrow \frac{1}{k} \cdot \frac{dy}{dx} + 0 = 1$$

$$\Rightarrow \frac{dy}{dx} = k$$

$$\cancel{k = y - kx + k^3}$$

$$y = \frac{dy}{dx} \cdot x + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$$

(iv) $y = ae^{-2x} + be^{-3x} \rightarrow ae^{-2x} = y - be^{-3x}$

$$\Rightarrow \frac{dy}{dx} = a \cdot e^{-2x} \cdot (-2) + b \cdot e^{-3x} \cdot (-3)$$

$$\Rightarrow \frac{dy}{dx} = -(2ae^{-2x} + 3be^{-3x})$$

$$\Rightarrow \frac{dy}{dx} = -\{2y - 2be^{-3x} + 3be^{-3x}\}$$

$$\Rightarrow \frac{dy}{dx} = -\{2y + be^{-3x}\} \rightarrow be^{-3x} = -2y - \frac{dy}{dx}$$

We are attitude forman whether we like it or not.

$$\Rightarrow \frac{d^2y}{dx^2} = -2\frac{dy}{dx} - \{b.e^{-3x}.(-3)\}$$

$$\Rightarrow -2\frac{dy}{dx} + 3be^{-3x}$$

$$= -2\frac{dy}{dx} + 3\left\{-2y - \frac{dy}{dx}\right\}$$

$$= -2\frac{dy}{dx} - 6y - 3\frac{dy}{dx}$$

$$\Rightarrow -6y - 5\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$$

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- Solution of First order differential equation:

$$\frac{dy}{dx} = f(x,y) \quad \text{——— (i)}$$

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{——— (ii)}$$

Methods

1. Separation of variables
2. Exact differential equation
3. Linear differential equation.

1.

Pr. 1 Solve the differential ~~equa~~ equation

$$x\frac{dy}{dx} + 1 - y$$

$$\rightarrow x\frac{dy}{dx} = y - 1$$

$$\Rightarrow \frac{dy}{y-1} = \frac{dx}{x}$$

Integrating both sides,

$$\int \frac{dy}{y-1} = \int \frac{dx}{x} + \ln c$$

$$\Rightarrow \ln(y-1) = \ln x + \ln c$$

$$\Rightarrow \ln\left[\frac{y-1}{x}\right] = \ln c \Rightarrow \frac{y-1}{x} = c \Rightarrow \boxed{y = 1 + cx}$$

Pr. 2 Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$\Rightarrow \frac{dy}{dx} = \frac{e^x}{e^y} + x^2 \cdot \frac{1}{e^y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y}\left[e^x + x^2\right]$

$\Rightarrow dy \cdot e^y = dx \cdot (e^x + x^2)$

$\Rightarrow e^y dy = e^x dx + x^2 dx$

$\Rightarrow \boxed{e^y = e^x + \frac{x^3}{3} + c}$

2. Exact D.E.

$M(x,y)dx + N(x,y)dy = 0$ — (i)

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then eq^n (i) is said to be exact differential equation.

∴ The general solution of eq^n (i) may express as

$\int M(x,y)dx + \int (\text{Terms of } N \text{ not containing } x)\,dy = c$ — (ii)

(y = constant)

Ex. 1. Solve $(x+y-3)dx + (x-y-5)dy = 0$

$\rightarrow$ Here, $M(x,y) = x+y-3$ and $N(x,y) = x-y-5$

$$\Rightarrow \frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given differential equation is exact.
The general solⁿ is

$$\int_{(y=\text{const.})} (x+y-3)dx + \int (-y-5)dy = c$$

$$\Rightarrow \frac{x^2}{2} + xy - 3x - \frac{y^2}{2} - 5y = c. \text{ (Ans.)}$$

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Solve : $y^2dx+(x^2-xy)dy=0$ —— (i)

→ Here $M(x,y)=y^2$ and $N(x,y)=x^2-xy$

$$\frac{\partial M}{\partial y}=2y, \quad \frac{\partial N}{\partial x}=2x-y$$

$$\because \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

The equation has not an exact solution.

Now, $M(x,y)x+N(x,y)y=\cancel{xy^2}+x^2y-\cancel{xy^2}=x^2y$

$$F(x,y)=\frac{1}{Mx+Ny}=\frac{1}{x^2y}$$

↓

Integrating Factor

Multiplying $F(x,y)$ in eq^n (i), we have

$$\frac{y}{x^2}dx+\left(\frac{1}{y}-\frac{1}{x}\right)dy=0 \quad \text{—— (ii)}$$

$$M_1=\frac{y}{x^2} \text{ and } N_1=\frac{1}{y}-\frac{1}{x}.$$

$$\frac{\partial M_1}{\partial y}=\frac{1}{x^2} \text{ and } \frac{\partial N_1}{\partial x}=\frac{1}{x^2}$$

$$\therefore \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

$\therefore$ Eq^n (ii) is exact differential equation.

The general solution is

$$\int \frac{y}{x^2}\,dx + \int \frac{1}{y}\,dy = c$$

$y = \text{const.}$

$$\Rightarrow y\int \frac{dx}{x^2} + \ln|y| = c$$

$$\Rightarrow y \cdot \frac{x^{-1}}{-1} + \ln|y| = c$$

$$\Rightarrow -\frac{y}{x} + \ln|y| = c.$$

• Rule 1: Obtaining I.F. of the differential equation when $Mx + Ny = 0$ is homogeneous in x and y, of same degree then $F(x,y) = \frac{1}{Mx+Ny}$, provided $Mx+Ny \neq 0$.

H.W.

(i) $x^2y\,dx - (x^3+y^3)\,dy = 0$

(ii) $(x^2y - 2xy^2)\,dx - (x^2 - 3x^2y)\,dy = 0.$

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ODE Regarding

$$M\,dx + N\,dy = 0, \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Rule II: If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \phi(x)$, then $e^{\int \phi(x)dx}$ is an I.F.

Rule III: If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \phi(y)$, then $e^{\int \phi(y)dy}$ is an I.F.

Problem: Solve: $\underbrace{(3x^2y^4 + 2xy)}_{M}dx + \underbrace{(2x^3y^3 - x^2)}_{N}dy = 0.$ —(i)

$\rightarrow \frac{\partial M}{\partial y} = 3x^2 \cdot 4y^3 + 2x = 12x^2y^3 + 2x$. $\quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$.

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow$ it is not exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(2x^3y^3 - x^2)}\left[12x^2y^3 + 2x - 6x^2y^3 + 2x\right]$$

$$= \cancel{\frac{4x}{2x^3y^3 - x^2}} \quad \frac{6x^2y^3 + 4x}{2x^3y^3 - x^2}$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{M}\left(6x^2y^3 - 2x - 12x^2y^3 - 2x\right)$$

$$= \frac{-6x^2y^3 - 4x}{3x^2y^4 + 2xy}$$

$$= \frac{-4x}{yx(3xy^3 + 2)} = \frac{-2x(3xy^3 + 2)}{2xy(3xy^3 + 2)}$$

$$= -2/y$$

Everyone was to blame but no one was responsible.

$\therefore$ I.F. $= e^{-\int 2/y\, dy} = e^{-2\log y} = e^{\log 1/y^2} = 1/y^2$

Now multiplying both sides of eq^n (i) by $1/y^2$ we get,

$$\left(3x^2y^2 + 2\frac{x}{y^2}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

$$\int \left(3x^2y^2 + \frac{2x}{y^2}\right)dx = x^2y^3 + x^2/y^2$$

$\therefore$ the general solution is given by,

$$x^2y^3 + \frac{x^2}{y^2} + c = 0.$$

<u>Problem:</u> $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0.$ —(i)

$\rightarrow \frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = -2xy$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-x^2y}(2xy + 2xy)$$

$$= -\frac{4xy}{x^2y}$$

$$= -\frac{4}{x} = \phi(x).$$

I.F. $= e^{\int -\frac{4}{x}dx} = e^{-4\int \frac{dx}{x}} = e^{-4\log x} = e^{\log x^{-4}}$

$= x^{-4}$

I.F. $\times$ eq^n (i),

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx - \frac{y}{x^2}dy = 0.$$

$$\int\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx$$

$$= y^2 \int \frac{dx}{x^3} - \int \frac{e^{1/x^3}}{x^4} dx$$

$$= y^2 \cdot \frac{x^{-2}}{-2} - \int\left[-\frac{1}{3} dt \cdot e^t\right]$$

$$= -\frac{1}{2} \cdot \frac{y^2}{x^2} + \frac{1}{3} \int e^t dt$$

$$= -\frac{1}{2} \cdot \frac{y^2}{x^2} + \frac{1}{3} e^{1/x^3}$$

x^{-3}

$\frac{x^{-2}}{-2}$

$t = \frac{1}{x^3}$

$\Rightarrow dt = -3/x^4 dx$

$\Rightarrow -\frac{1}{3} dt = \frac{dx}{x^4}$

$\therefore$ The general solution is : $\frac{e^{1/x^3}}{3} - \frac{y^2}{2x^2} + c = 0$.

<u>Rule IV:</u>

$$y f(xy) dx + x f(xy) dy = 0.$$

$M = y f(xy)$, $N = x f(xy)$

If $Mx - Ny \neq 0$, then $I.F. = \frac{1}{Mx - Ny}$

Problem:

Solve: $\underbrace{(xy\sin xy + \cos xy)\,y}_{M}\,dx + \underbrace{(xy\sin xy - \cos xy)\,x}_{N}\,dy = 0$ — (i)

$$M_x - N_y = (x^2 y\sin xy + x\cos xy)_y - (xy^2\sin xy - y\cos xy)_x$$

$$= \cancel{\cos xy(x+y) + \sin xy(x^2y - xy^2)}$$

$$= x^2y^2\cancel{\sin xy} + xy\cos xy - x^2y^2\cancel{\sin xy} + xy\cos xy$$

$$= 2xy\cos xy$$

$$\text{I.F.} = \frac{1}{2xy\cdot\cos xy}$$

I.F. × (i),

$$\left(\frac{\tan xy}{2} + \frac{1}{2xy}\right)y\,dx + \left(\frac{\tan xy}{2} - \frac{1}{2xy}\right)x\,dy = 0$$

$$\int \frac{1}{2}\left(\tan xy + \frac{1}{xy}\right)y\,dx$$

$$= \frac{1}{2}\int\left(y\tan xy + \frac{1}{x}\right)dx$$

$$= \frac{1}{2}\left[y\int \tan xy\,dx + \int\frac{dx}{x}\right]$$

$$= \frac{y}{2}\log|\sec xy|\cdot\frac{1}{y} + \frac{1}{2}\log|x|$$

$$\int -\frac{1}{2}\cdot\frac{\cancel{x}}{\cancel{x}y}\,dy$$

$$= -\frac{1}{2}\int\frac{dy}{y}$$

$$= -\frac{1}{2}\log|y|$$

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$\therefore$ The general solution is : $\frac{y}{2}(\log\sec xy)\cdot\frac{1}{y} + \frac{1}{2}\log x - \frac{1}{2}\log y + c = 0.$

<u>Solve:</u> $(1+xy)y\,dx + (1-xy)x\,dy = 0 \longrightarrow (i)$

$$Mx - Ny = (y + xy^2)x + (x - x^2y) = 2x^2y^2$$

$$= xy + x^2y^2 + xy - x^2y^2 = 2xy$$

$$\therefore I.F. = \frac{1}{2x^2y^2}$$

I.F. $\times$ (i),

$$\frac{(1+xy)}{2x^2y}dx + \frac{(1-xy)}{2xy^2}dy = 0$$

$$\Rightarrow \frac{1}{2x}dx + \frac{y}{2}dx + \frac{1}{2y}dy - \frac{x}{2}dy = 0.$$

$$\int \frac{1}{2x}dx = \frac{1}{2}\log|x|$$

$$\int \frac{y}{2}dx = \frac{y}{2}\int dx = \frac{xy}{2}$$

$$\int \frac{dy}{2y} = \frac{1}{2}\int\frac{dy}{y} = \frac{1}{2}\log|y|$$

$\therefore$ General solution : $\frac{1}{2}\log|x| + \frac{xy}{2} + \frac{1}{2}\log|y| + c = 0$

$$\Rightarrow \frac{1}{2}[\log x + \log y] + \frac{xy}{2} + c = 0$$

$$\Rightarrow \log x + \log y + xy = -2c$$

$$\Rightarrow \frac{1}{2x^2y}dx + \frac{1}{2x}dx + \frac{1}{2xy^2}dy - \frac{1}{2y}dy = 0$$

$$\int \frac{dx}{2x^2y} = \frac{1}{2y}\int \frac{dx}{x^2} = \frac{1}{2y}\cdot\frac{x^{-1}}{-1} = -\frac{1}{2xy}$$

$$\int \frac{dx}{2x} = \frac{1}{2}\int \frac{dx}{x} = \frac{1}{2}\log x.$$

$$\int -\frac{dy}{2y} = -\frac{1}{2}\int \frac{dy}{y} = -\frac{1}{2}\log y.$$

$\therefore$ General solution: $\frac{1}{2}\log x - \frac{1}{2xy} - \frac{1}{2}\log y + c = 0$

Linear ODE

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{--- (i)}$$

$\hookrightarrow$ linear in y.

$$\Rightarrow dy + P(x)y\,dx = Q(x)dx$$

$$\Rightarrow \cancel{(P(x)y - Q(x))dx}$$

$$\Rightarrow (Q(x) - P(x)y)dx - dy = 0.$$

$$M = Q(x) - P(x)y \;\therefore\; \frac{\partial M}{\partial y} = -P(x)$$

$$N = -1 \;\therefore\; \frac{\partial N}{\partial x} = 0.$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -1(-P(x) - 0) = P(x)$$

$$\therefore \text{I.F.} = e^{\int P(x)dx}$$

Multiplying I.F. both sides of (i), we get

$$\underbrace{e^{\int P dx}\cdot\frac{dy}{dx} + Pye^{\int P dx}} = Qe^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

Integrating both sides, we get

$$ye^{\int P dx} = \int Qe^{\int P dx}dx + c$$

$$\Rightarrow y = e^{-\int P dx} \int Q e^{\int P dx} + c e^{-\int P dx}. \text{ (Ans.)}$$

Problem:

Solve: $\cos^2 x \frac{dy}{dx} + y = \tan x$ —— (i)

$\longrightarrow$ linear in y

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\cos^2 x} = \tan x \cdot \sec^2 x$$

$$I.F. = e^{\int P dx} = e^{\int \sec^2 x dx} = e^{\tan x}.$$

I.F. $\times$ (i) and integrating,

$$y \cdot e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} dx$$

Let $\tan x = z \Rightarrow \sec^2 x dx = dz$

$$y \cdot e^{\tan x} = \int z \cdot e^z dz$$

$$= z \cdot e^z - \int 1 \cdot e^z dz + c$$

$$= ze^z - e^z + c = e^z (z - 1) + c$$

$$\Rightarrow y \cdot e^{\tan x} = e^{\tan x} (\tan x - 1) + c.$$

$$\Rightarrow y = \tan x - 1 + c e^{-\tan x}. \text{ (Ans.)}$$

★ Linear in x:

$$\frac{dx}{dy} + S(y)x = T(y)$$

$$I.F. = e^{\int S(y)dy}.$$

Problem:

Solve: $(1+y^2)dx = (\tan^{-1}y - x)dy$.

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \quad \text{—— (i)}$$

$$I.F. = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

I.F. × (i) and integrating, we get —

$$x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \cdot e^{\tan^{-1}y}\, dy.$$

Let $\tan^{-1}y = z$

$$\Rightarrow \frac{1}{1+y^2}dy = dz$$

$$x \cdot e^{\tan^{-1}y} = \int z \cdot e^z\, dz$$

$$\Rightarrow x \cdot e^{\tan^{-1}y} = e^z(z-1) + c$$

$$\Rightarrow x \cdot e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

$$\Rightarrow x = (\tan^{-1}y - 1) + c \cdot e^{-\tan^{-1}y}. \quad \text{(Ans.)}$$

Solve: $\frac{dy}{dx} + y = \frac{1}{1+e^{2x}}$ — (i)

$$I.F. = e^{\int dx} = e^{x}.$$

I.F. × (i) and integrating,

$$y \cdot e^{x} = \int \frac{e^{x}}{1+e^{2x}} \cdot dx$$

$$= \int \frac{dz}{2e^{x}} \cdot \frac{1}{1+z}$$

$$= \int \frac{dz}{1+z^{2}}$$

$$= \tan^{-1} z + c = \tan^{-1}(e^{x}) + c$$

$$\Rightarrow y = e^{-x} \tan^{-1} e^{x} + c \cdot e^{-x}. \text{ (Ans.)}$$

$e^{x} = z$

$\Rightarrow e^{x}\, dx = dz$

$\Rightarrow e^{x} \cdot dx = \frac{dz}{2}$

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Solve: $(x^2y^2+xy+1)y\,dx + (x^2y^2 - xy+1)x\,dy = 0$

$$I.F. = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

$$\frac{x^2y^3 + xy^2 + y}{2x^2y^2}dx + \frac{x^3y^2 - x^2y + x}{2x^2y^2}dy = 0.$$

$$\Rightarrow y\,dx + \frac{dx}{x} + \frac{dx}{x^2y} + x\,dy - \frac{dy}{y} + \frac{dy}{xy^2} = 0$$

$$\Rightarrow (y\,dx + x\,dy) + \frac{dx}{x} - \frac{dy}{y} + \frac{y\,dx + x\,dy}{x^2y^2} = 0$$

$$\Rightarrow d(xy) + \frac{dx}{x} - \frac{dy}{y} + \frac{d(xy)}{(xy)^2} = 0$$

Integrating,

$$xy + \ln|x| - \ln|y| - \frac{1}{xy} = c.$$

Bernoulli's Equation

$$\frac{dy}{dx} + Py = Qy^n$$

$$\Rightarrow y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \quad \text{——(i)}$$

Put $z = y^{1-n}$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx} \Rightarrow \frac{1}{1-n}\frac{dz}{dx} = y^{-n}\frac{dy}{dx}.$$

$$(i) \Rightarrow \frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\Rightarrow \frac{dz}{dx} + (1-n)Pz = (1-n)Q$$ — linear in z.

<u>Problem:</u>

$$x\frac{dy}{dx} + y = y^2 \log x.$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = y^2 \log x/x.$$

$$\Rightarrow y^{-2}\frac{dy}{dx} + \frac{y^{-1}}{x} = \log x/x \quad \text{—(i)}$$

Put $z = y^{-1}$.

$$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow -\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$$

$$(i) \Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{\log x}{x}$$ — linear in z.

$$I.F. = e^{-\int dx/x}$$
$$= e^{-\log x} = x^{-1} = 1/x$$

Multiplying and integrating we get,

$$z \cdot \frac{1}{x} = -\int \frac{\log x}{x^2}\,dx$$

$t = \log x$

$\Rightarrow dt = \frac{dx}{x}$

$\Rightarrow x = e^t.$

$$\Rightarrow \frac{z}{x} = -\int \frac{t}{e^t}\,dt = -\int t e^{-t}\,dt$$

$$= -\left[-te^{-t} + \int 1 \cdot e^{-t}\,dt\right]$$

$$= te^{-t} + e^{-t} = e^{-t}(t+1) + c$$

MONDAY

$$\Rightarrow z/x = e^{-\log x}(1+\log x) = \frac{1}{x}(1+\log x) + c$$

$$\Rightarrow z = 1 + \log x + cx$$

$$\Rightarrow 1/y = 1 + \log x + cx.$$

Q.5.

Solve:

$$x\frac{dy}{dx} - 2y = xy^4$$

$$\Rightarrow \frac{dy}{dx} - \frac{2y^{-3}}{x} = 1 \quad \text{—— (i)}$$

$$z = y^{-3} \Rightarrow \frac{dz}{dx} = -3.y^{-4}\frac{dy}{dx}$$

$$\Rightarrow -\frac{1}{3}\frac{dz}{dx} = y^{-4}\frac{dy}{dx}$$

$$\Rightarrow -\frac{1}{3}\frac{dz}{dx} - \frac{2z}{x} = 1$$

$$\Rightarrow \frac{dz}{dx} + \frac{6z}{x} = -3 \quad \text{—— (ii)}$$

$$I.F. = e^{6\int\frac{dx}{x}} = e^{6\log x} = x^6$$

$$z.x^6 = -3\int x^6 dx + c$$

$$\Rightarrow zx^6 = -\frac{3}{7}x^7 + c$$

$$\Rightarrow z = -\frac{3}{7}x + cx^{-6}$$

$$\Rightarrow 1/y^3 = -\frac{3}{7}x + cx^{-6} \quad \text{(Ans.)}$$

Q5.

Solve: $\frac{dy}{dx} + \frac{y\log x}{x} = \frac{y(\log y)^2}{x^2}$

$$\Rightarrow \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x\log y} = \frac{1}{x^2} \quad \text{--- (i)}$$

Put, $z = \frac{1}{\log y}$

$$\Rightarrow \frac{dz}{dx} = \frac{-1}{(\log y)^2} \times \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow -\frac{dz}{dx} = \frac{1}{y(\log y)^2} \cdot \frac{dy}{dx}$$

$$(i) \Rightarrow -\frac{dz}{dx} + \frac{z}{x} = 1/x^2$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -1/x^2$$

— linear in z.

$$I.F. = e^{-\int dx/x} = e^{-\log x} = 1/x.$$

$$z.\,1/x = -\int -\frac{1}{x^3}\,dx$$

$$\Rightarrow z/x = x^{-2}/-2 + c$$

$$\Rightarrow z/x = -\frac{1}{2x^2} + c$$

$$\Rightarrow \frac{1}{x\log y} = -\frac{1}{2x^2} + c.$$

Qs. $\frac{dy}{dx} + \frac{\sin 2y}{x} = x^3 \cos^2 y$ $\qquad 2\tan y \sec^2 y \frac{dy}{dx}$

$\Rightarrow \sec^2 y \frac{dy}{dx} + \frac{\tan^2 y}{x} = x^3$

$z = \tan^2 y \Rightarrow \frac{dz}{dx} = 2\tan y \cdot \sec^2 y \frac{dy}{dx}$

$\Rightarrow \frac{1}{2\tan y} \cdot \frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$

$\Rightarrow \frac{1}{2\sqrt{z}} \cdot \frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$

$\Rightarrow \frac{1}{2\sqrt{z}} \cdot \frac{dz}{dx} + \frac{z}{x} = x^3$

$\Rightarrow \frac{dz}{dx} + \frac{z}{x \cdot 2\sqrt{z}} =$

Qs. $\frac{dy}{dx} + \frac{\sin 2y}{x} = x^3 \cos^2 y$

$\Rightarrow \sec^2 y \frac{dy}{dx} + \frac{2\sin y \cos y}{x \cos^2 y} = x^3$

$\Rightarrow \sec^2 y \frac{dy}{dx} + \frac{2}{x} \tan y = x^3$ —— (i)

$z = \tan y \Rightarrow \frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$

(i) $\Rightarrow \frac{dz}{dx} + \frac{2z}{x} = x^3$

$I.F. = e^{2\int dx/x} = x^2.$

52 18 19 20 21 22 23 24 4 22 23 24 25 26 27 28
53 25 26 27 28 29 30 31 5 29 30 31

FM-II

16/02/2024

Second order linear ODE

$$\frac{d^2y}{dx^2} + \underbrace{a_1(x)}_{\text{Coeff.}}\frac{dy}{dx} + \underbrace{a_2(x)}_{\text{Coeff}}y = f(x) \quad \text{—— (i)}$$

Homogeneous part: → Right side is equal to zero.

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0. \quad \text{—— (ii)}$$

2nd order ODE with constant coefficient,

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = f(x). \quad \text{—— (iii)}$$

Examples

(i) $\frac{d^2y}{dx^2} + \left(\frac{x^2}{x+1}\right)\frac{dy}{dx} + (\sin x)y = e^{x^2}$

(ii) $\frac{d^2y}{dx^2} + \left(\frac{x^2}{x+1}\right)\frac{dy}{dx} + (\sin x)y = 0$

(iii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - \frac{5}{2}y = e^{x^2}$

Homogeneous 2nd order linear ODE with constant coeff.:

(i) $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$ —— (i) Characteristic or auxilliary eqn

Let $y = e^{mx} \Rightarrow \frac{dy}{dx} = me^{mx} \Rightarrow \frac{d^2y}{dx^2} = m^2e^{mx}$.

(i) $\Rightarrow m^2e^{mx} + a_1me^{mx} + a_2e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 + a_1m + a_2) = 0 \Rightarrow \boxed{m^2 + a_1m + a_2 = 0}$ —— (ii)

$[\because e^{mx} \neq 0]$

Write down ideas, they get lost like good pens.

Let, m_1 and m_2 are roots of (ii).

Case : 1

m_1 and m_2 are real with $m_1 \neq m_2$

$y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$

The general solution is : $y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{linear combination}}$

$\Rightarrow y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

Case : 2

m_1 and m_2 are both real and $m_1 = m_2 = m$ (say)

$y_1 = e^{mx}$ and $y_2 = xe^{mx}$

The general solution to the ODE is :

$y = (c_1 + c_2 x) e^{mx}$.

Case : 3

Complex roots : $m = a \pm ib$ [$\because$ complex roots occurs in conjugates]

Then the general solution is :

$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$.

Qs. Solve: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0.$ ——(i)

The auxilliary eqn of (i) is: $m^2 + 3m - 4 = 0$

$\Rightarrow (m+4)(m-1) = 0$

$\Rightarrow m = 1; m = -4$

Then the general solution of (i) is:

$y = c_1 e^{x} + c_2 e^{-4x}.$

Qs. Solve: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$ ——(i)

The characteristic eqn of (i) is: $m^2 + 4m + 4 = 0$

$\Rightarrow m^2 + (2+2)m + 4 = 0$

$\Rightarrow m^2 + 2m + 2m + 4 = 0$

$\Rightarrow m(m+2) + 2(m+2) = 0$

$\Rightarrow (m+2)(m+2) = 0$

$\Rightarrow m = -2; m = -2.$

$\therefore$ The general solution is:

$\cancel{y = (-2 - 2x)e}$

$y = (c_1 + c_2 x)e^{-2x}.$

Qs. Solve: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$ —(i)

The auxilliary eqn of (i) is:

$m^2 + 3m + 4 = 0$

$\therefore m = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$

$= -\frac{3}{2} \pm \frac{\sqrt{7}}{2} i$

$\therefore$ The general solution is:

$y = e^{-\frac{3}{2}x}\left(c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right).$

Non-Homogeneous 2nd order linear ODE with constant coefficient :

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x)$$

Symbolic operator

$$\frac{dy}{dx} \equiv Dy$$

$$D \equiv \frac{d}{dx}$$

$$D^2 \equiv \frac{d^2}{dx^2}$$

$$D^3 \equiv \frac{d^3}{dx^3}$$

$$\vdots$$

$$D^n \equiv \frac{d^n}{dx^n}$$

$$D^{-1} = \frac{1}{D} \Rightarrow \text{integration}$$

* Find $D^2(\sin x + e^{x^2})$

$$= D(\cos x + 2xe^{x^2})$$

$$= -\sin x + 2(e^{x^2} + 2x^2 e^{x^2})$$

$$= -\sin x + 2e^{x^2}(1 + 2x^2)$$

* Find $D^{-2}(\sin x)$

$$= D^{-1}(-\cos x)$$

$$= -\sin x$$

* Find $D^{-3}(1/x)$

$$x^{-1} =$$

$$= D^{-2} \int \frac{dx}{x}$$

$$\cancel{= D^{-2}(\log x) = D^{-1} \int \log x \, dx = D^{-1}(1/x)}$$

$$\cancel{= \int \frac{dx}{x} = \log x}$$

$$= D^{-2}(\log x) = D^{-1} \int \log x \, dx$$

EM-II

23.02.2024

Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{3x}$

Homogeneous part is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$.

The auxilliary eqn is: $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$.

Complementary Function $= (c_1 + c_2 x)e^x$.

Particular Integral $= \frac{1}{D^2 - 2D + 1} \cdot (e^{3x})$

$= \frac{1}{3^2 - 2\cdot 3 + 1}(e^{3x})$

$= \frac{1}{4}(e^{3x})$

$\boxed{y = C.F. + P.I.}$

$\therefore$ The general solution is: $y = (c_1 + c_2 x)e^x + \frac{e^{3x}}{4}$.

★ (i) $\frac{1}{\phi(D^2)}(\sin ax)$ or $(\cos ax)$

$= \frac{1}{\phi(-a^2)} \sin ax$ or $\cos ax$ $[\phi(-a^2) \neq 0]$

(ii) $\phi(-a^2) = 0$

$\frac{1}{\phi(D^2)}(\sin ax)$ or $(\cos ax)$

$= x \cdot \frac{1}{f'(D)} \sin ax$ or $\cos ax$

where $f(D) = \phi(D^2)$.

e.g., ① $\frac{1}{D^2+9}(\cos 3x)$ $\qquad f(D) = D^2 + 9 \Rightarrow f'(D) = 2D$.

$= x \cdot \frac{1}{2D}(\cos 3x) = \frac{x}{2}\int \cos 3x \, dx = \frac{x}{2} \cdot \frac{\sin 3x}{3}$

$= \frac{x \sin 3x}{6}$

$\cancel{+ a^2} = \cancel{+} (2)^2 = \cancel{+4}$

(2) $\frac{1}{D^2+16}(\sin 4x)$

$f(D) = D^2 + 16$

$f'(D) = 2D$

$= x \cdot \frac{1}{2D}(\sin 4x)$

$= \frac{x}{2}\int \sin 4x \, dx$

$= \frac{x}{2}(-\cos 4x)/4$

$= -\frac{x\cos 4x}{8}$.

(3) $\frac{1}{D^2-2D+1}(\sin 2x)$

$f'(D) = 2D - 2$

$= \frac{1}{-2^2-2D+1}(\sin 2x) = -\frac{1}{2D+3}(\sin 2x)$

$= -\frac{2D-3}{(2D+3)(2D-3)}(\sin 2x)$

$= -\frac{2D-3}{4D^2-9}(\sin 2x)$

$= -\frac{2D-3}{4.(-2^2)-9}(\sin 2x)$

$= -\frac{2D-3}{-25}(\sin 2x)$

$= \frac{1}{25}(2D-3)(\sin 2x)$.

$= \frac{1\times 2}{25}\cos 2x \times 2 - \frac{3\sin 2x}{25}$

$= \frac{4\cos 2x}{25} - \frac{3\sin 2x}{25}$.

Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \cos 2x$

$\Rightarrow m = 1, 1$

$C.F. = (c_1 + c_2 x)\, e^x$

$P.I. = \frac{1}{D^2 - 2D + 1}(\cos 2x)$

$= \frac{1}{25}(2D - 3)(\cos 2x)$

$= \frac{1 \times 2}{25}(-\sin 2x) \times 2 - \frac{3}{25}\cos 2x$

$= -\frac{4\sin 2x}{25} - \frac{3}{25}\cos 2x.$

$\therefore$ The general solution is: $y = (c_1 + c_2 x)e^x - \frac{4\sin 2x}{25} - \frac{3\cos 2x}{25}.$

Rule III

$R(x)$ is polynomial in x.
Then expand $1/f(D)$ following binomial expansion rule in assending power of D.

$$\frac{1}{f(D)}(R(x)) = (f(D))^{-1}(R(x)).$$

NOTE:

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots \infty$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots \infty$

$(1+x)^{-2} = 1 - 2x + \frac{2(2-1)}{2!}x^2 \ldots\ldots \infty$

$(1-x)^{-2} = 1 + 2x + \frac{2(2-1)}{2!}x^2 + \ldots\ldots \infty$

$(1+x)^{-p} = 1 - px + \frac{p(p-1)}{2!}x^2 - \frac{p(p-1)(p-2)}{3!}x^3 + \ldots\ldots \infty$

NOVEMBER 2022

WK	S	M	T	W	T	F	S
45			1	2	3	4	5
46	6	7	8	9	10	11	12
47	13	14	15	16	17	18	19
48	20	21	22	23	24	25	26
49	27	28	29	30			

DECEMBER 2022

WK	S	M	T	W	T	F	S
49					1	2	3
50	4	5	6	7	8	9	10
51	11	12	13	14	15	16	17
52	18	19	20	21	22	23	24
53	25	26	27	28	29	30	31

Solve: $\frac{1}{D^2+4}(x^2) = \frac{1}{4(\frac{D^2}{4}+1)}(x^2)$

$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}(x^2)$

$= \frac{1}{4}\left[1 - \frac{D^2}{4} + \frac{D^4}{4^2} - \cdots \infty\right](x^2)$

$= \frac{1}{4}\left[1 - \frac{D^2}{4}\right](x^2)$

$= \frac{x^2}{4} - \frac{1}{16}\frac{d^2}{dx^2}(x^2) = \frac{x^2}{4} - \frac{1}{16}\frac{d}{dx}(2x)$

$= \frac{x^2}{4} - \frac{1}{\cancel{16}_{8}}\cdot\cancel{2} = \frac{x^2}{4} - \frac{1}{8}$

EM - II

Qs. Solve $(D^2-1)y = \sin^2 x$

Auxilliary eqn : $m^2 - 1 = 0 \Rightarrow m = 1, -1$.

$C.F = c_1 e^{x} + c_2 e^{-x}$

$P.I. = \frac{1}{D^2-1}(\sin^2 x)$.

$= \frac{1}{D^2-1}\left(\frac{2\sin^2 x}{2}\right)$

$= \frac{1}{2}\left(\frac{1}{D^2-1}\right)(1 - \cos 2x)$

$= \frac{1}{2}\left(\frac{1}{D^2-1}\right)(1) - \frac{1}{2}\left(\frac{1}{D^2-1}\right)(\cos 2x)$.

$= -\frac{1}{2}(1-D^2)^{-1}(1) - \frac{1}{2}\left(\frac{1}{-2^2-1}\right)(\cos 2x)$.

$$= -\frac{1}{2}\left(1+D^2+D^4+\dots\infty\right)(1)+\frac{1}{10}\cos 2x$$

$$= -\frac{1}{2}\cdot 1+\frac{1}{10}\cos 2x$$

$\therefore$ The general solution is :

$$y = c_1e^{x}+c_2e^{-x}-1/2+1/10\cos 2x.$$

Qs. Solve : $(D^2-1)y=\cos^2 x$ $\quad m=-1,1.$

$$C.F. = c_1e^{x}+c_2e^{-x}$$

$$P.I. = \frac{1}{D^2-1}(\cos^2 x)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-1}(1+\cos 2x)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-1}(1)+\frac{1}{2}\cdot\frac{1}{D^2-1}(\cos 2x)$$

$$= \frac{1}{2}\cdot(D^2-1)^{-1}(1)+\frac{1}{2}\cdot\frac{1}{-2^2-1}\cos 2x$$

$$= -\frac{1}{2}\left(1+D^2+\dots\infty\right)(1)-\frac{1}{10}\cos 2x$$

$$= -\frac{1}{2}-\frac{1}{10}\cos 2x$$

$$y = c_1e^{x}+c_2e^{-x}-\frac{1}{2}-\frac{1}{10}\cos 2x.$$

Qs. Find $\frac{1}{D^2-1}(x+x^2)$

$= -(1-D^2)^{-1}(x+x^2)$.

$= -(1+D^2+D^4+\dots\infty)(x+x^2)$

$= -[x+x^2+2]$

Rule: IV

$R(x) = e^{ax} f(x)$

Then $1/f(D)\left[e^{ax} f(x)\right]$

$= e^{ax} \cdot \frac{1}{f(D+a)} f(x)$

Examples

(i) Find $\frac{1}{D^2+4} xe^x$

$= e^x \frac{1}{(D+1)^2+4}(x)$

$= e^x \frac{1}{D^2+2D+5}(x)$

$= \frac{e^x}{5} + \frac{1}{1+\frac{D^2+2D}{5}}(x)$

$= \frac{e^x}{5}\left[1+\frac{D^2+2D}{5}\right]^{-1}(x)$

$$= \frac{e^x}{5}\left[1 - \frac{D^2+2D}{5} + \cdots\cdots \infty\right](x)$$

$$= \frac{e^x}{5}\left(x - \frac{2}{5}\right)$$

(ii) Solve,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$$

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1.$$

$$C.F. = c_1e^x + c_2xe^x.$$

$$P.I. = \frac{1}{(D^2 - 2D + 1)}(xe^x)$$

$$= \frac{1}{(D-1)^2}(xe^x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x)$$

$$= e^x \cdot \frac{1}{D^2}(x)$$

$$= e^x \cdot \frac{1}{D}\left(\int x\,dx\right)$$

$$= e^x \cdot \frac{1}{D} \cdot \frac{x^2}{2}$$

$$= e^x/2 \cdot \frac{1}{D}\left(\int x^2\right)$$

$$= \frac{e^x}{2} \cdot \frac{x^3}{3}$$

$$= \frac{1}{6}x^3e^x.$$

EM-II 01.03.2024

• D'Alambert's Theorem

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

Q. $\frac{1}{D^2+4}(x\sin x)$

Imaginary part of $\frac{1}{D^2+4}(xe^{ix})$

$\frac{1}{D^2+4}(xe^{ix})$

$= e^{ix}\frac{1}{(D+i)^2+4}(x)$

$= e^{ix}\frac{1}{D^2+2Di+i^2+4}(x) = e^{ix}\frac{1}{D^2+2iD+3}(x)$

$= \frac{e^{ix}}{3}\left(\frac{1}{1+\frac{D^2+2iD}{3}}\right)(x)$

$= \frac{e^{ix}}{3}\left(1+\frac{D^2+2iD}{3}\right)^{-1}(x)$

$= \frac{e^{ix}}{3}\left[1-\frac{D^2+2iD}{3}+\dots\infty\right](x)$

$= \frac{e^{ix}}{3}\left(x-\frac{2i}{3}\right)$

$= \frac{x}{3}(\cos x+i\sin x) - \frac{2}{9}i(\cos x+i\sin x)$ —— (i)

The imaginary part of (i) $= \frac{x}{3}\sin x - \frac{2}{9}\cos x$.

Note: $\frac{1}{D^2+4}(x\cos x)$

~~Imaginary~~ Real part of $\frac{1}{D^2+4}(\cos x \cdot x)$

$= \frac{x}{3}\cos x + \frac{2}{9}\sin x$.

Rule: IV.
$R(x) = e^{ax}f(a)$
Then $\frac{1}{F(D)}[e^{ax}f(x)]$
$= e^{ax}\frac{1}{F(D+a)}f(x)$

Q. Find $\frac{1}{D^2+1}e^x\sin x$

$= e^x\frac{1}{(D+1)^2+1}\sin x$

$= e^x\frac{1}{D^2+2D+2}\sin x$

$= e^x\frac{1}{-1^2+2D+2}\sin x = e^x\frac{1}{2D+1}\sin x$

$= e^x\frac{2D-1}{4D^2-1}\sin x = e^x\frac{2D-1}{4(-1^2)-1}\sin x$

$= -\frac{e^x}{5}(2D-1)\sin x = -\frac{e^x}{5}(2\cos x-\sin x)$.

EM-II 05/03/2024

Cauchy-Euler's

• ODE with variable coefficient

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0. \quad \text{—— (i)}$$

Put $x = e^z \Rightarrow z = \log x \longrightarrow \frac{dz}{dx} = \frac{1}{x}$.

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\cdot\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \quad \text{—— (ii)}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right).$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}. \quad \text{—— (iii)}$$

$D \equiv \frac{d}{dx}$, $D' = \frac{d}{dz}$

(i) $\Rightarrow (x^2D^2 + xD + 1)y = 0$ —— (iv)

(ii) $\Rightarrow xD \equiv D'$

(iii) $\Rightarrow x^2D^2 \equiv (D')^2 - D' = D'(D'-1)$

(iv) $\Rightarrow [D'(D'-1) + D' + 1]y = 0$

$\Rightarrow ((D')^2 + 1)y = 0.$

NOTE: $x^3D^3 \equiv D'(D'-1)(D'-2)$

$x^4D^4 \equiv D'(D'-1)(D'-2)(D'-3)$.

Qs. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \log x$. —— (i)

Put $x = e^z$

$\therefore z = \log x$

Then from (i),

$[D'(D'-1) + 4D' + 2]y = z$, where $D' \equiv \frac{d}{dz}$

$\Rightarrow [(D')^2 + 3D' + 2]y = z.$

Auxiliary eqn: $m^2 + 3m + 2 = 0$

$\Rightarrow (m+2)(m+1) = 0$

$\Rightarrow m = -1, -2$.

C.F. $= c_1e^{-z} + c_2e^{-2z}$.

$$\text{P.I.} = \frac{1}{D'^2 + 3D' + 2}(z) = \frac{1}{2}\cdot\left[1 + \frac{D'^2 + 3D'}{2}\right]^{-1}(z)$$

$$= \frac{1}{2}\left[1 - \frac{D'^2 + 3D'}{2} + \cdots\cdots \infty\right](z)$$

$= \frac{z}{2} - \frac{3}{4}$.

The general solution is: $y = c_1 e^{-z} + c_2 e^{-2z} + \frac{z}{2} - \frac{3}{4}$.

$\Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x} + \frac{1}{2}\log x - \frac{3}{4}$.

Q.5. $(x^2D^2 - xD - 3)y = x^2 \log x$ —— (i)

Put $x = e^z$

$\therefore z = \log x$

(i) $\Rightarrow (D'(D'-1) - D' - 3)y = e^{2z}.z$

$(D')^2 - 2D' - 3)y = ze^{2z}$

Auxiliary eqn: $m^2 - 2m - 3 = 0$

$\Rightarrow (m-3)(m+1) = 0$

$\Rightarrow m = -1, 3$

C.F. $= c_1 e^{-z} + c_2 e^{3z}$

$y = c_1 e^{-z} + c_2 e^{3z} - \frac{1}{3} e^{2z}\left(z + \frac{2}{3}\right)$

$= \frac{c_1}{x} + c_2 x^3 - \frac{1}{3} x^2 (2/3 + \log x)$.

EM-II

08.03.2024

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad \text{—— (i)}$$

$$\frac{d^2y}{dx^2} + (1+e^x)\frac{dy}{dx} + e^x y = \tan x \quad [\text{Example of (i)}]$$

Method of Variation of Parameters

Consider the ODE (i). Let, ~~y(x)~~ $y_1(x)$ and $y_2(x)$ be the Independent solution of (i).

Then, the particular integral of (i) is

$$y_p = y_1 \int \frac{y_2 R(x)}{W(y_1, y_2)} - y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)}$$

where, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$.

Q.5. Solve: $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ —— (i)

The homogeneous part is, $\frac{d^2y}{dx^2} - y = 0$

Auxilliary eqn: $m^2 - 1 = 0$

$\Rightarrow m^2 = 1$

$\Rightarrow m = \pm 1$.

C.F. $= c_1 e^x + c_2 e^{-x}$.

Let, the general solution of (i) be

$$y = \phi(x) e^x + \psi(x) e^{-x}.$$

$$\frac{dy}{dx} = \phi(x) e^x - \psi(x) e^{-x} + \phi'(x) e^x + \psi'(x) e^{-x}.$$

Let us assume that,

$$\phi'(x) e^x + \psi'(x) e^{-x} = 0. \quad \text{—— (ii)}$$

$$\therefore \frac{dy}{dx} = \phi(x) e^x - \psi(x) e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \phi(x) e^x + e^x \cdot \phi'(x) - \{\psi(x) e^{-x} (-1) + e^{-x} \psi'(x)\}$$

$$= \phi(x) e^x + \phi'(x) e^x + \psi(x) e^{-x} - \psi'(x) e^{-x}.$$

Putting the value of $\frac{d^2y}{dx^2}$ in the eqn (i),

$$\phi(x) e^x + \phi'(x) e^x + \psi(x) e^{-x} - \psi'(x) e^{-x} - y = \frac{2}{1+e^x}$$

$$\Rightarrow \cancel{\phi(x) e^x} + \phi'(x) e^x + \cancel{\psi(x) e^{-x}} - \psi'(x) e^{-x} - \cancel{\phi(x) e^x} - \cancel{\psi(x) e^{-x}} = \frac{2}{1+e^x}$$

$$\Rightarrow \phi'(x)e^{x} - \psi'(x)e^{-x} = \frac{2}{1+e^{x}} \quad \text{--- (iii)}$$

Now, solving eqn (ii) and (iii),

$$\phi'(x)e^{x} + \cancel{\psi'(x)e^{-x}} = \frac{2}{1+e^{x}}$$

$$\phi'(x)e^{x} - \cancel{\psi'(x)e^{-x}} = 0$$

$$\cancel{2}\phi'(x)e^{x} = \frac{\cancel{2}}{1+e^{x}}$$

$$\Rightarrow \phi'(x) = \frac{1}{e^{x}+e^{2x}} = \frac{e^{-x}}{1+e^{x}}$$

$$\therefore \frac{e^{-x}}{1+e^{x}} \cdot e^{x} + \psi'(x)e^{-x} = \frac{2}{1+e^{x}}$$

$$\Rightarrow \frac{1}{1+e^{x}} + \psi'(x)e^{-x} = \frac{2}{1+e^{x}}$$

$$\Rightarrow \psi'(x)e^{-x} = \frac{2-1}{1+e^{x}} = \frac{1}{1+e^{x}}$$

$$\Rightarrow \psi'(x) = \frac{1}{e^{-x}+1} = \frac{e^{-x}}{1+e^{-x}}.$$

$$\therefore \phi(x) = \int \frac{e^{-x}}{1+e^{x}}\,dx$$

$$\boxed{\begin{aligned} &e^{-x} = z \Rightarrow dz = -e^{-x}dx \\ &\Rightarrow \cancel{dx = e^{-x}dz = \frac{dz}{z}}. \end{aligned}}$$

$$= -\int \frac{1}{(1+1/z)} \cdot \frac{dz}{z}$$

$$= \int \frac{\cancel{dz}}{z^{\cancel{2}}(1+z)}$$

$$= -\int \frac{z\,dz}{1+z} = -\int \frac{(1+z)-1}{(1+z)}\,dz$$

$$= -\int dz + \int \frac{dz}{1+z} = -z + \log(1+z) + c_1$$

$$= -e^{-x} + \log(1+e^{-x}) + c_1.$$

$$\psi(x) = -\int \frac{e^{x}}{1+e^{x}}\,dx = -\log(1+e^{x}) + c_2$$

$\therefore$ The general solution is,

$$y = \phi(x)e^{x} + \psi(x)e^{-x}$$

$$= \{-e^{-x} + \log(1+e^{-x}) + c_1\}e^{x} + \{-\log(1+e^{x}) + c_2\}e^{-x}$$

$$= \underline{-1} + \underline{e^{x}c_1} + \underline{e^{x}\log(1+e^{-x})} - e^{-x}\log(1+e^{x}) + \underline{e^{-x}c_2}$$

$$= c_2e^{-x} + c_1e^{x} - 1 + e^{x}\log(1+e^{-x}) - e^{-x}\log(1+e^{x}).$$

H.W. Solve by the method of variation of parameters,

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x \quad \text{—— (i)}$$

→ Homogeneous part : $\frac{d^2y}{dx^2} + 4y = 0$

Auxilliary eqn $\Rightarrow m^2 + 4 = 0$

$\Rightarrow m^2 = -4$

$\Rightarrow m = \pm\sqrt{-4} = \pm 2i = 0 \pm 2i$

Then the general solution is : $y = e^{0.x}(c_1 \cos 2x + c_2 \sin 2x)$

Let, the general solution of eqn (i) be

$$y = \phi(x)\cos 2x + \psi(x)\sin 2x$$

$$\Rightarrow \frac{dy}{dx} = \phi(x)\{\sin 2x.(-2)\} + \cos 2x.\phi'(x) + \psi(x)\{\cos 2x.(2)\} + \sin 2x.\psi'(x)$$

Let us assume that,

$$\phi'(x)\cos 2x + \psi'(x)\sin 2x = 0 \quad \text{—— (ii)}$$

$$\therefore \frac{dy}{dx} = 2\cos 2x\,\psi(x) - 2\sin 2x\,\phi(x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2[\cos 2x.\psi'(x) + \psi(x)\{\sin 2x(-2)\}] - 2[\sin 2x.\phi'(x) + \phi(x)\{\cos 2x(2)\}]$$

$$= 2\cos 2x\,\psi'(x) - 4\sin 2x\,\psi(x) - 2\sin 2x\,\phi'(x) - 4\cos 2x\,\phi(x)$$

Putting the value of $\frac{d^2y}{dx^2}$ in the eqn (i) we get,

$$2\cos 2x\,\psi'(x) - 4\sin 2x\,\psi(x) - 2\sin 2x\,\phi'(x) - 4\cos 2x\,\phi(x) + 4y = 4\tan 2x$$

$$\Rightarrow 2\cos 2x\,\psi'(x) - 2\sin 2x\,\phi'(x) - \cancel{4\sin 2x\,\psi(x)} - \cancel{4\cos 2x\,\phi(x)} + \cancel{4\cos 2x\,\phi(x)} + \cancel{4\sin 2x\,\psi(x)} = 4\tan 2x$$

$$\Rightarrow 2(\cos 2x\,\psi'(x) - \sin 2x\,\phi'(x)) = 4\tan 2x$$

$$\Rightarrow \cos 2x\,\psi'(x) - \sin 2x\,\phi'(x) = 2\tan 2x \quad \text{—— (iii)}$$

Now, solving (ii) and (iii) ——

$$\cos 2x\, \phi'(x) + \sin 2x\, \psi'(x) = 0 \quad \text{———} \times \cos 2x$$
$$\cos 2x\, \psi'(x) - \sin 2x\, \phi'(x) = 2\tan 2x \quad \text{—} \times \sin 2x$$

$$\cos^2 2x\, \phi'(x) + \cancel{\sin 2x \cos 2x\, \psi'(x)} = 0$$
$$\underset{(+)}{-} \sin^2 2x\, \phi'(x) \underset{(-)}{+} \cancel{\sin 2x \cos 2x\, \psi'(x)} = \underset{(-)}{\frac{2\sin^2 2x}{\cos 2x}}$$

$$\phi'(x)\left[\cos^2 2x + \sin^2 2x\right] = -2\sin 2x \tan 2x$$

$$\Rightarrow \phi'(x) = -2\sin 2x \tan 2x$$

From (ii),

$$-2\sin 2x \tan 2x \cdot \cos 2x + \psi'(x) \sin 2x = 0$$

$$\Rightarrow \psi'(x) = 2\sin 2x$$

$$\therefore \phi(x) = -2\int \sin 2x \tan 2x \, dx$$

$$2\sin^2 x = 1 - \cos 2x$$
$$\Rightarrow 2\sin^2 2x = 1 - \cos 4x$$

$$= -2\int \frac{\sin^2 2x}{\cos 2x}\, dx$$

$$= -2\int \frac{1 - \cos 4x}{2\cos 2x}\, dx$$

$$= -2\left[\int \sec 2x\, dx - \int \cos 2x\, dx\right]$$

$$= -2\left[\frac{\log|\sec 2x + \tan 2x|}{2} - \frac{\sin 2x}{2}\right] + c_1$$

$$\psi(x) = 2\int \sin 2x\, dx = \cancel{2}\left(-\frac{\cos 2x}{\cancel{2}}\right) + c_2$$

$\therefore$ The general solution is : $y = \phi(x)\cos 2x + \psi(x)\sin 2x$

$$\Rightarrow y = \{\sin 2x - \log|\sec 2x + \tan 2x| + c_1\}\cos 2x + \{-\cos 2x + c_2\}\sin 2x$$

$$\Rightarrow y = \cancel{\sin 2x \cos 2x} - \log|\sec 2x + \tan 2x|\cos 2x + c_1 \cos 2x - \cancel{\sin 2x \cos 2x} + c_2 \sin 2x$$

$$\Rightarrow y = c_1 \cos 2x - \log|\sec 2x + \tan 2x|\cos 2x + c_2 \sin 2x.$$

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$$\frac{d^3y}{dx^3} - y = e^{2x}.$$

Auxiliary eqn is : $m^3 - 1 = 0$

$\Rightarrow (m-1)(m^2+m+1) = 0$

$\Rightarrow m = 1$ or $m^2+m+1 = 0$

$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$

$\therefore$ The roots are $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$

$C.F. = c_1 e^x + e^{-x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

$P.I. = \frac{1}{D^3-1}(e^{2x}) = \frac{1}{2^3-1}e^{2x} = \frac{1}{7}e^{2x}.$

$\therefore$ The general solution is :

$$y = c_1 e^x + e^{-x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{7}e^{2x}.$$

Qs. $(D-1)(D-2)(D+3)y = 0$

$\rightarrow (m-1)(m-2)(m+3) = 0$

$\Rightarrow m = 1, 2, -3$

$y = c_1 e^x + c_2 e^{2x} + c_3 e^{-3x}.$

Qs. $(D-1)^3 y = 0$

$\rightarrow m = 1, 1, 1.$

$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

$= (c_1 + c_2 x + c_3 x^2)e^x.$

Qs. $(D-2)^4 y = 0$

$\rightarrow y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^{2x}.$

Qs. $(D^2-1)^2 y = 0$

$\rightarrow \cancel{D^4 - 2D^2 + 1 = 0}$

$(m^2-1)^2 = 0$

$\Rightarrow (m-1)^2(m+1)^2 = 0$

$\Rightarrow m = 1, 1, -1, -1.$

$y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x}.$

Qs. $(D^2+1)^2 y = 0$

$\rightarrow \quad (m^2+1)^2 = 0$

$\Rightarrow (m^2+1)(m^2+1) = 0$

$\cancel{\Rightarrow m^2 = -1} \quad \cancel{\Rightarrow m = \mp 1} \quad \cancel{m^2 = -1} \quad \cancel{\Rightarrow m = \mp 1}$

$\Rightarrow m = \pm i, \pm i.$

$y = (c_1 \cos x + c_2 \sin x) + x(c_3 \cos x + c_4 \sin x).$

Qs. $(D^2+4)^3 y = 0$

$\rightarrow (m^2+4)^3 = 0 \Rightarrow (m^2+4)(m^2+4)(m^2+4) = 0$

$\Rightarrow m = \pm 2i, \pm 2i, \pm 2i.$

$y = \cancel{(c_1 + c_3 x) \cos 2x + (c_2 + c_4 x)}$

$(c_1 + c_3 x + c_5 x^2) \cos 2x + (c_2 + c_4 x + c_6 x^2) \sin 2x$