-: Vector Calculus :-

Scalar function:- Any real-valued function which is defined at each points in a certain domain in space is known as scalar function.

Ex:- $f(x,y,z) = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$ is a scalar function.

Note:- Scalar functions are also known a scalar field in space.

Vector function:- Let D be a domain, A function $\vec{F} = \vec{F}(P) = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ defined at each point $P \in D$ is called a vector function.

In that case, we say that $\vec{F}$ is a vector field in D.

Ex:- $\vec{F}(x,y,z) = (x+y+z)\hat{i} + xyz\,\hat{j} + (x^2-y^2)z\,\hat{k}$ is a vector function.

Level Surfaces:-

Let $f(x,y,z)$ be a real-valued continuous scalar function in a domain D. Then $f(x,y,z) = $ a constant $= c$ defines the equation of a surface and is known as level surface.

Ex:- i) $f(x,y,z) = x^2+y^2+z^2$

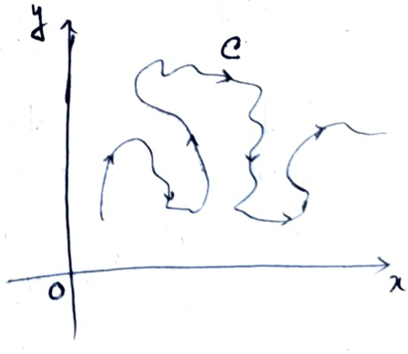
then $x^2+y^2+z^2 = c^2$ ~~is~~ are level surfaces which are sphere.

ii) $f(x,y,z) = z-(x^2+y^2)$

then $x^2+y^2 = z-c$, $z \geq c$ represent level surfaces which are paraboloid.

** Parametric representation of curves :-

Parametric representation is very important concept in vector analysis. It helps us to represent a multiple variable function into a function of parameters.



Suppose that a particle moves along the curve C given in the figure. It is impossible to describe the curve C by a function $y = f(x)$. But the x and y-coordinates of the particle are function of time and so we can write $x = h(t)$ and $y = g(t)$. Such a pair of equations is a very convenient way to represent a curve. It also helps us to analyze a complex function.

Parametric equation:- Suppose that x and y are both given as a function of a third variable t (called a parameter) by the equn $x = h(t)$, $y = g(t)$, then they are called parametric equations

For each value of t, we can determine the point (x, y) on the curve C and we can trace the curve. This curve is known as parametric curve.

Note that, t is a parameter, it need not be t all time we can use any notation to represent parameter like, θ, α etc.

Parametric representation of position vector :-

Position vector in two-dimension is given by $\vec{r} = x\hat{i} + y\hat{j}$.

" " " three " " " " $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

If parametric equations are

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

then parametric form of position vectors are.

$$\vec{r} = f(t)\hat{i} + g(t)\hat{j} \quad \text{(in two-dimension)}$$

$$\vec{r} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}) \quad \text{(in three ")}$$

Parametric representation of a straight line:-

Let $A \equiv (a_1, a_2, a_3)$ and $B \equiv (b_1, b_2, b_3)$ be two points in 3-dimension Then the parametric form of the position vector of a point on the straight line $\overline{AB}$ is given by.

$$\vec{r} = [a_1 + t(b_1 - a_1)]\hat{i} + [a_2 + t(b_2 - a_2)]\hat{j} + [a_3 + t(b_3 - a_3)]\hat{k}$$

Ex:- Parametric form of position vector on the straight line joining the points $(1, -1, 3)$ and $(3, 2, 1)$ is.

$$\vec{r} = (1 + 2t)\hat{i} + (-1 + 3t)\hat{j} + (3 - 2t)\hat{k}.$$

* Parametric form of a straight line in 2-dimension given by $ax + by = c$ is

$$\vec{r} = t\hat{i} + \left(\frac{c - at}{b}\right)\hat{j},$$

taking $x = t$, $y = \frac{c - at}{b}$ $\left[\text{Solving } y = \frac{c - ax}{b}\right]$.

Parametric representation of Some ~~stan~~ standard curve (2-D):-

Circle : $x^2 + y^2 = a^2$.

$$x = a\cos(t), \quad y = a\sin(t), \quad 0 \le t \le 2\pi.$$

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x = a\cos(t), \quad y = b\sin(t), \quad 0 \le t \le 2\pi.$$

Parabola :- $y^2 = 4ax$

$$x = at^2, \quad y = 2at,$$

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

Parametric representation of Surface :-

To represent a surface in parametric form, we need two parameters. Parametric form of some standard surfaces are given below:

Sphere :- Equation : $x^2+y^2+z^2=a^2$

Parametric form : $x = a\cos\theta\cos\phi$.

$y = a\sin\theta\cos\phi$

$z = a\sin\phi$, $0\le\theta\le 2\pi$, $-\frac{\pi}{2}\le\phi\le\pi/2$

Here θ and ϕ are the parameters.

Ellipsoid :- Equation : $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Parametric form: $x = a\cos\theta\cos\phi$

$y = b\sin\theta\cos\phi$

$z = c\sin\phi$. $0\le\theta\le 2\pi$, $-\pi/2\le\phi\le\pi/2$

Paraboloid :- Equation: $z = x^2+y^2$

Parametric form: $x = u\cos\theta$

$y = u\sin\theta$ $0\le\theta\le 2\pi$.

$z = u^2$,

Cylinder :- Equation: $x^2+y^2=a^2$, $z=u$.

Parametric form: $x = a\cos\theta$,

$y = a\sin\theta$, $0\le\theta\le 2\pi$.

$z = u$,

*** **Note!-**

i) Any curve or surface can be represented by its parametric form of the position vector of a point on the curve or surface.

ii) If equation of a curve is given then to find the parametric form, we can take one variable as parameter and find the value of the other variable in terms of the parameter by solving the given equation of the curve.

Derivative of a vector function:-

We know that any vector function can be represented by its position vector $\vec{r}$. Hence derivative of a vector function is nothing ~~but~~ but the derivative of position vector $\vec{r}$. Also the derivative represent the tangent vector to the curve C.

Let vector function be $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Then $\dfrac{d\vec{r}}{dt} = \vec{r}'(t) = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k}$.

Ex:- 1. Represent the parabola $y = 1 - 2x^2$, $-1 \le x \le 1$ in parametric form. Hence find $r'(1/\sqrt{2})$.

Sol:- Let $x = \sin t$, then $y = 1 - 2\sin^2 t = \cos 2t$

$\therefore$ Parametric form is given by.

$\vec{r}(t) = \sin t\,\hat{i} + \cos 2t\,\hat{j}$, $-\pi/2 \le t \le \pi/2$.

$\therefore \vec{r}'(t) = \cos t\,\hat{i} - 2\sin 2t\,\hat{j}$

$\therefore$ if $x = \frac{1}{\sqrt{2}}$ then $\sin t = 1/\sqrt{2} \Rightarrow t = \pi/4$.

$\therefore \vec{r}'(\pi/4) = \frac{1}{\sqrt{2}}\hat{i} - 2\hat{j}$

2. Find the tangent vector to the curve given by $x = t^3$, $y = (1+t)/t$, $z = 1 + t^2$ at $t = 2$. Hence, find the parametric representation of the tangent vector.

Sol:- Given that, $\vec{r}(t) = t^3\hat{i} + (1+t)/t\,\hat{j} + (1+t^2)\hat{k}$

$\therefore \vec{r}'(t) = 3t^2\hat{i} + \left(-\frac{1}{t^2}\right)\hat{j} + 2t\hat{k}$.

$\therefore \vec{r}'(2) = 12\hat{i} - \frac{1}{4}\hat{j} + 4\hat{k}$.

$r(2) = 8\hat{i} + \frac{3}{2}\hat{j} + 5\hat{k}$.

$\therefore$ Tangent vector is passing through $(8, 3/2, 5)$ and has slope $(12, -\frac{1}{4}, 4)$. Hence the parametric form of tangent vector at $t=2$ is given by.

$$\vec{r} = t\left(12, -\tfrac{1}{4}, 4\right) + (8, 3/2, 5) = (8+12t)\hat{i} + \left(\tfrac{3}{2} - \tfrac{1}{4}t\right)\hat{j} + (5+4t)\hat{k}$$

Note:-

i) $(\vec{u}\cdot\vec{v})' = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$

ii) $(\vec{u}\times\vec{v})' = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$

Smooth curve:- Let $\vec{r}(t)$ denote the position vector of a point P on the curve C. Let $\vec{r}(t)$ have continuous first order derivative. Then $\vec{r}(t)$ is called a smooth function on an interval (a,b) if $\vec{r}'(t) \neq 0$ on (a,b). In that case, the curve C is called smooth curve.

Length of a Space Curve:-

Let the space curve C be given by its parametric form as $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \le t \le b$. Then the length of the curve is given by.

$$l = \int_a^b \left[(x'(t))^2 + (y'(t))^2 + (z'(t))^2\right]^{\frac{1}{2}} dt$$

or, $$l = \int_a^b \left[\vec{r}'(t) \cdot \vec{r}'(t)\right]^{\frac{1}{2}} dt.$$

Ex:- Find the length of the helix given by,

$$\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 5t\,\hat{k}, \quad 0 \le t \le 2\pi.$$

Sol:-
$$\vec{r}'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 5\hat{k}.$$

$\therefore$ Length $= \int_0^{2\pi} (4\cos^2 t + 4\sin^2 t + 25)^{\frac{1}{2}} dt$

$= \int_0^{2\pi} (4+25)^{1/2} dt = \int_0^{2\pi} \sqrt{29}\, dt$

$= \left[\sqrt{29}\, t\right]_0^{2\pi} = 2\sqrt{29}\,\pi.$

Note:- Length of the curve is also known as arc-length

i.e. $$S = \int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Which is arc length of the curve given by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ from a point $t=a$ to any point $t=t$.

Note that elementary arc-length is given by.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \quad [\text{3-D case}]$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \quad [\text{2-D case}]$$

Ex:- i) Evaluate $\int_C x^2 y\, ds$ where $C: x = 3\cos t,\ y = 3\sin t,\ 0 \le t \le \frac{\pi}{2}$

(ii) Evaluate $\int_C (x^2 + yz)\, ds$ where $C: x = 4y,\ z = 3$ from $(2, \frac{1}{2}, 3)$ to $(4, 1, 3)$.

Vector Integral Calculus

15.05.23

Smooth curve: Let $\vec{r}(t)$ denote the position vector ~~denote~~ of point P ~~of~~ on the curve C.
Let $\vec{r}(t)$ have continuous first order derivative, the C is called a smooth curve if

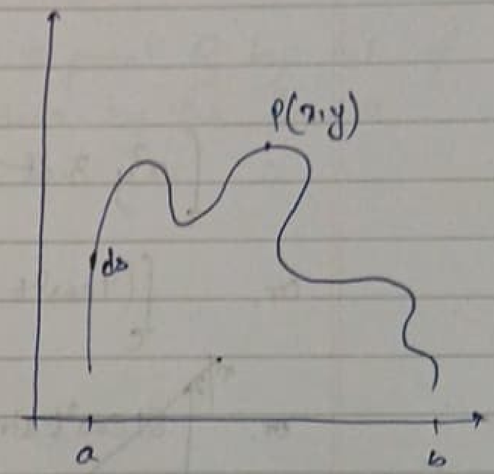
$$\frac{d\vec{r}}{dt} \neq 0. \text{ on the curve domain.}$$

Length of a space curve:-
Let a space curve C be given by its parametric form $\vec{r}(t)$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$a \leq t \leq b$$

$$S = \int ds = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$



$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

Ex Helix: $\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 5t\,\hat{k}$, $0 \leq t \leq 2\pi$

$x(t) = 2\cos t \qquad y(t) = 2\sin t \qquad z(t) = 5t$

$\frac{dx}{dt} = -2\sin t \qquad \frac{dy}{dt} = 2\cos t \qquad \frac{dz}{dt} = 5$

$$S = \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t + 25}\, dt = \int_0^{2\pi} \sqrt{29}\, dt \quad \cancel{3t}$$

$$= 2\pi\sqrt{29} \text{ unit}$$

Ex Evaluate $\int_C x^2 y \, ds$, $C: x = 3\cos t \quad y = 3\sin t$

$0 \le t \le \pi/2$

Given, $x = 3\cos t \qquad \frac{dx}{dt} = -3\sin t$

$\frac{dy}{dt} = 3\cos t$

now, ds [2D case]

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{9\sin^2 t + 9\cos^2 t}\; dt$$

$$= 3\, dt.$$

$\therefore \int_C x^2 y \; 3.\, dt$

or, $\int_C (9\cos^2 t \cdot 3\sin t \cdot 3)\, dt$ or, $81 \int_0^{\pi/2} \cos^2 t \sin t \, dt$

or, $\int_0^{\pi/2} 81 \cos^2 t \sin t \, dt$

or, $81 \left[\sin t \int_0^{\pi/2} \cos^2 t \, dt - \int_0^{\pi/2} \left\{ \frac{d}{dt}(\sin t) \cdot \int_0^{\pi/2} \cos^2 t \, dt \right\} dt \right]$

or, $81 \left[\sin t \left[\frac{\sin^3 t}{3} \right]_0^{\pi/2} - \int_0^{\pi/2} \cos t \right.$

Let, ~~cot~~ $\cos t = u \Rightarrow u^2 = \cos^2 t$

$\therefore du = -\sin t \, dt \Rightarrow \sin t \, dt = -du$

$\therefore 81 \int_1^0 -u^2 \, du$ or, $81 \int_0^1 u^2 \, du$ or, $81 \left[\frac{u^3}{3} \right]_0^1$

or, $\frac{81}{3} = 27.$