Fourier Transform

Periodic function :-

A function $f(x)$ is said to be periodic function of period T if $f(x+T) = f(x)$ for all real number x.

Suppose, $\sin(0) = 0$
$\sin(2\pi) = 0$
$\sin(4\pi) = 0$

Here, the function $f(x) = \sin x$ is a periodic function of period 2π. This is also called sinusoidal periodic function.

In general $f(x+T) = f(x+2T) \cdots = f(x+nT)$

In Taylor series or Maclaurin series we can expand the function if it's derivatives are continuous but if the derivatives are not continuous then we need to use Fourier series to expand the function $f(x)$. In this series we can expand $f(x)$ on an interval which is an infinite series containing sine and cosine of the variable x.

Thus the definition of Fourier series of $f(x)$ on an interval $C < x < C + 2\pi$, C is a real number is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Here, a_0, a_n and b_n are called Fourier coefficient and they are expressed by the following relations:-

$$a_0 = \frac{1}{\pi}\int_{C}^{C+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{C}^{C+2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{C}^{C+2\pi} f(x) \sin nx\,dx$$

Now we need to discuss few conditions -

i) If $c = 0$ then the interval is $0 < x < 2\pi$ then we can obtain a_0, a_n, b_n by putting $c = 0$ in the limits of the interval.

ii) If $c = -\pi$ then the interval is $-\pi < x < \pi$ and Here, we need to discuss two cases :-

a) If $f(x)$ is odd function:- $[f(-x) = -f(x)]$

Here, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \right]$$

Putting $x = -t$ in first integral,

$$= \frac{1}{\pi} \left[-\int_{\pi}^{0} f(-t)\,dt + \int_{0}^{\pi} f(x)\,dx \right]$$

$$= \frac{1}{\pi}\left[\int_{\pi}^{0} f(t)\,dt + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{0}^{\pi} f(t)\,dt + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= 0 \quad \left[\text{In definite integral we can change variable.}\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

Put, $x = -t$ for first integral

$$= \frac{1}{\pi}\left[-\int_{\pi}^{0} f(-t)\cos n(-t)\,dt + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[+\int_{\pi}^{0} f(t)\cos nt\,dt + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{0}^{\pi} f(t)\cos nt\,dt + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

Put $x = -t$, for first integral,

$$= \frac{1}{\pi}\left[-\int_{\pi}^{0} f(-t)\sin n(-t)\,dt + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{0}^{\pi} f(t)\sin n(-t)\,dt + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

So, the fourier series in this case is -

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\,dx \quad \text{where, } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

This is also called Fourier Sine Series.

b) If $f(x)$ is even function :- $[f(-x) = f(x)]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{+\pi}^{0} f(-t)\,dt + \int_{0}^{\pi} f(x)\,dx\right] \quad \left[\because \text{Put } x=-t \text{ for first integral}\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} f(t)\,dt + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{\pi}^{0} f(-t)\cos n(-t)\,dt + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

[Put $x = -t$ for first integral]

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} f(t)\cos nt\,dt + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

Again,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{\pi}^{0} f(-t)\sin(-t)n\,dt + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{0}^{\pi} f(t)\sin nt\,dt + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= 0$$

So, the fourier series in this case is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

This is called Fourier Cosine series.

SOME IMPORTANT FACTS :-

i) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$, $n \in \mathbb{Z}$

ii) $\int_{0}^{2\pi} \sin nx\,dx = 0$

iii) $\int_{0}^{2\pi} \cos nx\,dx = 0$

iv) $\int_{0}^{2\pi} \sin^2 nx\,dx = \pi$

v) $\int_{0}^{2\pi} \cos^2 nx\,dx = \pi$

vi) $\int_{0}^{2\pi} \sin nx \sin mx\,dx = 0$

vii) $\int_{0}^{2\pi} \cos nx \cos mx\,dx = 0$

viii) $\int_{0}^{2\pi} \sin nx \cos mx\,dx = 0$

ix) $\int_{0}^{2\pi} \cos nx \sin mx\,dx = 0$

To find Fourier Co-efficient and Euler's formula :-

Here the Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Integrating both sides from c to $c+2\pi$

$$\int_c^{c+2\pi} f(x)\,dx = \frac{1}{2} a_0 \int_c^{c+2\pi} dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin nx\,dx$$

$$= \frac{1}{2} a_0 (c+2\pi - c) + 0 + 0$$

$$= a_0 \pi$$

where, $a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\,dx$

Again multiply each side by $\cos nx$ in equation (1) and integrating from c to $c+2\pi$

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{1}{2} a_0 \int_c^{c+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos^2 nx\,dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \cos nx \sin nx\,dx$$

$$= 0 + a_n \pi + 0$$

So, $a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\,dx$

Again multiply each side by $\sin nx$ in equation ①
and integrating from c to $c+2\pi$

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_c^{c+2\pi}\sin nx\,dx + \sum_{n=1}^{\infty}a_n\int_c^{c+2\pi}\cos nx\sin nx\,dx + \sum_{n=1}^{\infty}b_n\int_c^{c+2\pi}\sin^2 nx\,dx$$

$$= 0 + 0 + b_n\pi$$

so, $$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Here the formula of a_0, a_n and b_n is called
Euler's formula.

DIRICHLET'S CONDITION :-

Any function $f(x)$ can be expressed as

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n\cos nx + \sum_{n=1}^{\infty}b_n\sin nx$$ where a_0, a_n, b_n are constants.

i) $f(x)$ is single valued in $(c, c+2\pi)$

ii) $f(x)$ is periodic function of period 2π.

iii) $f(x)$ and $f'(x)$ piecewise continuous on $(c, c+2\pi)$

Then the fourier series with its coefficients converges to -

a) $f(x)$ if x is a point of continuity

b) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Expand the function $f(x) = x\sin x$ as a Fourier series ~~thus~~ in the interval $-\pi \le x \le \pi$ Hence deduces that

$$\frac{1}{1\times 3} - \frac{1}{3\times 5} + \frac{1}{5\times 7} - \frac{1}{7\times 9} + \cdots = \frac{\pi - 2}{4}$$

Here, $f(-x) = -x\sin(-x) = x\sin x = f(x)$, Thus $f(x)$ is even function.

So, $b_n = 0$. Then The fourier series ~~is~~

$$f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Here, $a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$

$$= \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\times\pi$$

$$= 2$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$

$$= \frac{1}{\pi}\int x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx - \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi} - \frac{1}{\pi}\left[-x\frac{\cos(n-1)x}{(n-1)} + \frac{\sin(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{(n+1)}\right] - \frac{1}{\pi}\left[-\frac{\pi\cos(n-1)\pi}{(n-1)}\right]$$

$$= \cancel{\frac{\cos(n+1)\pi}{(n+1)} + \cos}$$

$$= \frac{\cos(n-1)\pi}{(n-1)} - \frac{\cos(n+1)\pi}{(n+1)}$$

$$= \frac{1}{n-1} - \frac{1}{n+1} \text{ if } n \text{ is odd but } n \ne 1$$

$$= -\frac{1}{n-1} + \frac{1}{n+1} \text{ if } n \text{ is even}$$

$$= \begin{cases} \frac{2}{n^2-1} & \text{if } n \text{ is odd but } n \neq 1 \\ \frac{-2}{n^2-1} & \text{if } n \text{ is even} \end{cases}$$

If $n = 1$ then,

$$a_1 = \frac{2}{\pi} \int_0^\pi x \sin x \cos x \, dx$$

$$= \frac{1}{\pi} \int_0^\pi x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[-\frac{x \cos 2x}{2} - \int -\frac{\cos(2x)}{2} dx \right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{x \cos(2x)}{2} + \frac{1}{4} \sin 2x \right]_0^\pi$$

$$= \frac{1}{\pi} \times \left(-\frac{\pi}{2}\right) = -\frac{1}{2}$$

Therefore, $x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \cdots\right]$

Taking $x = \frac{\pi}{2}$

$$\frac{\pi}{2} - 1 = -2\left[\frac{-1}{3} + \frac{1}{4^2-1} \cdots\right]$$

$$\frac{\pi - 2}{4} = \frac{1}{3} - \frac{1}{15} - \cdots$$

$$= \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5}$$

- Find the Fourier series to represent e^{ax} in the interval $-\pi \leq x \leq \pi$

Here, the Fourier series is -

$$f(x) = e^{ax} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} dx$

$$= \frac{1}{a\pi}\left[e^{ax}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{a\pi}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\{a \cos nx + n \sin nx\}\right]_{-\pi}^{\pi}$$

$\left[\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{(a^2+b^2)}[a \cos bx - b \sin bx]\right.$

$$= \frac{1}{\pi}\left[\frac{n e^{a\pi}\sin(n\pi) + a e^{a\pi}\cos(n\pi)}{a^2+n^2} + \frac{n\sin(n\pi) - a\cos(n\pi)}{n^2 e^{a\pi} + a^2 e^{a\pi}}\right]$$

$= \frac{3\pi^2}{12}$ 4

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi}\left((ae^{2\pi a} + a)\sin(\pi n) + (1 - e^{2\pi a})n\cos(n\pi)\right)}{\pi(n^2+a^2)}$$

3) Obtain the Fourier series of e^{-x} in the interval $0 < x < 2\pi$. ⇒ [Hint - Find a_0, a_n, b_n]

4) Obtain the Fourier series of x^2 in $-\pi < x < \pi$. Hence Show that

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{6}$ [Put $x=\pi$]

$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$ [Put $x=0$]

$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$ [add above two]

Hint, Here $f(x) = x^2$ is even so, $b_n = 0$. Then we need to find a_0, a_n.

5) Find the Fourier series of $x - x^2$ from $x = -\pi$ to $x = \pi$ Hence deduce -

$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots = \frac{\pi^2}{12}$

Here, The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$= -\frac{2\pi^2}{3}$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - \int(1-2x)\frac{\sin nx}{n}\,dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - \left[-\frac{1}{n}\int(2x-1)\cos nx\right]dx\right]$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - \left[-\frac{1}{n}\left[\frac{(2x-1)\sin nx}{n} - \int\frac{2\sin nx}{n}\,dx\right]\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - \left[-\frac{1}{n}\left[\frac{(2x-1)\sin nx}{n} - \frac{2\cos nx}{n^2}\right]\right]\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx = \frac{(-2)(-1)^n}{n}$$

For deduction $x=0$

c) Obtain the Fourier series of $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x < 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

Hence deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$

For, $-\pi < x \le 0$, $f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi}$

Now, $0 \le -x \le \pi$

Again, $0 \le x \le \pi$ then $-\pi \le -x \le 0$, $f(-x) = 1 + \frac{2(-x)}{\pi}$
$= 1 - \frac{2x}{\pi}$

$\therefore$ $f(x)$ is even function

so, $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 1\,dx - \frac{4}{\pi^2}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[x\right]_0^{\pi} - \frac{4}{\pi^2}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= 2-2 = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= -\frac{2}{\pi^2}\int_0^{\pi}(2x-\pi)\cos nx\,dx$$

$$= -\frac{2}{\pi^2}\left[\frac{(2x-\pi)\sin nx}{n} - \int 2\frac{\sin nx}{n}dx\right]_0^{\pi}$$

$$= -\frac{2}{\pi^2}\left[\frac{(2x-\pi)\sin nx}{n} - \frac{2}{n^2}\int \sin(u)\,du\right]_0^{\pi} \quad \left[\because u = nx \Rightarrow \frac{du}{dx} = n \Rightarrow dx = \frac{1}{n}du\right]$$

$$= -\frac{2}{\pi^2}\left[\frac{(2x-\pi)\sin nx}{n} + \frac{2\cos nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{2}{\pi^2}\left[\frac{\pi\sin n\pi}{n} + \frac{2\cos n\pi}{n^2} - \frac{2}{n^2}\right]$$

$$= -\frac{2}{\pi^2}\left[\frac{\pi n\sin n\pi + 2\cos n\pi - 2}{n^2}\right]$$

$$= \frac{-2n\pi\sin n\pi - 4\cos(n\pi) + 4}{n^2\pi^2} = -\frac{4(-1)^n}{\pi^2 n^2} + \frac{4}{\pi^2 n^2}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos nx \quad [\text{Put } x=0]$$

$$= \frac{4}{\pi^2}\cdot 2 + \frac{4}{9\cdot\pi^2}\cdot 2$$

$$= \frac{8}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

7) Express $f(x) = |x|$, $-\pi < x < \pi$, as Fourier series and Hence show that,

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$$

Here, $f(-x) = |-x| = x = |x| = f(x)$

Hence, $f(x)$ is even.

So, $b_n = 0$

$$\therefore a_0 = \frac{2}{\pi}\int_0^\pi |x|\,dx \quad \left[\text{because in } 0 \le x \le \pi \text{ Here } x \text{ is even}\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi$$

$$= \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi |x| \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} - \frac{1}{n^2}\int \sin(u)\,du\right]_0^\pi \quad \left[\because u = nx \Rightarrow dx = \frac{1}{n}du\right]$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\pi \sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & n \text{ is even} \\ \dfrac{-4}{n^2\pi^2}, & n \text{ is odd} \end{cases}$$

$$\therefore |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \cdots\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2}$$

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8) Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$

Let, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(2\pi^2 - \frac{4\pi^2}{2}\right)$$

$$= -\frac{1}{2\pi}\left(2\pi^2 - \frac{4\pi^2}{2}\right)$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\pi\frac{\sin 2n\pi}{n} - \frac{\cos 2n\pi}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right] = 0$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[-(\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \times \frac{2\pi}{n} = \frac{1}{n}$$

Then fourier series is →

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

9) Obtain the Fourier Series of $f(x) = \frac{1}{4}(\pi - x)^2, 0 < x < 2\pi$

Hence obtain,

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\cdots = \frac{\pi^2}{6}$ [Put $x=0$]

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$ [Put $x=\pi$]

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$ [add above two]

10) Show that for $-\pi < x < \pi$

$$\sin ax = \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} \cdots\cdots\right)$$

[Hint :→ $f(x) = \sin ax$]

11) Obtain Fourier expansion for $\sqrt{1-\cos x}$ in the interval $-\pi < x < \pi$

HW

Here, we have $= \sqrt{1-\cos x} = \sqrt{2\sin^2\frac{x}{2}}$

$$f(-x) = \sqrt{2\sin^2\left(\frac{-x}{2}\right)}$$

$$= +\sqrt{2}\sin\frac{x}{2}$$

$f(x)$ is ~~odd~~ even function. Therefore $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin\frac{x}{2}\,dx$$

FOURIER SERIES FOR Discontinuous Function

The process in this case is same as before but at the point of discontinuity we need to find the average of left hand and right hand limit.

let, $x = a$ be the point of discontinuity

Then, $f(a) = \frac{f(a+0) + f(a-0)}{2}$

12) Obtain the fourier series for the function.

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$$

and hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} = \frac{\pi^2}{8}$

We know that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} (-x)\,dx\right]$$

$$= -\pi$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} x \cos nx\,dx + \int_{0}^{\pi} (-x)\cos nx\,dx\right]$$

$$= \frac{2}{n^2\pi}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = 0$$

$$\therefore f(x) = \frac{-\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} \cdots\right)$$

at the point of discontinuity,

$$f(0) = \frac{f(0+) + f(0-)}{2} = \frac{0+0}{2} = 0$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots$$

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19) Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$

Let, $f(x) = e^{-x} = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

Here,

$$a_0 = \frac{1}{l} \int_{-l}^{l} e^{-x} dx = \frac{1}{l} \left[-e^{-x} \right]_{-l}^{l} = \frac{1}{l} \left[e^{l} - e^{-l} \right] = \frac{2 \sinh l}{l}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} e^{-x} \cos \frac{n\pi x}{l} dx = \frac{1}{l} \left[\frac{e^{-x}}{1 + \frac{n^2\pi^2}{l^2}} \left(\frac{n\pi}{l} \sin \frac{n\pi x}{l} - \cos \frac{n\pi x}{l} \right) \right]_{-l}^{l}$$

20) Find the Fourier Series expansion of the function

$f(x) = x - x^2 \, , \, -1 < x < 1$

Let, $f(x) = x - x^2 = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

Here,
$l = 1$ then, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

Now,

$$a_0 = \int_{-1}^{1} (x - x^2)\, dx = \int_{-1}^{1} x\, dx - \int_{-1}^{1} x^2\, dx$$

$$= -\frac{2}{3}$$

$$a_n = \int_{-1}^{1} (x - x^2) \cos(n\pi x)\, dx$$

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$$+\sum_{n=1} \left(\frac{\quad}{n\pi} \right)$$

21) Given $f(x) = \begin{cases} 0, & 0<x<c \\ 1, & c<x<2c \end{cases}$ expand $f(x)$ in a Fourier series of period $2c$.

Let, $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right)$

So,

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx = \frac{1}{c}\int_c^{2c} dx = \frac{1}{c}(2c-c) = 1$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx = \frac{1}{c}\int_c^{2c}\cos\left(\frac{n\pi x}{c}\right)dx$$

$$= \frac{1}{c}\left[\frac{\sin\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)}\right]_c^{2c}$$

$$= \frac{1}{n\pi}\times 0 = 0$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

$$= \frac{1}{c}\int_c^{2c}\sin\left(\frac{n\pi x}{c}\right)dx$$

$$= \frac{1}{c}\left[\frac{-\cos\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)}\right]_c^{2c} = -\frac{1}{n\pi}\left[\cos(2n\pi) - \cos(n\pi)\right]$$

$$= -\frac{1}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \frac{-2}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

22) Find a Fourier series for the function

$$f(x) = \begin{cases} 0 & , \text{when } -2 < x < -1 \\ k & , \text{when } -1 < x < 1 \\ 0 & , \text{when } 1 < x < 2 \end{cases}$$

The Fourier series expansion is →

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Here, $l = 2$. So,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

Now,

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} k\, dx = \frac{k}{2} [1 - (-1)]$$

$$= k$$

HALF RANGE SERIES :-

If we want to find to expansion of the function $f(x)$ in the range $(0, \pi)$ where, the Fourier series is of the period 2π, more generally, If the range is $(0, l)$ for the period $2l$ then there are two distinct half range series. They are Half range cosine series,

$$f(x) = \frac{a_0}{2l} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where, $a_0 = \frac{2}{l} \int_0^l f(x)\,dx$

$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

and half range sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where, $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

26) If $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x; & \pi/2 < x < \pi \end{cases}$ then show that

i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$

ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$

i) Here, $l = \pi$ and ~~here~~ we use half range sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

where, $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin(nx)\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\,dx\right]$$

H.T 24) Find the half range Sine Series of $f(x) = x^2$ in the interval $0<x<3$.

H.T 25) Find the half range Sine Series of $f(x) = x$ in the interval $0<x<2$.

* 26) Find the half range Cosine Series of $f(x) = x$ in the interval $0<x<2$.

H.T 27) Obtain the half range Sine Series of $f(x) = e^x$ in $0<x<1$.

* 28) Find the half range cosine Series for the function $f(x) = (x-1)^2$ in the interval $0<x<1$. Also Show that,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6} \quad \& \quad \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}.$$

$\therefore f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\left(\frac{\pi x}{2}\right)}{1^2} + \frac{\cos\left(\frac{3\pi x}{2}\right)}{3^2} + \cdots\cdots\right]$

* 26) Find the half range cosine series of $f(x) = x$ in the interval $0 < x < 2$.

$\Rightarrow$ Here we have, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$ (as $l = 2$)

Now, $a_0 = \frac{2}{2}\int_0^2 x\,dx = \left.\frac{x^2}{2}\right]_0^2 = 2.$

$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx = \left[x\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} + \frac{\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$

$= \frac{4}{n\pi}\sin n\pi + \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}.$

$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} -\frac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

28) Here we have Fourier Cosine Series in $0<x<1$. Then we have,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{1} \quad (\text{as } l=1).$$

Here, $a_0 = \frac{2}{1}\int_0^1 (x-1)^2 dx = 2\left[\frac{x^3}{3} - \cancel{2}\frac{x^2}{\cancel{2}} + x\right]_0^1$

$= 2\left(\frac{1}{3} - 1 + 1\right) = \frac{2}{3}$

$a_n = 2\int_0^1 (x-1)^2 \cos n\pi x \, dx$

$= 2\left[(x-1)^2 \frac{\sin n\pi x}{n\pi} - 2\int (x-1)\frac{\sin n\pi x}{n\pi} dx\right]_0^1$

$= -4\left[(x-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + \frac{\sin n\pi x}{n^3\pi^3}\right]_0^1$

$= -4\left[\frac{-1}{n^2\pi^2}\right] = \frac{4}{n^2\pi^2}$

$\therefore f(x) = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \cdots\right]$

if we put $x=0$ then

$1 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right]$

$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots$

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Put $x=1$

$$0 = \frac{1}{3} + \frac{4}{\pi^2}\left[-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}\cdots\right]$$

$$\frac{\pi^2}{12} = \left[1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\cdots\cdots\right]$$

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