EM-II $\qquad$ 03/05/2024

- Green's Theorem for plane

If S be a closed region bounded by a simple closed curve C and M, N are two continuous functions of x, y which are having continuous first order partial derivatives then

$$\int_C M dx + N dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

where C traversed in counter clockwise sense.

Q. Verify Green's Theorem for

$\int_C \{(x^2+xy)\,dx + x\,dy\}$ where C is the curve enclosing the region bounded by the parabola $y=x^2$ and the straight line $y=x$.

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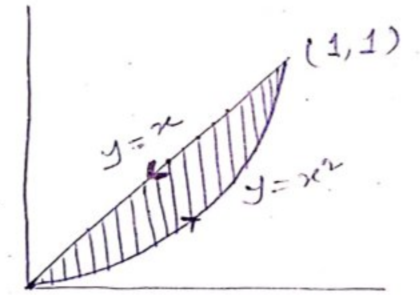

$\int_{C_1} \{(x^2+xy)dx + x\,dy\}$ where $C_1 : y=x^2$

$= \int_{C_1} \{(x^2+x.x^2)dx + x.\,2x\,dx\}$

$= \int_{x=0}^{1} (x^2+x^3)\,dx + \int_{x=0}^{1} 2x^2\,dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1 + 2\left[\frac{x^3}{3}\right]_0^1$

$= \frac{1}{3} + \frac{1}{4} + \frac{2}{3}$

$= \frac{4+3+8}{12}$

$= \frac{15}{12}$

$= \frac{5}{4}$.

Again, $\int_{c_2} [(x^2+xy)dx + xdy]$ where $c_2 : y = x$

$$= \int_{x=1}^{0} [(x^2+x^2)dx + xdx]$$

$$= 2\int_1^0 x^2 dx + \int_1^0 x dx$$

$$= 2\left[\frac{x^3}{3}\right]_1^0 + \left[\frac{x^2}{2}\right]_1^0$$

$$= 2\left[0 - \frac{1}{3}\right] + \left[0 - \frac{1}{2}\right] = -\frac{2}{3} - \frac{1}{2} = \frac{-4-3}{6} = -\frac{7}{6}.$$

Now, $c = c_1 + c_2$ is a simple closed curve enclosing the region S.

$$\therefore \int_c [(x^2+xy)dx + xdy] = \frac{5}{4} - \frac{7}{6} = \frac{1}{12}.$$

Now, here $M = (x^2+xy)$ and $N = x$

$\therefore$ By Green's theorem, we have

$$\int_c (Mdx + Ndy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_S (1-x)\, dxdy$$

$$= \int_{x=0}^{1} \left(\int_{y=x^2}^{x} (1-x)dy\right) dx$$

$$= \int_{x=0}^{1} \left[(1-x)\int_{y=x^2}^{x} dy\right] dx$$

$$= \int_0^1 \left[(1-x)(x-x^2)\right] dx$$

$$= \int_0^1 \left[x - x^2 - x^2 + x^3\right] dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^3}{3}\right]_0^1 + \left[x^4/4\right]_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{6-8+3}{12}.$$

$$= \frac{1}{12}.$$

Hence, Green's Theorem is verified.

Q. Evaluate using Green's Theorem $\int_C (y\,dx + 2x\,dy)$ where C is the boundary of the square $0 \leq x \leq 1,\ 0 \leq y \leq 1$ traversed in counter clockwise sense.

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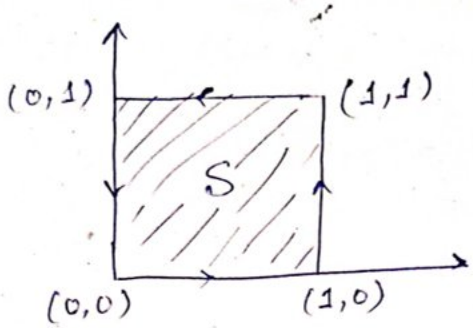

Here, $M = y$ and $N = 2x$

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$\Rightarrow \int_C y\,dx + 2x\,dy = \iint_S (2 - 1)\,dx\,dy$$

$$= \iint_S dx\,dy = \int_{x=0}^{1} \int_{y=0}^{1} dx\,dy$$

$$= \int_0^1 (y)_0^1\,dx = (+1) \int_0^1 dx$$

$$= (+1)\,(x)_0^1 = +1$$

Q. Using Green's Theorem, evaluate $\int_C (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy$ where c is the circle $x^2 + y^2 = 9$ traversed counterclockwise. [0]

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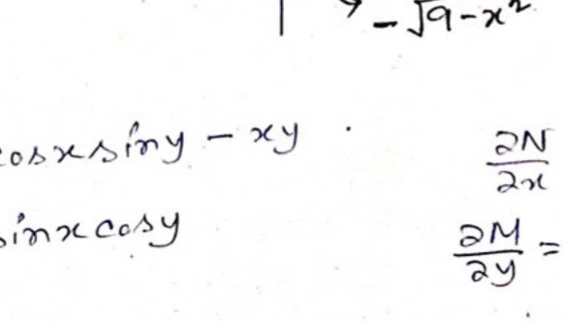

$M = \cos x \sin y - xy$ $\qquad \dfrac{\partial N}{\partial x} = \cos y \cos x$

$N = \sin x \cos y$ $\qquad \dfrac{\partial M}{\partial y} = \cos x \cos y - x$

$$\int_C (\cos x \sin y - xy)\,dx + \sin x \cos y\, dy$$

$$= \iint_S (\cos x \cos y - \cos x \cos y + x)\, dx\, dy$$

$$= \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x)\, dx\, dy$$

$$= \int_{-3}^{3} \left[x \left(2\sqrt{9-x^2} \right) \right] dx$$

$$= \int_{-3}^{3} \left[2x\sqrt{9-x^2} \right] dx$$

$$\sqrt{9-x^2} = z$$

$$\Rightarrow \frac{1}{2}(9-x^2)^{-1/2}(-2x)\,dx = dz$$

$$\Rightarrow \frac{1}{\sqrt{9-x^2}} \cdot 2x\,dx = -2\,dz$$

$$\Rightarrow 2x\,dx = -2\sqrt{9-x^2}\,dz$$

$$= \int_{-3}^{3} \left[-2\sqrt{9-x^2} \cdot \sqrt{9-x^2} \right] dz$$

$$= \int_{-3}^{3} \left[-2z^2 \right] dz$$

$$= -\frac{2}{3}\left[z^3 \right]_{-3}^{3} = -\frac{2}{3}\left[27 - 27 \right] \neq 0,$$

$$x^2 = z$$

$$\Rightarrow 2x\,dx = dz$$

$$= \int_{-3}^{3} \left[\sqrt{9-z} \right] dz$$

$$= \left[(9-z)^{3/2} \times \frac{2}{3} \times \frac{1}{(-1)} \right]_{-3}^{3}$$

$$= -\frac{2}{3}\left[(9-x^2)^{3/2} \right]_{-3}^{3}$$

$$= -\frac{2}{3}\left[(9-9)^{3/2} - (9-9)^{3/2} \right]$$

$$= 0.$$