INEXACT DIFFERENCIAL EQUATION

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Form: $M\,dx + N\,dy = 0$

For exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Inexact DE: $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

Ex: $y\,dx - x\,dy = 0$ — (1)

Here, $M = y$ $N = -x$

now, $\dfrac{\partial M}{\partial y} = \dfrac{\partial y}{\partial y} = 1$

and, $\dfrac{\partial N}{\partial x} = \dfrac{\partial(-x)}{\partial x} = -1$.

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

$\therefore y\,dx - x\,dy = 0$ is Inexact.

~~Ex~~: now, multiplying $\frac{1}{y^2}$ to (1);

$$\frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow \frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0$$

now, $\dfrac{\partial M}{\partial y} = \dfrac{\partial(y^{-1})}{\partial y} = -\dfrac{1}{y^2}$

$\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(-\dfrac{x}{y^2}\right) = -\dfrac{1}{y^2}$

now, $\dfrac{y\,dx - x\,dy}{y^2} = 0$ is exact where $\dfrac{1}{y^2}$ is the 'Integrating Factor'.

Integrating factor : For the D.E of the form
$$M\,dx + N\,dy = 0$$
if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then equation is a inexact DE.

To make it exact, we multiply the equation by $f(x,y)$. Here 'f' is known as integrating factor.

Finding IF :

For an ~~integrating~~ inexact dE
$$M\,dx + N\,dy = 0$$

i) if. $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = h(x)$

then, $I.F = e^{\int h(x)\,dx}$

ii) if. $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$

then, $I.F = e^{\int -g(y)\,dy}$

Procedure for solving Inexact DE :

Step 1: Compare the given DE with std. form $M dx + N dy = 0$ and obtain the value of M & N

Step 2: Calculate the value of $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$ and get the value of $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$.

Step 3: Find the appropriate Integrating factor by dividing the difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ by M or N.

Then, multiply the IF with the DE and Hence obtain an exact DE, as $M'dx + N'dy = 0$, where $M' = M \cdot IF$ and $N' = N \cdot IF$

Step 4 Solve the exact DE $M'dx + N'dy = 0$ using earlier procedure.

Ex: $(4xy + 3y^2 - x)\,dx + x(x+2y)\,dy = 0$

Sol^n: Here, comparing given DE to $M dx + N dy = 0$

$M = 4xy + 3y^2 - x$

$N = x^2 + 2xy$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(4xy + 3y^2 - x) = 4x + 6y$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2xy) = 2x + 2y$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, given DE is inexact

$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x + 6y - 2x + 2y = 2x + 4y$

$= 2(x+2y)$

now, $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{\cancel{2/x}\ N}$

$= \cancel{\partial M} \quad \dfrac{2(x+2y)}{x(x+2y)} = \dfrac{2}{x}$

$\therefore \; IF = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$\therefore$ The exact DE will be :

$x^2(4xy + 3y^2 - x)\,dx + x^3(x+2y)\,dy = 0$

$M' = x^2(4xy + 3y^2 - x)$

$N' = x^3(x+2y)$

now, $\int M' dx = \int x^2(4xy + 3y^2 - x)\,dx$

$= \int (4x^3y + 3x^2y^2 - x^3)\,dx$

$= \dfrac{4yx^4}{4} + \dfrac{3y^2x^3}{3} - \dfrac{x^4}{4} = x^4y + \cancel{3}y^2x^3 - \dfrac{x^4}{4}$

and, $\int N' dx = \int (x^4 + 2x^3y)\,dy$

$= x^4y + \dfrac{2x^3y^2}{2} = x^4y + x^3y^2$

$\therefore$ Solution of DE is : $x^4y + x^3y^2 - \dfrac{x^4}{4} = c$

Integrating Factor

Question $y\,dx + (-x-y^2)\,dy = 0$

→ Here, comparing given DE to $M\,dx + N\,dy = 0$

$\therefore M = y$

$N = (-x-y^2)$

$\frac{\partial M}{\partial y} = \frac{\partial y}{\partial y} = 1$ $\qquad$ $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-x-y^2)$

$= -1$

$\therefore \frac{\partial M}{\partial y} \,!= \frac{\partial N}{\partial x}$, given DE is inexact

now, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1-(-1) = 2.$

also, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2}{y}$

$\therefore \text{IF} = e^{-\int(\frac{2}{y})dy} = e^{-2\ln y} = e^{\ln y^{-2}} = \frac{1}{y^2}$

now,

$M' = M \times \text{IF} = y \times \frac{1}{y^2} = \frac{1}{y}$

$N' = N \times \text{IF} = -(x+y^2) \times \frac{1}{y^2} = -\frac{x+y^2}{y^2}$

now, $M'dx + N'dy = 0$

or, $\frac{1}{y}dx - \frac{x+y^2}{y^2}dy = 0.$ —— (1)

now,

$$M' = \frac{1}{y} \qquad N' = \frac{-x-y^2}{y^2}$$

$$\int M'\,dx = \int \frac{1}{y}\,dx = \frac{x}{y}$$

$$\int N'\,dy = \int\left(-\frac{x}{y^2} - \frac{y^2}{y^2}\right)dy = \frac{x}{y} - y$$

$\therefore$ Solution of (1) is $\frac{x}{y} - y = c$.

Question $(3y - e^x)\,dx + x\,dy = 0$

ans

$$M = 3y - e^x$$
$$N = x$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3y - e^x) = 3.$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x) = 1.$$

$\because \frac{\partial M}{\partial y} \mathrel{!=} \frac{\partial N}{\partial x}$ given DE is inexact.

now, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3 - 1 = 2$

also,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2}{x}$$

$$\therefore \text{IF} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2.$$

$M' = 3x^2y - x^2e^x$

$N' = x^3$

$\int M'dx = \int(3x^2y - x^2e^x)dx = x^3y - \left(x^2e^x - 2(e^x x - e^x)\right)$

$\int N'dy = \int x^3 dy = x^3y$

$\therefore$ General solution $= x^3y - x^2e^x - 2(e^x x - e^x)$

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$\int x^2e^x dx \longrightarrow$ integration by parts

$\Rightarrow x^2e^x - 2(e^x x - e^x) \rightarrow$ while doing this $\int x \cdot e^x dx = (x-1)e^x$

$\hookrightarrow (x^2 - 2x + 2)e^x$

[also integration by parts]