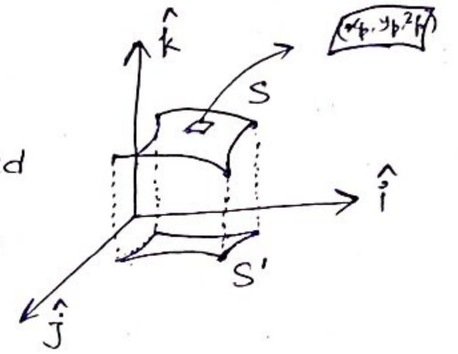
• EM - II 19.04.2024

Normal Surface Integral

Let S be a surface on the space.
Let S' be the orthogonal projection of S on xy-plane.
Let, $f(x,y,z)$ be a function defined on S.
Let, the surface S be discretized in n elementary surfaces ΔS_p, $p = 1, 2, \ldots\ldots, n$

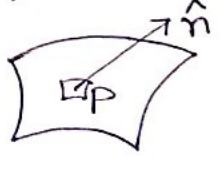


Let, (x_p, y_p, z_p) be a point on ΔS_p.

Then the surface integral of $f(x,y,z)$ over the surface S, is defined as $\lim_{\Delta S_p \to 0} \sum_{p=1}^{n} f(x_p, y_p, z_p)\, \Delta S_p$

$$= \iint_S f(x,y,z)\, ds$$

* Now, let $\vec{F}(x,y,z)$ be a vector point function defined on S.



Let, $\hat{n}$ be the outward normal at a point P on the surface.
Let, ΔS_p be an elementary surface area at P with direction $\hat{n}$.

Then, $\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n}\, ds$ —— (i)

is called flux of $\vec{F}$ along S or normal surface integral of $\vec{F}$ over S or surface integral over S.

Corrolary: $d\vec{s} = ds_1 \hat{i} + ds_2 \hat{j} + ds_3 \hat{k}$

$= dydz\,\hat{i} + dxdz\,\hat{j} + dxdy\,\hat{k}$

(i) implies,

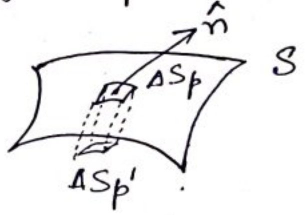
$$\iint_S \vec{F}\cdot d\vec{s}$$

$$= \iint_S (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (dydz\,\hat{i} + dxdz\,\hat{j} + dxdy\,\hat{k})$$

$$= \boxed{\iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)} \quad ***$$

* Let, $\Delta S_p'$ be the projection of ΔS_p on the xy-plane.

The $\Delta S_p' = \Delta x \Delta y$



Again, $\Delta S_p' = |\hat{n}\cdot\hat{k}|\,\Delta S$

Taking $\Delta S \rightarrow 0$

$\therefore \Delta S_p', \Delta x, \Delta y \rightarrow 0$.

$ds' = dxdy = |\hat{n}\cdot\hat{k}|\,ds$

$$\Rightarrow ds = \frac{dxdy}{|\hat{n}\cdot\hat{k}|}$$

Then $\iint_S \vec{F}\cdot\hat{n}\,ds$ implies $\iint_S \vec{F}\cdot\hat{n}\,\frac{dxdy}{|\hat{n}\cdot\hat{k}|}$

$$\therefore \boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \vec{F}\cdot\hat{n}\cdot\frac{dxdy}{|\hat{n}\cdot\hat{k}|}}$$

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$\hat{i}$ = direction along yz plane

$\hat{j}$ = direction along xz plane

$\hat{k}$ = direction along xy plane.

* Eq^n for 3D plane: $ax + by + cz = d$

$$\Rightarrow \frac{x}{(d/a)} + \frac{y}{(d/b)} + \frac{z}{(d/c)} = 1$$

$(d/a, 0, 0)$, $(0, d/b, 0)$, $(0, 0, d/c)$.

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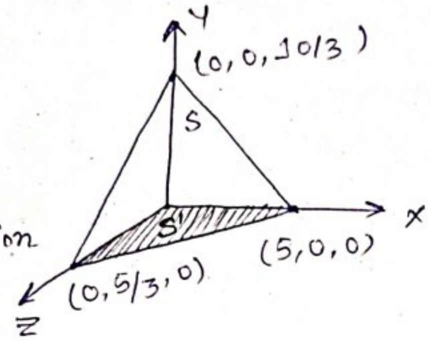
Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 6z\hat{i} - 4\hat{j} + y\hat{k}$ and S is the part of the plane $2x+6y+3z=10$ on the first octant.

$\longrightarrow$

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_{S'} \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

where S' is projection of S on the xy-plane.



$2x+6y+3z=10$

$\Rightarrow z = \cancel{\ldots}$

$\Rightarrow z = \frac{1}{3}(10-2x-6y)$

$\hat{n}$ = outward normal vector to the surface S.

$$= \frac{\vec{\nabla}\cdot(2x+6y+3z)}{|\vec{\nabla}\cdot(2x+6y+3z)|} = \frac{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+6y+3z)}{\left|\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+6y+3z)\right|}$$

$$= \frac{2\hat{i}+6\hat{j}+3\hat{k}}{|2\hat{i}+6\hat{j}+3\hat{k}|} = \frac{2}{7}\hat{i} + \frac{6}{7}\hat{j} + \frac{3}{7}\hat{k}$$

Now, $\vec{F}\cdot\hat{n} = (6z\hat{i} - 4\hat{j} + y\hat{k})\cdot\left(\frac{2}{7}\hat{i} + \frac{6}{7}\hat{j} + \frac{3}{7}\hat{k}\right)$

$$= \frac{12}{7}z - \frac{24}{7} + \frac{3y}{7}$$

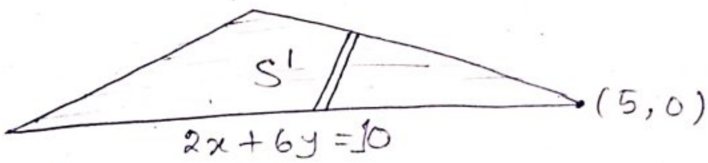
$$= \cancel{\frac{12}{7}\left(\frac{2}{3}(\ldots)\right) - \frac{24}{7} + \frac{3y}{7}}$$

$$= \frac{1}{7}(16 - 8x - 21y).$$

$|\hat{n}\cdot\hat{k}|$

$= \left|\left(\frac{2}{7}\hat{i} + \frac{6}{7}\hat{j} + \frac{3}{7}\hat{k}\right)\cdot\hat{k}\right|$

$= \frac{3}{7}$



$$\int_{x=0}^{5}\int_{y=0}^{1/6(10-2x)} \frac{1}{7}(16-8x-21y)\,\frac{dx\,dy}{|3/7|}$$

$$= \frac{1}{7}\times\frac{7}{3}\int_{x=0}^{5}\left[16y - 8xy - \frac{21}{2}y^2\right]_0^{1/6(10-2x)} dx$$

$$= \frac{1}{3}\int_{x=0}^{5}\left[\frac{16}{6}(10-2x) - \frac{8}{6}x(10-2x) - \frac{21}{12}(10-2x)^2\right]dx$$

$$= -\frac{1}{18}\int_{x=0}^{5}(15+42x-9x^2)\,dx = -\frac{25}{2}.$$

Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the part of the plane $2x+y+2z=6$ on the first octant. [81]

$$2z = 6 - 2x - y$$
$$\Rightarrow z = 3 - x - \frac{y}{2}.$$

$\rightarrow$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S'} \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\hat{n} = \frac{\vec{\nabla}\cdot(2x+y+2z)}{|\vec{\nabla}\cdot(2x+y+2z)|}$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}.$$

$$\vec{F}\cdot\hat{n} = \{(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\}\cdot\{\tfrac{2}{3}\hat{i} + \tfrac{1}{3}\hat{j} + \tfrac{2}{3}\hat{k}\}$$

$$= \frac{2x}{3} + \frac{2y^2}{3} - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2x}{3} + \frac{2y^2}{3} - \frac{2}{3}x + \frac{4}{3}y\left(3 - x - \frac{y}{2}\right)$$

$$= \cancel{\frac{2x}{3}} + \frac{2y^2}{3} - \cancel{\frac{2}{3}x} + \underline{4y} - \frac{4}{3}xy - \underline{\frac{2}{3}y}$$

$$= \frac{10}{3}y - \frac{4}{3}xy + \frac{2}{3}y^2.$$

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Q. $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \, d\vec{s}$ where the surface s is the sphere $x^2+y^2+z^2=1$ in the first octant.

$\rightarrow$

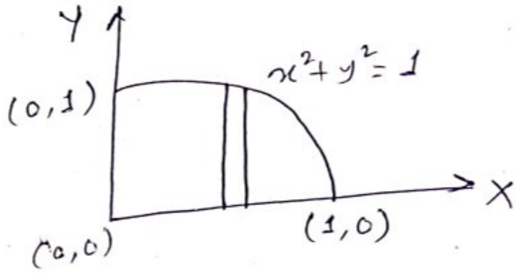
$$\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \, d\vec{s}$$

$$= \iint_{S'} (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot \hat{n} \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \text{———} (i)$$

$$\hat{n} = \frac{\vec{\nabla}(x^2+y^2+z^2-1)}{|\vec{\nabla}(x^2+y^2+z^2-1)|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{|\sqrt{4(x^2+y^2+z^2)}}$$

$$= \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4\cdot 1}} \quad [\because x^2+y^2+z^2 = 1 \text{ on } s]$$

$$= x\hat{i}+y\hat{j}+z\hat{k}$$



From (i),

$$I = \iint_{S'} 3xyz \cdot \frac{dx\,dy}{z}$$

$$= 3\iint_{S'} xy \, dx\,dy$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy \, dy\,dx$$

$$= 3\int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{1-x^2}} dx = \frac{3}{2}\int_{x=0}^{1} x(1-x^2)\,dx$$

$$= \frac{3}{2}\left[\int_{x=0}^{1} x\,dx - \int_{x=0}^{1} x^3\,dx\right]$$

$$= \frac{3}{2}\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1\right]$$

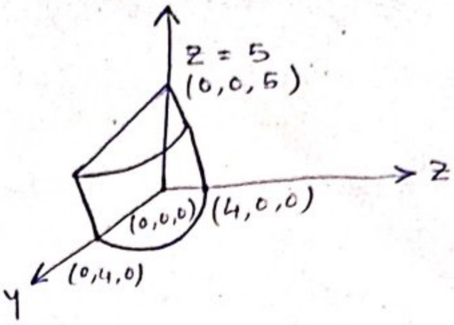
$$= \frac{3}{2}\left[\frac{1}{2}\times 1 - \frac{1}{4}\times 1\right]$$

$$= \frac{3}{4} - \frac{3}{8} = \frac{6-3}{8} = \frac{3}{8}.$$

Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the curved surface of the cylinder $x^2+y^2=16$ in the first octant between $z=0$ to $z=5$.

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_{S'} \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$



$$\hat{n} = \frac{\vec{\nabla}(x^2+y^2-16)}{|\vec{\nabla}(x^2+y^2-16)|}$$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4\times 16}} = \frac{x}{4}\hat{i}+\frac{y}{4}\hat{j}$$

$$\therefore\ I = \iint_{S'} \vec{F}\cdot\hat{n} = \iint_{S'} (z\hat{i}+x\hat{j}-3y^2z\hat{k})\cdot\left(\frac{x}{4}\hat{i}+\frac{y}{4}\hat{j}\right)\frac{dy\,dz}{x/4}$$

$$= \iint_{S'} \left(\frac{xz}{4}+\frac{xy}{4}\right)\cdot\frac{dy\,dz}{x/4}$$

$$= \iint_{S'} (z+y)\,dy\,dz$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} (z+y)\,dy\,dz$$

$$= \int_{y=0}^{4} \left[\left(\frac{z^2}{2}\right)_0^5 + y\cdot(z)_0^5\right] dy$$

$$= \int_{y=0}^{4} \left[\frac{25}{2} + y\cdot 5\right] dy$$

$$= \frac{25}{2}(y)_0^4 + 5\cdot\left(\frac{y^2}{2}\right)_0^4$$

$$= \frac{25}{\cancel{2}}\times\overset{2}{\cancel{4}} + \frac{5}{\cancel{2}}\times\overset{8}{\cancel{16}} = 50+40 = 90.$$

* Volume Integral

Let $f(x,y,z)$ be a scalar point function defined on a volume V.

Let us sub-divide the volume V into n small elementary volumes ΔV_p $(p=1,2,\dots,n)$

Consider the limit if exist,

$$\lim_{n\to\infty}\sum_{p=1}^{n} f(x_p,y_p,z_p)\,\Delta V_p,\ \text{where } (x_p,y_p,z_p)\in\Delta V_p$$

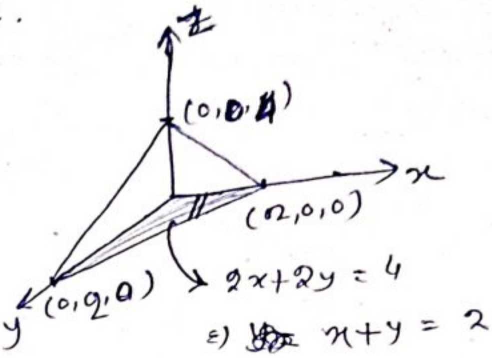
then
$$\iiint_V f(x,y,z)\,dV = \lim_{n\to\infty}\sum_{p=1}^{n} f(x_p,y_p,z_p)\,\Delta V_p.$$

Note: Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

$$\therefore \iiint_V \vec{F}\cdot dv = \iiint_V F_1\,dv\,\hat{i} + \iiint_V F_2\,dv\,\hat{j} + \iiint_V F_3\,dv\,\hat{k}$$

Q. Find $\iiint_V \vec{\nabla}\vec{F}\,dv$, over the region bounded by the co-ordinate planes and the plane $2x+2y+z=4$, where $\vec{F} = (2x^2 - z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$.

$$\frac{x}{2} + \frac{y}{2} + \frac{z}{4} = 1$$



$2x+2y=4$

$\Rightarrow x+y = 2$

$\Rightarrow y = 2-x$

$\vec{\nabla}\cdot\vec{F} = 2x$

$$\iiint_V 2x\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left[2x(z)_0^{4-2x-2y}\right]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left[2x(4-2x-2y)\right]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left[8x - 4x^2 - 4xy\right]dy\,dx$$

$$= \int_{x=0}^{2}\left[8x\cdot(y)_0^{2-x} - 4x^2(y)_0^{2-x} - 4x\left[y^2/2\right]_0^{2-x}\right]$$

$$= \int_{x=0}^{2}\left[8x(2-x) - 4x^2(2-x) - 2x(2-x)^2\right]$$

$$= \int_{x=0}^{2}\left[16x - 8x^2 - 8x^2 + 4x^3 - 2x(4-4x+x^2)\right]$$

$$= \int_{x=0}^{2}\left[16x - 16x^2 + 4x^3 - 8x + 8x^2 - 2x^2\right]$$

$$= \int_{x=0}^{2}\left[8x - 10x^2 + 4x^3\right]$$

$$= 8\cdot\left[x^2/2\right]_0^2 - 10\left[x^3/3\right]_0^2 + 4\left[\frac{x^4}{4}\right]_0^2$$

$$= 4\times4 - 10/3\times8 + 16 = 32 - \frac{80}{3} = \frac{96-80}{3}$$

$$= \frac{16}{3}.$$

$$\int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x \, dz\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \left[2x(4-2x-2y) \right] dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \left[8x - 4x^2 - 4xy \right] dy\, dx$$

$$= \int_{x=0}^{2} \cancel{\int_{y=0}^{2-x}} \left[8x(y)_0^{2-x} - 4x^2(y)_0^{2-x} - 4x\left(y^2/2\right)_0^{2-x} \right] dx$$

$$= \int_{x=0}^{2} \left[8x(2-x) - 4x^2(2-x) - 2x(2-x)^2 \right] dx$$

$$= \int_{x=0}^{2} \left[16x - 8x^2 - 8x^2 + 4x^3 - 2x(4-4x+x^2) \right] dx$$

$$= \int_{x=0}^{2} \left[\underline{16x} - 8\cancel{x^2} - 8x^2 + \overset{4}{\cancel{8}}x^3 - \underline{8x} + 8\cancel{x^2} - 2x^3 \right] dx$$

$$= \int_{x=0}^{2} \left[16x - 8x - 8x^2 + 2x^3 \right] dx$$

$$= \int_{x=0}^{2} \left[8x - 8x^2 + 2x^3 \right] dx$$

$$= 2\int_{x=0}^{2} \left[4x - 4x^2 + x^3 \right] dx$$

$$= 2\left[4\left(x^2/2\right)_0^2 - 4\left(x^3/3\right)_0^2 + \left(x^4/4\right)_0^2 \right]$$

$$= 2\left[4/2 \times 4 - \frac{4}{3} \times 8 + \frac{1}{4} \times 16 \right]$$

$$= 2\left[8 + 4 - \frac{32}{3} \right]$$

$$= 2\left[12 - \frac{32}{3} \right]$$

$$= 2(4/3)$$

$$= \frac{8}{3}$$