20/2/23

Non-homogeneous first order ODE :

A first order non-homogeneous ODE of the form:

$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ is solvable by using variable.

To apply the variable seperation method, we need to take some special transformations.

Depending on the co-efficients of x & y, we have 2 cases.

Case - I : if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then we take the transformation

$X = x + h \quad ; \quad Y = y + k$

such that,

$c_1 - a_1h - b_1k = 0$ — (i)

and, $c_2 - a_2h - b_2k = 0$ — (ii)

from,

(i) & (ii); find values of h & k.

Also, transform the equation in terms of X & Y which is a homogeneous equation

Then, we can apply the process of solution for homogeneous equation.

Ex: $\frac{dy}{dx} = -\frac{x - 2y + 1}{4x - 3y - 6} = \frac{-x + 2y - 1}{4x - 3y - 6}$ — (1)

Sol. Let us take, $X = x + h$

$Y = y + k$

Then, to make the equation a homogenous equation, we must have,

$$-1-(-1)h-(2)k=0 \Rightarrow h-2k=1$$
$$-6-(4)h-(-3)k=0 \Rightarrow -4h+3k=6$$

now, $h=-3$ and $k=-2$

$\therefore X = x-3 \Rightarrow x = X+3$ also, $dx = dX$ and $dy = dY$

$Y = y-2 \Rightarrow y = Y+2$

$\therefore$ (1) $\Rightarrow \dfrac{dY}{dX} = \dfrac{-X+2Y}{4X-3Y}$, now this is a homogeneous equation

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Let, $Y = vX$ $\therefore \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\cancel{\therefore v + X\dfrac{dv}{dX}}$ $\dfrac{dY}{dX} = \dfrac{-X+2Y}{4X-3Y}$

or, $dY(4X-3Y) = (-X+2Y)dX$

or, $4XdY - 3YdY = -XdX + 2YdX$

or, $4XdY + XdX = 2YdX + 3YdY$

or, $X(4dY + dX) = Y(2dX + 3dY)$

let, $v = y/x \Rightarrow y = vx \quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

now, $\frac{dy}{dx} = \frac{-x+2y}{4x-3y}$

$\therefore v + x\frac{dv}{dx} = \frac{-1+2v}{4-3v}$

or, $x\frac{dv}{dx} = \frac{-1+2v}{4-3v} - v = \frac{3v^2-2v-1}{4-3v}$

or, $\frac{4-3v}{3v^2-2v-1}\,dv = \frac{dx}{x}$

or, $\int \frac{4-3v}{(3v+1)(v-1)}\,dv = \int \frac{dx}{x}$

or, $\int \left(\frac{-15}{4(3v+1)} + \frac{1}{4(v-1)}\right) dv = \ln x + c$

or, $-\frac{5}{4}\ln|3v+1| + \frac{1}{4}\ln|v-1| = \ln x + c$

or, $-\frac{5}{4}\ln\left[3(y/x)+1\right] + \frac{1}{4}\ln\left[\frac{y}{x}-1\right] = \ln x + c$

or, $-\frac{5}{4}\ln\left(\frac{3y+x}{x}\right) + \frac{1}{4}\ln\left(\frac{y-x}{x}\right) = \ln x + c$

or, $-\frac{5}{4}\ln\left(\frac{3y-6+x-3}{x-3}\right) + \frac{1}{4}\ln\left(\frac{y-2-x+3}{x-3}\right) = \ln(x-3) + c$

or, $-5/4\ln\left(\frac{3y+x-9}{x-3}\right) + \frac{1}{4}\ln\left(\frac{y-x+1}{x-3}\right) = \ln(x-3) + c$

Partial Fraction Decomposition.

$$f(x) = \frac{ax+b}{a_1x^2+b_1x+c_1}$$

if: $$\frac{x^2+x+1}{(x^2+2x+1)(x-1)}$$

$$= \frac{Ax+B}{x^2+2x+1} + \frac{C}{x-1}$$

$$f(x) = \frac{2x+3}{(x-1)(x-3)}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$\therefore 2x+3 = A(x-2)+B(x-1)$$

$$x=1 \Rightarrow 5 = A(1-2)+B(1-1) \Rightarrow A=-5$$

$$x=2 \Rightarrow 7 = A(2-2)+B(2-1) \Rightarrow B=7$$

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Case - II : if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then we take the transformation

$$a_1 x + b_1 y = z$$

Ex: $\frac{dy}{dx} = \frac{y-x}{y-x+2}$ ..... (i)

Here, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ as $\frac{-1}{-1} = \frac{1}{1}$

$\therefore$ We take the transformation, $y - x = z$

$$\Rightarrow \frac{dy}{dx} - 1 = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} + 1$$

$\therefore$ (i) $\Rightarrow \frac{dz}{dx} + 1 = \frac{z}{z+2}$

or, $\frac{dz}{dx} = \frac{z}{z+2} - 1 = \frac{-2}{z+2}$

or, $(z+2)\,dz = -2\,dx$

or, $\int (z+2)\,dz = -2\int dx$

or, $\frac{z^2}{2} + 2z = -2x + c$

or, $\frac{(y-x)^2}{2} + 2(y-x) + 2x = c$

H/W i) $\frac{dy}{dx} = \frac{2x+y-1}{4x+2y+5}$

ii) $(2x+2y+1)dx + (x+y-1)dy = 0$.

H/W

i) $\frac{dy}{dx} = \frac{2x+y-1}{4x+2y+5}$ ------ (1)

Here, $\frac{2}{4} = \frac{1}{2}$

$\therefore$ Let $2x+y = z \longrightarrow 2 + \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dz}{dx} - 2 = \frac{dy}{dx}$

so, $4x + 2y = 2z$.

$\therefore$ (1) $\rightarrow \quad \frac{dz}{dx} - 2 = \frac{z-1}{2z+5}$

or, $\frac{dz}{dx} = \frac{z-1+2(2z+5)}{2z+5} = \frac{5z+9}{2z+9}$

or, $\frac{2z+9}{5z+9}\,dz = dx$ — Integrating both sides

or, $\int \frac{2z+9}{5z+9}\,dz = \int dx$

or, $\int \frac{2z}{5z+9}\,dz + \frac{9}{5}\int \frac{5}{5z+9}\,dz = \ln x + C$

or, $2\int \frac{z}{5z+9}\,dz + \frac{9}{5}\ln|5z+9| = \ln x + C$

or, $2\int\left(\frac{1}{5} - \frac{9}{5(5z+9)}\right)dz + \frac{9}{5}\ln(5z+9) = \ln x + C$

or, $2\left(\frac{1}{5}z - \frac{9}{25}(\ln|5z+9|)\right) + \frac{45}{25}\ln(5z+9) = \ln x + C$

or, $\frac{2}{5}z + \frac{27}{25}\ln|5z+9| = \ln x + C$

or, $\frac{2(2x+y)}{5} + \frac{27}{25}\ln|10x+5y+9| = \ln x + C$

ii) $(2x+2y+1)dx + (x+y-1)dy = 0$

or, $(2x+2y+1)dx = -(x+y-1)dy$

or, $\frac{dy}{dx} = -\frac{2x+2y+1}{x+y-1} = \frac{-2x-2y-1}{x+y-1}$ — (1)

Here, $\frac{-2}{1} = \frac{-2}{1}$

let, $x+y = z \longrightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore -2x-2y = -2z$

or, $\frac{dy}{dx} = \frac{dz}{dx} - 1$

$\therefore$ (1) $\rightarrow \frac{dz}{dx} - 1 = \frac{-2z-1}{z-1}$

or, $\frac{dz}{dx} = \frac{-2z-1}{z-1} + 1 = \frac{-2z-1+z-1}{z-1} = \frac{-z-2}{z-1}$

or, $\cancel{z+2}\ \frac{1-z}{z+2}dz = dx$

or, $\int \frac{1-z}{z+2}dz = \int dx$

or, $\int \frac{1}{z+2}\,dz - \int \frac{z}{z+2}\,dz = \ln x + \ln C$

or, $\ln|z+2| - \int\left(1-\frac{2}{z+2}\right)dz = \ln Cx$

or, $\ln|z+2| - z + 2\ln|z+2| = \ln Cx$

or, $\ln(x+y+2) + 2\ln(x+y+2) - x - y = \ln Cx$

or, $3\ln(x+y+2) - x - y = \ln Cx$