14/2/23

Infinite Series

Let $\{U_n\}$ be the sequence of real number, then the expression: $u_1 + u_2 + u_3 \ldots \infty$ is called infinite series and it is denoted as $\sum u_n$.

To find whether the series is convergent or not, we need to construct, $S_1 = u_1$

$S_2 = u_1 + u_2$

$\vdots$

where $\{S_n\}$ is called sequence of partial sum.

If the sequence $\{S_n\}$ is convergent, then the series is also convergent.

Let us ~~cond~~ consider a series :-

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots \infty$$

Here,

$$u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

now, $$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\therefore \lim_{n\to\infty} = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

Thus, the series is convergent and $\sum u_n$ converges to 1.

Divergent Series: A series $\sum u_n$ is said to be divergent if $\lim_{n\to\infty} S_n = \pm\infty$

Ex: $S_n = 1+2+3+\dots\dots+n$

$$S_n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = \infty$$

Oscillatory Series: A series which is neither convergent nor divergent is called oscillatory series.

Ex: $1+(-1)+1+(-1)+1+(-1)+\dots\infty$

Geometric Series: The series $1+x+x^2+x^3+\dots\infty$ is known as geometric series. This series is convergent if $-1<x<1$ and divergent if $x \geq 1$.

If $x=-1$, then the series is oscillatory

Ex: i) $1+\frac{1}{2}+\frac{1}{2^2}+\dots$

$\therefore x = 1/2 < 1 \rightarrow$ convergent.

ii) $1+2+2^2+2^3+\dots$

$\therefore x = 2 > 1 \rightarrow$ divergent

17/02/23

Series

p-series.

The series of the form $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots + \infty$ is called p-series. This series is convergent if $p \geq 1$ and divergent if $p < 1$.

Properties of series :

- If the series $\sum u_n$ is convergent and converges to 's' then the series $\sum K u_n$ is also convergent and converges to 'Ks'.
- If 2 series $\sum u_n$ and $\sum v_n$ converging to 's' and 't' respectively then $\sum (u_n + v_n)$ converges to 's+t'.

Note: If $\sum u_n$ is a convergent series then for its 'n^{th} term' - u_n
$\lim_{n \to \infty} u_n = 0$.

→ Let us consider a series $1 + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 3 \cdot 5} + \dots + \infty$.

Check whether it is convergent or not.

ans

$$u_n = \frac{1}{1.3.5 \dots (2n-1)} < \frac{1}{1.2.2 \dots (n-1) \text{ times}} = \frac{1}{2^{n-1}}$$

Let, $\frac{1}{2^{n-1}} = v_n$

$$\therefore \sum_{n=1}^{\infty} v_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots \infty$$

$\therefore \sum v_n$ is a geometric series of $x = 1/2 < 1$,
and is thus convergent.

and since, $u_n < v_n$

by process of comparision test ;

$\sum u_n$ is also convergent.

→ Comparision Test :

Let $\sum u_n$ and $\sum v_n$ be 2 series of positive terms and there exist an integer 'N' such that –

$$u_n \leq k v_n \ , \ \forall n \geq N.$$

then, $\sum u_n$ is convergent if $\sum v_n$ is so and $\sum v_n$ is divergent if $\sum u_n$ is so.

• Homework

Q. 1. Use Ratio Test

(i) $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots\cdots$

(ii) $\sum \frac{n!\,2^n}{n^n}$

(iii) $(1/3)^2+(1.2/3.5)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\cdots\cdots$

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• Absolutely convergent series:

$\sum U_n$

$\sum |U_n|$ is convergent

Example:

$\sum (-1)^{n-1}\frac{1}{n^2}$

$\sum |U_n| = \sum \frac{1}{n^2}$

• Conditionally Convergence Series:

A series $\sum U_n$ is said to be a conditionally convergence series if $\sum U_n$ is convergent but $\sum |U_n|$ is not convergent.

• Example of a conditionally convergence series:

$\sum (-1)^{n-1}\frac{1}{n} = \sum (-1)^{n-1} U_n$

$= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$

$U_n=\frac{1}{n}$, $\{1/n\}$

$\sum |U_n| = \sum 1/n$

Being p.series with $P=1$, $\sum 1/n$ is divergent.

Qs: Show that the series $\sum \frac{\cos nx}{n^2}$ is absolutely convergent.

→ $\sum \frac{\cos nx}{n^2} = \sum U_n$

$\sum |U_n| = \sum \left|\frac{\cos nx}{n^2}\right|$

$|U_n| = \left|\frac{\cos nx}{n^2}\right| \leq 1/n^2$

$\Rightarrow V_n = \frac{1}{n^2}$

$\boxed{|U_n| \leq V_n}$ where $\sum V_n$ is convergent.

$\sum |U_n|$ is convergent [By using the comparison test]

Alternating Series

A series is said to be alternating series if the terms are alternatively positive & negative.

Ex :- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

Notation: $\sum (-1)^{n-1} U_n$

Leibnitz theorem

An alternating series $\sum(-1)^{n-1}$ is convergent under two conditions.

i) $\sum U_n$ is monotonically decreasing series.

ii) $\lim\limits_{n\to\infty} U_n = 0$

$U_n = \frac{1}{n}$

$U_{n+1} = \frac{1}{n+1}$ $\qquad U_n > U_{n+1} \ \forall\ n \in N$

$\sum U_n$ is monotonically decreasing & $\lim\limits_{n\to\infty} U_n = 0$

$\therefore$ By Leibnitz Theorem, the series $\sum(-1)^{n-1} U_n$ is convergent.

Q) Examine the convergence of the Series

$$2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\cdots$$

$$\boxed{U_n = \frac{1+n}{n}}$$

$$U_{n+1} = \cancel{\frac{2+n}{n+1}} \quad \frac{1+(n+1)}{n+1}$$

$U_n > U_{n+1}$

$U_n - U_{n+1} > 0$

$$U_n - U_{n+1} = \frac{1+n}{n} - \frac{n+2}{n+1}$$

$$= \frac{(1+n)^2 - n(n+2)}{n(n+1)}$$

$$= \frac{1+\cancel{n^2}+\cancel{2n}-\cancel{n^2}-\cancel{2n}}{n(n-1)}$$

$$= \frac{1}{n(n-1)} > 0 \quad n \in N$$

$\sum U_n$ is a monotonically decreasing series

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 1 + 0 = 1$$

The series is Not Convergent.

By Leibnitz theorem, we can say the series is not convergente as $\lim_{n\to\infty}$ is not 0.

$\boxed{HW}$ $\frac{1}{2^3} - \frac{1+2}{3^3} + \frac{1+2+3}{4^3} - \frac{1+2+3+4}{5^3} + \cdots$

Test the Convergence of the series by Leibnitz test.

__Absolutely Convergent Series:__

A series $\sum U_n$ is said to be absolutely convergent if $\sum |U_n|$ is convergent.