06/02/23

Ordinary Differential Equations.

Differential Equations - An equation which contains derivatives of various order and the variables is called a differential equation.

Differential Equations (DE)

- Ordinary Differential Equation (ODE)
- Partial Differential Equation (PDE)

ODE - A differential equation with only one independent variable, is called Ordinary Differential Equation.

PDE - A differential equation with more than one independent variables and partial derivatives of dependent variable with respect to them, is called Partial Differential Equation.

Ex: i) $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 - y^4 = \sin t \longrightarrow$ ODE

ii) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 4 \longrightarrow$ PDE

Linear Ordinary Differential Equation :

An ODE is called linear, if the dependent variable and its derivatives occur only in first degree, and no products of the dependent variable and its derivative occur.

Non-Linear ODE :- Otherwise the DE is known as non-Linear ODE.

Ex : i) $y'' + 3y' + 5y = e^x \cdot \cos x \longrightarrow$ Linear

ii) $y'' + y \cdot y' + xy = 0 \longrightarrow$ Non-Linear

iii) $x^2 (y')^4 + y = \sin x \longrightarrow$ Non-Linear

iv) $x^3 y''' + x^2 y'' + (x-1) y' = \sin x \longrightarrow$ Linear

v) $y' = \sqrt{x^2 + y} \longleftrightarrow (y')^2 = x^2 + y \longrightarrow$ Non-Linear

* Order of a Differential Equation.

The order of a differential equation is the order of the highest order derivative involved in the equation.

Degree of a Differential Equation.

The degree of a differential equation is the degree/power of the highest order derivative involved in the equation after the equation is made free from radicals and fractions in its derivatives

Ex : i) $y''' + (y'')^3 + xy'^2 + y^5 = e^{3x}$ [Order - 3, Degree - 1, NL]

ii) $(y'')^{3/2} + y' + y^2 = \sin x \Rightarrow (y'')^{3/2} = \sin x - y' - y^2$

$\rightarrow (y'')^3 = (\sin x - y' - y^2)^2$ [Order - 2, Degree - 3, NL]

Formulation of differential equations :

- Differential equations are formed by eliminating arbitary constants from a relation in the variables and constants
- If we have a relation containing n arbitrary constants, then we have to differenciate the relation n-times, which will produce n equations.
- Eliminating n arbitrary constents from n equations, we will get a differential equation.

Ex: $y = A\cos x + B\sin x$, A & B are arbitrary constant

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x = -(A\cos x + B\sin x) = -y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0.$$

It is a second order, first degree, linear differential equation.

7/2/23

Formulation of ODE.

Question $y = A\cos(mx) + B\sin(mx)$ where A, B are arbitrary constants.

- Sol^n

$$y' = -Am\sin mx + Bm\cos mx$$
$$y'' = -Am^2\cos mx - Bm^2\sin mx$$

$$y'' = -m^2(A\cos mx + B\sin mx)$$

$$y'' = -m^2 y$$

or, $y'' + m^2 y = 0$ → 2nd Order
Degree 1.
Linear

Question $y = e^x(A\cos x + B\sin x)$

$$y' = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) \quad —(1)$$

$$y'' = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$$

$$\cancel{y'' = 2e^x(-A\sin x + B\cos x) + y - y}$$

$$y'' = y + e^x(-A\sin x + B\cos x) + e^x(-A\sin x + B\cos x) - y$$

From (1),

$$y' - y = e^x(-A\sin x + B\cos x)$$

$\therefore \; y'' = 2(y' - y)$

or, $y'' - 2y' + 2y = 0 \longrightarrow$ 2^{nd} Order
Degree 1
Linear

Question: Find the DE corresponding to the relation - $xy = Ae^{x} + Be^{-x}$

Soln

$xy = Ae^{x} + Be^{-x}$

or, $y + xy' = Ae^{x} - Be^{-x}$

or, $y' + xy'' + y' = Ae^{x} + Be^{-x}$

or, $y' + xy'' + y' = xy$

or, $2y' + xy'' - xy = 0 \longrightarrow$ 2^{nd} Order
Degree 1
Linear

H/W i) $y = A\sec x + B\tan x$ ans: $y'' - \tan x\, y' - y\sec^2 x = 0$
ii) $ax^2 + by^2 = 1$. ans: $xyy'' + xy'^2 - yy' = 0$

Solution of a DE :-

A function $y = f(x)$ is called solution of a DE if y is a continuous function and differenciable upto required order and if we substitute the value of y and its derivative in the given equation, then the equation is reduced to an identity.

i) General solution or complete solution.
ii) Particular solution
iii) Singular solution

Ex: i) $y'' + y = 0$

General solution :- $y = A\cos x + B\sin x$

if $A = 1$, $B = 2$:
Particular solution :- $y = \cos x + 2\sin x$

ii) $(y')^2 + xy' = y$

General solution :- $y = Ax + A^2$

if $A = 1$; Particular solution :- $y = x + 1$.

Singular Solution :- $4y + x^2 = 0$

$\Rightarrow y = -\frac{x^2}{4}$

H/W

i) $y = A\sec x + B\tan x$

$y' = A\sec x\tan x + B\sec^2 x$

$y' = \sec x(A\tan x + B\sec x)$

$y'' = \sec x\tan x(A\tan x + B\sec x) + \sec x(A\sec^2 x + B\sec x\tan x)$

$= y'\tan x + \sec^2 x(A\sec x + B\tan x)$

$= y'\tan x + y\sec^2 x$

or, $y'' - y'\tan x - y\sec^2 x = 0$

ii) $ax^2 + by^2 = 1$. — differenciating wrt x.

$\Rightarrow 2ax + 2byy' = 0$

or, $ax + byy' = 0$ — (i)

now, $y' = -\frac{ax}{by}$ — (ii)

differenciating (i);

$\Rightarrow a + b(yy'' + y'y') = 0$

or, $yy'' + (y')^2 = -\frac{a}{b}$ — (iii)

Plugging eq. (iii) in eq. (ii); we get,

$$\Rightarrow y' = \frac{x}{y}\left(yy'' + (y')^2\right)$$

or, $yy' = xyy'' + x(y')^2$

or, $xyy'' + x(y')^2 - yy' = 0.$

13/2/23

Initial Value Problem (IVP) - A differential equation with the condition for an initial value of independent variable is called initial value problem.

Ex: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = \sin x$

for $x=0,\ y=0$

Boundary Value Problem (BVP) - A differential equation is called boundary value problem if the conditions are imposed on the dependant variables at the boundary points of the domain of independant variable.

* Variable seperation method for solving first order first degree differential equation.

Represented as : $\frac{dy}{dx} = f(x, y)$

Say, $M(x,y)\,dx + N(x,y)\,dy = 0$

We can apply the variable seperation method if M is a function of x only $M = f(x)$ and N is a function of y only $N = f(y)$.

Then, we can easily integrate the terms indivisualy and get the solution of the problem.

General look : $M(x)\,dx + N(y)\,dy = 0$

Ex: $(1+x^2)\,dy - (1+y^2)\,dx = 0$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or, $\tan^{-1} y = \tan^{-1} x + C$, where C is an arbitrary constant.

Question $(1+e^x)y\,dy - (1+y)e^x\,dx = 0$

or, $(1+e^x)y\,dy = (1+y)e^x\,dx$

or, $\frac{y}{(1+y)}\,dy = \frac{e^x}{(1+e^x)}\,dx$ — integrating both sides.

or, $\int \frac{y}{(1+y)}\,dy = \int \frac{e^x}{(1+e^x)}\,dx$

$\int \frac{e^x}{1+e^x}\,dx$

or, $\int \frac{dt}{t}$ $\Rightarrow 1+e^x = t$ $\Rightarrow e^x\,dx = dt$

$= \ln t$

or, $\int \left(\frac{1+y}{1+y} - \frac{1}{1+y}\right) dy = \int \left(\frac{1+e^x}{1+e^x} - \frac{1}{1+e^x}\right) dx + C$

or, $\int dy - \int \frac{dy}{1+y} = \int dx - \int \frac{dx}{1+e^x} + C$

$\boxed{\int \frac{dx}{1+e^x}} = x - \ln(1+e^x) + C$

or, $y - \ln(1+y) = \ln(1+e^x) + C$

Question $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

or, $x\sqrt{1-y^2}\,dx = -y\sqrt{1-x^2}\,dy$

or, $\frac{x}{\sqrt{1-x^2}}\,dx = -\frac{y}{\sqrt{1-y^2}}\,dy$ — integrating both sides.

or, $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\int \frac{y}{\sqrt{1-y^2}}\,dy$

Let, $t = \sqrt{1-x^2}$

or, $t^2 = 1-x^2$

~~or, 2t dx 2x~~

or, $2t\,dt = -2x\,dx$

or, $t\,dt = -x\,dx$

or, $-t\,dt = x\,dx$

Similarly, $-z\,dz = y\,dy$

where, $z = \sqrt{1-y^2}$

or, $\int \frac{-t\,dt}{t} = -\int \frac{-z\,dz}{z}$

or, $-t = +z + c$

or, $t + z = c$

or, $\sqrt{1-x^2} + \sqrt{1-y^2} = c$

H/W

i) $\tan x \sin^2 y\,dx + \cos^2 x \cot y\,dy = 0$

ii) $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

iii) $\frac{dy}{dx} = e^{2x-y} + x^3 e^{-y}$

H/W i) $\tan x \sin^2 y \, dx + \cos^2 x \cot y \, dy = 0$

$\Rightarrow \tan x \sin^2 y \, dx = -\cos^2 x \cot y \, dy$

or, $-\frac{\tan x}{\cos^2 x} dx = \frac{\cot y}{\sin^2 y} dy$

or, $-\frac{\sin x}{\cos^3 x} dx = \frac{\cos y}{\sin^3 y} dy$

~~Let~~, or, $\int -\frac{\sin x \, dx}{\cos^3 x} = \int \frac{\cos y \, dy}{\sin^3 y}$

Let, $u = \cos x$ $\qquad$ $\omega = \sin y$

$du = -\sin x \, dx$ $\qquad$ $d\omega = \cos y \, dy$

or, $\int \frac{du}{u^3} = \int \frac{d\omega}{\omega^3}$

or, $\int u^{-3} du = \int \omega^{-3} d\omega$

or, $\frac{u^{-2}}{-2} = \frac{\omega^{-2}}{-2} + c$

or, $\frac{1}{u^2} = \frac{1}{\omega^2} + (-2)c$

or, $\frac{1}{\cos^2 x} = \frac{1}{\sin^2 y} + K$ $\qquad$ Let, $K = -2C$.

or, $\sec^2 x - \operatorname{cosec}^2 y = K$.

ii) $3e^x \tan y \, dx + (1-e^x)\sec^2 y \, dy = 0$

$$\Rightarrow \frac{3e^x}{(e^x-1)} dx = \frac{\sec^2 y \, dy}{\tan y}$$

or, $$\int \frac{3e^x \, dx}{(e^x-1)} = \int \frac{\sec^2 y}{\tan y} dy$$

Let, $e^x - 1 = z$ $\qquad\qquad$ $\tan y = k$

or, $e^x dx = dz$ $\qquad\qquad$ $\sec^2 y \, dy = dk$

or, $$\int \frac{3\,dz}{z} = \int \frac{dk}{k}$$

or, $3 \log z = \log k + \log c$

or, $\log z^3 = \log kc$

or, $z^3 = kc$

or, $(e^x - 1)^3 = c.\tan y$

$\int \tan x \, dx$

$\int \frac{\sin x}{\cos x} dx$

iii) $\frac{dy}{dx} = e^{2x-y} + x^3 e^{-y}$.

$\Rightarrow \frac{dy}{dx} = \frac{e^{2x}}{e^y} + \frac{x^3}{e^y}$

or, $e^y dy = (e^{2x} + x^3) dx$

or, $\int e^y dy = \int (e^{2x} + x^3) dx$

or, $e^y = \frac{e^{2x}}{2} + \frac{x^4}{4} + C$

or, $4e^y = 2e^{2x} + x^4 + 4C$

let $4C = K$

or, $4e^y - 2e^{2x} - x^4 = K$.