

Mathematics

A Reference Manual

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Abstract

This document covers Calculus, Linear Algebra, Math Analysis and Differential Equations.

Calculus 3

This section largely follows Chapters 12-16 in Calculus Early Transcendentals.

Chapter 12: Vectors and Geometry of Space

Distance Formula in 3D

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad (1)$$

Equation of a Sphere

r: Radius of sphere

C(h,k,l): Center of sphere

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \quad (2)$$

Unit Vectors

Suppose that $\vec{a} \neq 0$ and that \vec{u} has the same direction of \vec{a} :

$$u = \frac{1}{|a|}a = \frac{a}{|a|} \quad (3)$$

Dot Products

$$a \cdot b = a_1b_1 + a_2b_2 + \dots a_nb_n \quad (4)$$

Properties include:

1. Output is a scalar.
2. $a \cdot a = |a|^2$
3. $a \cdot b = b \cdot a$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$
5. $ca \cdot b = c(b \cdot a) = a \cdot (cb)$

Theorem: If θ is the angle between two vectors, then

$$a \cdot b = |a||b| \cos \theta \quad (5)$$

$$\cos \theta = \frac{a \cdot b}{|a||b|} \quad (6)$$

Two vectors are orthogonal if $a \cdot b = 0$

Projections

Scalar projection of \vec{b} onto \vec{a} ... aka component of b along a :

$$\text{comp}_{\vec{a}} \vec{b} = \frac{a \cdot b}{|a|} \quad (7)$$

Vector projection of \vec{b} onto \vec{a} :

$$\text{proj}_{\vec{a}} \vec{b} = \left(\frac{a \cdot b}{|a|} \right) \frac{a}{|a|} = \left(\frac{a \cdot b}{|a|^2} \right) a \quad (8)$$

Associated Inequalities and Identities

Cauchy-Schwartz Inequality: $|a \cdot b| \leq |a||b|$

Triangle Inequality: $(a + b) \leq |a| + |b|$

Parallelogram Identity: $|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$

Cross Product

Suppose $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ or $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ then

$$a \times b = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \quad (9)$$

Which really comes from the following. Instead do this:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Expanding this determinant, we get:

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Properties:

1. The resultant vector is orthogonal to both \mathbf{a} and \mathbf{b} .
2. $a \times b = |a||b| \sin \theta$
3. vectors \mathbf{a} and \mathbf{b} are parallel if $a \times b = 0$
4. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
5. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
6. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
7. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
8. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
9. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Triple Product

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (10)$$

Volume of a parallelepiped: $V = |a \cdot (b \times c)|$

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Equations of Lines and Planes

$$\vec{v} = \langle a, b, c \rangle \quad \vec{r} = \langle x, y, z \rangle \quad \vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

Suppose that \vec{a} and \vec{v} are parallel vectors, and t is a scalar. Then:

$$\vec{a} = t\vec{v} \quad (11)$$

$$\vec{r} = r_0 + t\vec{v} \quad (12)$$

Which also means:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad (13)$$

Line segment from \vec{r}_0 to \vec{r}_1 :

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1 \quad (14)$$

Let \mathbf{n} be a vector orthogonal (normal ie. 90 deg) to a plane P, then:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad (15)$$

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \quad (16)$$

Scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with \mathbf{n} :

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (17)$$

Two planes are parallel if their normal vectors are parallel.

Cylinders and Quadric Surfaces

Insert images of surfaces here

Chapter 13: Vector Functions