Mathematics
A Reference Manual

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Abstract

This document covers Calculus, Linear Algebra, Math Analysis and Differential Equations.

Calculus 3

This section largely follows Chapters 12-16 in Calculus Early Transcendentals.

Chapter 12: Vectors and Geometry of Space

Distance Formula in 3D

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1}$$

Equation of a Sphere

r: Radius of phere
C(h,k,l): Center of sphere

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \tag{2}$$

Unit Vectors

Suppose that $\vec{a} \neq 0$ and that $\vec{u}$ has the same direction of $\vec{a}$:

$$u = \frac{1}{|a|}a = \frac{a}{|a|} \tag{3}$$

Dot Products

$$a \cdot b = a_1b_1 + a_2b_2 + ...a_nb_n \tag{4}$$

Properties include:
1. Output is a scalar. 2. $a \cdot a = |a|^2$ 3. $a \cdot b = b \cdot a$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ 5. $ca \cdot b = c(b \cdot a) = a \cdot (cb)$

Theorem: If θ is the angle between two vectors, then

$$a \cdot b = |a||b|\cos\theta \tag{5}$$

$$\cos\theta = \frac{a \cdot b}{|a||b|} \tag{6}$$

Two vectors are orthogonal if $a \cdot b = 0$

Projections
Scalar projection of $\vec{b}$ onto $\vec{a}$... aka component of b along a:

$$\text{comp}_{\vec{a}}\vec{b} = \frac{a \cdot b}{|a|} \tag{7}$$

Vector projection of $\vec{b}$ onto $\vec{a}$:

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{a \cdot b}{|a|}\right)\frac{a}{|a|} = \left(\frac{a \cdot b}{|a|^2}\right)a \tag{8}$$

Associated Inequalities and Identities

Cauchy-Schwartz Inequality: $|a \cdot b| \leq |a||b|$

Triangle Inequality: $(a + b) \leq |a| + |b|$

Parallelogram Identity: $|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$

Cross Product

Suppose $a = \langle a_1, a_2, a_3\rangle$ and $b = \langle b_1, b_2, b_3\rangle$ or $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ then

$$a \times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle \tag{9}$$

Which really comes from the following. Instead do this:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Expanding this determinant, we get:

$$\vec{a} \times \vec{b} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Properties:

1. The resultant vector is orthogonal to both a and b.
2. $a \times b = |a||b| \sin\theta$
3. vectors a and b are parallel if $a \times b = 0$
4. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
5. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
6. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
7. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
8. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
9. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Triple Product

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{10}$$

Volume of a parallelepiped: $V = |a \cdot (b \times c)|$
Insert image here

Equations of Lines and Planes

$\vec{v} = \langle a, b, c \rangle \qquad \vec{r} = \langle x, y, z \rangle \qquad \vec{r_0} = \langle x_0, y_0, z_0 \rangle$

$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$

$x = x_0 + at \qquad y = y_0 + at \qquad z = z_0 + at$

Suppose that $\vec{a}$ and $\vec{v}$ are parallel vectors, and t is a scalar. Then:

$$\vec{a} = t\vec{v} \tag{11}$$

$$\vec{r} = r_0 + t\vec{v} \tag{12}$$

Which also means:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{13}$$

Line segment from $\vec{r_0}$ to $\vec{r_1}$:

$$\mathbf{r}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1} \qquad 0 \le t \le 1 \tag{14}$$

Let **n** be a vector orthogonal (normal ie. 90 deg) to a plane P, then:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0 \tag{15}$$

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0} \tag{16}$$

Scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with $\mathbf{n}$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{17}$$

Two planes are parallel if their normal vectors are parallel.

Cylinders and Quadric Surfaces
See image.

Chapter 13: Vector Functions

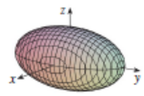
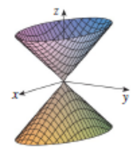
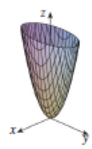
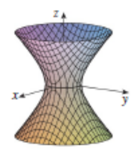
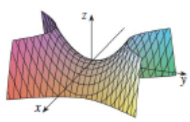
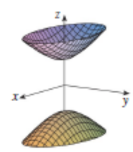
Plenty of parametric stuff. Lets begin.

Space Curves
If $x = f(t) \quad y = g(t) \quad z = h(t) \quad$ then $\quad r(t) = \langle x, y, z \rangle$

Arc Length

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \tag{18}$$

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	**Cone** 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	**Hyperboloid of One Sheet** 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	**Hyperboloid of Two Sheets** 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Or simply $L=\int_a^b |r'(t)|dt$