Mathematics
A Reference Manual

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Abstract

This document covers Calculus, Linear Algebra, Math Analysis and Differential Equations.

Calculus 3

This section largely follows Chapters 12-16 in Calculus Early Transcendentals.

Chapter 12: Vectors and Geometry of Space

Distance Formula in 3D

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1}$$

Equation of a Sphere
r: Radius of phere
C(h,k,l): Center of sphere

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \tag{2}$$

Unit Vectors
Suppose that $\vec{a} \neq 0$ and that $\vec{u}$ has the same direction of $\vec{a}$:

$$u = \frac{1}{|a|}a = \frac{a}{|a|} \tag{3}$$

Dot Products

$$a \cdot b = a_1b_1 + a_2b_2 + ...a_nb_n \tag{4}$$

Properties include:
1. Output is a scalar. 2. $a \cdot a = |a|^2$ 3. $a \cdot b = b \cdot a$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ 5. $ca \cdot b = c(b \cdot a) = a \cdot (cb)$

Theorem: If θ is the angle between two vectors, then

$$a \cdot b = |a||b| \cos\theta \tag{5}$$

$$\cos\theta = \frac{a \cdot b}{|a||b|} \tag{6}$$

Two vectors are orthogonal if $a \cdot b = 0$

Projections

Scalar projection of $\vec{b}$ onto $\vec{a}$... aka component of b along a:

$$\text{comp}_{\vec{a}}\vec{b} = \frac{a \cdot b}{|a|} \tag{7}$$

Vector projection of $\vec{b}$ onto $\vec{a}$:

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{a \cdot b}{|a|}\right)\frac{a}{|a|} = \left(\frac{a \cdot b}{|a|^2}\right) a \tag{8}$$

Associated Inequalities and Identities

Cauchy-Schwartz Inequality: $|a \cdot b| \leq |a||b|$

Triangle Inequality: $(a + b) \leq |a| + |b|$

Parallelogram Identity: $|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$

Cross Product

Suppose $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ or $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ then

$$a \times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \tag{9}$$

Which really comes from the following. Instead do this:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Expanding this determinant, we get:

$$\vec{a} \times \vec{b} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Properties:

1. The resultant vector is orthogonal to both a and b.
2. $a \times b = |a||b| \sin \theta$
3. vectors a and b are parallel if $a \times b = 0$
4. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
5. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
6. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
7. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
8. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
9. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Triple Product

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{10}$$

Volume of a parallelepiped: $V = |a \cdot (b \times c)|$

Equations of Lines and Planes