Package Report

math_tools

Andrea Gotelli

Laboratoire des Sciences du Numérique de Nantes LS2N

École centrale de Nantes, Nantes
IMT-Atlantique, Nantes
France

Contents

List of Figures

Chapter 1

Lie Algebra Operations

Note that we use ${}^{a}R_b$ to express the orientation of the frame $\mathcal{F}_b$ with respect to the frame $\mathcal{F}_a$and $r_{b/a}$as the position of frame $\mathcal{F}_b$ with respect to the frame $\mathcal{F}_a$. Being $\mathcal{F}_a$ a moving or a fixed frame, when expressing a relative pose ${}^{a}g_b$ we refer a as the fixed frame and b as the moving one, to express the dependence of a for the relative pose of b.

Given the homogeneous tranformation from frame $\mathcal{F}_a$ to frame $\mathcal{F}_b$, namely ${}^{a}g_b$, defined as ${}^{a}g_b = \langle {}^{a}R_b,\ r_{b/a} \rangle$. This create a 4×4 matrix acting as a map in the Lie group.

$$ {}^{a}g_b = \begin{bmatrix} {}^{a}R_b & r_{b/a} \\ 0_{[3\times1]} & 1 \end{bmatrix} \tag{1.1} $$

This operator uniquely defines the pose of a frame with respect to a reference or euclidean frame, namely $\mathcal{F}_w$ or $\mathcal{F}_e$.

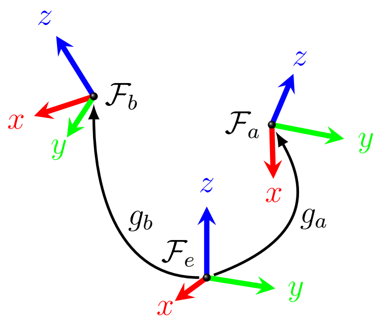

Figure 1.1: Representation of two generic frames : $\mathcal{F}_a$ and $\mathcal{F}_b$ in space, with the reference or Euclidean frame $\mathcal{F}_e$.

Notation rules for homogeneous matrices A generic homogeneous transformation is expressed as ${}^{a}g_{b}$. However, if the fixed frame is the euclidean one, thus having $a = e$, then we directly express the pose as g_b omitting the suffix e.

We can define g as the homogeneous matrix acting on homogeneous vectors $v^T = \begin{bmatrix} v_x & v_y & v_z & 1 \end{bmatrix}$. Such that $v_{b/a} = {}^{a}g_{b}{}^{b}v_{b}$ where $v_{b/a}$ is the expression of v_b with respect of $\mathcal{F}_a$. The action of g on the group is to move and rotate homogeneous vector or matrices belonging to $\mathbb{R}^{[4\times 1]}$ and $\mathbb{R}^{[4\times 4]}$ respectively.

From this definition we can define the derivatives of this operator with respect to time expressed with respect to the reference frame for the two frames.

$$\begin{aligned} \frac{d}{dt}(g_a) &= H_a \\ \frac{d}{dt}(g_b) &= H_b \end{aligned} \tag{1.2}$$

These two derivatives can be expressed in local coordinates for every frame, by reprojecting the derivative into the local coordinates frame.

$$\begin{aligned} g_a^{-1}\frac{d}{dt}(g_a) &= \eta_a \\ g_b^{-1}\frac{d}{dt}(g_b) &= \eta_b \end{aligned} \tag{1.3}$$

The twist η_a and η_b expresses the velocities of the frame $\mathcal{F}_a$, respectively $\mathcal{F}_b$, in the local coordinate frame or, "in their point of view".

If we consider the relative pose ${}^{a}g_{b}$, the derivative with respect to time, expressed in the loccal coordinates frame, is a twist in $se(3)$ defined as follows.

$${}^{b}\eta_{b} = \left[{}^{a}g_{b}{}^{-1}\frac{d}{dt}\left({}^{a}g_{b}\right)\right]^{\vee} = \left[{}^{b}g_{a}\frac{d}{dt}\left({}^{a}g_{b}\right)\right]^{\vee} \tag{1.4}$$

Where we can define ${}^{b}\eta_{b}$ as the twist of $\mathcal{F}_b$ with respect to the twist of $\mathcal{F}_a$ expressed in the coordinates of $\mathcal{F}_b$.

Starting from the time derivative, we can extend the formulation for other derivatives using the poincarré commutation. we then define the derivative with respect of the material coordinates $\frac{\partial g}{\partial X} = g'$ while we define a variation of the pose as Δg

$$\begin{aligned} {}^{b}\eta_{b} &= \left[{}^{b}g_{a}{}^{a}\dot{g}_{b}\right]^{\vee} \\ {}^{b}\xi_{b} &= \left[{}^{b}g_{a}{}^{a}g_{b}'\right]^{\vee} \\ {}^{b}\Delta\zeta_{b} &= \left[{}^{b}g_{a}\Delta{}^{a}g_{b}\right]^{\vee} \end{aligned} \tag{1.5}$$

Notations rules for Lie algebra elements For an element in the Lie algebra we express a twist η finish this part

When we have a series of connected bodies, we will need to propagate the kinematics along the chain. When we have a twist in a body $\mathcal{B}_b$, namely η_b, expressed in the local coordinates of frame $\mathcal{F}_b$ we then might need to propagate this twist to the body $\mathcal{B}_a$ with relative pose from $\mathcal{F}_b$ to $\mathcal{F}_a$ given by ${}^a g_b$. In this case we use the adjoint operator $Ad_{{}^a g_b}$ defined as follows :

$$Ad_{{}^a g_b} = \begin{bmatrix} {}^a R_b & 0_{[3\times3]} \\ \hat{r}_{b/a}{}^a R_b & {}^a R_b \end{bmatrix} \tag{1.6}$$

The adjoint operator Ad_g is a representation. It represents the action of g, being an homogeneous matrix belonging to the group of homogeneous transformation $SE(3)$, to the Lie algebra $se(3)$. Reminding that g operates on $SE(3)$ hemogeneous elemants like vector or matrices having dimensions of $\mathbb{R}^{[4\times1]}$ and $\mathbb{R}^{[4\times4]}$ respectively, the adjoint operator represents this action on Lie algabra vectors and matrices belonging to $se(3)$ with dimension of $\mathbb{R}^{[6\times1]}$ and $\mathbb{R}^{[6\times6]}$ respectively. Taking a generic frame $\mathcal{F}_b$ with its twist in local coordinates η_b we can give an example of the adjoint operator as follows.

$$\eta_a = Ad_{{}^a g_b}\eta_b \quad = \begin{bmatrix} {}^a R_b & 0_{[3\times3]} \\ \hat{r}_{b/a}{}^a R_b & {}^a R_b \end{bmatrix} \begin{bmatrix} \Omega_b \\ V_b \end{bmatrix} \tag{1.7}$$

1.0.1 Properties of Adg

$$Ad_g\,[\eta_1, \eta_2] = [Ad_g\eta_1, Ad_g\eta_2] \tag{1.8}$$

But also

$$[\eta_1, \eta_2] = ad_{\eta_1}\eta_2 \tag{1.9}$$

We then have

$$Ad_g ad_{\eta_1}\eta_2 = ad_{Ad_g\eta_1} Ad_g\eta_2 \tag{1.10}$$

If we need to compute the acceleration than we need to derivate this relation. We thus have that

$$\begin{aligned} \dot{\eta}_a &= \frac{d}{dt}\left(Ad_{{}^a g_b}\eta_b\right) \\ &= \dot{Ad}_{{}^a g_b}\eta_b + Ad_{{}^a g_b}\dot{\eta}_b \end{aligned} \tag{1.11}$$

The derivative of the adjoint operator, namely $\dot{Ad}_{{}^a g_b}$, is defined as follows

$$\dot{Ad}_{{}^a g_b} = Ad_{{}^a g_b} ad_{\eta_{b/a}} \tag{1.12}$$

Where the $ad_{\eta_{b/a}}$ operator accounts for the contribution of the twist in the derivative with respect to time; This operator is defined as :

$$ad_{\eta_{b/a}} = \begin{bmatrix} \hat{\Omega}_{b/a} & 0 \\ \hat{V}_{b/a} & \hat{\Omega}_{b/a} \end{bmatrix} \tag{1.13}$$

If we now want to determine the tangent kinematics we first need to introduce a variation of the pose due to a change in the generalised coordinates as follows.

$$\Delta\zeta_a = g_a^{-1}\Delta g_a \tag{1.14}$$

We can compute the tangent of the twist as follows

$$\Delta\eta_a = g_a^{-1}\Delta\frac{d}{dt}\left(g_a\right) \tag{1.15}$$

We can now project this twist as usual and as we described before

$$\Delta\eta_a = \Delta\left(Ad_{{}^a g_b}\eta_b\right) \quad = \Delta Ad_{{}^a g_b}\eta_b + Ad_{{}^a g_b}\Delta\eta_b \tag{1.16}$$

Similarly to what we introduced before, we can define the operator $\Delta Ad_{{}^a g_b}$.

$$\Delta Ad_{{}^a g_b} = Ad_{{}^a g_b} ad_{\Delta\zeta_{b/a}} \tag{1.17}$$

We now have that, given a relative pose ${}^a g_b$ we can compute $\Delta\zeta_{b/a}$ as

$$\Delta\zeta_{b/a} = \Delta\zeta_b - \Delta\zeta_a \tag{1.18}$$

We want to show that $\dot{Ad}_g^{-T} = -Ad_g^{-T}ad_\eta^T = -ad_{Ad_g\eta}^T Ad_g^{-T}$ Where we need this ?

We need to invoke the duality between twists and wrenches in order to find that, for every virtual twist $\delta\zeta$

$$\begin{aligned} \delta\zeta^T\left(Ad_g^{-T}ad_\eta^T F\right) &= \delta\zeta^T\left(ad_{Ad_g\eta}^T Ad_g^{-T}F\right) \\ \left(ad_\eta Ad_g^{-1}\delta\zeta\right)^T F &= \left(Ad_g^{-1}ad_{Ad_g\eta}\delta\zeta\right)^T F \\ &= \left(Ad_g^{-1}ad_{Ad_g\eta}Ad_g Ad_g^{-1}\delta\zeta\right)^T F \end{aligned} \tag{1.19}$$

Now using the property (1.10) we can simplify the right part of Equation (1.19) as follows

$$\begin{aligned} \left(ad_\eta Ad_g^{-1}\delta\zeta\right)^T F &= \left(Ad_g^{-1}Ad_g ad_\eta Ad_g^{-1}\delta\zeta\right)^T F \\ &= \left(ad_\eta Ad_g^{-1}\delta\zeta\right)^T F \end{aligned} \tag{1.20}$$

As Equations (1.19) and (1.20) are valied for every wrench F and twist $\delta\zeta$, we have proven the realation

$$\dot{Ad}_g^{-T} = -Ad_g^{-T} ad_\eta^T = -ad^T_{Ad_g\eta} Ad_g^{-T} \tag{1.21}$$

1.0.2 Definition of relative acceleration

Having two bodies namely β_k and β_j, with $j > k$, we can define the relative twist as $\eta_{j/k}$ where in the pedix j/k the / means with respect to; in this case it expressed the twist η_j with respect to the frame attached to the body β_k.

If we know the twist of the body β_k, expressed in its local coordinates, as η_k, then we can compute the twist of body β_j expresses in the local coordinated frame attached to the body β_j as

$$\eta_j = Ad_{{}^j g_k}\eta_k + \eta_{j/k} \tag{1.22}$$

We can differentiate this equation in order to obtain

$$\begin{aligned} \dot{\eta}_j &= Ad_{{}^j g_k}\dot{\eta}_k + \dot{Ad}_{{}^j g_k}\eta_k + \dot{\eta}_{j/k} \\ &= Ad_{{}^j g_k}\dot{\eta}_k + Ad_{{}^j g_k} ad_{\eta_{k/j}}\eta_k + \dot{\eta}_{j/k} \end{aligned} \tag{1.23}$$

Equation (1.23) needs the knowledge of the relative twist $\eta_{k/j}$ expressing the twist of the body k as seen in the coordinate frame attached to body j. In our Newton-Euler formalism, we have the opposite quantity $\eta_{j/k}$. We thus need to find an expression of Equation (1.23) that uses $\eta_{j/k}$ instead of $\eta_{k/j}$.

We start from the identity ${}^j g_k {}^k g_j = \mathbb{1}$. If we differentiate it with respect to time we obtain

$$\begin{aligned} {}^j\dot{g}_k {}^k g_j &= -{}^j g_k {}^k\dot{g}_j \\ &= -{}^j\hat{\eta}_{j/k} \end{aligned} \tag{1.24}$$

By multiplying the left hand side of Equation (1.24) by ${}^j g_k {}^k g_j$ we obtain

$$\begin{aligned} {}^j g_k {}^k g_j {}^j\dot{g}_k {}^k g_j &= -{}^j\hat{\eta}_{j/k} \\ {}^j g_k \hat{\eta}_{k/j} {}^k g_j &= \end{aligned} \tag{1.25}$$

Using the relation $[g\hat{\eta}g^{-1}]^\vee = Ad_g\eta$ **proof required!**in the left hand side of Equation (1.25) we obtain the following expression.

$$\begin{aligned} {}^j g_k {}^k g_j {}^j\dot{g}_k {}^k g_j &= -{}^j\hat{\eta}_{j/k} \\ Ad_{{}^j g_k}\eta_{k/j} &= -{}^j\eta_{j/k} \\ \eta_{k/j} &= -Ad^{-1}_{{}^j g_k} {}^j\eta_{j/k} \end{aligned} \tag{1.26}$$

From the right hand side of Equation (1.26) we can apply the relation $Ad_g^{-1} = Ad_{g^{-1}}$ in order to obtain

$$\begin{aligned} \eta_{k/j} &= -Ad^{-1}_{{}^j g_k}{}^j\eta_{j/k} \\ &= -Ad_{{}^j g_k^{-1}}{}^j\eta_{j/k} \\ &= -Ad_{{}^k g_j}{}^j\eta_{j/k} \end{aligned} \tag{1.27}$$

Substituing Equation (1.27) into Equation (1.23) we obtain the following expression.

$$\dot{\eta}_j = Ad_{{}^j g_k}\dot{\eta}_k + Ad_{{}^j g_k} ad_{-Ad_{{}^k g_j}{}^j\eta_{j/k}}\eta_k + \dot{\eta}_{j/k} \tag{1.28}$$

Introducing the relation $Ad_g Ad_g^{-1} = \not\Vdash$ in Equation (1.28) we obtain

$$\begin{aligned} \dot{\eta}_j &= Ad_{{}^j g_k}\dot{\eta}_k - Ad_{{}^j g_k} ad_{Ad_{{}^k g_j}{}^j\eta_{j/k}} Ad_{{}^k g_j} Ad^{-1}_{{}^k g_j}\eta_k + \dot{\eta}_{j/k} \\ &= Ad_{{}^j g_k}\dot{\eta}_k - Ad_{{}^j g_k} ad_{Ad_{{}^k g_j}{}^j\eta_{j/k}} Ad_{{}^k g_j} Ad_{{}^j g_k}\eta_k + \dot{\eta}_{j/k} \end{aligned} \tag{1.29}$$

Lemma 1 We want to show that

$$ad_{Ad_g\eta_1} Ad_g \eta_2 = Ad_g ad_{\eta_1}\eta_2 \tag{1.30}$$

For this we use the definition of ad_η with the Lie brackets

$$ad_{\eta_1}\eta_2 = [\hat{\eta}_1, \hat{\eta}_2]^\vee \tag{1.31}$$

Now as $[Ad_g\hat{\eta}_1, Ad_g\hat{\eta}_2]^\vee = Ad_g\,[\hat{\eta}_1, \hat{\eta}_2]^\vee$ we have that

$$\begin{aligned} [Ad_g\hat{\eta}_1, Ad_g\hat{\eta}_2]^\vee &= ad_{Ad_g\eta_1} Ad_g\eta_2 \\ Ad_g\,[\hat{\eta}_1, \hat{\eta}_2]^\vee &= Ad_g ad_{\eta_1}\eta_2 \end{aligned} \tag{1.32}$$

Thus we have proven that $Ad_g ad_{\eta_1}\eta_2 = ad_{Ad_g\eta_1} Ad_g\eta_2$. $\therefore$

We can use the previus Lemma 1 in order Equation (1.29) having $Ad_{{}^j g_k}\eta_k$ as η_2 and $ad_{Ad_{{}^k g_j}{}^j\eta_{j/k}} Ad_{{}^k g_j}$ as $ad_{Ad_g\eta_1} Ad_g$ with $\eta_1 = {}^j\eta_{j/k} = \eta_{j/k}$. We thus have that $Ad_g ad_{\eta_1}\eta_2 = Ad_{{}^k g_j} ad_{\eta_{j/k}} Ad_{{}^j g_k}\eta_k$ and this can be used in the following expression

$$\begin{aligned} \dot{\eta}_j &= Ad_{{}^j g_k}\dot{\eta}_k - Ad_{{}^j g_k}\left[ad_{Ad_{{}^k g_j}{}^j\eta_{j/k}} Ad_{{}^k g_j} Ad_{{}^j g_k}\eta_k\right] + \dot{\eta}_{j/k} \\ &= Ad_{{}^j g_k}\dot{\eta}_k - Ad_{{}^j g_k}\left[Ad_{{}^k g_j} ad_{\eta_{j/k}} Ad_{{}^j g_k}\eta_k\right] + \dot{\eta}_{j/k} \\ &= Ad_{{}^j g_k}\dot{\eta}_k - ad_{\eta_{j/k}} Ad_{{}^j g_k}\eta_k + \dot{\eta}_{j/k} \end{aligned} \tag{1.33}$$

Using the defintion of twist in Equation (1.22) we can obtain the relation $Ad_{j_{g_k}}\eta_k = \eta_j - \eta_{j/k}$. Using this last relation in Equation (1.33) we obtain :

$$\dot{\eta}_j = Ad_{j_{g_k}}\dot{\eta}_k - ad_{\eta_{j/k}}\left(\eta_j - \eta_{j/k}\right) + \dot{\eta}_{j/k} \tag{1.34}$$

Lemma 2 We want to proove that $ad_{\eta_i}\eta_i = 0$. If we take the defintion with Lie brackets be obtain : $ad_{\eta_i}\eta_i = [\eta_i, \eta_i] = \eta_i\eta_i - \eta_i\eta_i = 0 \therefore$

We can use the lemma 2 to obtain the following :

$$\dot{\eta}_j = Ad_{j_{g_k}}\dot{\eta}_k - ad_{\eta_{j/k}}\eta_j + \dot{\eta}_{j/k} \tag{1.35}$$

knowing that $ad_{\eta_1}\eta_2 = -ad_{\eta_2}\eta_1$ We obtain the final formulation

$$\dot{\eta}_j = Ad_{j_{g_k}}\dot{\eta}_k + ad_{\eta_j}\eta_{j/k} + \dot{\eta}_{j/k} \tag{1.36}$$

1.0.3 Delta dot Ad g

Starting from the realation

$$\dot{g} = g\hat{\eta} \tag{1.37}$$

We can obtain the variational part

$$\Delta g = g\Delta\hat{\zeta} \tag{1.38}$$

And similary extend this to the Adjoint operator using Equation (1.17).

$$\Delta Ad_g = Ad_g ad_{\Delta\zeta} \tag{1.39}$$

Time derivating this equation we obtain :

$$\Delta\dot{A}d_g = \dot{A}d_g ad_{\Delta\zeta} + Ad_g ad_{\Delta\dot{\zeta}} \tag{1.40}$$

Where using the definition in Equation (1.12) we obtain

$$\Delta\dot{A}d_g = Ad_g ad_\eta ad_{\Delta\zeta} + Ad_g ad_{\Delta\dot{\zeta}} \tag{1.41}$$

We need now to investigate the term $ad_{\Delta\dot{\zeta}}$. Using the the following eral-tion starting from the definition of the material derivative of twist $\frac{d\eta}{dX} = \eta'$.

$$\begin{aligned} \eta' &= -ad_\xi\eta + \dot{\xi} \\ &= ad_\eta\xi + \dot{\xi} \end{aligned} \tag{1.42}$$

Now using the commutation of Poincarré, we can change from a material derivative to a variation obtaining the followiing

$$\Delta\eta = ad_\eta \Delta\zeta + \Delta\dot{\zeta} \tag{1.43}$$

And thus

$$\Delta\dot{\zeta} = \Delta\eta - ad_\eta \Delta\zeta \tag{1.44}$$

Using Equation (1.44) into Equation (1.41) we obtain the following.

$$ad_{\Delta\dot{\zeta}} = ad_{(\Delta\eta - ad_\eta \Delta\zeta)} \tag{1.45}$$

Substituing Equation (1.45) into Equation (1.41) we obtain the following expression

$$\begin{aligned} \Delta\dot{Ad}_g &= Ad_g ad_\eta ad_{\Delta\zeta} + Ad_g ad_{\Delta\eta} - Ad_g ad_{ad_\eta \Delta\zeta} \\ &= Ad_g \left(ad_\eta ad_{\Delta\zeta} + ad_{\Delta\eta} - ad_{ad_\eta \Delta\zeta} \right) \end{aligned} \tag{1.46}$$

1.0.4 Tangent Kinematics

Here we want to derivate the formulation for the tangent kinematics of a Cosserat rod.

Delta eta

Knowing, from the analogy with the twist (Equation (1.22)), that we can express the variation on the pose for a body j knowing the one of a body k, with $j > k$, and their relative pose variation $\Delta\zeta_{j/k}$ as follows.

$$\Delta\zeta_j = Ad_{{}^j g_k} \Delta\zeta_k + \Delta\zeta_{j/k} \tag{1.47}$$

Now for what concerns the twist, taking the variation of Equation (1.22) we obtain.

$$\begin{aligned} \Delta\eta_j &= \Delta \left(Ad_{{}^j g_k} \eta_k \right) + \Delta\eta_{j/k} \\ &= \Delta Ad_{{}^j g_k} \eta_k + Ad_{{}^j g_k} \Delta\eta_k + \Delta\eta_{j/k} \end{aligned} \tag{1.48}$$

Using the expression in Equation (1.39) we obtain

$$\Delta\eta_j = Ad_{{}^j g_k} ad_{\Delta\zeta_{k/j}} \eta_k + Ad_{{}^j g_k} \Delta\eta_k + \Delta\eta_{j/k} \tag{1.49}$$

We now need to find a link between $\Delta\zeta_{k/j}$ and $\Delta\zeta_{j/k}$ as it is the latter to be available in the Newton-Euler algorithm. For this we process as we did before : taking the variation of the identity ${}^j g_k {}^k g_j = \mathbb{1}$ we obtain the following expression.

$$\begin{aligned} \Delta^j g_k{}^k g_j &= -{}^j g_k \Delta^k g_j \\ &= -\Delta\zeta_{j/k} \end{aligned} \tag{1.50}$$

We can add the identity ${}^j g_k{}^k g_j$ on the left side of Equation (1.50) to obtain

$${}^j g_k{}^k g_j \Delta^j g_k{}^k g_j = -\Delta\zeta_{j/k} \tag{1.51}$$

Now reminding the defintion of $\Delta\zeta$ from Equation (1.14) we obtain

$${}^j g_k \Delta\hat{\zeta}_{k/j}{}^k g_j = -\Delta\zeta_{j/k} \tag{1.52}$$

We now have that $\left[g\Delta\hat{\zeta}g^{-1}\right] = Ad_g \Delta\zeta$ so we obtain

$$\begin{aligned} Ad_{{}^j g_k} \Delta\hat{\zeta}_{k/j} &= -\Delta\zeta_{j/k} \\ \Delta\hat{\zeta}_{k/j} &= -Ad^{-1}_{{}^j g_k} \Delta\zeta_{j/k} \\ &= -Ad_{{}^j g_k^{-1}} \Delta\zeta_{j/k} \\ &= -Ad_{{}^k g_j} \Delta\zeta_{j/k} \end{aligned} \tag{1.53}$$

Using Equation (1.53) into Equation (1.49) we obtain the following expression.

$$\Delta\eta_j = Ad_{{}^j g_k} ad_{-Ad_{{}^k g_j} \Delta\zeta_{j/k}} \eta_k + Ad_{{}^j g_k} \Delta\eta_k + \Delta\eta_{j/k} \tag{1.54}$$

We can now introduce the identity $Ad_{{}^k g_j} Ad_{{}^j g_k}$ to obtain

$$\Delta\eta_j = -Ad_{{}^j g_k} ad_{Ad_{{}^k g_j} \Delta\zeta_{j/k}} Ad_{{}^k g_j} Ad_{{}^j g_k} \eta_k + Ad_{{}^j g_k} \Delta\eta_k + \Delta\eta_{j/k} \tag{1.55}$$

using the Lemma 1 described before, we have that $ad_{Ad_{{}^k g_j} \Delta\zeta_{j/k}} Ad_{{}^k g_j} Ad_{{}^j g_k} \eta_k = Ad_{{}^k g_j} ad_{\Delta\zeta_{j/k}} Ad_{{}^j g_k} \eta_k$ thus we obtain

$$\begin{aligned} \Delta\eta_j &= -Ad_{{}^j g_k} Ad_{{}^k g_j} ad_{\Delta\zeta_{j/k}} Ad_{{}^j g_k} \eta_k + Ad_{{}^j g_k} \Delta\eta_k + \Delta\eta_{j/k} \\ &= Ad_{{}^j g_k} \Delta\eta_k - ad_{\Delta\zeta_{j/k}} Ad_{{}^j g_k} \eta_k + \Delta\eta_{j/k} \end{aligned} \tag{1.56}$$

Now reminding that from Equation (1.22) $Ad_{{}^j g_k} \eta_k = \eta_j - \eta_{j/k}$. using this equivalence in Equation (1.57) and applying directly the Lemma 2 we obtain the final formulation.

$$\Delta\eta_j = Ad_{{}^j g_k} \Delta\eta_k + ad_{Ad_{{}^j g_k} \eta_k} \Delta\zeta_{j/k} + \Delta\eta_{j/k} \tag{1.57}$$

Delta dot eta

Starting from the definition of the derivative of the local twist

$$\dot{\eta} = Ad_{j_{g_k}} \dot{\eta}_k + ad_{\eta_j} \eta_{j/k} + \dot{\eta}_{j/k} \tag{1.58}$$

If we take the variation of this equation we obtain

$$\Delta\dot{\eta} = Ad_{j_{g_k}} \Delta\dot{\eta}_k + \Delta Ad_{j_{g_k}} \dot{\eta}_k + ad_{\Delta\eta_j} \eta_{j/k} + ad_{\eta_j} \Delta\eta_{j/k} + \Delta\dot{\eta}_{j/k} \tag{1.59}$$

Now we remind that $\Delta Ad_{j_{g_k}} \dot{\eta}_k = Ad_{j_{g_k}} ad_{\Delta\zeta_{k/j}} \dot{\eta}_k$. In this cas we need to apply the formulation for changing from $\Delta\zeta_{k/j}$ to $\Delta\zeta_{j/k}$.

We can directlu use Equation (1.53) to obtain that

$$Ad_{j_{g_k}} ad_{\Delta\zeta_{k/j}} \dot{\eta}_k = -Ad_{j_{g_k}} ad_{Ad_{k_{g_j}} \Delta\zeta_{j/k}} \dot{\eta}_k \tag{1.60}$$

Adding the identity $Ad_{k_{g_j}} Ad_{j_{g_k}}$ we obtain

$$-Ad_{j_{g_k}} ad_{Ad_{k_{g_j}} \Delta\zeta_{j/k}} \dot{\eta}_k = -Ad_{j_{g_k}} ad_{Ad_{k_{g_j}} \Delta\zeta_{j/k}} Ad_{k_{g_j}} Ad_{j_{g_k}} \dot{\eta}_k \tag{1.61}$$

We can now call $\eta_1 = \Delta\zeta_{j/k}$ and $\eta_2 = Ad_{j_{g_k}} \dot{\eta}_k$ we thus have

$$-Ad_{j_{g_k}} \left[ad_{Ad_{k_{g_j}} \eta_1} Ad_{k_{g_j}} \eta_2 \right] = -Ad_{j_{g_k}} \left[Ad_{k_{g_j}} ad_{\eta_1} \eta_2 \right] \tag{1.62}$$

We can now simplify to

$$\begin{aligned} -Ad_{j_{g_k}} Ad_{k_{g_j}} ad_{\eta_1} \eta_2 &= -ad_{\eta_1} \eta_2 \\ &= ad_{\eta_2} \eta_1 \\ &= ad_{Ad_{j_{g_k}} \dot{\eta}_k} \Delta\zeta_{j/k} \end{aligned} \tag{1.63}$$

We thus established once again that

$$Ad_{j_{g_k}} ad_{\Delta\zeta_{k/j}} \dot{\eta}_k = ad_{Ad_{j_{g_k}} \dot{\eta}_k} \Delta\zeta_{j/k} \tag{1.64}$$

Using now Equation (1.64) into Equation (1.59) we obtain

$$\Delta\dot{\eta} = Ad_{j_{g_k}} \Delta\dot{\eta}_k + ad_{Ad_{j_{g_k}} \dot{\eta}_k} \Delta\zeta_{j/k} + ad_{\Delta\eta_j} \eta_{j/k} + ad_{\eta_j} \Delta\eta_{j/k} + \Delta\dot{\eta}_{j/k} \tag{1.65}$$

Or, similarly

$$\Delta\dot{\eta} = Ad_{j_{g_k}} \Delta\dot{\eta}_k + ad_{Ad_{j_{g_k}} \dot{\eta}_k} \Delta\zeta_{j/k} - ad_{\eta_{j/k}} \Delta\eta_j + ad_{\eta_j} \Delta\eta_{j/k} + \Delta\dot{\eta}_{j/k} \tag{1.66}$$