# Extracting information from a set of observations

# Last lecture

We talked about how to interpret an observation (data point)

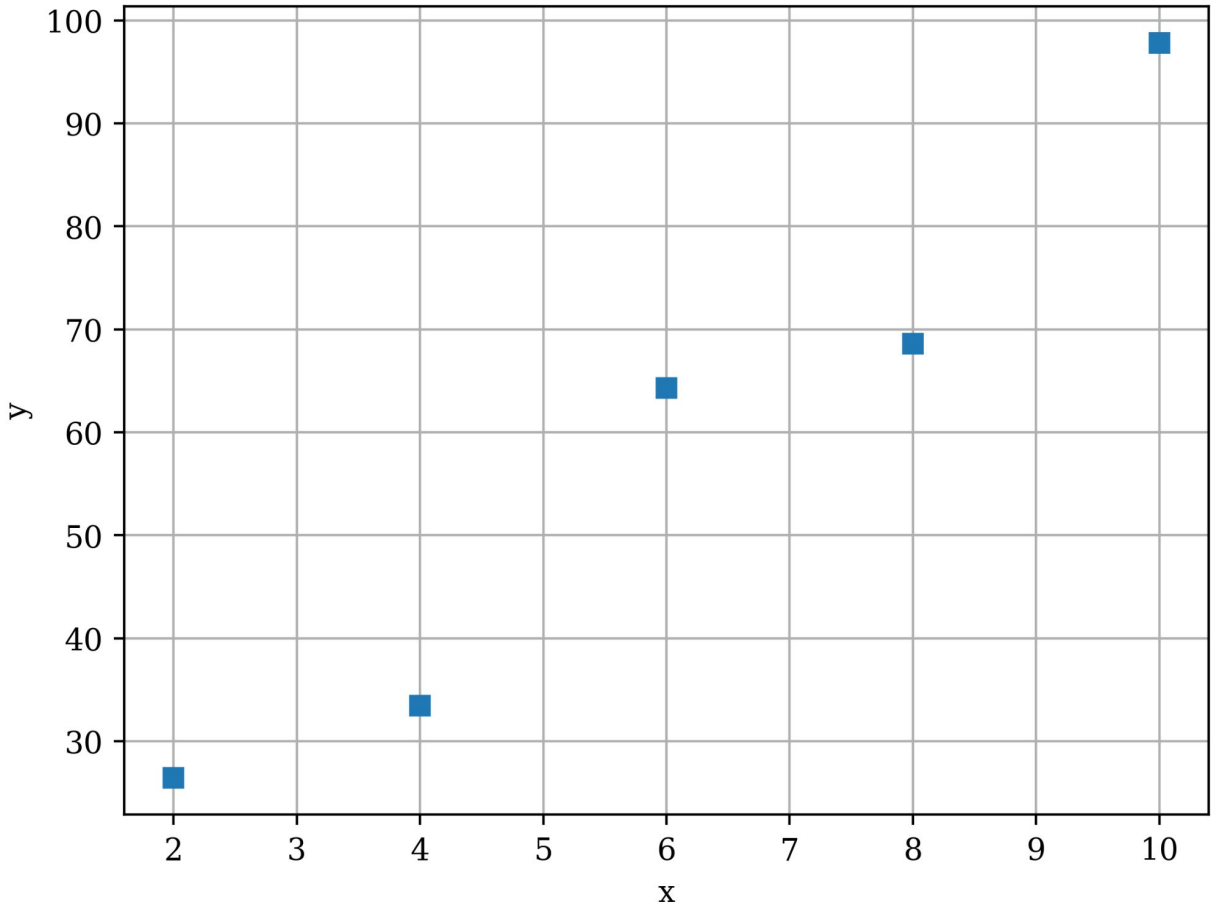
Today -

We will talk about how to extract information from a set of observations

Linear-ish looking data points

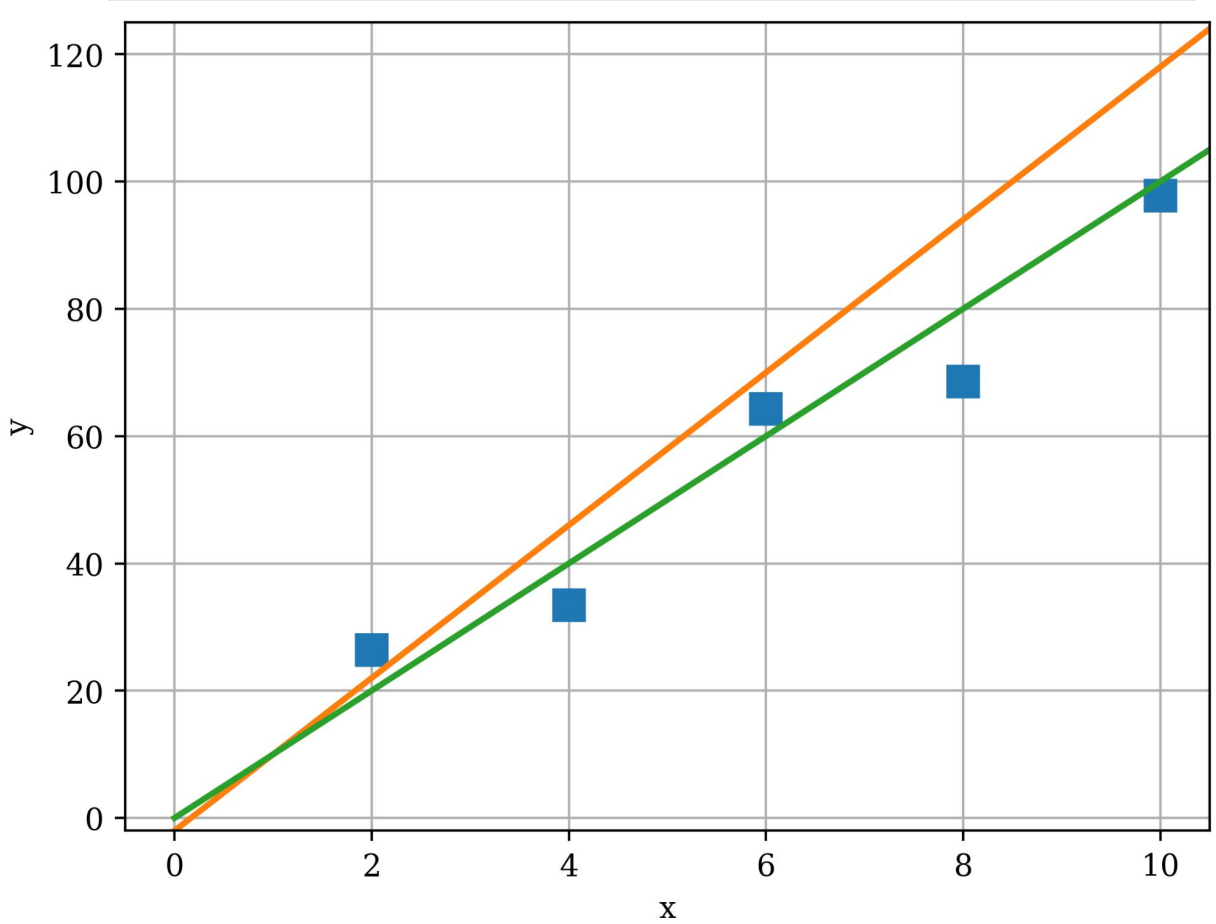
Want to find a linear trend line that describes it well

How can we do this?



How can we choose between 2 lines

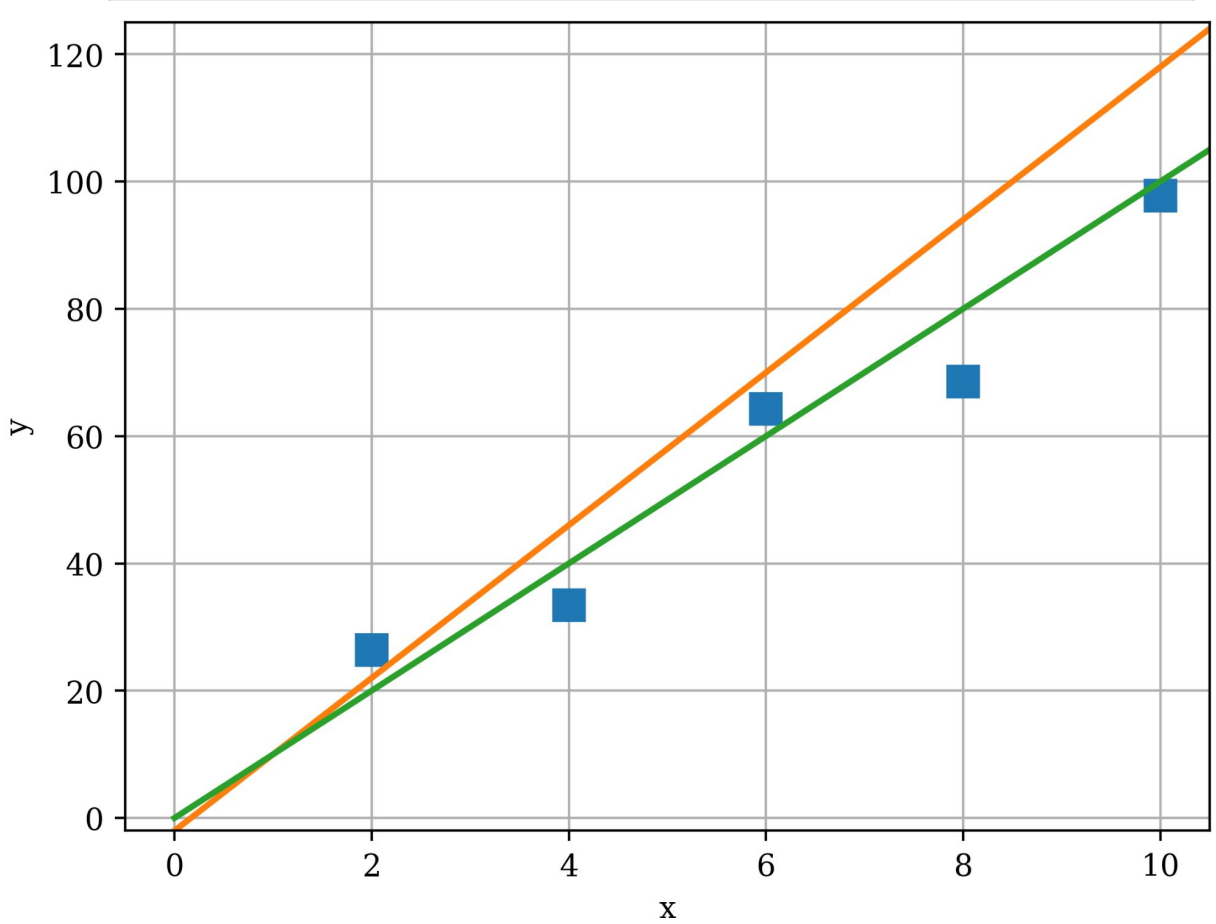
Which describes the data better?

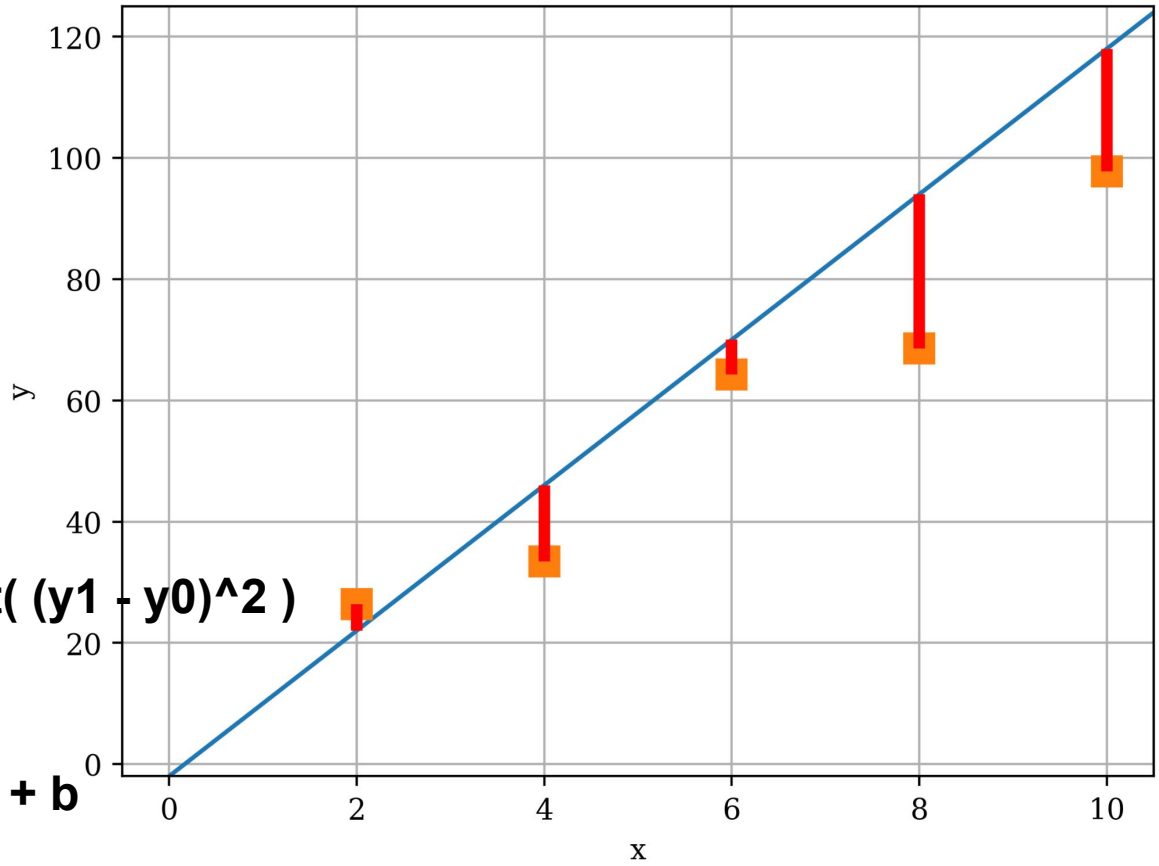


How can we choose between 2 lines

Which describes the data better?

How about the one closest to the data points?

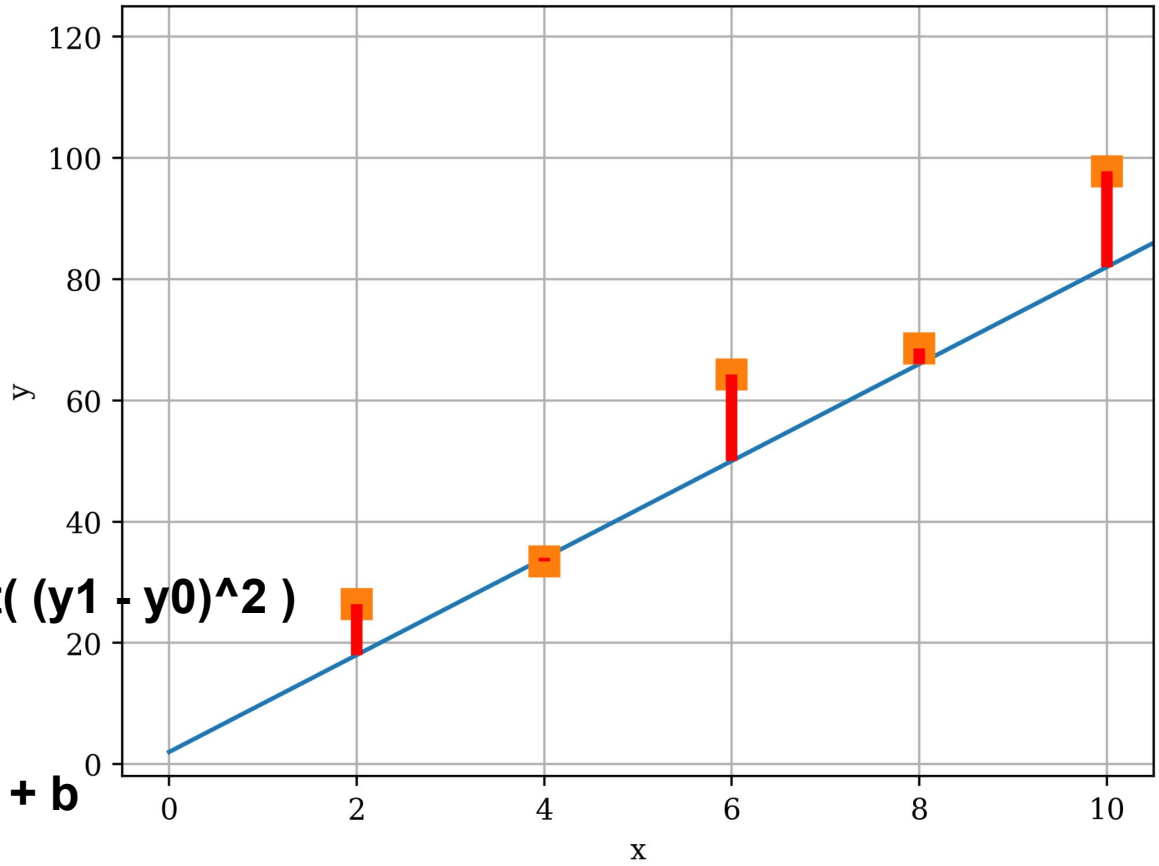





**distance = sqrt( (y1 - y0)^2 )**

**y0 = data**

**y1 = line = m*x + b**

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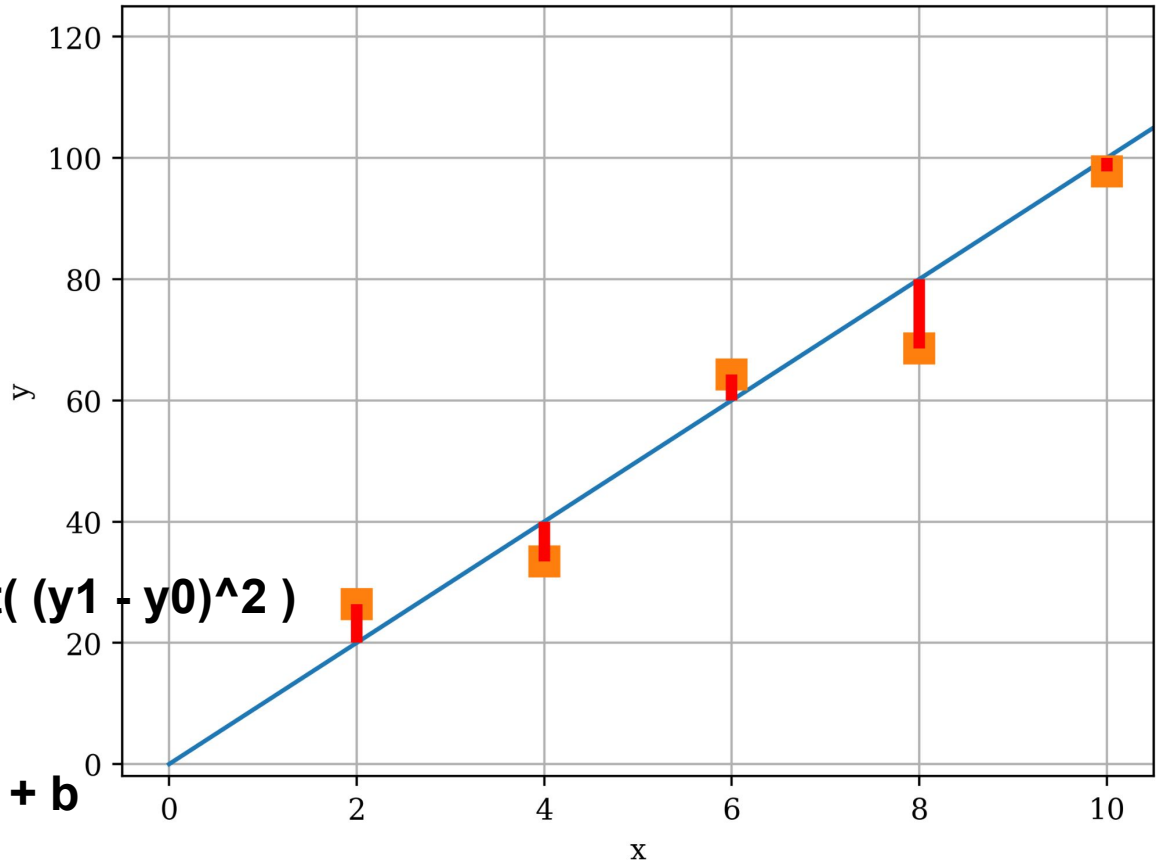


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**y0 = data**

**y1 = line = m*x + b**

# Least Squares

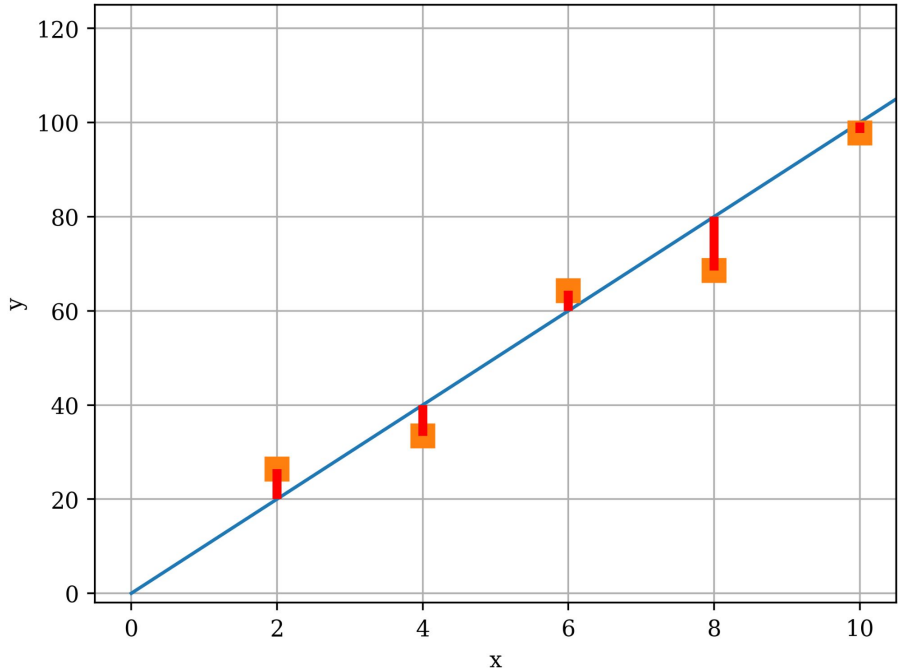
This is known as least squares fitting

a very common method

$S = \Sigma_i \, ( y(x_i) - y_i )^2$

Then find the form of y(x) that minimizes S

Here y(x) = m*x + b



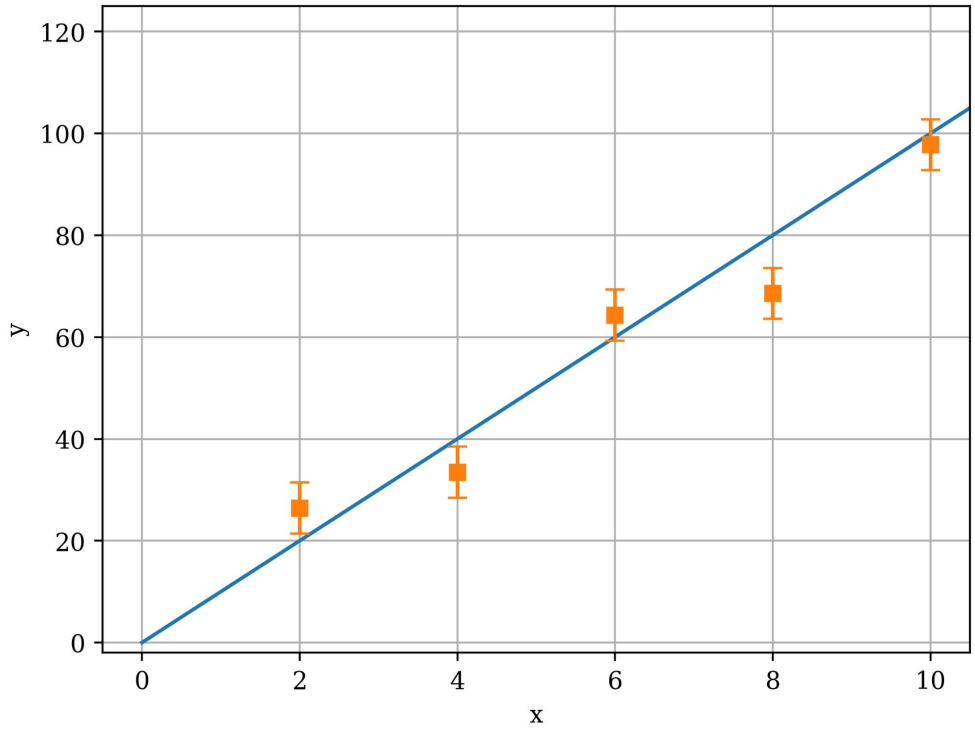
# Let’s make this realistic

These data points are observations, they should have some error to them

Let’s say they have a Gaussian error

Here’s 1 sigma error bars

How can we take into account the error doing least squares?



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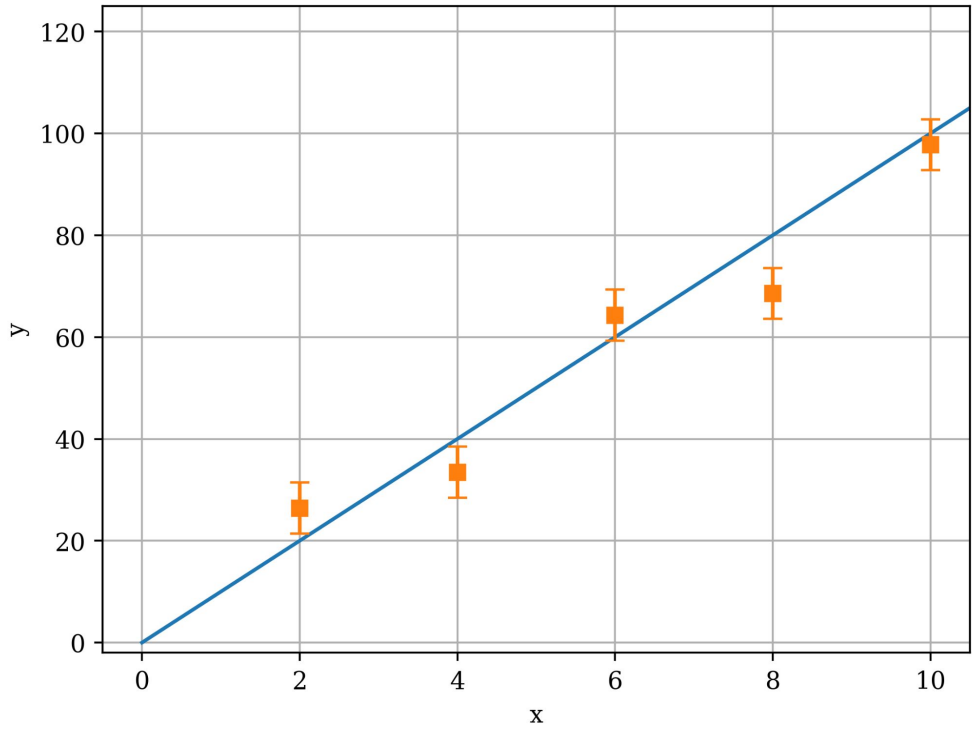
How can we take into account the error doing least squares?

Let’s divide the distance by σ

$s_i = ( (y(x_i) - y_i ) / \sigma_i )^2$

It’s now how many σ’s is it from the line

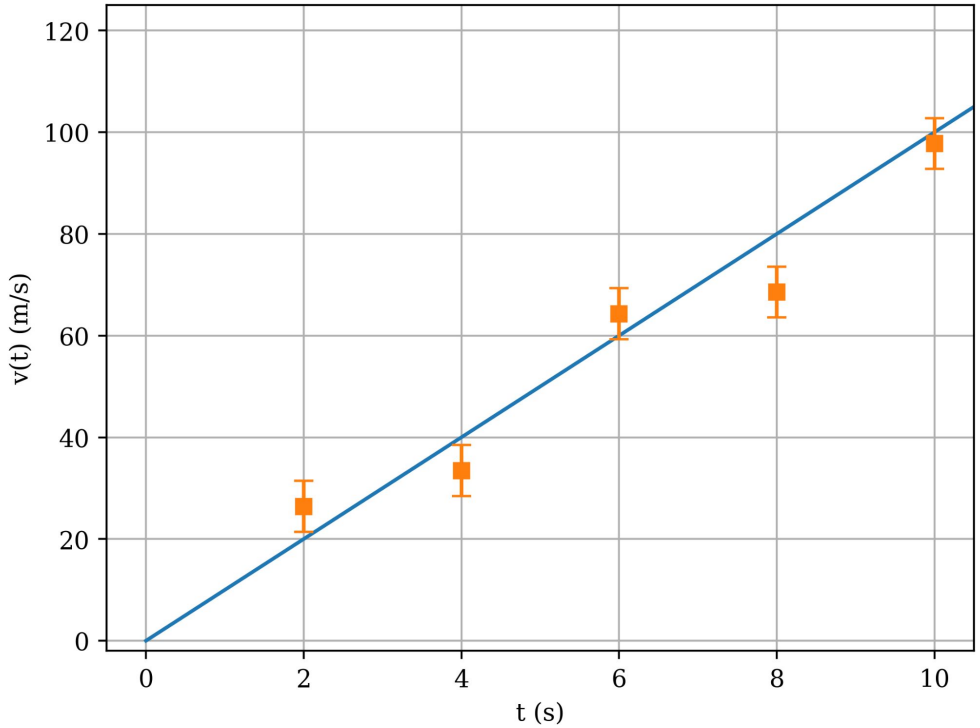


# Let’s make this realistic

More realism

Let’s say the y-axis is the speed of a car (dragster, this is pretty fast) and the x-axis is time

The speed is measured via a radar gun every 2 s, with a 1 sigma error of 5 m/s



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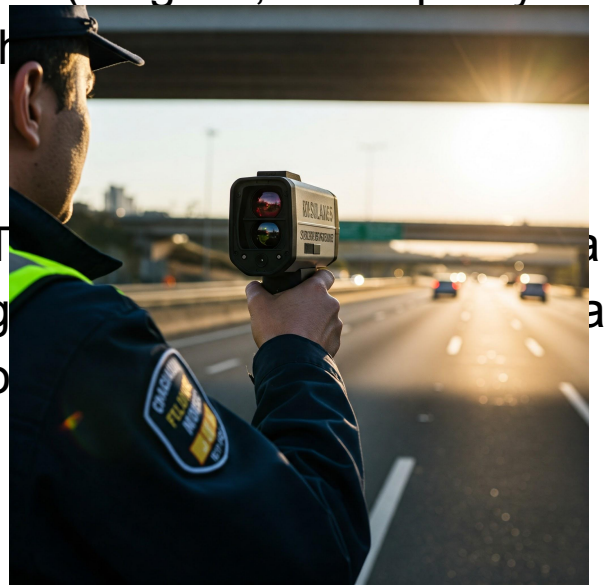

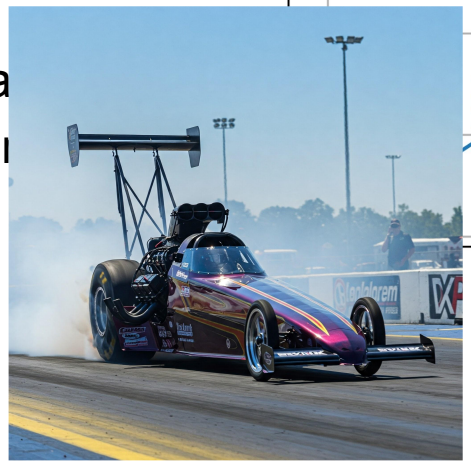

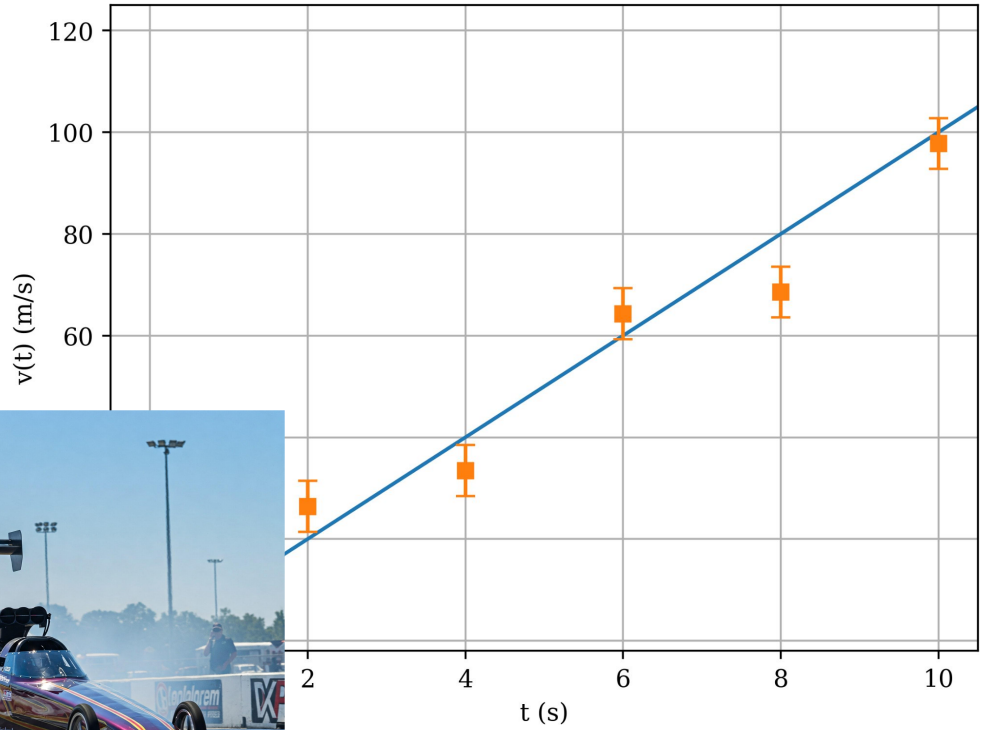

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In this case, what does the slope represents?

More realism

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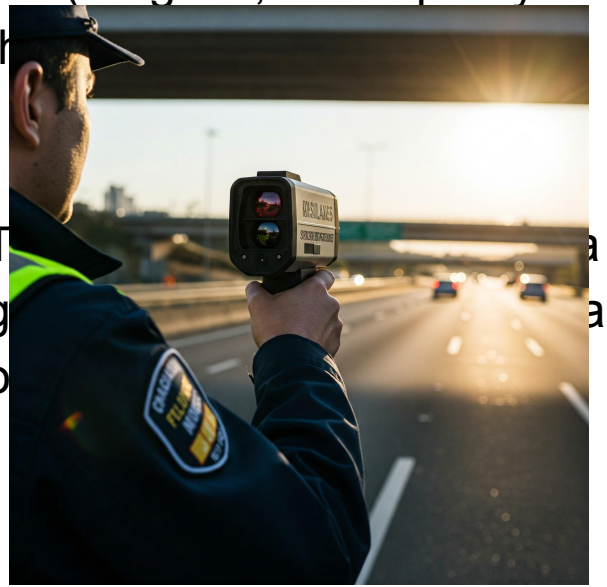

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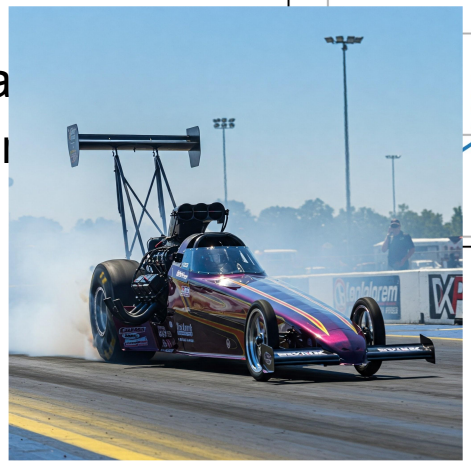
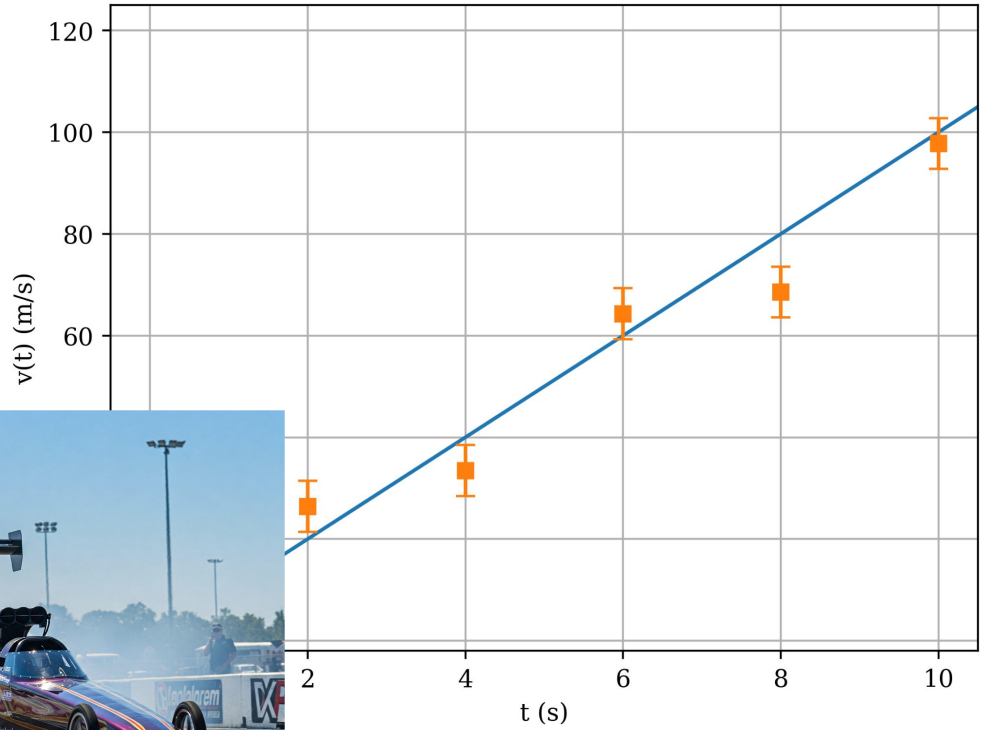




If the errors are Gaussian and

$y(x_i)$ gives us close to the “true” value

$(y(x_i) - y_i) / \sigma_i$ should follow a Standard Normal distribution (mu=0, σ=1)

$Z_i = (y(x_i) - y_i) / \sigma_i$

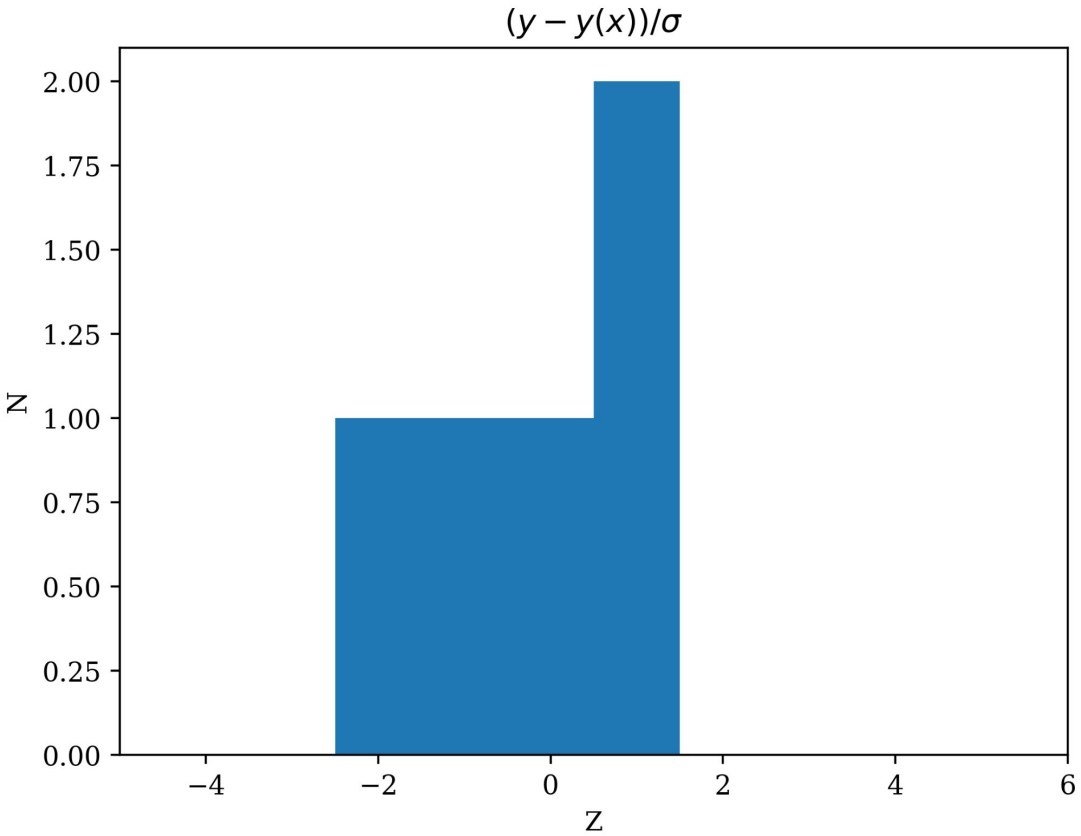
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}$$

S = the sum of a bunch of normally distributed values squared

Let’s plot the Z values

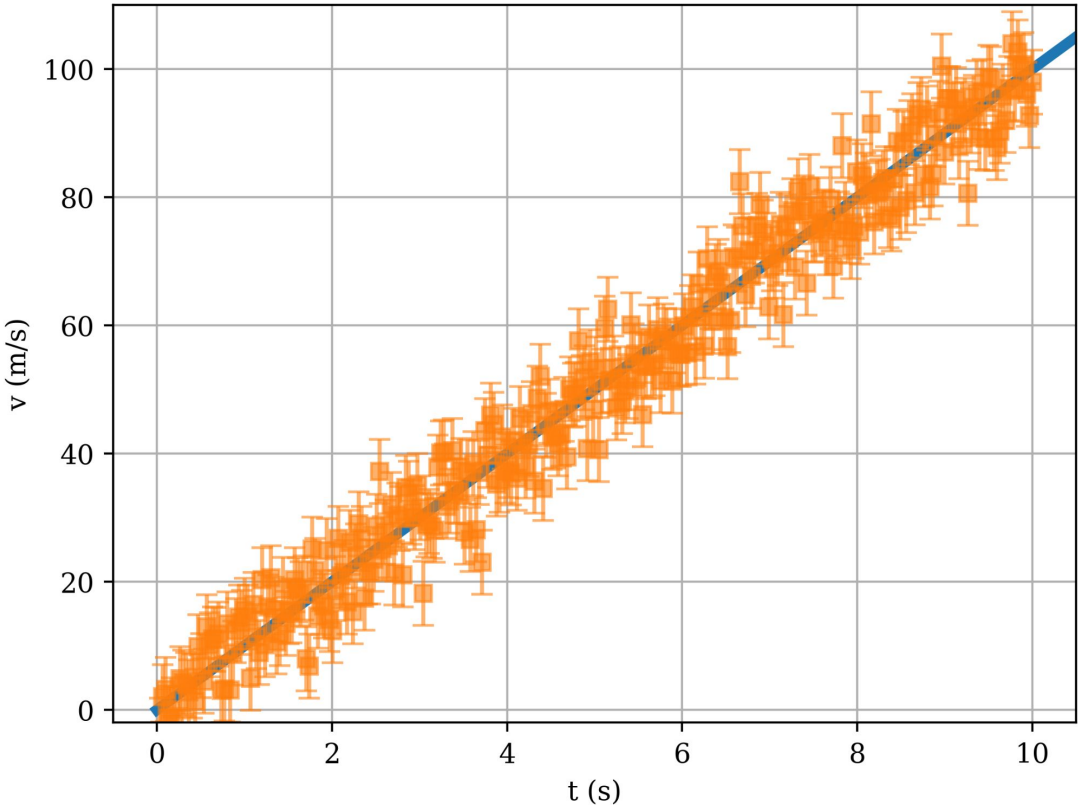
Clearly see a Gaussian,

right?



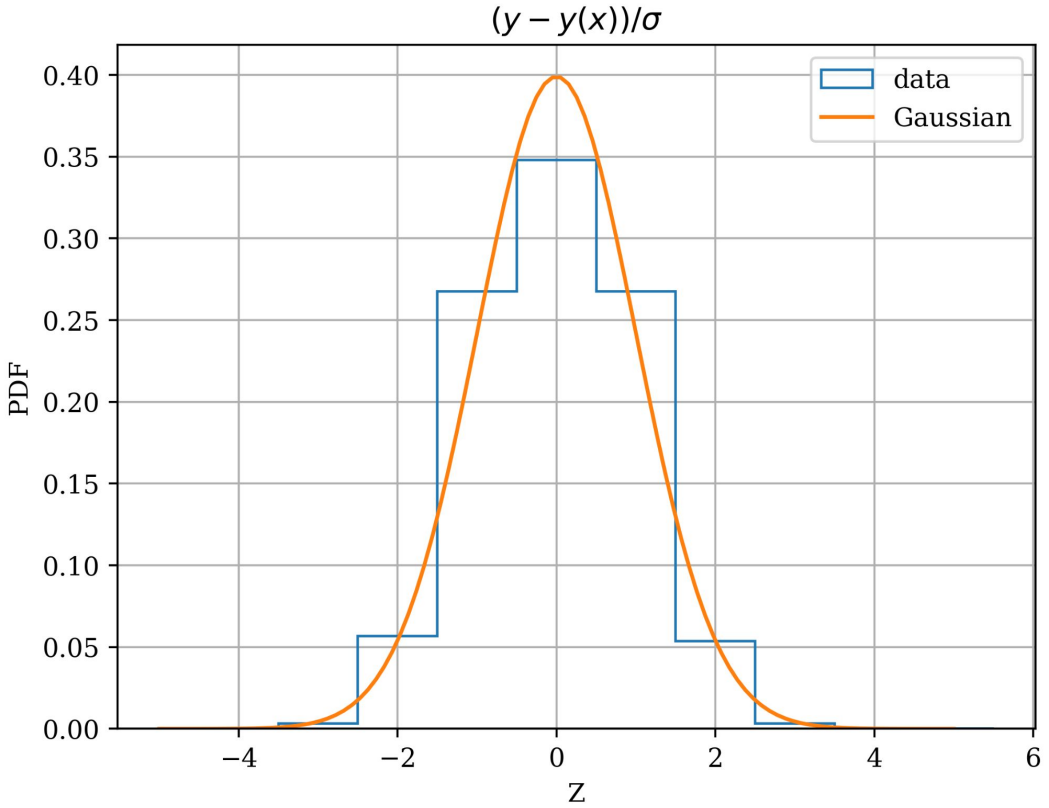
Let’s repeat this with more data points

Measure the speed 30 times a second



Let’s repeat this with more data points

Would be more exact with even more data, but actually looks like a standard normal now



So now, S or sum of squares is

S = the sum of a bunch of normally distributed values squared

What's special about that?

Well that sum happens to follow a specific probability distribution

The chi2 distribution $\chi^2$

The $\chi^2$ PDF depends on the number of values in the sum, k

The function is very complicated

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$$\frac{1}{2^{k/2}\Gamma(k/2)}\, x^{k/2-1} e^{-x/2}$$

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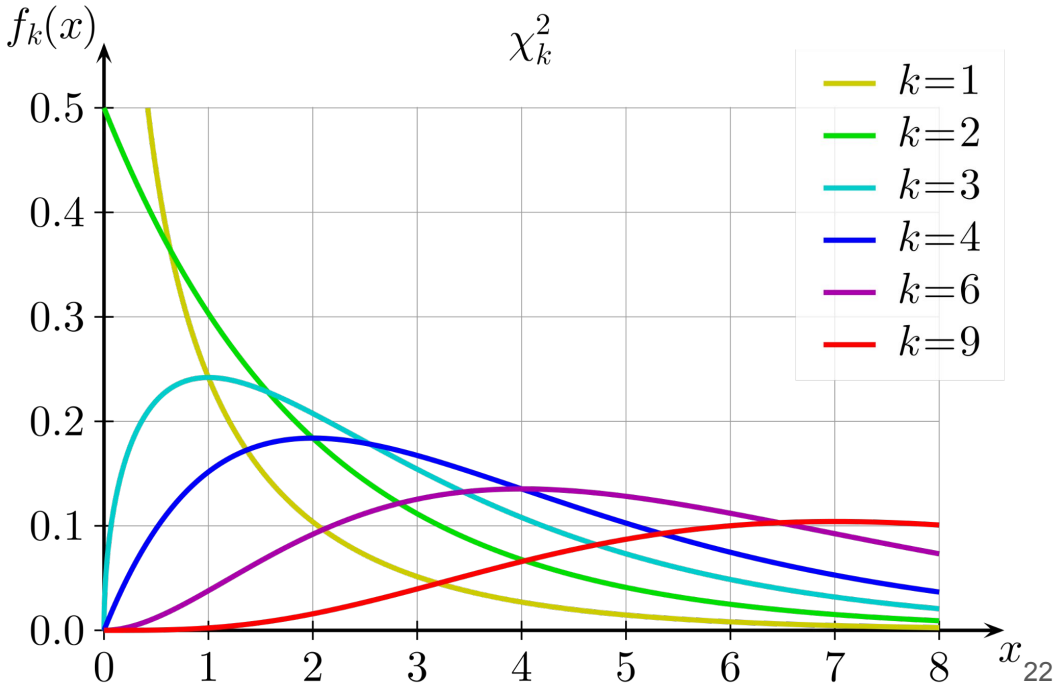

Now what does that mean S should follow a certain distribution?

S itself is an observation

Every time we rerun this experiment we will get a different S

Those values of S will follow a $\chi^2$ distribution

Ok so let's have the dragster do this 10,000 times

Accelerating at exactly 10 m/s2 each time

Each time we take our 5 measurements

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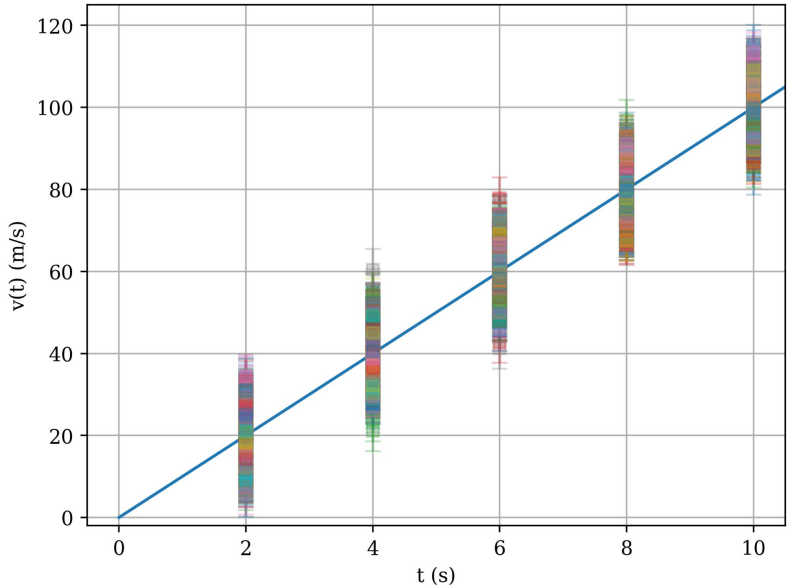

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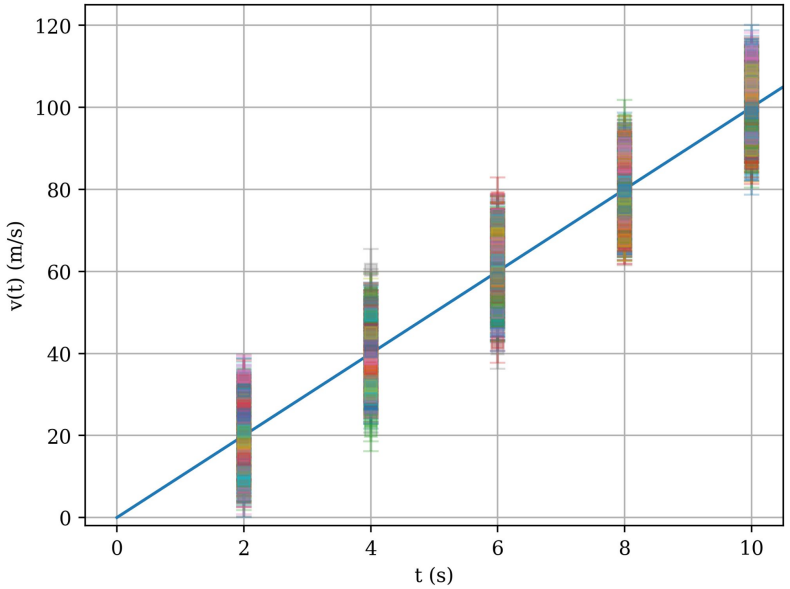
And calculate S

$S = \Sigma_i \left( (v(t_i) - v_i ) / \sigma_i \right)^2$

$v(t_i) = a*t_i$ , where a = 10 m/s2



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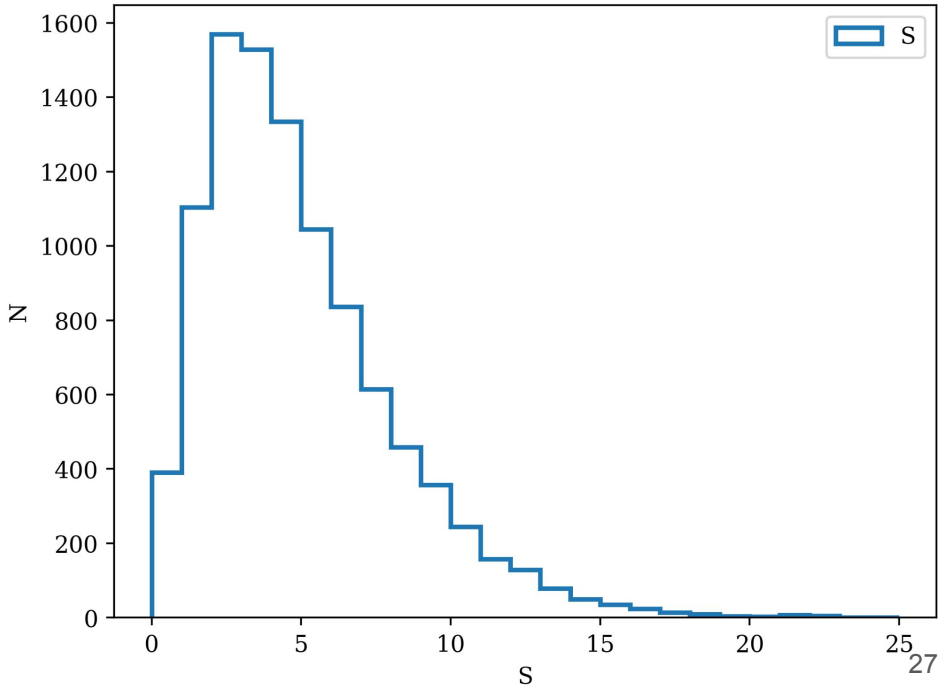

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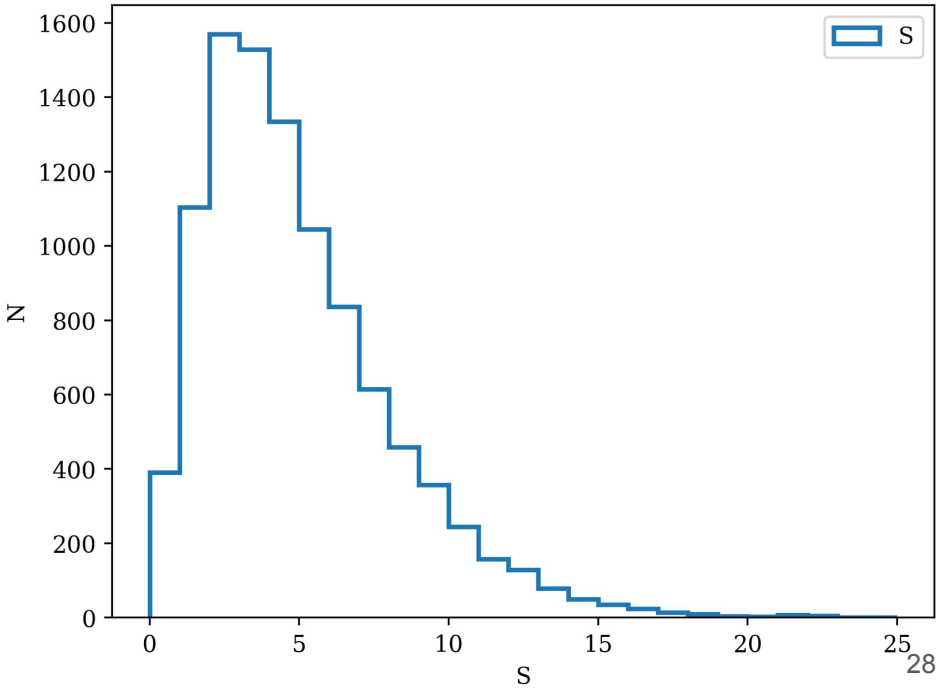
Now let's see how it compares to a $\chi^2$

What should k be?



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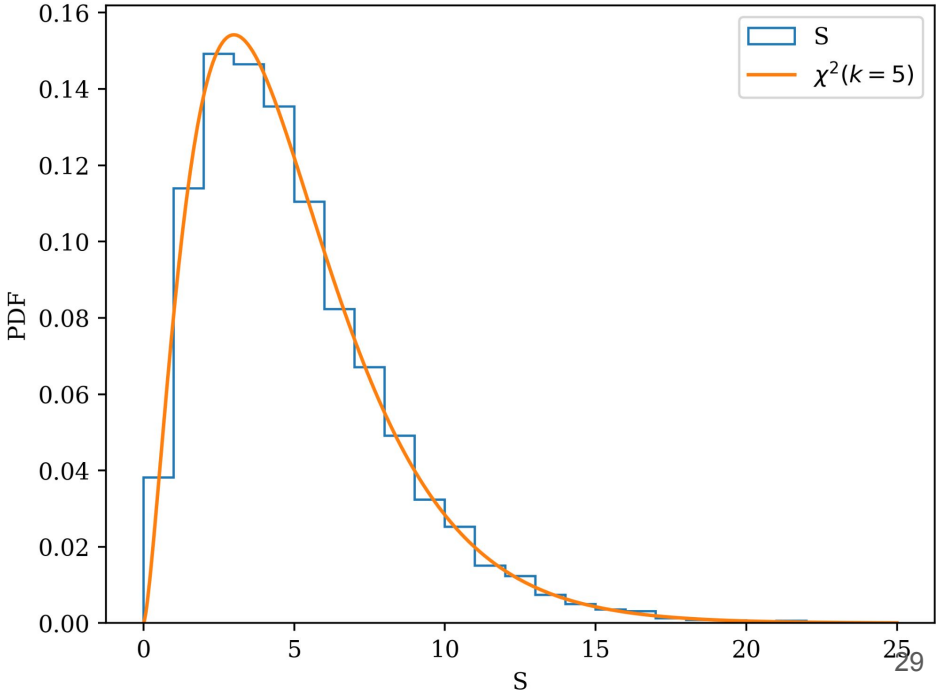
5 numbers in sum, k = 5



# Goodness of fit

S (or sometimes it's just called $\chi^2$ ), not only tells us what the best parameters are to describe our data

It also tells us how well the function describes our data

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This of course describes the data well

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The smaller the $\chi^2$ value the closer the function is to the data, but how big is too big?

# Goodness of fit

$P(\chi^2 > 11.07, k=5) = \int_{11.07}^{\infty} p(\chi^2, k=5)\, d(\chi^2) = 0.95$

This integral is also known as the survival function stats.chi2.sf(11.07)

This means that 95% of the time $\chi^2$ should be < 11.07

So if $\chi^2$ is above that there's a chance it does not fit well

# Goodness of fit

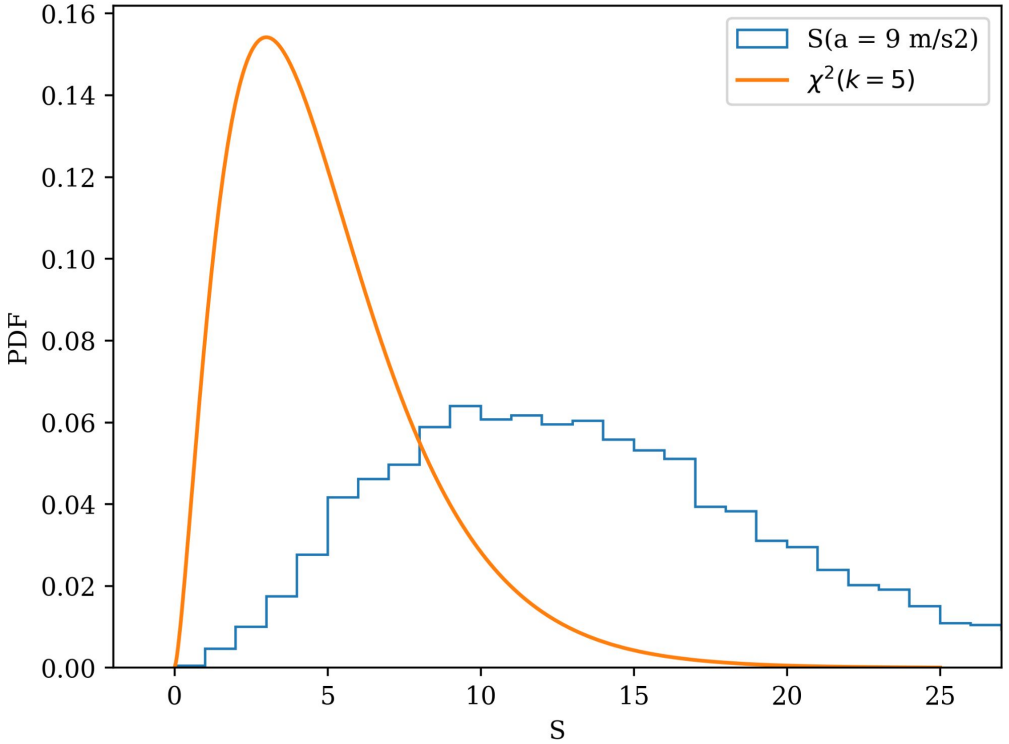
Let’s repeat the experiments and set $a = 9\ m/s^2$ when calculating $\square^2$

# Goodness of fit

Let's repeat the experiments and set a = 9 m/s$^2$ when calculating $\chi^2$

Distributions do not match any more

61.5% of the time $\chi^2$ was > 11.07



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And if we instead set a = 8 $m/s^2$

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# Goodness of fit

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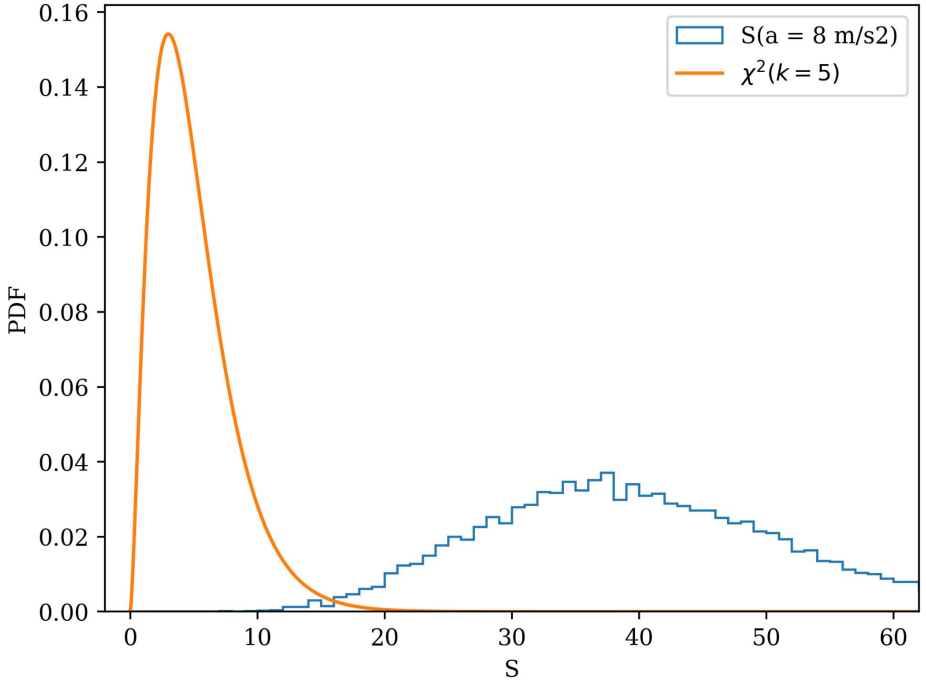
Distributions do not match at all

99.9% of the time $\chi^2$ was > 11.07



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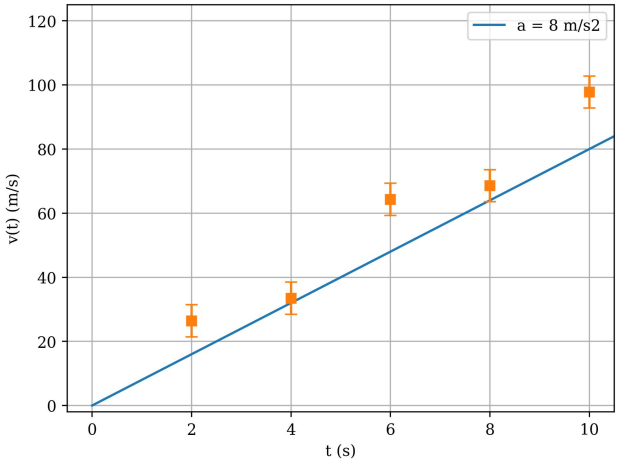


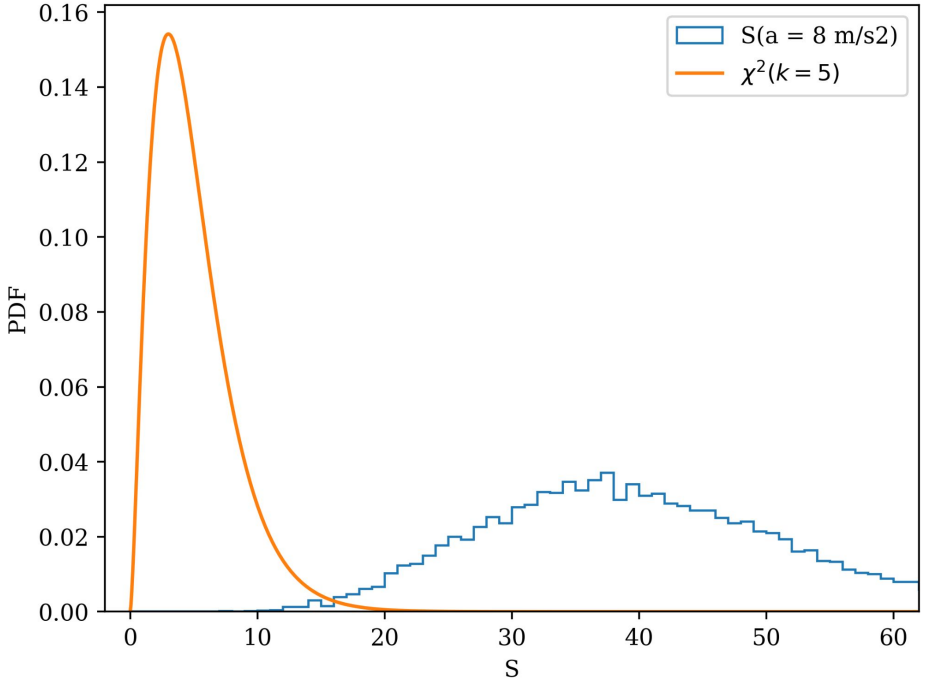

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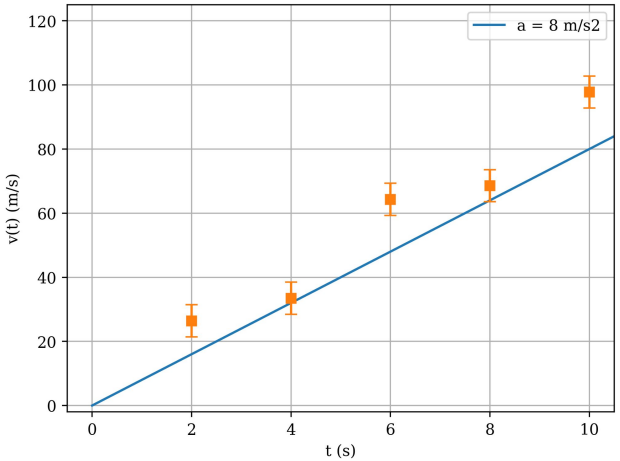


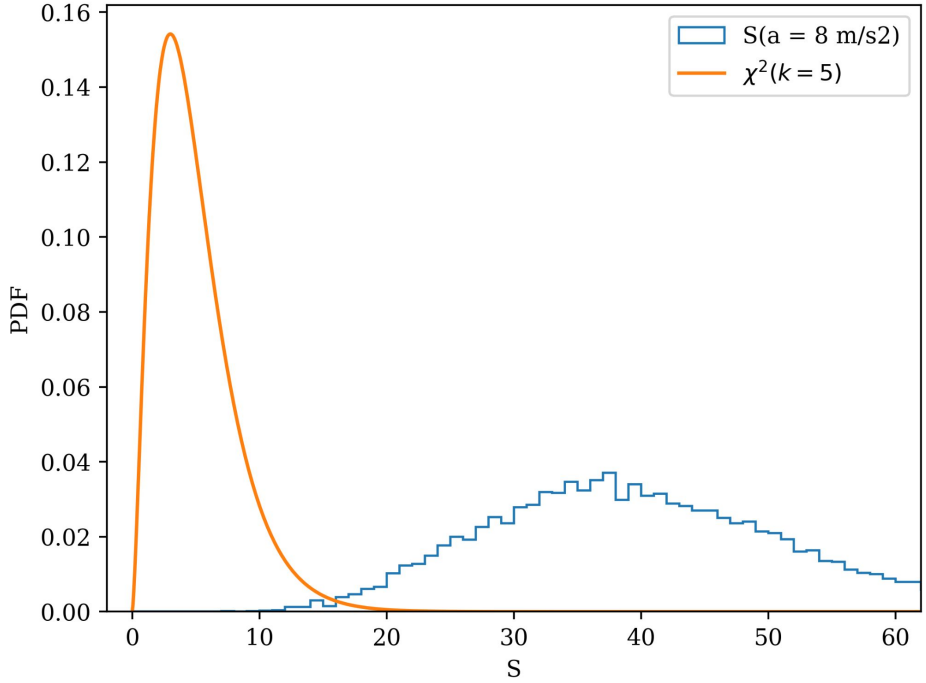


**We'll go more in depth on this next class**

# How to actually fit to the data

So far we haven't actually "fit" the data, we've only assumed some values of the slope

We started with a method called "least squares",

The optimal set of parameters will minimize S, or $\square^2$

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- Calculus
- Brute force
- A little bit of both - numerical minimization, we'll come back to this

# Calculus!

$$S = \sum_i \left( y_i - \left( m \cdot x_i \right) \right)^2$$

Assume b=0

$$\frac{\partial S}{\partial m} = 0 = -2 \sum_i \left( y_i - \left( m \cdot x_i \right) \right) x_i$$

$$0 = \sum_i \left( y_i x_i \right) - m \sum_i \left( x_i^2 \right)$$

$$m = \frac{\sum_i \left( y_i x_i \right)}{\sum_i \left( x_i^2 \right)}$$

An analytical solution like this really only exists for a linear function

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$$m = \frac{\sum_i \left( y_i x_i \right)}{\sum_i \left( x_i^2 \right)}$$

# Calculus!

$$S = \sum_i \left( \frac{y_i}{\sigma_i} - \frac{\left( m \cdot x_i \right)}{\sigma_i} \right)^2$$

$$m = \frac{\sum_i \left( y_i x_i / \sigma_i^2 \right)}{\sum_i \left( x_i / \sigma_i \right)^2}$$

With sigma included

$$m = \sum y_i x_i / \sum x_i^2$$

$m = \sum y_i x_i / \sum x_i^2$

m = (26.43x2 + 33.47x4 + 64.31x6 + 68.59x8 + 97.80x10) / (2^2 + 4^2 + 6^2 + 8^2 + 10^2)

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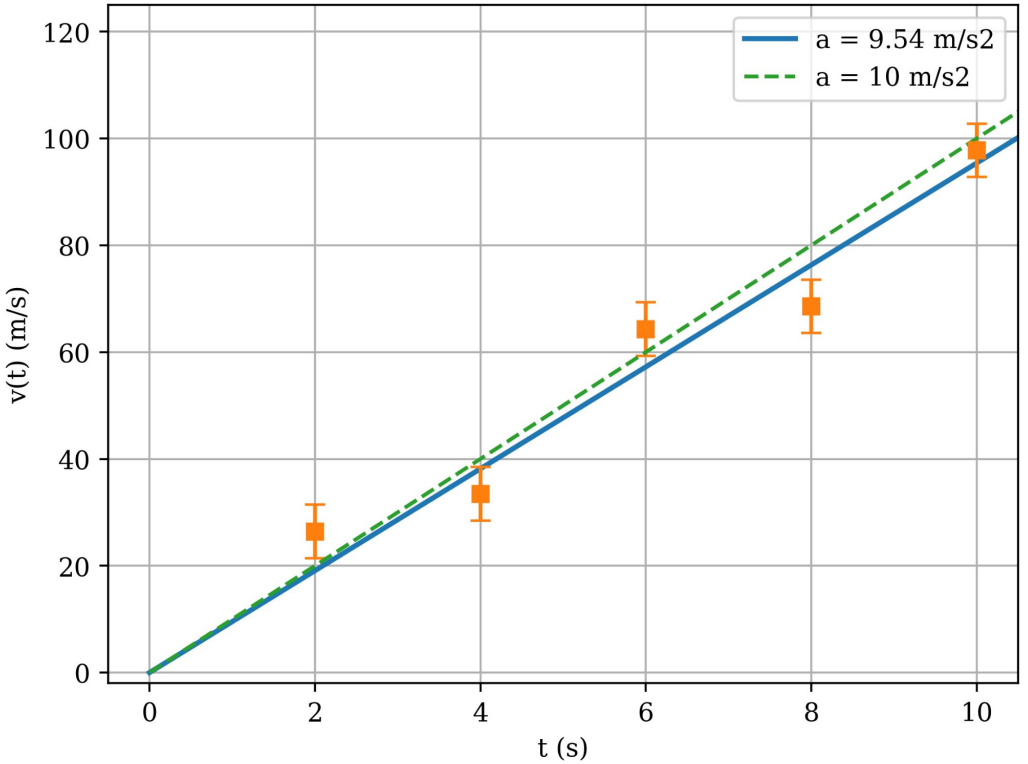

# Is this actually a minimum? What does S(m) look like?

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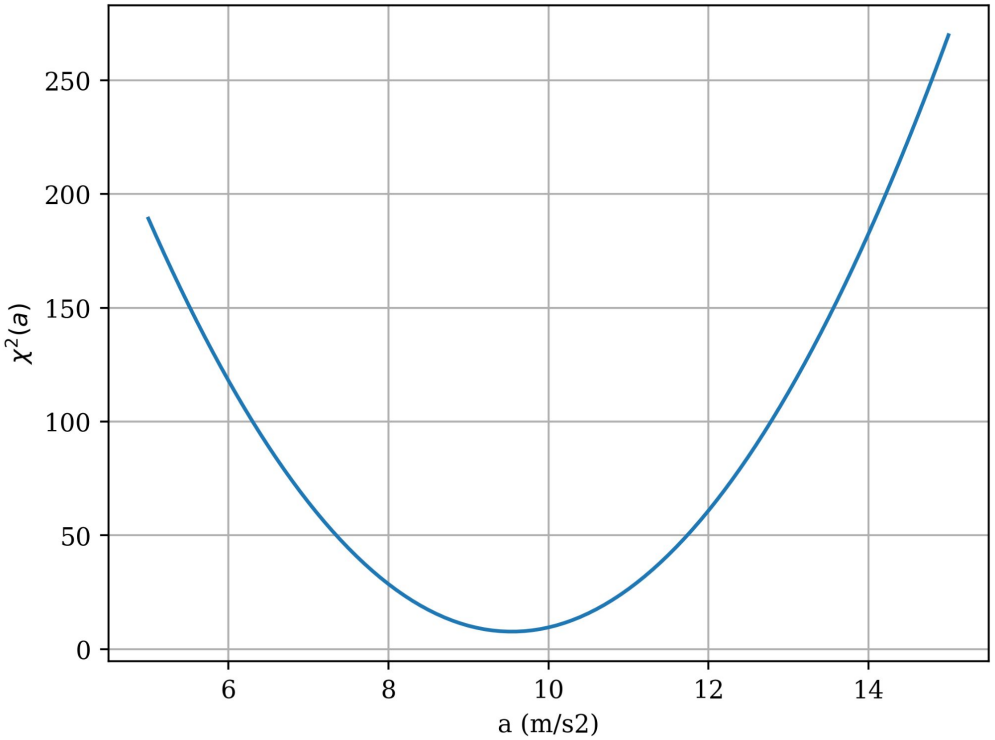
Yes, it is a minimum

S(m) is

-

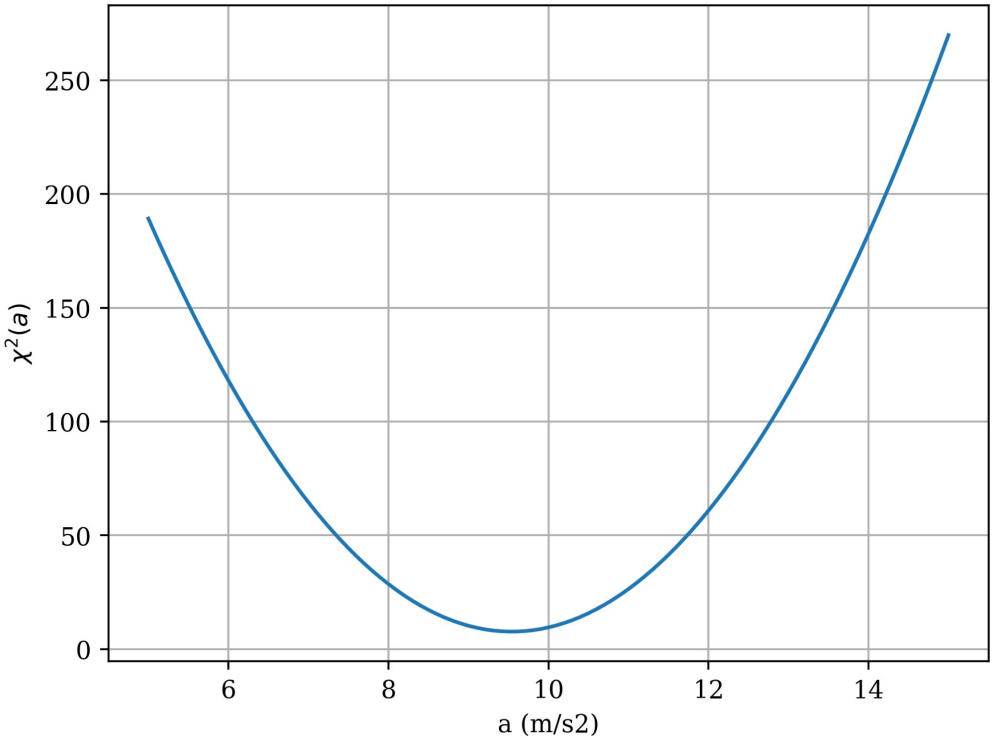


# Is this actually a minimum? What does S(m) look like?

Yes, it is a minimum

S(m) is

- parabolic
- has a clear single minimum
- continuous



# Brute Force!

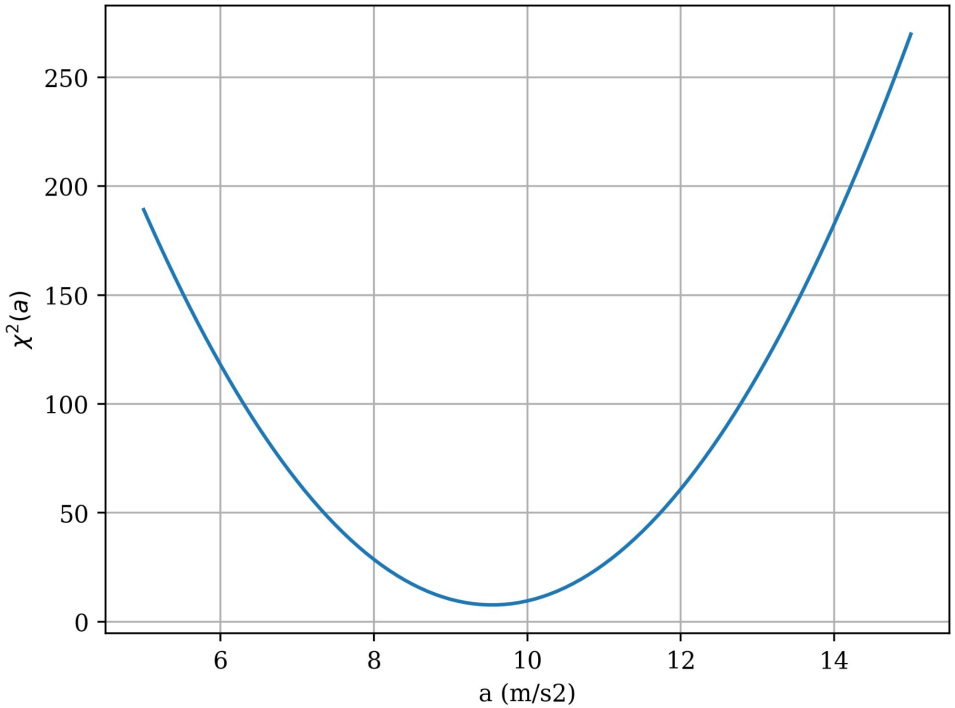
The brute force method works just like this

```
a_arr = np.linspace(0, 100, big_integer)

for i in range(big_integer):

        Chi2_arr[i] = np.sum(((y - a_arr[i]*t)/sigma)**2)

a_best = a_arr[np.argmin(Chi2_arr)]
```



# Brute Force!

Obvious downsides -

Takes a lot more compute power

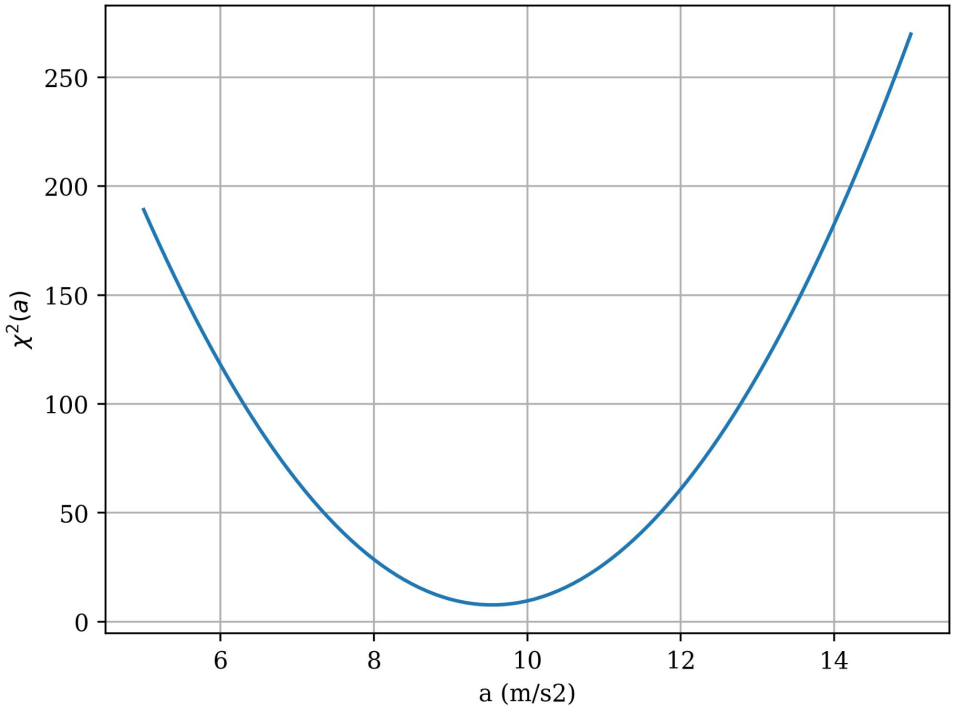
Not as accurate

- only as accurate as step size
- Need to know bounds to start and end at

Pros -

Very simple to do!

There doesn't need to be an analytical solution



# Next Time!

- Here we found the “best-fit” parameter for a model using data
- Next time we will find our error PDF for that parameter
- We will also assess whether a model, even with its “best-fit” parameters describes the data well