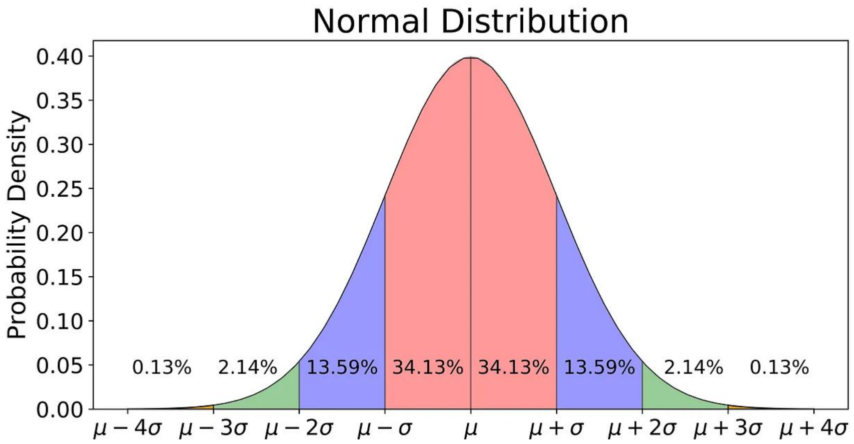
More Statistics and How to Count

Last time

- We left off talking about the
  - Normal/Gaussian distribution
  - Central limit theorem
    - Saw this in action in the tutorial



How to interpret observations/measurements

Measurements of non-discrete parameters are never exact

There will always be some error to a measurement

How to interpret observations/measurements

Measurements of non-discrete parameters are never exact

There will always be some error to a measurement

There are 2 types of errors -

- Random: makes fluctuations of measurements above and below the actual value
  - These should cancel out over many observations
  - Creates a spread in your measurements that should average to actual value
- Systematic: creates an offset in one direction relative to the actual value
  - These do not average away over many observations
  - Creates a bias in your measurements

How to interpret observations/measurements

Measurements of non-discrete parameters are never exact

There will always be some error to a measurement

There are 2 types of errors -

- **Random**: makes fluctuations of measurements above and below the actual value
    - These should cancel out over many observations
    - Creates a spread in your measurements that should average to actual value
- Systematic: creates an offset in one direction relative to the actual value
    - These do not average away over many observations
    - Creates a bias in your measurements

We’re going to focus on random errors

Taking a measurement

Random errors can often be assumed to be “Gaussian”

Errors are often reported as “1-sigma” errors

What does that actually mean though?

Say you try to measure the energy of a photon with a detector

The detector has a known uncertainty to its measurements

Say you try to measure the energy of a photon with a detector

The detector has a known uncertainty to its measurements

- $\sigma = 1$ keV

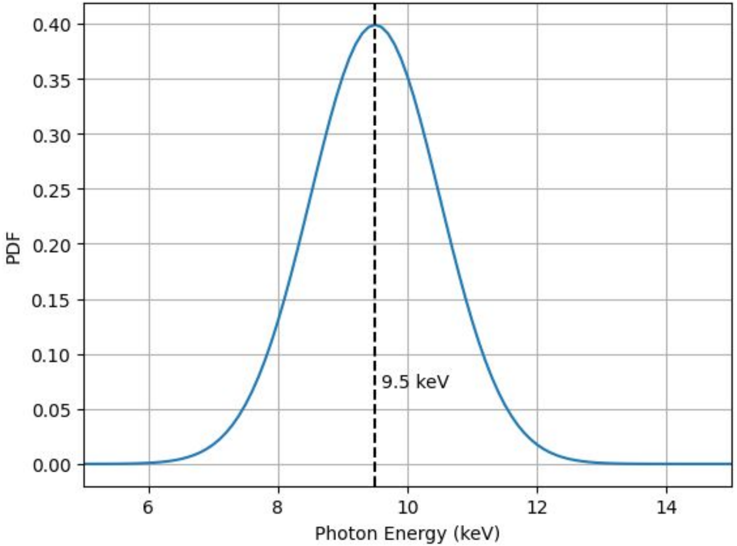
A photon comes in and the measured energy is 9.5 keV

- How do we interpret that?
- Is that actual energy definitely 9.5 keV?
- If not, what is it?

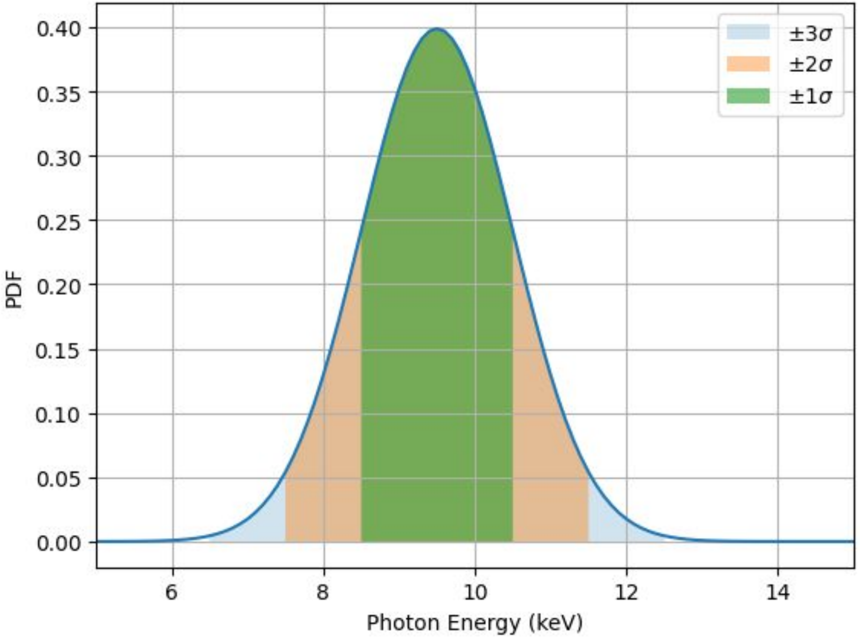
We don’t know the exact photon energy

What we do know now is the probability distribution of what the exact photon energy could be

PDF = Normal Distribution = $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$



Remember we can integrate our PDF to get probability in an interval



$$P(\,8.5\,keV \le E \le 10.5\,keV) = \int_{8.5}^{10.5} p(\,x)\,dx$$

$$P(\,(\mu-\sigma) \le x \le (\mu+\sigma)\,) = \int_{(\mu-\sigma)}^{(\mu+\sigma)} N(\mu,\sigma;\,x)\,dx = \sim 0.68$$

P(8.5 keV <= E <= 10.5 keV) = 0.683 1-sigma
P(7.5 keV <= E <= 11.5 keV) = 0.955 2-sigma
P(6.5 keV <= E <= 12.5 keV) = 0.997 3-sigma

Error Bars / Confidence Intervals

- Errors on a measurement are usually reported in a range
  - E = 9.5 ± 1 keV
  - E = [8.5 keV, 10.5 keV] - 68% credible interval
  - E = [7.5 keV, 11.5 keV] - 95% credible interval
  - …

- But it’s important to remember these are a snapshot of a full distribution

Taking more measurements

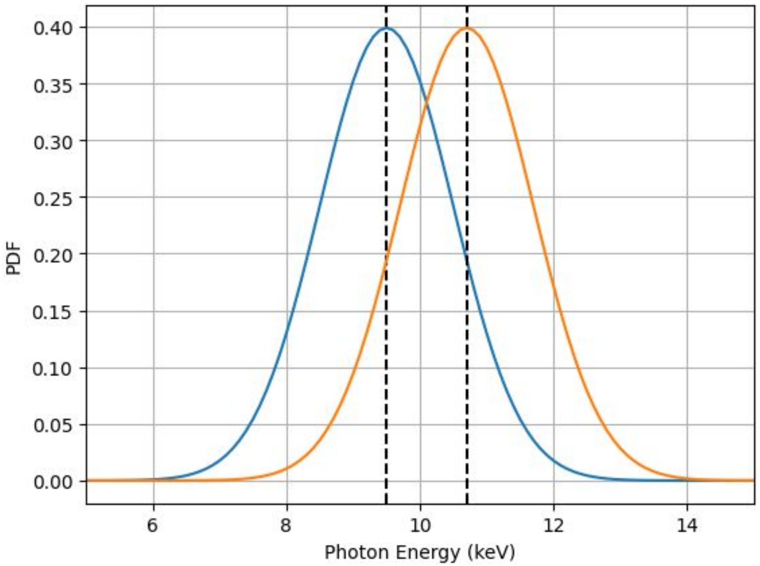
Say we have a machine that emits photons at only one energy

We make measurements of the photon energy for multiple photons

Taking more measurements

Say we have a machine that emits photons at only one energy

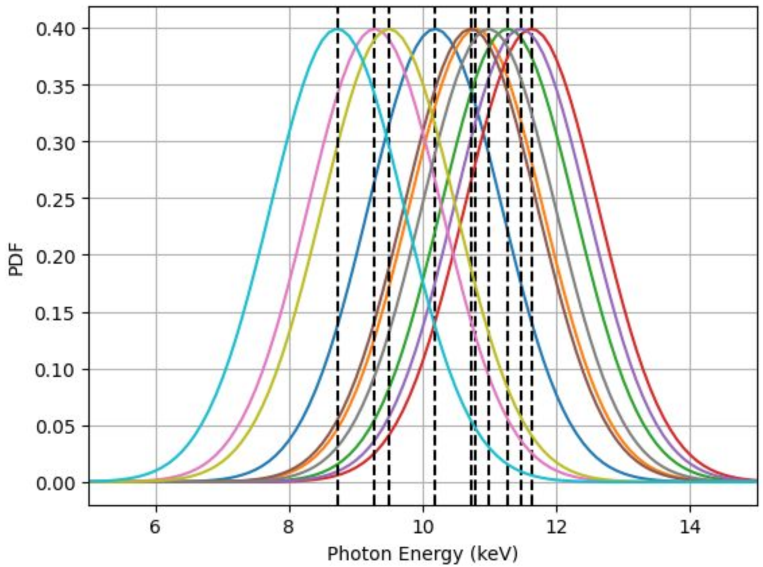
We make measurements of the photon energy for multiple photons



Taking more measurements

Say we have a machine that emits photons at only one energy

We make measurements of the photon energy for multiple photons



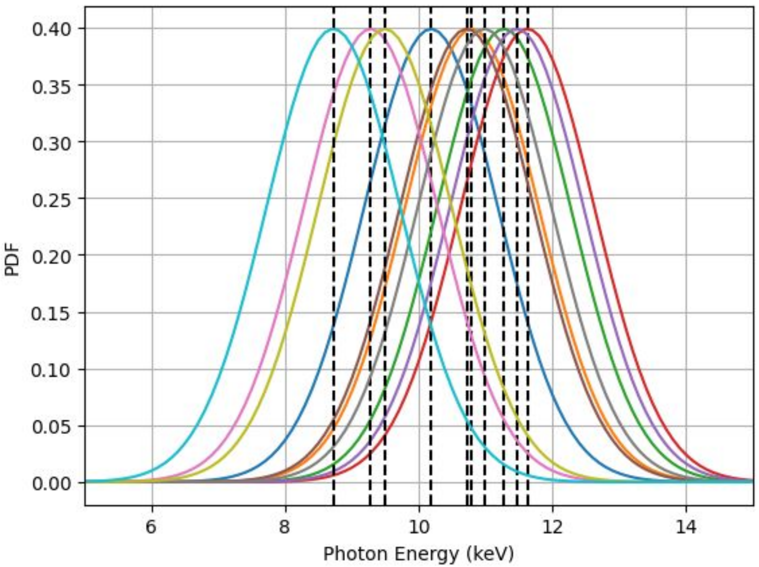
Taking more measurements

Say we have a machine that emits photons at only one energy

We make measurements of the photon energy for multiple photons

This should improve our knowledge, right?

- We should be more certain about what the photon energy is?



Taking more measurements

Say we have a machine that emits photons at only one energy

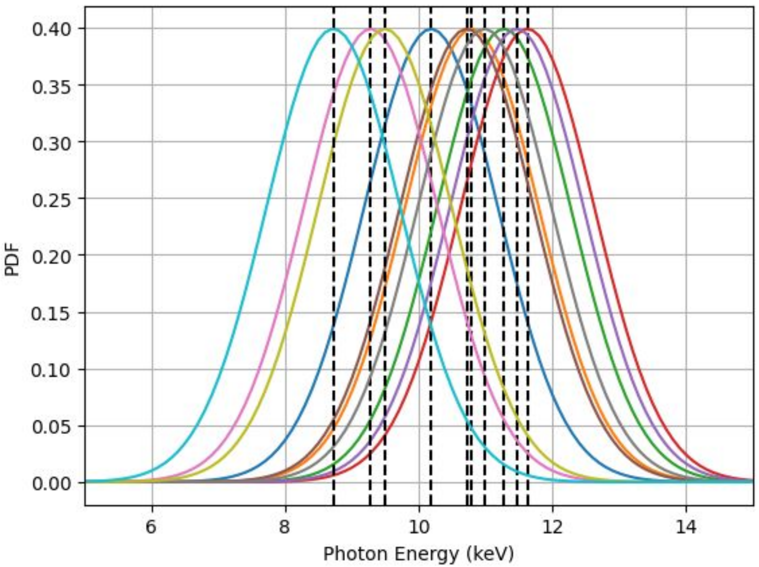
We make measurements of the photon energy for multiple photons

This should improve our knowledge, right?

- We should be more certain about what the photon energy is?

But how?

How do we combine this information?



Answer - Math

Let’s skip the doing math part and go straight to the answer

After multiple measurements the new most likely answer is

- The average of the measurements

The average will tend towards the actual energy as you take more and more measurements

What about our error?

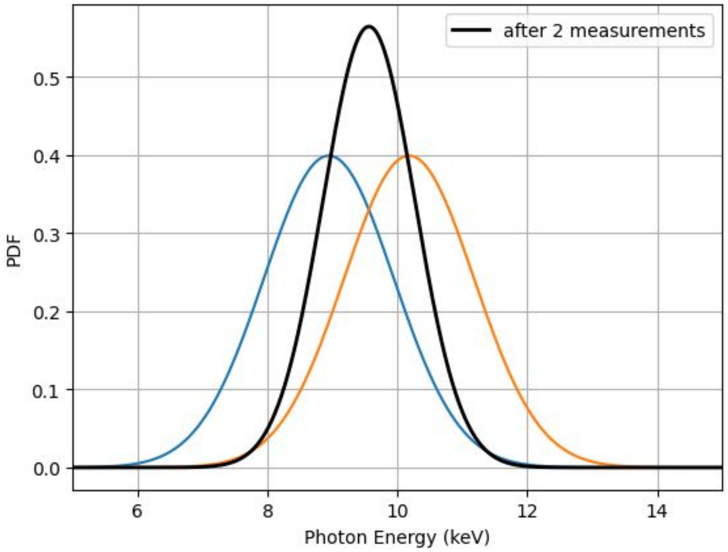
Now we're measuring an average

What's the error PDF from a measurement of a mean?

- Gaussian again
- μ = your measured average
- σ = (uncertainty on single measurement) / $\sqrt{n}$
    - n = number of measurements

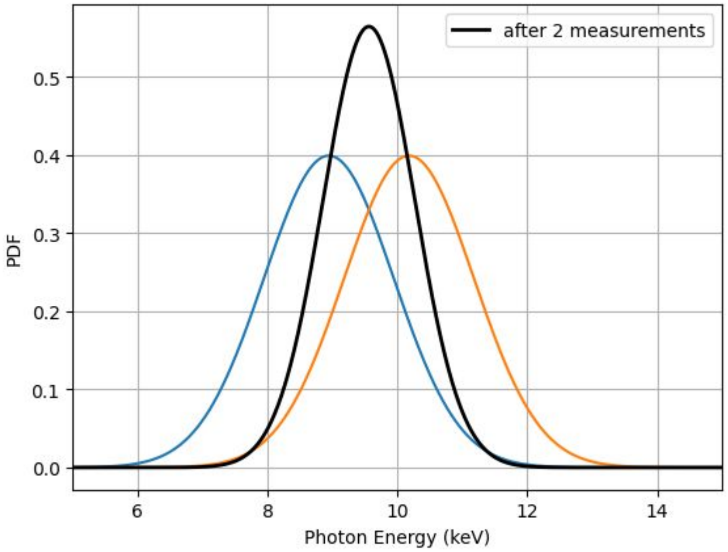
Your error PDF will get narrower as you take more measurements

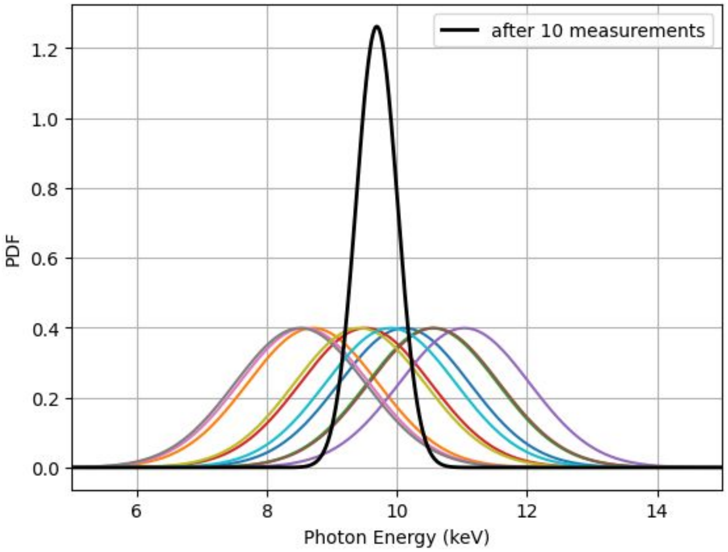
- 1 sigma error bar $\propto 1 / \sqrt{n}$



After 2 measurements

E = 9.56 ± 0.71 keV



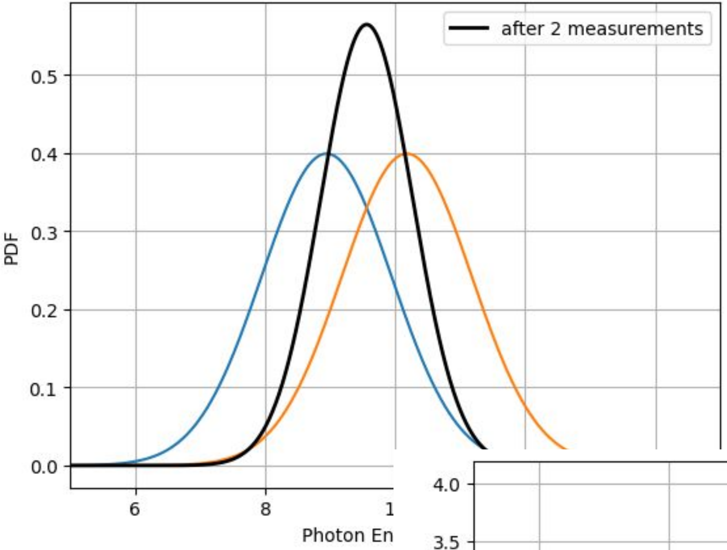


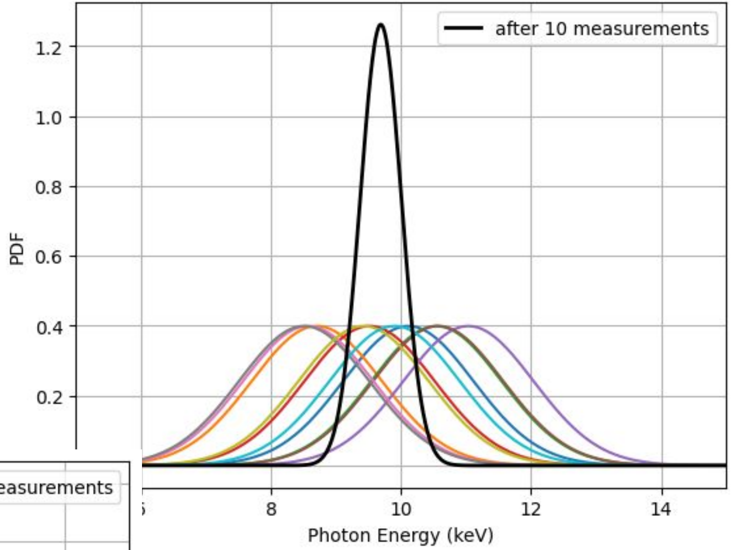
After 2 measurements

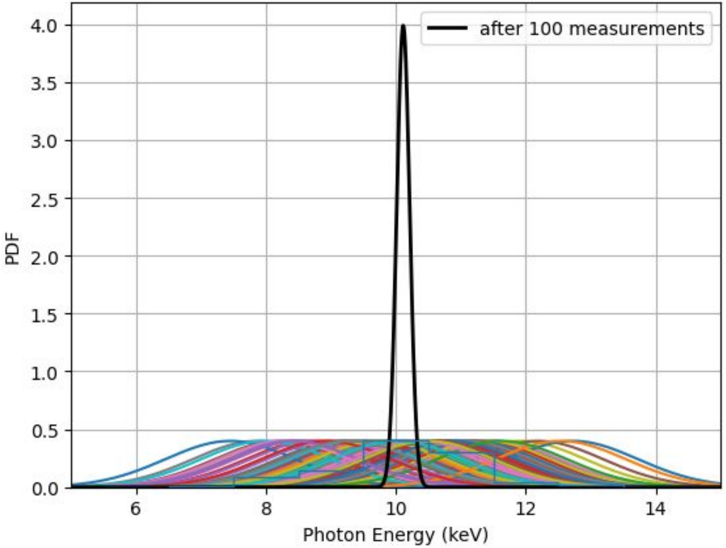
E = 9.56 ± 0.71 keV

After 10

E = 9.69 ± 0.32 keV







After 2 measurements

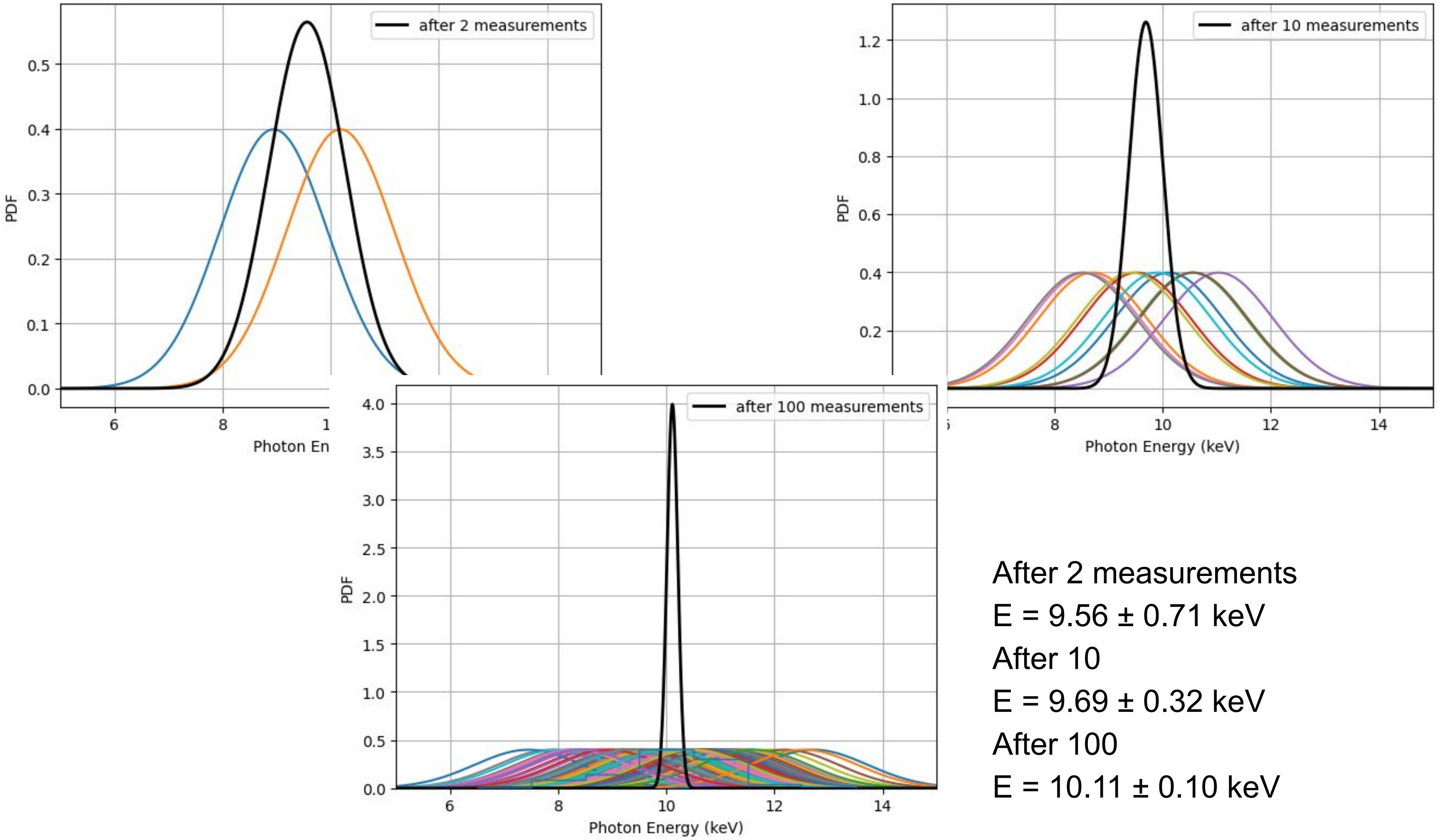
E = 9.56 ± 0.71 keV

After 10

E = 9.69 ± 0.32 keV

After 100

E = 10.11 ± 0.10 keV



After 2 measurements

E = 9.56 ± 0.71 keV

After 10

E = 9.69 ± 0.32 keV

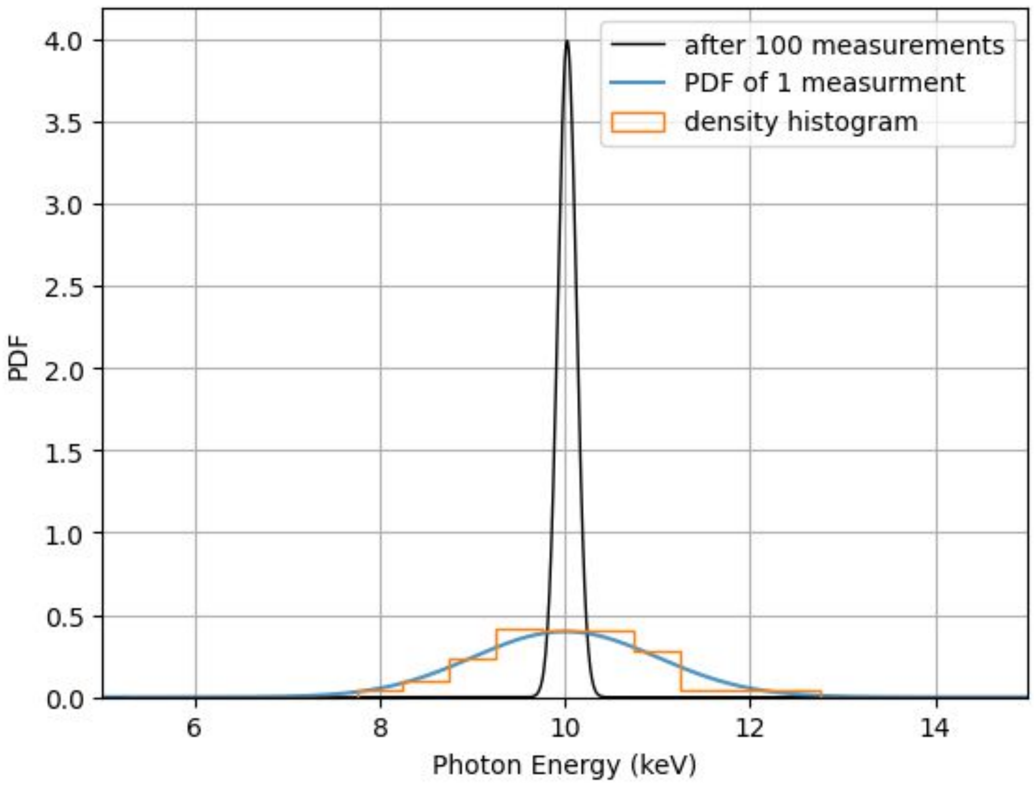
After 100

E = 10.11 ± 0.10 keV

Remember that the distribution of measurements still has σ = 1 keV

It's only our knowledge from combining the measurements that's getting narrower

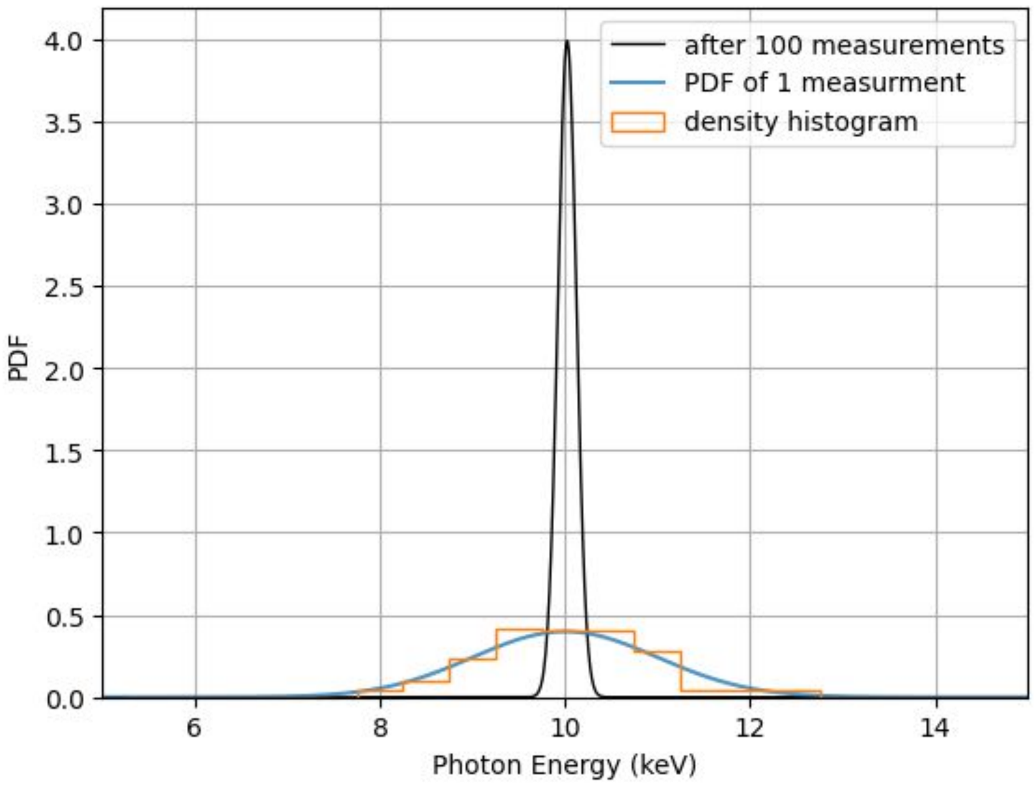
- The narrow PDF is really the PDF of the mean of the measurements



Flashback to last time from our probability theory perspective

Each measurement was like sampling from the true underlying distribution

- Our narrow PDF is us narrowing in on the true value of the mean for that true underlying distribution



Counting Statistics

A lot of astronomy is counting things, especially photons

If a source has a steady flux the average number of photons per time stays the same

But, when each photon arrives (or is emitted) is a random process.

Counting Statistics

A lot of astronomy is counting things, especially photons

If a source has a steady flux the average number of photons per time stays the same

But, when each photon arrives (or is emitted) is a random process.

Say the rate of detection is on average 1 per minute

Photon 1 has no bearing on photon 2

- Photon 2 does not wait 1 minute after photon 2

Counting Statistics

A lot of astronomy is counting things, especially photons

If a source has a steady flux the average number of photons per time stays the same

But, when each photon arrives (or is emitted) is a random process.

Say the rate of detection is on average 1 per minute

Photon 1 has no bearing on photon 2

- Photon 2 does not wait 1 minute after photon 2

For a given rate of random arrivals, the number that will arrive in a given time is a probabilistic distribution

- Found via math, which we'll skip again

Poisson

This is known as a Poisson process

The probability distribution of the number arrivals (or counts) in a given amount of time is given by the Poisson distribution

Counts are integers, which are **not continuous**, they are **discrete**

Poisson distribution is a probability mass function (PMF)

- From last class, a PMF gives us the probability for each possible outcome

Poisson

This is known as a Poisson process

The probability distribution of the number arrivals (or counts) in a given amount of time is given by the Poisson distribution

Counts are integers, which are not continuous, they are discrete

Poisson distribution is a probability mass function (PMF)

- From last class, a PMF gives us the probability for each possible outcome

$$P(N;\mu) = \frac{(\mu)^N e^{-\mu}}{N!}$$

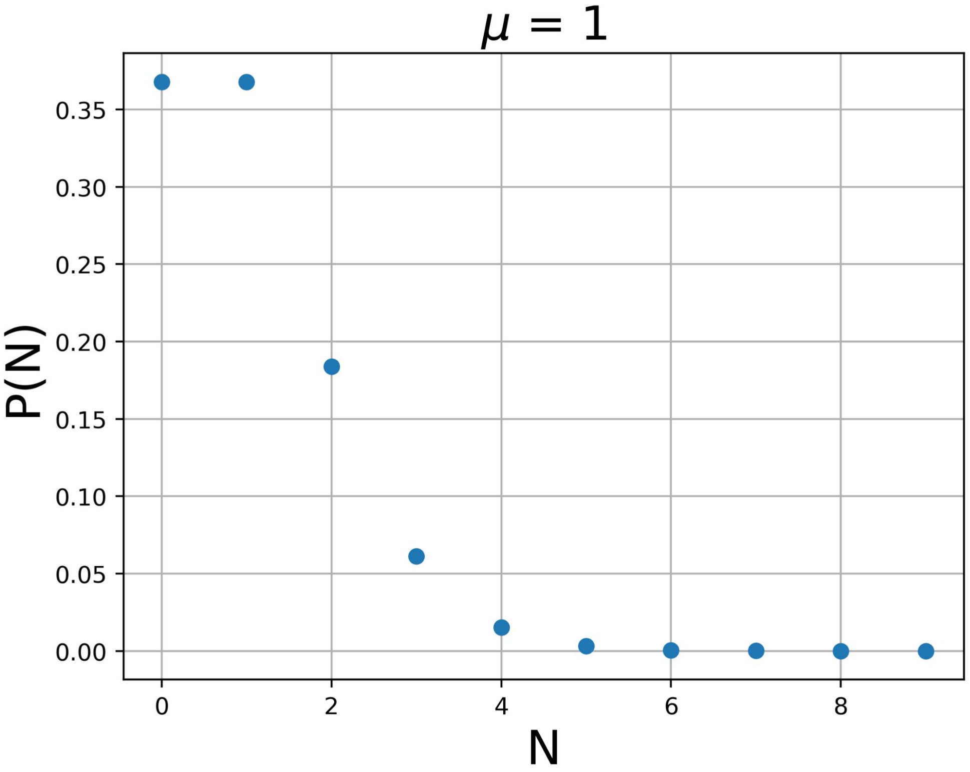
N = number of counts or arrivals
μ = average counts
μ = average rate * time
! means factorial

$P(0;\mu=1) = 0^1\ e^{-1} / 0!$

$P(0;\mu=1) = e^{-1} = 0.368$

$P(1;\mu=1) = 1^1\ e^{-1} / 1!$

$P(1;\mu=1) = e^{-1} = 0.368$



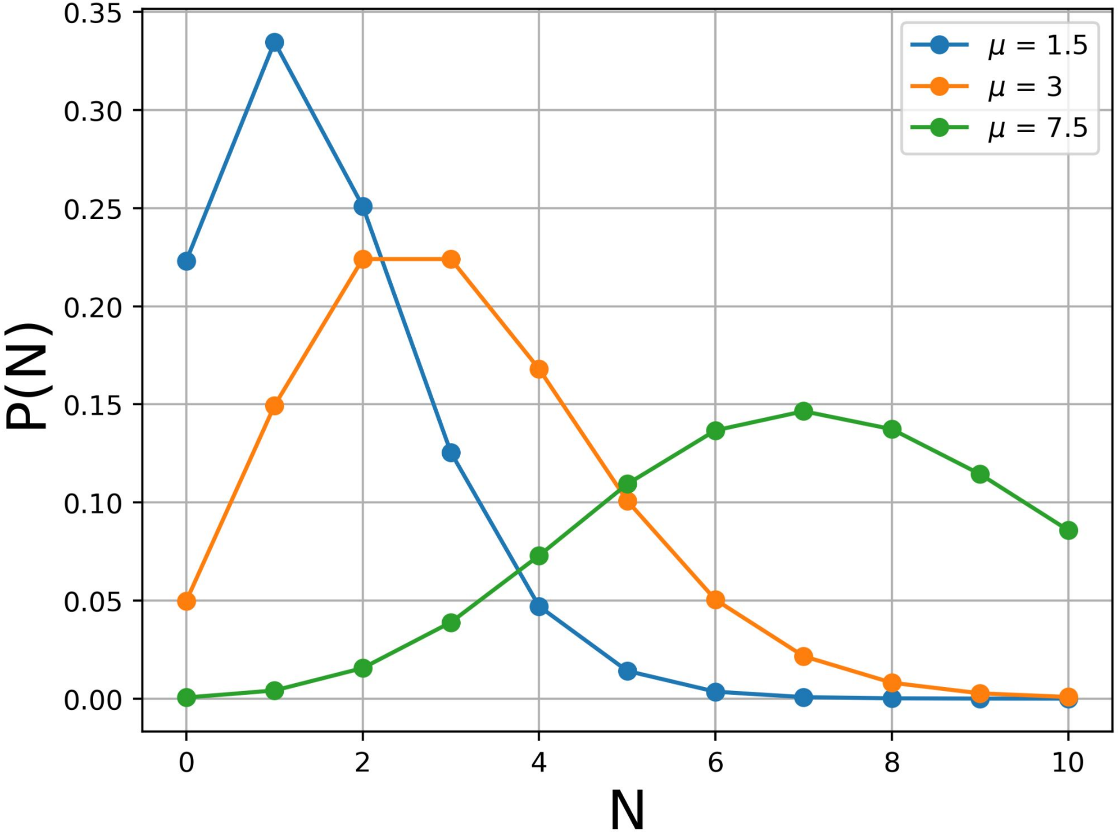
μ does not have to be an integer, and usually isn't

N has to be a counting number

- Integer >= 0

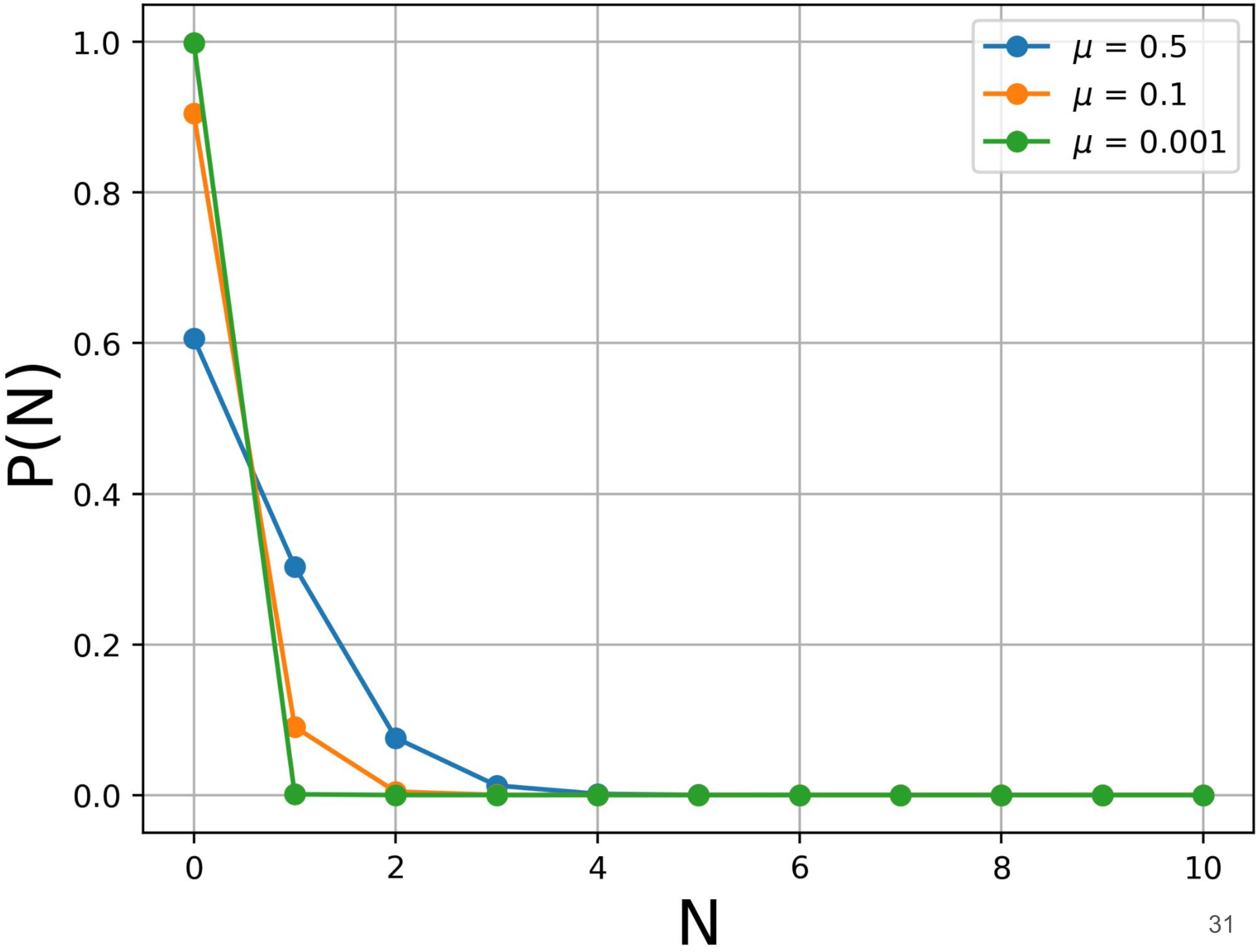
The mean of the Poisson distribution is μ

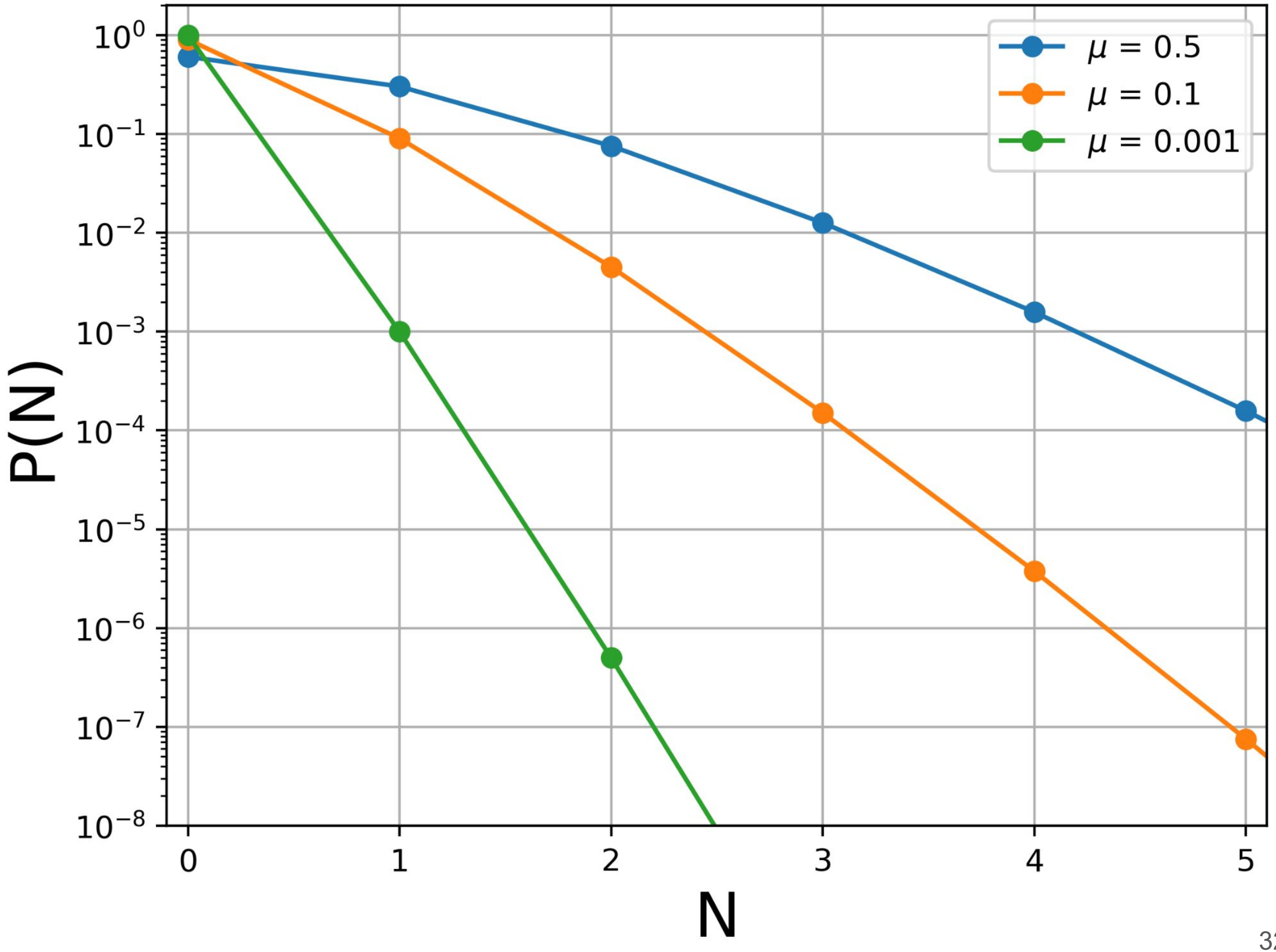
The variance is also μ



μ can be < 1

It can also be << 1





10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
P(N)
0
1
2
3
4
5
N
μ = 0.5
μ = 0.1
μ = 0.001

$P(1;\mu) = \mu^1 e^{-\mu} / 1!$

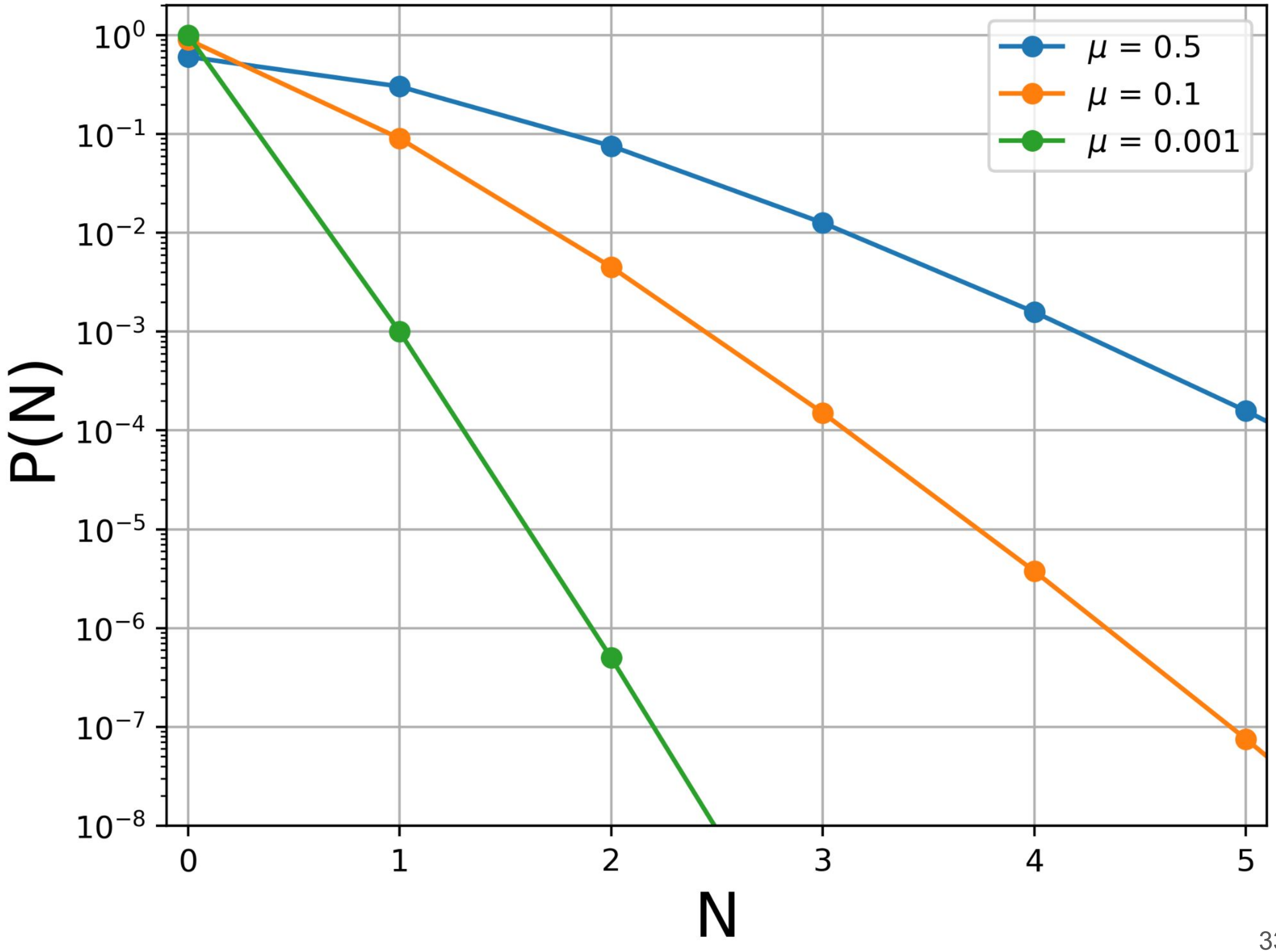
when $\mu << 1$

$e^{-\mu} \sim e^0 = 1$

$P(1;\mu) \sim \mu$

$P(1;\mu=10^{-3}) \sim 10^{-3}$

Useful for rare things



The Poisson distribution gives us the distribution of probabilities of the possible outcomes (how many counts), given a known rate

What if instead we want to measure the rate

- Like we measured the energy of the photon

This will be partly the same

- The best estimate will still be your observation
    - Or the mean over several observations
- Your estimate will still improve with more observations

But, the error on the estimate will be a little different

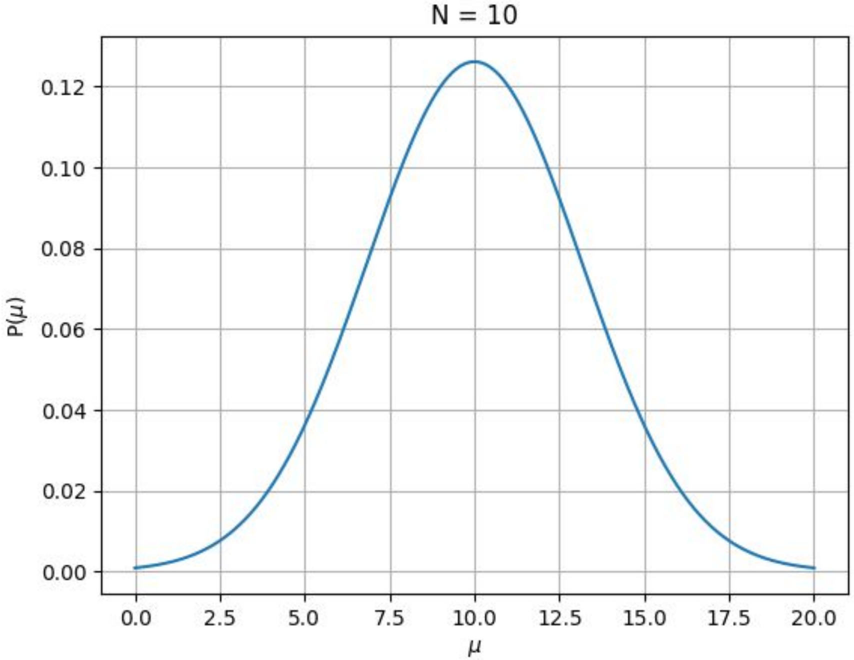
Poisson rate error

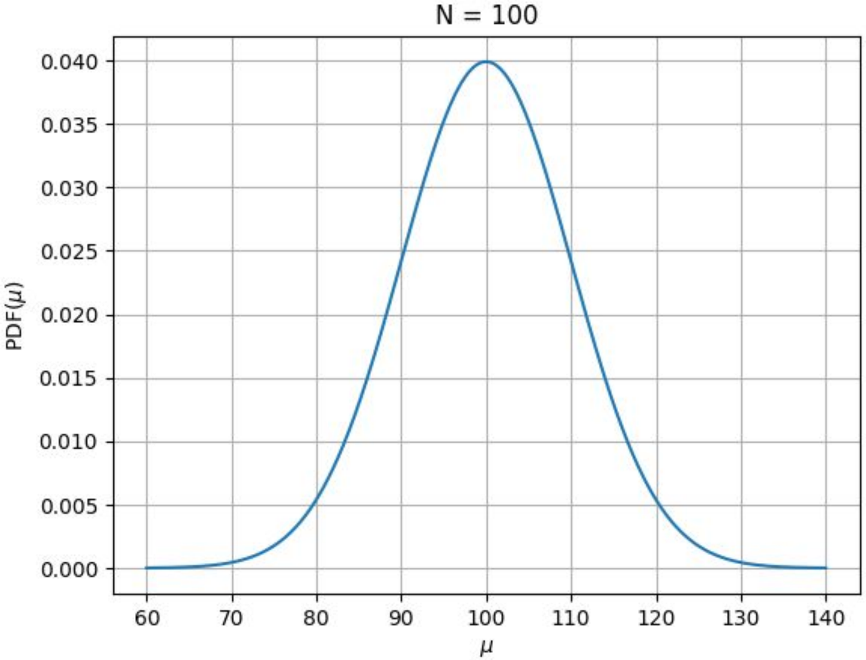
- The error PDF for a Poisson rate is complicated
- We're actually not going to use the exact error PDF
- We're going to rely on an approximation that works well if there's enough counts

- At enough counts, we can use the central limit theorem
- This means the error PDF will be approximately Gaussian
    - Gaussian is continuous though?
    - Well so is what we're estimating

Gaussian approximation of Poisson

- At enough counts, we can use the central limit theorem
- This means the error PDF will be approximately Gaussian
  - Gaussian is continuous though?
  - Well so is what we’re estimating
  - We’re measuring a rate, not an integer number of counts
- The variance of a Poisson distribution is the same as its mean
  - $\mu = N$
  - $\sigma = \sqrt{N}$


N = 10
0.12
0.10
0.08
0.06
0.04
0.02
0.00
P(μ)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
μ


N = 100
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
PDF(μ)
60
70
80
90
100
110
120
130
140
μ

What if we average together multiple observations?

Well that’s the same as taking a longer observation

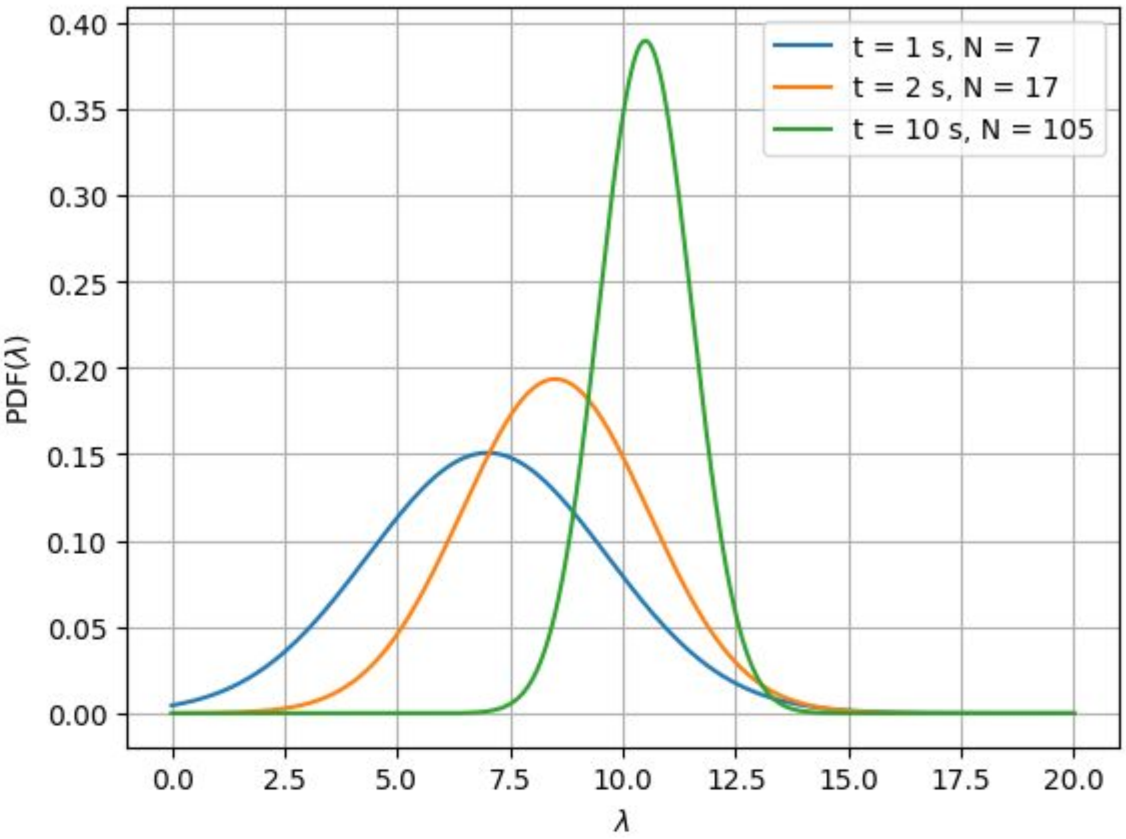
Let’s say

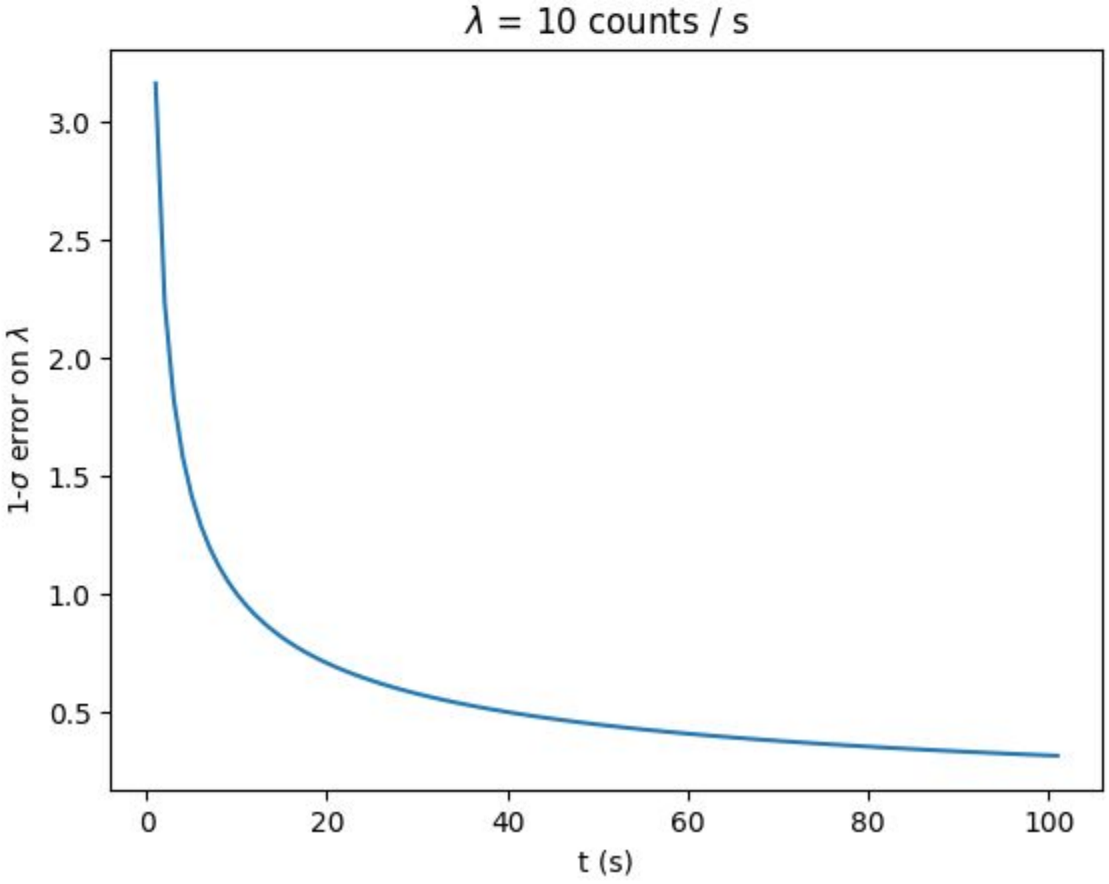
$\boldsymbol{\lambda}$ is our rate (counts / s)

$\mu = \boldsymbol{\lambda} * t$

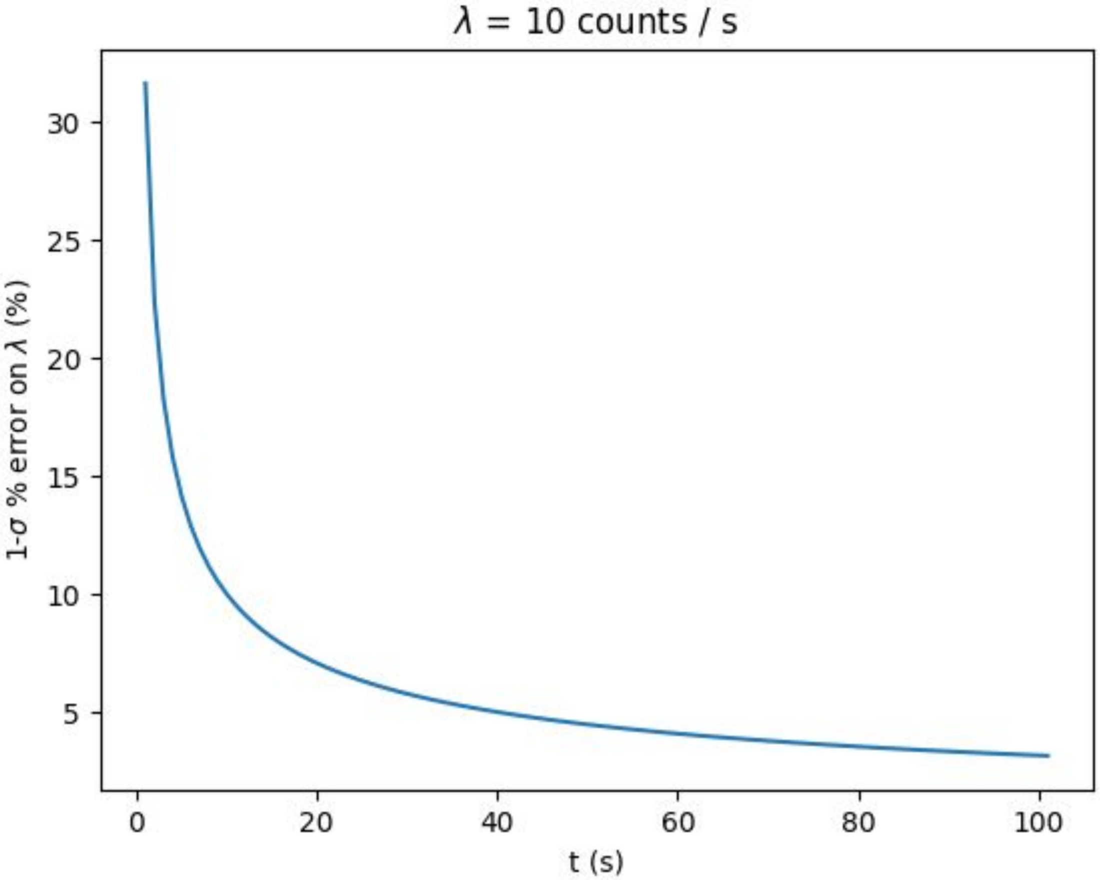
Then to have an error PDF as a function of $\boldsymbol{\lambda}$ instead of μ

- Divide everything by t
    - Gaussian mean = N / t
    - $\sigma = N^{0.5} / t$


t = 1 s, N = 7
t = 2 s, N = 17
t = 10 s, N = 105
PDF(λ)
λ
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0



λ = 10 counts / s
1-σ error on λ
3.0
2.5
2.0
1.5
1.0
0.5
0
20
40
60
80
100
t (s)


λ = 10 counts / s
1-σ % error on λ (%)
30
25
20
15
10
5
0
20
40
60
80
100
t (s)

Background

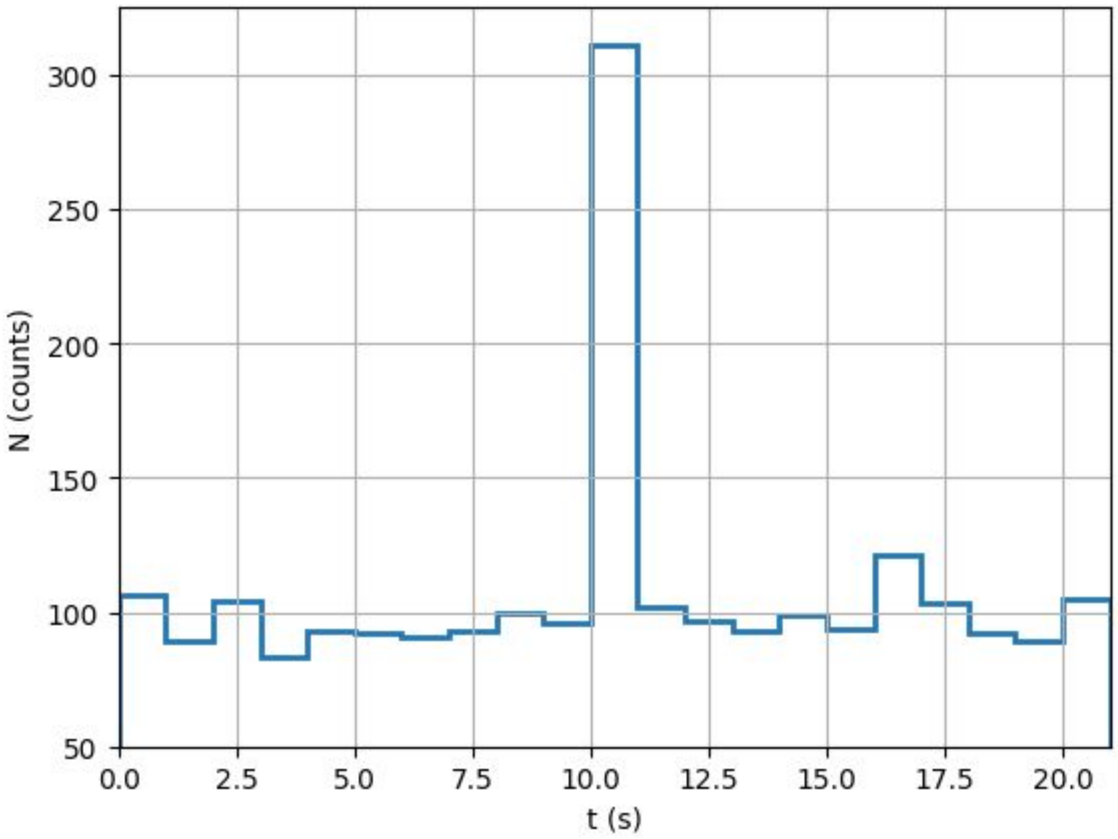
Everything we’ve talked about so far is measuring a signal

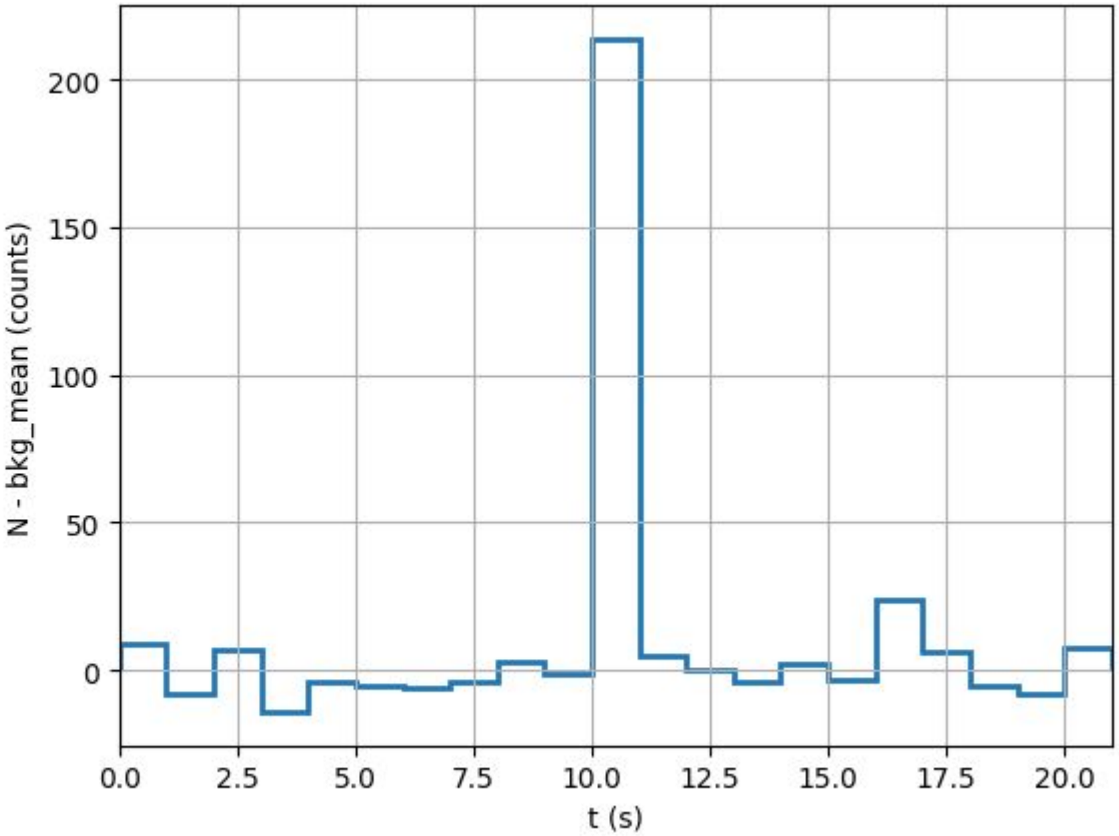
There will pretty much always also be background

- Light from other sources, diffuse light, electronics noise, …

How do we deal with background?

- We need to estimate what the background is somehow
- You can “subtract out” the background


N (counts)
t (s)
300
250
200
150
100
50
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0



N - bkg_mean (counts)
0
50
100
150
200
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t (s)

What's the answer and error?

- Remember what's counted is the total counts in that time bin
  - The Poisson error is then sqrt(N)

What’s the answer and error?

- Remember what’s counted is the total counts in that time bin
    - The Poisson error is then sqrt(N)
- So our answer for what is the signal rate
    - $\langle\lambda\rangle$ = (N - bkg_mean) / t = 213.9 counts / s
    - $\sigma = \sqrt{N}$ = 17.6 counts / s
    - λ = 213.9 +/- 17.6 counts / s (1 sigma error)

What’s the answer and error?

- Remember what’s counted is the total counts in that time bin
    - The Poisson error is then sqrt(N)
- So our answer for what is the signal rate
    - $<\lambda> = (N - \text{bkg\_mean}) / t = 213.9$ counts / s
    - $\sigma = \sqrt{N} = 17.6$ counts / s
    - $\lambda = 213.9 +/- 17.6$ counts / s (1 sigma error)

- Disclaimer - we ignored the error on the bkg rate, but that was small

Noise

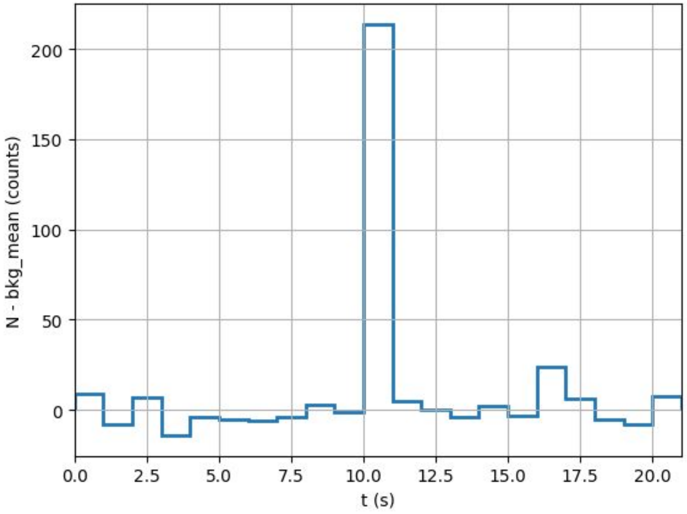
The mean of the bkg is pretty well known

But in each individual 1 s bin, there's a lot of fluctuations

These statistical fluctuations are called noise

The noise is often calculated as the standard deviation of these fluctuations

Noise ~ $\sqrt{(\text{bkg\_mean})}$



Signal to noise (S/N)

The signal to noise ratio (S/N) is measure that’s often used to describe the strength of a measurement

- How much does it rise above the noise?

It’s often used to tell how well we can analyse the observation or how much information we can get out of it

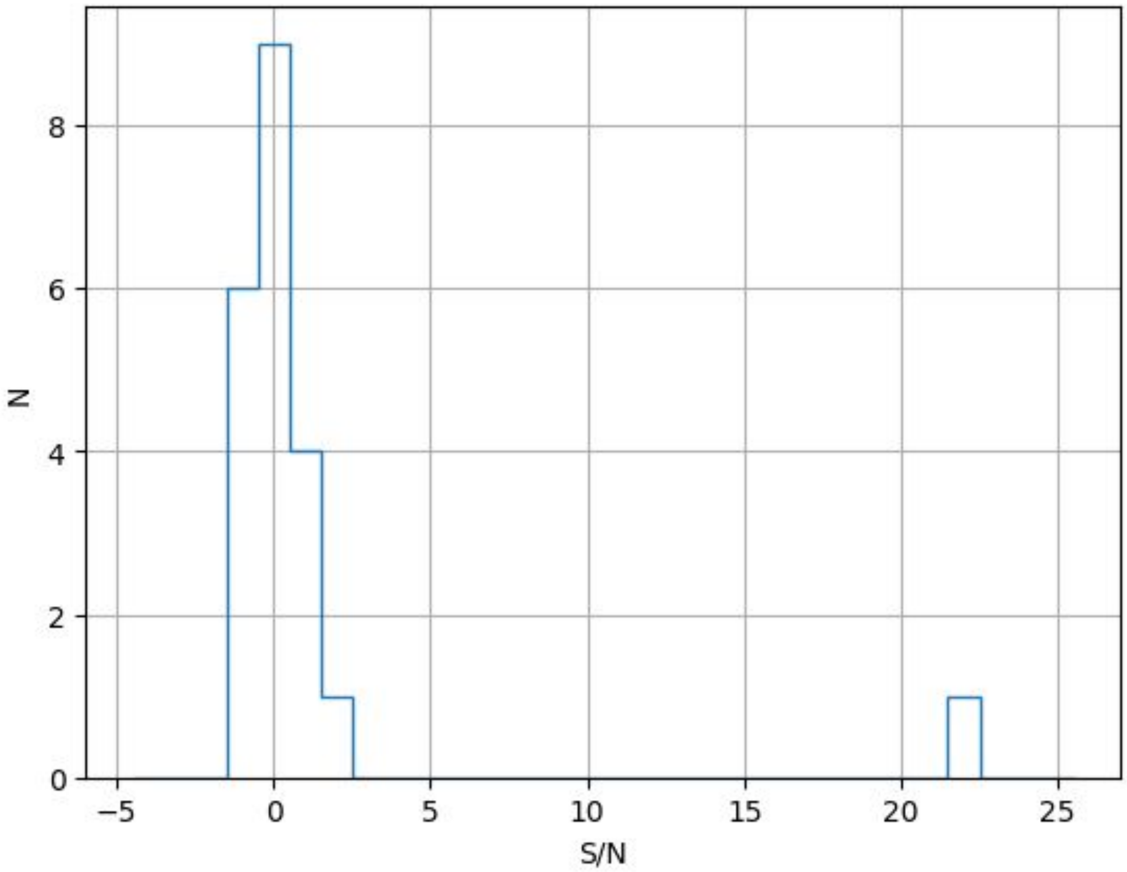
$$S/N = \text{signal} / \text{noise} = N_{signal} / \sqrt{(\text{bkg\_mean})}$$

Is this real or is this just, background fluctuations

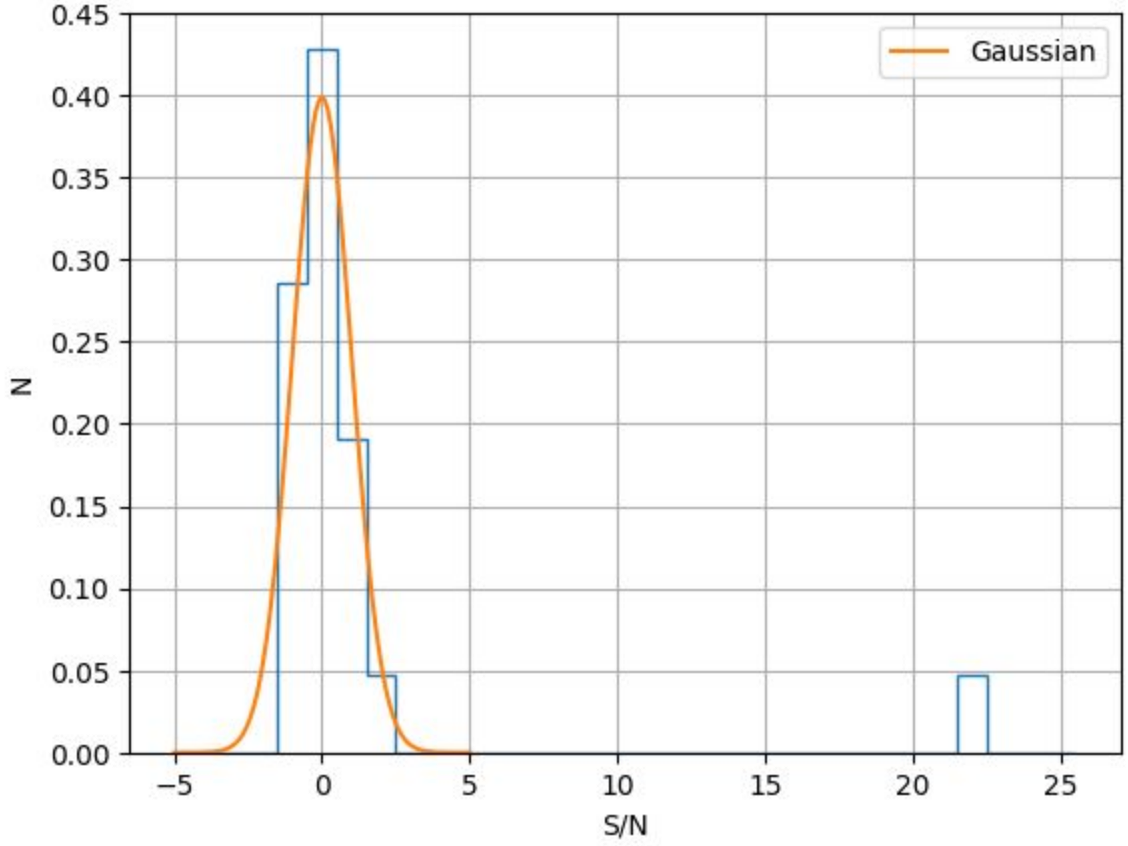
The S/N is also often used to discriminate between what are background fluctuations and what is an actual detection of signal

If there is only bkg fluctuations, the S/N should follow a Gaussian distribution

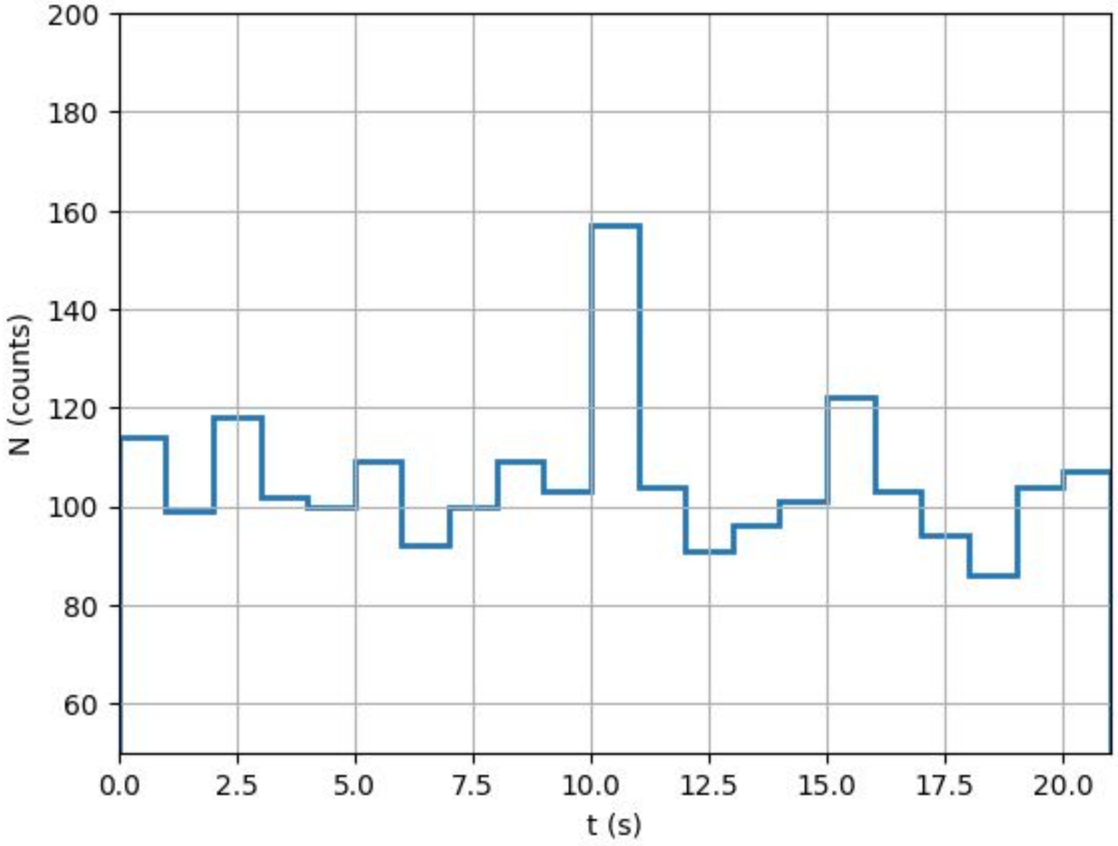
If there is signal, it should be an outlier

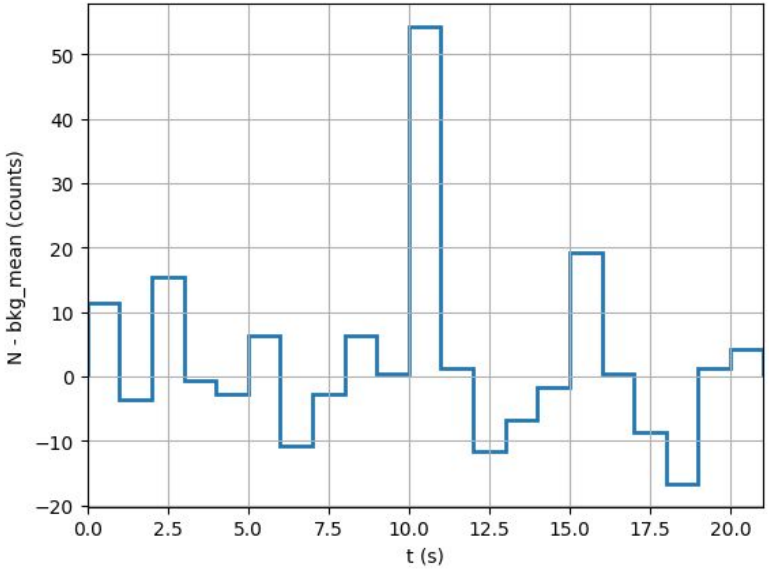


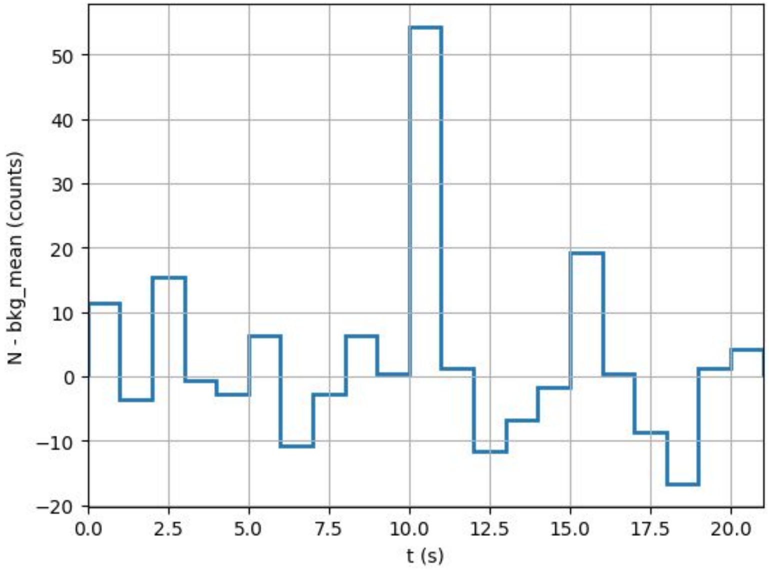
N
S/N
0
2
4
6
8
−5
0
5
10
15
20
25


0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
N
−5
0
5
10
15
20
25
S/N
Gaussian

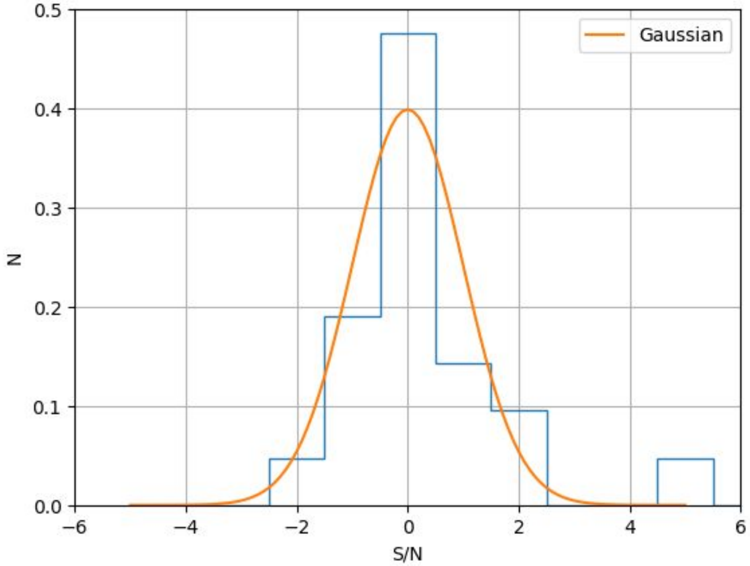
What if it’s weaker?




N - bkg_mean (counts)
t (s)
50
40
30
20
10
0
−10
−20
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

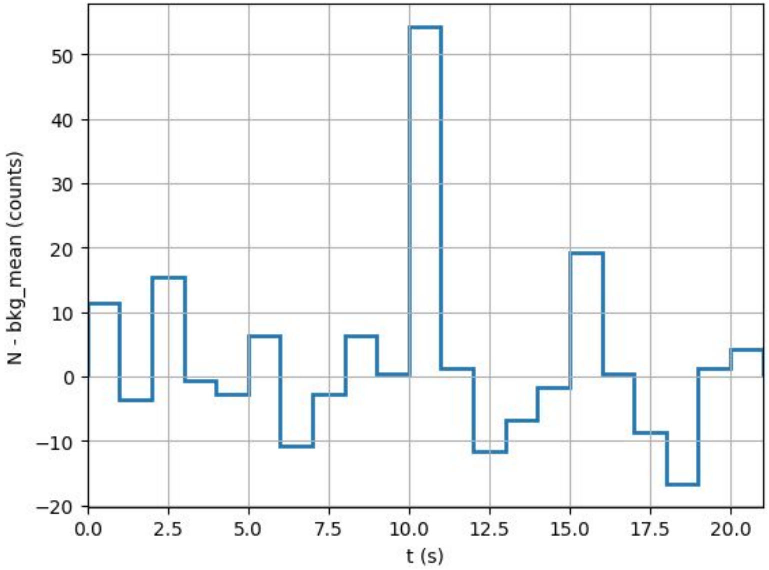


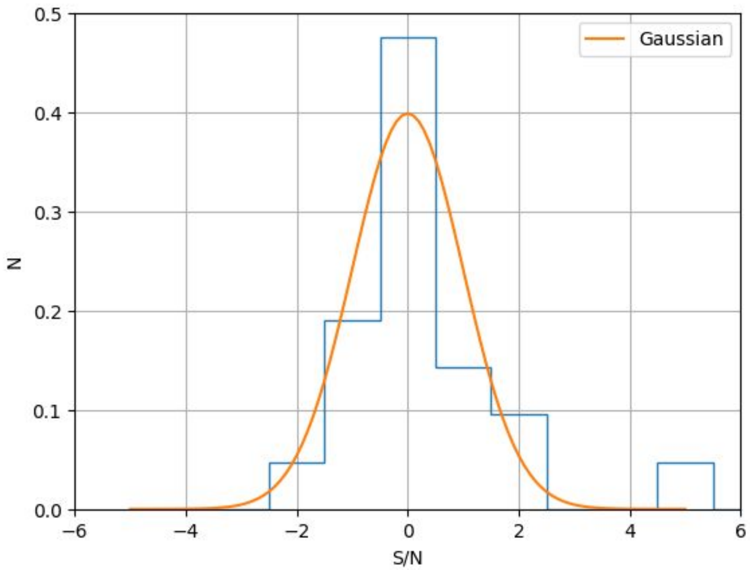
N - bkg_mean (counts)
t (s)
50
40
30
20
10
0
−10
−20
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0



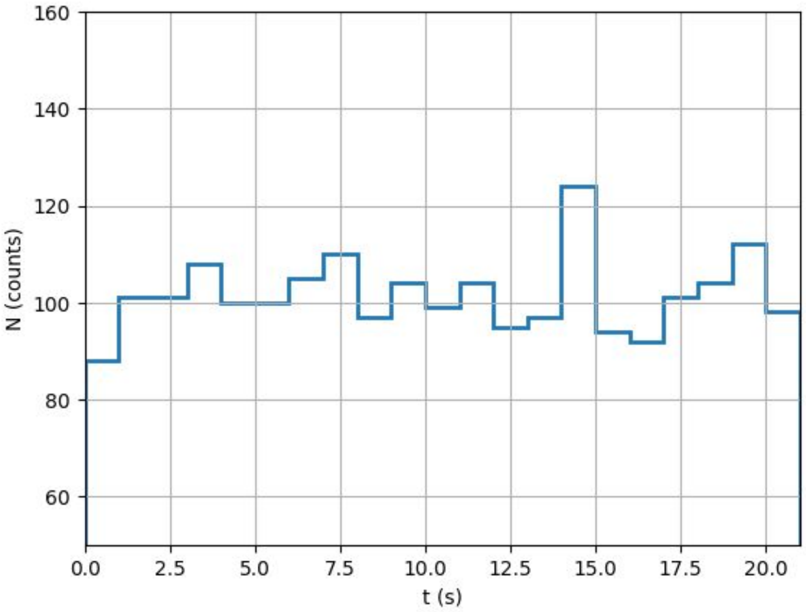
Gaussian
N
S/N
0.5
0.4
0.3
0.2
0.1
0.0
−6
−4
−2
0
2
4
6

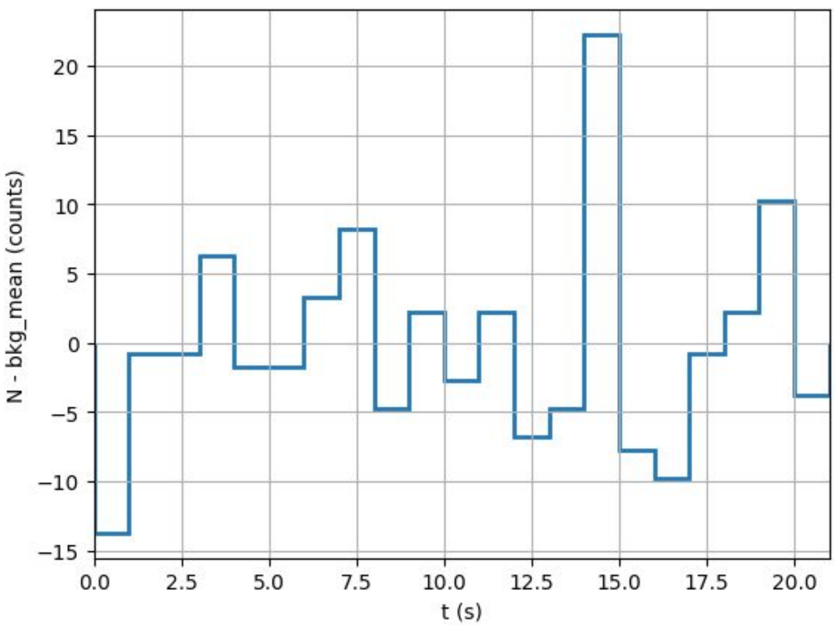
Is it real or bkg?





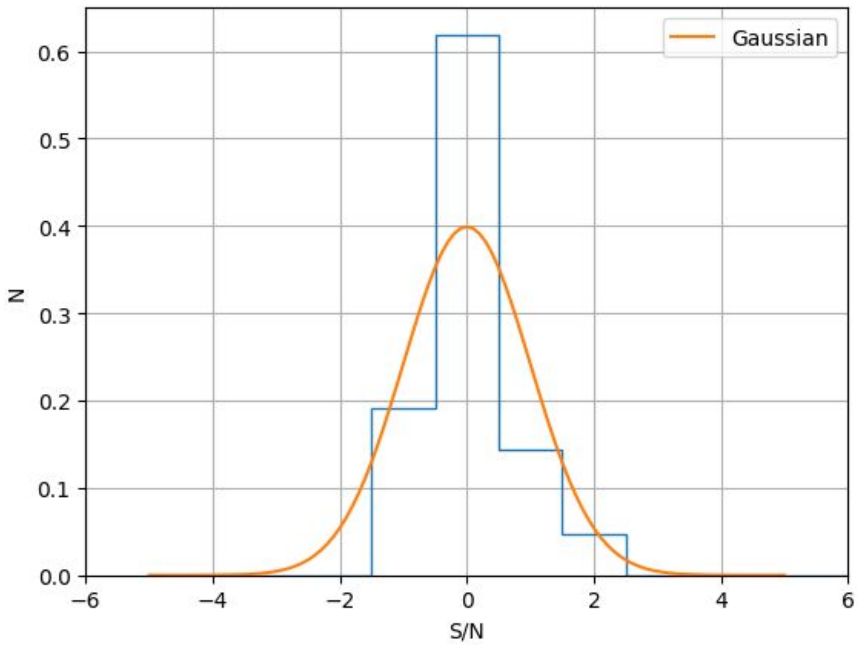
How about now?





Max S/N is 2.2

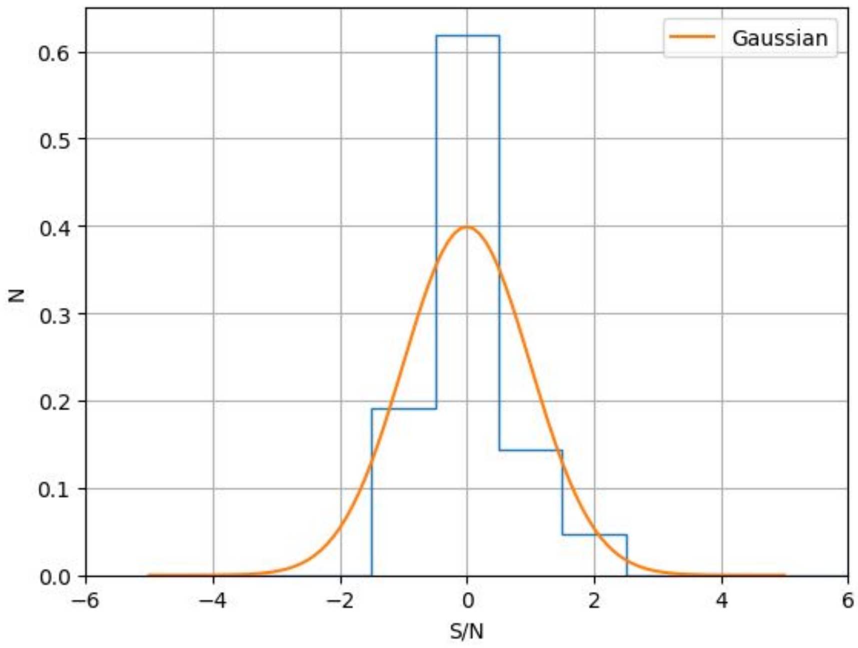
What's the probability of measuring a S/N of 2.2 or higher?



Max S/N is 2.2

What's the probability of measuring a S/N of 2.2 or higher?

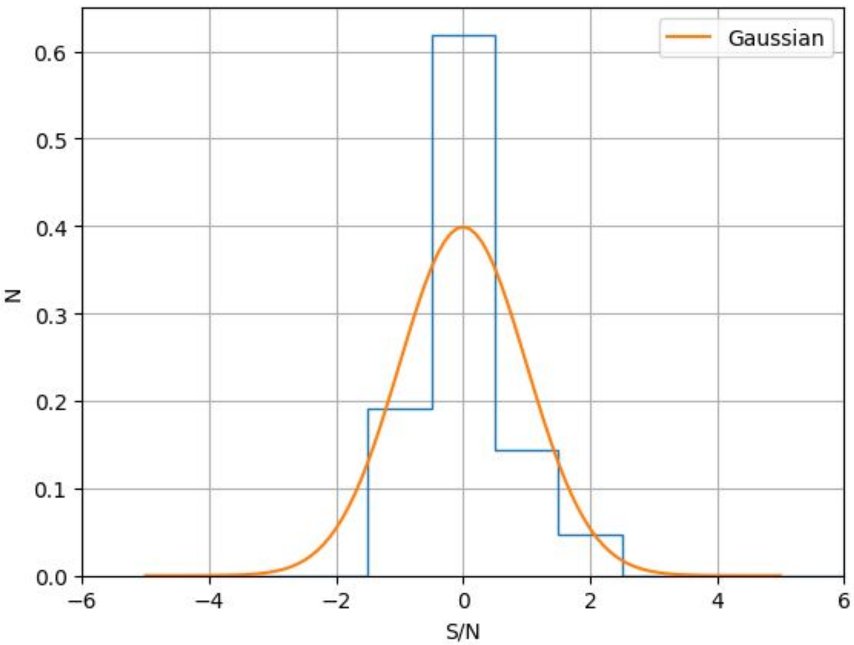
$$P(x > 2.2) = \int_{2.2}^{\infty} N(x; \mu = 0, \sigma = 1)\, dx = \sim 0.014$$



Max S/N is 2.2

What's the probability of measuring a S/N of 2.2 or higher?

$$P(x>2.2) = \int_{2.2}^{\infty} N(x;\mu=0,\sigma=1)\, dx = \sim 0.014$$



Is this likely to be real then?

$$P(x > 2.2) = \int_{2.2}^{\infty} N(x; \mu = 0, \sigma = 1)\, dx = \sim 0.014$$

Is this likely to be real then?

This is for a single observations, we looked at 21

We need to account for the number of trials (or experiments)

This is an approximation,

21 x 0.014 = ~0.3

$$P(x > 2.2) = \int_{2.2}^{\infty} N(x; \mu = 0, \sigma = 1)\, dx = \sim 0.014$$

Is this likely to be real then?

stats.norm.sf(2.2)
or
1 - stats.norm.cdf(2.2)

This is for a single observations, we looked at 21

We need to account for the number of trials (or experiments)

This is an approximation,

21 x 0.014 = ~0.3