Problem 1.

$$U_t - U_{xx} = (\pi^2 - 1)e^{-t}\sin(\pi x) \qquad (x,t) \in (0,1)\times(0,1) \qquad BCs: \begin{array}{l} u(x,0) = \sin(\pi x) \\ u(0,t) = u(1,t) = 0 \end{array}$$

Weak Form:

Multiply both sides by test function $v(x)$:

$$(u_t - u_{xx})v(x) = f(x,t)v(x)$$

$$\iint_\Omega u_t v(x)\,dx\,dt - \iint_\Omega u_{xx} V(x)\,dx\,dt = \iint_\Omega f(x,t)v(x)$$

IBP:

$$u = v(x) \qquad v = u_x$$
$$du = v_x(x) \qquad dv = u_{xx}$$

$$-\underbrace{\cancel{\int_{\partial\Omega} v(x) u_x \, db}^{\,0}}_{BCs} + \int_\Omega u_x v_x(x)\,dx\,dt = \iint f(x)v(x)\,dz$$

$$\iint_\Omega u_t v\,dx\,dt + \iint_\Omega u_x v_x\,dx\,dt =$$

$$\iint_\Omega u_t v\,dx\,dt + \iint_\Omega u_x v_x\,dx =$$

$$\iint_\Omega u_t v\,dx\,dt + \iint_\Omega u_x v_x\,dx - \iint_\Omega f(x)v(x)\,dx = 0 \quad \text{Weak Form}$$

* Forward euler to approx u_t:

$$R_i = \int_a^b \frac{u(t+\Delta t) - u(t)}{\Delta t} v\,dx + \int_a^b u_x v_x\,dx - \int_a^b f(x,t)v\,dx$$

* Gelerkin expansion

$$R_i = \frac{1}{\Delta t}\int_0^1 \sum_{j=1}^n u_j^{n+1}\phi_j\phi_i\,dx - \frac{1}{\Delta t}\int_a^b \sum_{j=1}^n u_j^n \phi_j\phi_i\,dx + \int_a^b \sum_{j=1}^n u_j^n \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x}dx - \int_a^b f(x,t)\phi_i(x)\,dx$$

$$= \frac{1}{\Delta t}\underbrace{\sum_{j=1}^n u_j^{n+1}}_{u^{n+1}}\underbrace{\int_a^b \phi_j\phi_i\,dx}_{M} - \frac{1}{\Delta t}\underbrace{\sum_{j=1}^n u_j^n}_{\vec{u}^n}\underbrace{\int_a^b \phi_j\phi_i\,dx}_{M} + \underbrace{\sum_{j=1}^n u_j^n}_{u^n}\underbrace{\int_a^b \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x}dx}_{K} - \underbrace{\int_a^b f(x,t+\Delta t)\phi_i(x)\,dx}_{\vec{F}^n}$$

$$\vec{u}^{n+1} = \vec{u}^n - \Delta t M^{-1}K u^n + \Delta t M^{-1}\vec{F}^n$$
$$= [I - \Delta t M^{-1}K]u^n + \Delta t M^{-1}\vec{F}^n$$

★ BCs: $u(x,0) = \sin(\pi x)$
$u(0,t) = u(1,t) = 0$

test = trial $\Rightarrow \phi(0) = \phi(1) = 0 \Rightarrow R_1 = R_N = 0$

★ Calculating M matrix:

$$\int_{x_i}^{x_{i+1}} \phi_i(x)\,\phi_j(x)\,dx \Rightarrow \int_{-1}^{1} \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\partial x}{\partial \xi}\,d\xi \qquad i \neq j$$

$$\frac{h}{8}\int \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right) d\xi$$

$$\frac{h}{8}\int_{-1}^{1} 1 + \cancel{\xi} - \cancel{\xi} - \xi^2\,d\xi$$

$$\frac{h}{8}\left[\xi - \frac{\xi^3}{3}\right|_{-1}^{1}$$

$$1 - \frac{1}{3} - \left(-1 - \frac{-1}{3}\right)$$

$$\frac{h}{\cancel{8}} \cdot \frac{\cancel{4}}{3} = \frac{h}{6}$$

$$\frac{h}{2}\int \left(\frac{1-\xi}{2}\right)\left(\frac{1-\xi}{2}\right) \qquad j = i \qquad \frac{h}{2}\int \left(\frac{1+\xi}{2}\right)\left(\frac{1+\xi}{2}\right)$$

$$\frac{h}{8}\int 1 - \xi - \xi + \xi^2 \qquad \frac{h}{8}\int 1 + \xi + \xi + \xi^2$$

$$\frac{h}{8}\int 1 - 2\xi + \xi^2 \qquad \frac{h}{8}\int 1 + 2\xi + \xi^2$$

$$\frac{h}{8}\left[\xi - \xi^2 + \frac{\xi^3}{3}\right|_{-1}^{1} \qquad \frac{h}{8}\left[\xi + \xi^2 + \frac{\xi^3}{3}\right|_{-1}^{1}$$

$$\frac{h}{\cancel{8}} \cdot \frac{\cancel{8}}{3} = \frac{h}{3} \qquad \frac{h}{\cancel{8}} \cdot \frac{\cancel{8}}{3} = \frac{h}{3}$$

$$M_{ii} = \frac{h}{3} + \frac{h}{3} = \frac{2h}{3}$$

★ Calculating K matrix

$$K = \int_{x_i}^{x_{i+1}} \frac{\partial \phi_i(x)}{\partial x}\frac{\partial \phi_j(x)}{\partial x} \Rightarrow \int_{-1}^{1} \left(\frac{\partial \hat{\phi}_1}{\partial \xi}\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \hat{\phi}_2}{\partial \xi}\frac{\partial \xi}{\partial x}\right)\frac{\partial x}{\partial \xi}\,\partial \xi \qquad i \neq j$$

$$\frac{\partial \hat{\phi}_1}{\partial \xi} = -\frac{1}{2} \qquad x(\xi) = (\xi + 1)\frac{h}{2} + 1$$

$$\frac{\partial \hat{\phi}_2}{\partial \xi} = \frac{1}{2} \qquad \frac{\partial x}{\partial \xi} = \frac{h}{2}$$

$$\frac{\partial \xi}{\partial x} = \frac{2}{h}$$

$$\int \left(-\frac{1}{\cancel{2}}\frac{\cancel{2}}{h}\right)\left(\frac{1}{\cancel{2}}\frac{\cancel{2}}{\cancel{h}}\right)\frac{\cancel{h}}{2}\,d\xi$$

$$-\frac{1}{2h}\left[\xi\right|_{-1}^{1}$$

$$= -\frac{1}{h}$$

$$\int_{-1}^{1}\left(\frac{\partial\hat{\phi}_1}{\partial\xi}\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\hat{\phi}_1}{\partial\xi}\frac{\partial\xi}{\partial x}\right) \qquad i=1$$

$$\int_{-1}^{1}\left(-\cancel{\frac{1}{2}}\,\frac{\cancel{2}}{h}\right)\left(-\frac{1}{2}\,\frac{2}{\cancel{h}}\right)\frac{\cancel{h}}{2}\,d\xi$$

$$\frac{\cancel{2}}{\cancel{2}h}\left[\xi\right|_{-1}^{1} = \frac{2}{h}$$

at $(0,0)$ and (N,N) both are equal $\frac{1}{h}$