Exercises

Convex Analysis and Optimization

Prof. Dr. Peter Ochs

www.mop.uni-saarland.de/teaching/CAO19

— Winter Term 2019 / 2020 —

Submission Instructions: Submit your solutions in the lecture hall before or directly after the lecture. *Clearly* write your *name* on the first sheet. Please use *A4 paper format* and *staple* all sheets together. Solutions that get separated and cannot be identified will not be evaluated.

— Assignment 7 —

Exercise 1. [3 + 3 + 2 + 3 + 3 + 1 = 15 points]

Let $C := \{x \in \mathbb{R}^N : \sum_{i=1}^N x_i \leq 1 \text{ and } x_i \geq 0\}$. Compute the following sets and functions and observe their relationships:

(a) Compute the support function σ_C of C.

(b) Compute the gauge function γ_C of C.

(c) Compute the following set

$$C^\circ := \{y \in \mathbb{R}^N : \sigma_C(y) \leq 1\}.$$

(d) Compute the support function σ_{C° of C°.

(e) Compute the gauge function γ_{C° of C°.

(f) Which objects compute the same?

Exercise 2. [5 + 5 = 10 points]

Let $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ be a proper convex function and $\bar{x}$ be a point at which f is finite.

(a) Find an example that shows

$$N_{\operatorname{epi} f}(\bar{x}, f(\bar{x})) \neq \operatorname{cl} \bigcup_{\lambda>0} \{\lambda(v,-1) \in \mathbb{R}^N \times \mathbb{R} : v \in \partial f(\bar{x})\}.$$

(b) Is the formula correct, if "$\neq$" is replaced by "$\subset$" or "$\supset$"?

Exercise 3. [4 + 3 + 5 + 3 = 15 points]

We continue Assignment 6, Exercise 1. Let $a := t_1 < \ldots < t_n =: b$, for some $n \in \mathbb{N}$, be breakpoints in some interval $[a,b]$. We associate a value $f_i \in \mathbb{R}$ with each of the breakpoints t_i, for $i = 1, \ldots, n$. Now, define a (piecewise affine) function $f\colon \mathbb{R} \to \overline{\mathbb{R}}$ by $f(t_i) = f_i$, for all $i = 1, \ldots, n$, and f is an affine function on (t_i, t_{i+1}), for $i = 1, \ldots, n-1$, which is extended to $\mathbb{R}$ by $f(t) = f_1 + s_-(t - t_1)$ on $(-\infty, t_1)$ and $f(t) = f_n + s_+(t - t_n)$ on $(t_n, +\infty)$, where $s_-, s_+ \in \overline{\mathbb{R}}$ are possibly "infinite" slopes. Let f be convex.

(a) Derive a formula for the subdifferential of f at any point $x \in \mathbb{R}$.

(b) Express your formula for the subdifferential at $\bar{x}$ using the value of the left derivative $f'_-(\bar{x}) := f'(\bar{x}; -1)$ and the right derivative $f'_+(\bar{x}) := f'(\bar{x}; +1)$, where

$$f'(\bar{x}; w) := \lim_{\tau \searrow 0} \frac{f(\bar{x} + \tau w) - f(\bar{x})}{\tau} .$$

(c) This is an extension of Exercise 1(c) of Assignment 6, where you are given the piecewise affine function and your task is to plot the subdifferential by filling in the missing pieces of code given in `ex07_01.py`.

(d) State the subdifferential of

$$f(x) = \delta_{[-1,1]}(x) + 5|x| .$$

Submission Instructions for the Coding Exercise:

- Create a `README.md` with your group and matriculation info.
- Use the `ex07_01.py` file provided.
- Make sure that the code can be executed using `python3 ex07_01.py`.
  - *Don't use exotic packages! (we check only with python3)*
- Compress the files to `zip` or `tar.gz` format on a standard Linux machine.
  - *Submissions that cannot be unpacked on a standard Linux machine will receive no points.*
  - *Compress the files using* `tar -czvf Ex07_Surname1_Surname_2.tar.gz FOLDER`.
- Send a *single* eMail *before the end of the lecture* on the submission date to the tutor

  Mahesh Chandra Mukkamala: `mukkamala@math.uni-sb.de`.

  - *Only the first eMail will be considered!*
  - *You won't get points for late submissions!*