Exercises

Convex Analysis and Optimization

Prof. Dr. Peter Ochs

www.mop.uni-saarland.de/teaching/CA019

— Winter Term 2019 / 2020 —

Submission Instructions: Submit your solutions in the lecture hall before or directly after the lecture. *Clearly* write your *name* on the first sheet. Please use *A4 paper format* and *staple* all sheets together. Solutions that get separated and cannot be identified will not be evaluated.

The lecture is cancelled on 21st and 22nd January. Please submit your sheets between 4-6 pm on 22nd January to Mahesh Chandra at Room 4.13 in Building E1.7.

— Assignment 12 —

Exercise 1. [6 points]

An *ℓ_1-regularized linear least squares problem* has the form

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax-b\|_2^2 + \lambda\|x\|_1 \,,$$

for some $A\in\mathbb{R}^{M\times N}$, $b\in\mathbb{R}^M$, and $\lambda>0$. Such a problem is also known as *LASSO problem*.
A similar type of problem is the *elastic net regularized linear least squares problem*, which has the form

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax-b\|_2^2 + \lambda_1\|x\|_1 + \frac{\lambda_2}{2}\|x\|_2^2 \,.$$

for some $A\in\mathbb{R}^{M\times N}$, $b\in\mathbb{R}^M$, and $\lambda_1,\lambda_2>0$.
Show that any elastic net regularized linear least squares problem can also be written as a ℓ_1-regularized linear least squares problem.

Exercise 2. [8 points]

Fix $m\geq 1$. Let $A_i\in\mathbb{R}^{M_i\times N_i}$ for $i=1,\dots,m$ with $M_i=N_{i+1}$ for $i=1,\dots,m-1$, $b\in\mathbb{R}^{M_1}$ and $\lambda_i>0$ for $i=1,\dots,m$. Consider the following minimization problem:

$$\min_{x_i\in\mathbb{R}^{N_i}} \frac{1}{2}\|x_1-b\|_2^2 + \sum_{i=1}^{m-1}\lambda_i\|A_ix_i-x_{i+1}\|_1 + \lambda_m\|A_mx_m\|_1 \,.$$

Rewrite the problem as a minimization problem of the following form:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|\tilde{P}x-\tilde{b}\|_2^2 + \|\tilde{A}x\|_1$$

for $\tilde{P}\in\mathbb{R}^{M\times N}$, $\tilde{b}\in\mathbb{R}^M$, and $\tilde{A}\in\mathbb{R}^{K\times N}$.

Exercise 3. [2 + 2 + 2 + 2 + 2 = 10 points]

Let $f\in\mathscr{F}_L^{1,1}(\mathbb{R}^N)$ with $L>0$ and $f^* := \min_{x\in\mathbb{R}^N} f(x)$ (we assume that a solution exists). We want to find an approximation $x\in\mathbb{R}^N$ that yields a predefined accuracy of $\varepsilon>0$ with respect to the function values i.e., $f(x)-f^*<\varepsilon$.

(a) Determine the least number of iteration that the lower complexity bound for this class of problems predicts for any method (satisfying the usual assumptions from the lecture) to achieve an accuracy of $\varepsilon > 0$.

(b) How many iterations does the gradient descent method with constant step size rule ($\alpha = 1/L$) need to guarantee an accuracy of ε.

(c) How many iterations does the accelerated gradient descent method from Nesterov ($\tau = 1/L$ and $t_k = (k+1)/2$) need to guarantee an accuracy of ε.

(d) Assume, we have achieved an accuracy of ε. How many (additional) iterations are required to achieve an accuracy $\varepsilon' := 10^{-p}\varepsilon$ for some $p \in \mathbb{N}$ using the gradient descent method and the accelerated gradient method.

(e) Now, we consider a certain optimization problem from the class described above, i.e., let $f \in \mathscr{F}_L^{1,1}(\mathbb{R}^N)$. Can you decide if the gradient descent method or the accelerated gradient descent method converges faster to the predefined accuracy? Justify your answer.

Exercise 4. [4 + 4 + 8 = 16 points]

This is in continuation of Exercise 4 of Sheet 10. The goal is minimize the following optimization problem

$$\min_{u \in [-1,1]^N} \langle c, u \rangle + ||Ku||_1 \,, \tag{1}$$

where $K \in \mathbb{R}^{M \times N}$ and $c \in \mathbb{R}^N$ for certain positive integers M and N. We solve the problem using Primal-Dual Hybrid Gradient (PDHG) Algorithm.

(a) Convert the above given problem to saddle point problem as per Lecture 24 by choosing g and f appropriately.

(b) Clearly derive the update steps of the PDHG algorithm.

(c) Fill in the missing parts of the code given in `ex12_01.py`.

Submission Instructions for the Coding Exercise:

- Create a `README.md` with your group and matriculation info.
- Use the `ex12_01.py` file provided.
- Make sure that the code can be executed using `python3 ex12_01.py`.
  - *Don't use exotic packages! (we check only with python3)*
- Compress the files to `zip` or `tar.gz` format on a standard Linux machine.
  - *Submissions that cannot be unpacked on a standard Linux machine will receive no points.*
  - *Compress the files using* `tar -czvf Ex12_Surname1_Surname_2.tar.gz FOLDER`.
- Send a *single* eMail *before the end of the lecture* on the submission date to the tutor

  Mahesh Chandra Mukkamala: `mukkamala@math.uni-sb.de`.

  - *Only the first eMail will be considered!*
  - *You won't get points for late submissions!*