Exercises

Convex Analysis and Optimization

Prof. Dr. Peter Ochs

www.mop.uni-saarland.de/teaching/CAO19

— Winter Term 2019 / 2020 —

Submission Instructions: Submit your solutions in the lecture hall before or directly after the lecture. *Clearly* write your *name* on the first sheet. Please use *A4 paper format* and *staple* all sheets together. Solutions that get separated and cannot be identified will not be evaluated.

— Assignment 9 —

Exercise 1. [3 + 4 + 4 + 4 + 25 = 40 points]

We are given N observations of dimension M, which we collect as columns of a matrix $A \in \mathbb{R}^{M\times N}$. Suppose the observations are linearly depended up to some gross outliers. We formulate an optimization problem that decomposes A into a matrix X that contains the outliers and a low-rank matrix Y that extracts the part of the observations that lies on a low dimensional subspace, i.e., $A \approx X + Y$.
This goal is formulated in the following *convex Robust PCA* problem:

$$\min_{X,Y\in\mathbb{R}^{M\times N}} \mu\|X\|_1 + \lambda\|Y\|_* + \frac{1}{2}\|X+Y-A\|_F^2 \,,$$

where

$$\|X\|_1 := \sum_{i=1}^{M}\sum_{j=1}^{N}|X_{i,j}| \qquad \text{(1-norm)}$$

$$\|B\|_F := \Big(\sum_{i=1}^{M}\sum_{j=1}^{N}|B_{i,j}|^2\Big)^{1/2} \qquad \text{(Frobenius norm)}$$

$$\|Y\|_* := \sum_{i=1}^{\min(M,N)} \sigma_i \qquad \text{(nuclear norm)}\,.$$

where $\sigma_1,\dots,\sigma_{\min(M,N)}$ are the singular values of Y, and $\lambda,\mu > 0$.
As a practical example, we apply the Robust PCA model to background subtraction in surveillance videos. Here the low rank component models the static background and outliers are given by the moving objects in the video.
Solve the following steps on a sheet of paper. You can use the Fact 1 below without proof.

(a) Is the objective strongly convex with respect to the Frobenius norm?

(b) Decompose the objective into a smooth convex function h and a non-smooth convex function g. Verify that the Proximal Gradient Descent algorithm can be applied.

(c) Determine the Lipschitz constant of ∇h.

(d) Derive the formula for the update step of the Proximal Gradient Algorithm.

Implement the algorithm for solving the problem.

- Download the program template and the dataset from the webpage.
  The dataset was extracted from `http://research.microsoft.com/en-us/um/people/jckrumm/wallflower/testimages.htm`.

(e) Fill the missing code (indicated by `TODO`) in the program template.

- Submit your code together with the 4 images that are written to the current folder following the instructions below.

Fact 1. The proximal mapping for the nuclear norm can be computed explicitly as follows:

$$\operatorname*{argmin}_Y \|Y\|_* + \frac{1}{2\tau}\|Y - \bar{Y}\|_F^2 = U\mathcal{S}_\tau(\Sigma)V^\top\,, \quad (\mathcal{S}_\tau(A))_{i,j} = \begin{cases} A_{i,j} - \tau\,, & \text{if } A_{i,j} > \tau\,; \\ A_{i,j} + \tau\,, & \text{if } A_{i,j} < -\tau\,; \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where $U\Sigma V^\top = \bar{Y}$ is the singular value decomposition of the matrix $\bar{Y}$.

Submission Instructions:

- Unpack the files using `tar -xzvf ex.tar.gz`.
- Fill-in the missing parts in the files that are provided as exercise. (Marked with `TODO`.)
- Make sure that the code can be executed using `python ex09.py`.
  - *Don't use exotic packages! (we check with python3)*
- Compress the files to `zip` or `tar.gz` format on a standard Linux machine.
  - *Submissions that cannot be unpacked on a standard Linux machine will receive no points.*
  - *Compress the files using* `tar -czvf Ex09_Surname1_Surname_2.tar.gz FOLDER`.
- If not done yet, rename the submission file to `Ex09_Surname1_Surname_2`.
- Send a *single* eMail *before the end of the lecture* on the submission date to the tutor

  Mahesh Chandra Mukkamala: `mukkamala@math.uni-sb.de`.

  - *Only the first eMail will be considered!*
  - *You won't get points for late submissions!*