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Homework 3

MAT 167
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Problem 1A

The goal is to prove that all eigenvalues of a symmetric matrix A is real.

First, assume that $\vec{x}$ is a non-zero vector.

Next, let $A\vec{x} = \lambda\vec{x}$.

$$\lambda\bar{\vec{x}}^T\vec{x} = \bar{\vec{x}}^T(\lambda\vec{x}) \tag{1}$$

$$= \bar{\vec{x}}^T A\vec{x}$$

$$= (A^T\bar{\vec{x}})^T\vec{x}$$

Since the matrix A is symmetric, A = A^T

$$= (A\bar{\vec{x}})^T\vec{x}$$

$$= (\bar{A}\bar{\vec{x}})^T\vec{x}$$

$$= (\bar{\lambda}\bar{\vec{x}})^T\vec{x}$$

$$= \bar{\lambda}\bar{\vec{x}}^T\vec{x}$$

Since $\vec{x}$ is a nonzero vector, that means $\bar{\vec{x}}^T\vec{x} \neq 0$. This also means $\lambda = \bar{\lambda}$ thus proving that the eigenvalues of a symmetric matrix A are real.

Problem 1B

The goal is to prove that eigenvectors $\vec{x_1}$ and $\vec{x_2}$ are orthogonal. Two vectors are orthogonal if and only if the dot product of the two results in 0.

$$< A\vec{x_1}, \vec{x_2} >=< \vec{x_1}, A^T\vec{x_2} > \tag{1}$$

From the problem statement, it is given that A is a symmetric matrix nxn matrix. Since A is a symmetric matrix, it follows that A = A^T.

From here, we can use the following relation to prove orthogonality. Note that the following lambdas refer to the eigenvalues. Also note that <> denotes dot product.

$$\lambda_1 < \vec{x_1}, \vec{x_2} > \tag{2}$$

$$=< \lambda_1\vec{x_1}, \vec{x_2} >$$

$$=< A\vec{x_1}, \vec{x_2} >$$

$$=< \overrightarrow{x_1}, A^T\overrightarrow{x_2} >$$

$$=< \overrightarrow{x_1}, A\overrightarrow{x_2} >$$

$$=< \overrightarrow{x_1}, \lambda_2\overrightarrow{x_2} >$$

$$= \lambda_2 < \overrightarrow{x_1}, \overrightarrow{x_2} >$$

Following the above,

$$(\lambda_1 - \lambda_2) < \overrightarrow{x_1}, \overrightarrow{x_2} >= 0 \tag{3}$$

Since λ_1 - $\lambda_2 \neq 0$, that means $< \overrightarrow{x_1}$, $\overrightarrow{x_2} >$ must $= 0$.

Problem 2

$$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$$

Problem 2A

Compute reduced QR factorization using the classical gram-schmidt algorithm.

$$r_{11} = ||a_1|| = \sqrt{1^2 + \epsilon^2 + 0^2} \approx 1 \tag{1}$$

$$q_1 = \frac{a_1}{r_{11}} = \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix} \tag{2}$$

$$r_{12} = q_1^T a_2 \tag{3}$$

$$\begin{bmatrix} 1 & \epsilon & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \epsilon \end{bmatrix} = 1$$

$$r_{22} = ||a_2 - r_{12} q_1|| \tag{4}$$

$$\left|\left| \begin{bmatrix} 1 \\ 0 \\ \epsilon \end{bmatrix} - 1 * \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix} 0 \\ -\epsilon \\ \epsilon \end{bmatrix} \right|\right|$$

$$= \sqrt{0^2 + \epsilon^2 + \epsilon^2} = \sqrt{2\epsilon^2} = \epsilon\sqrt{2}$$

$$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}} \tag{5}$$

$$= \frac{\begin{bmatrix} 1 \\ 0 \\ \epsilon \end{bmatrix} - \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix}}{\epsilon\sqrt{2}}$$

$$\begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

From the above calculations, we can construct the Q and R matrices.

$$\hat{Q} = \begin{bmatrix} 1 & 0 \\ \epsilon & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} 1 & 1 \\ 0 & \epsilon\sqrt{2} \end{bmatrix}$$

Problem 2B

Compute reduced QR factorization using the modified gram-schmidt algorithm.

$$r_{11} = \sqrt{1^2 + \epsilon^2 + 0^2} = 1 \tag{1}$$

$$q_1 = \frac{a_1}{r_{11}} = \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix} \tag{2}$$

$$r_{12} = 1 \tag{3}$$

$$v_2 = a_2 - q_1 \tag{4}$$

$$= \begin{bmatrix} 1 \\ 0 \\ \epsilon \end{bmatrix} - \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\epsilon \\ \epsilon \end{bmatrix}$$

$$r_{22} = \sqrt{0^2 + \epsilon^2 + \epsilon^2} = \epsilon\sqrt{2} \tag{5}$$

$$q_2 = \frac{v_2}{r_{22}} = \begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{6}$$

After having done the calculations by hand on paper, I noticed that the process for the QR factorization for this specific matrix results in the same calculations. Since the calculations are the same, the $\hat{Q}$ and $\hat{R}$ matrices are identical.

$$\hat{Q} = \begin{bmatrix} 1 & 0 \\ \epsilon & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} 1 & 1 \\ 0 & \epsilon\sqrt{2} \end{bmatrix}$$

Problem 2C

$$\vec{v_1} = sign(a_{11})||\vec{a_1}||\vec{e_1} + \vec{a_1} \tag{1}$$

$$= sign(1) * \sqrt{1^2 + \epsilon^2 + 0^2} * \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ \epsilon \\ 0 \end{bmatrix}$$

$$Q_1 = F_1 = I_3 - 2P_{\vec{v_1}} = I - 2\frac{\vec{v_1}\vec{v_1}^T}{\vec{v_1}^T\vec{v_1}} \tag{2}$$

$$\vec{v_1}^T\vec{v_1} = \begin{bmatrix} 2 & \epsilon & 0 \end{bmatrix} \begin{bmatrix} 2 \\ \epsilon \\ 0 \end{bmatrix} = 4 + \epsilon^2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{4+\epsilon^2} \begin{bmatrix} 4 & 2\epsilon & 0 \\ 2\epsilon & \epsilon^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_1 A = \begin{bmatrix} -1 & -\epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} -1-\epsilon^2 & -1 \\ 0 & -\epsilon \\ 0 & \epsilon \end{bmatrix}$$

$$\approx \begin{bmatrix} -1 & -1 \\ 0 & -\epsilon \\ 0 & \epsilon \end{bmatrix}$$

$$\vec{v_2} = sign(-\epsilon) \left\| \begin{bmatrix} -\epsilon \\ \epsilon \end{bmatrix} \right\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\epsilon \\ \epsilon \end{bmatrix} \tag{4}$$

$$= -1 * \sqrt{2\epsilon^2} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\epsilon \\ \epsilon \end{bmatrix}$$

$$= \begin{bmatrix} -\epsilon\sqrt{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -\epsilon \\ \epsilon \end{bmatrix}$$

$$= \begin{bmatrix} -\epsilon\sqrt{2} - \epsilon \\ \epsilon \end{bmatrix}$$

$$= \epsilon \begin{bmatrix} -(\sqrt{2}+1) \\ 1 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & F_{2_{11}} & F_{2_{12}} \\ 0 & F_{2_{21}} & F_{2_{22}} \end{bmatrix}, F_2 = I_2 - 2\frac{\vec{v_2}\vec{v_2}^T}{\vec{v_2}^T\vec{v_2}} \tag{5}$$

$$\vec{v_2}^T\vec{v_2} = \epsilon^2 \begin{bmatrix} -(\sqrt{2}+1) & 1 \end{bmatrix} \begin{bmatrix} -(\sqrt{2}+1) \\ 1 \end{bmatrix} = 2\epsilon^2(2+\sqrt{2})$$

$$\vec{v_2}\vec{v_2}^T = \epsilon^2 \begin{bmatrix} -(\sqrt{2}+1) \\ 1 \end{bmatrix} \begin{bmatrix} -(\sqrt{2}+1) & 1 \end{bmatrix}$$

$$= \epsilon^2 \begin{bmatrix} 3+2\sqrt{2} & -(\sqrt{2}+1) \\ -(\sqrt{2}+1) & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2\epsilon^2}{2\epsilon^2(2+\sqrt{2})} \begin{bmatrix} 3+2\sqrt{2} & -(\sqrt{2}+1) \\ -(\sqrt{2}+1) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{Q} = Q_1 Q_2 \tag{6}$$

$$= \begin{bmatrix} -1 & -\epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \frac{\epsilon}{\sqrt{2}} & \frac{-\epsilon}{\sqrt{2}} \\ -\epsilon & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{R} = Q_2^T Q_1^T A \tag{7}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & -\epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -\epsilon \\ 0 & \epsilon \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 0 & \epsilon\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

From the above calculations the Q and R matrices are as follows

$$\hat{Q} = \begin{bmatrix} -1 & \frac{\epsilon}{\sqrt{2}} & \frac{-\epsilon}{\sqrt{2}} \\ -\epsilon & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} -1 & -1 \\ 0 & \epsilon\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

Problem 2D

$$\text{CGS, MGS} = ||\hat{Q}^T\hat{Q} - I||_F \tag{1}$$

$$= \left\| \begin{bmatrix} 1 & \epsilon & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \epsilon & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|_F$$

$$= \left\| \begin{bmatrix} 1 & \frac{-\epsilon}{\sqrt{2}} \\ \frac{-\epsilon}{\sqrt{2}} & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|_F$$

$$= \left\| \begin{bmatrix} 0 & \frac{-\epsilon}{\sqrt{2}} \\ \frac{-\epsilon}{\sqrt{2}} & 0 \end{bmatrix} \right\|_F$$

$$= \sqrt{0^2 + (\frac{-\epsilon}{\sqrt{2}})^2 + (\frac{-\epsilon}{\sqrt{2}})^2 + 0^2}$$

$$= \sqrt{(\epsilon^2)}$$

$$= \epsilon$$

$$\text{Householder} = ||\hat{Q}^T\hat{Q} - I||_F \tag{2}$$

$$= \left|\left|\begin{bmatrix} -1 & -\epsilon & 0 \\ \frac{\epsilon}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-\epsilon}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & \frac{\epsilon}{\sqrt{2}} & \frac{-\epsilon}{\sqrt{2}} \\ -\epsilon & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|\right|_F$$

$$= \left|\left|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|\right|_F$$

$$= \left|\left|\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right|\right|_F$$

$$= \sqrt{0^2 + ... + 0^2}$$

$$= 0$$

Problem 3A

From the problem statement the formula for Ex is given as follows

$$Ex = \frac{1}{2}(x + Fx) \tag{1}$$

We must find a matrix F that manipulates the vector $[x_1 ... x_m]^T$ to become $[x_m ... x_1]^T$ by multiplying F to the left side of the first vector. The only matrix that can satisfy this condition is an opposite identity matrix. (Not sure what the exact term is)

$$F = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{2}$$

Knowing the 2 equations above, we can manipulate the first equation as follows.

$$Ex = \frac{1}{2}(I + F)x \tag{3}$$

Next, we can factor out the x's

$$E = \frac{1}{2}(I + F)$$

Next, we can test to see if E is indeed a projector. From the definition of projectors, for E to be a projector, E must be equal to E^2

$$E^2 = (\frac{1}{2}(I + F))^2 \tag{4}$$

$$= \frac{1}{4}(I + 2F + F^2)$$

For simplicity's sake, I'll be using 2x2 matrices to calculate the above formula.

$$= \frac{1}{4}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2\right)$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}(I + F)$$

From the above calculations, we have verified that E is indeed a projector. Next we must check if E is an orthogonal projector. According to the definition of orthogonal projectors, for E to be an orthogonal projector, E must be equal E^T

$$E^T = (\frac{1}{2}(I+F))^T \tag{5}$$

$$= \frac{1}{2}(I^T + F^T)$$

The transpose of an identity matrix is the identity matrix. In addition, the transpose of an opposite identity matrix is also an opposite identity matrix.

$$= \frac{1}{2}(I + F)$$

From the above calculations, we have verified that E is an orthogonal projector.

Problem 3B

The entries of E are as follows. It is a matrix where every diagonal element is 1 times $\frac{1}{2}$.

$$E = \frac{1}{2}\begin{bmatrix} 1 & 0 & \dots & \dots & 0 & 1 \\ 0 & \ddots & \dots & \dots & \iddots & 0 \\ \vdots & \vdots & 1 & 1 & \vdots & \vdots \\ \vdots & \vdots & 1 & 1 & \vdots & \vdots \\ 0 & \iddots & \dots & \dots & \ddots & 0 \\ 1 & 0 & \dots & \dots & 0 & 1 \end{bmatrix}$$

Problem 4F

	a. LU decomp	b. classical	c. modified
1	0.999999987553534	1.000013235262778	1.000000001037688
2	0.000003035555746	-0.002271878517024	-0.000000426525961
3	-8.000107223400610	-7.938616964071605	-7.999981225479608
4	0.001423704116279	-0.658829494464540	-0.000317425652966
5	10.655373774743218	14.318222026838642	10.669411140608812
6	0.045442453037987	-11.751149036327023	-0.013695419296221
7	-5.801951544721462	17.501943903943026	-5.647518454486244
8	0.276178684126201	-28.490805171617495	-0.074362540544609
9	1.265787064885740	22.922986699238752	1.692331412997670
10	0.266400072869003	-8.928073659072817	0.007068809438282
11	-0.486139651857658	1.344638932084307	-0.374710650307755
12	0.109029202866436	0.028288274126652	0.088131160218759

	d. householder	e. x = A\b
1	1.000000000996607	1.000000000996607
2	-0.000000422742862	-0.000000422743364
3	-7.999981235690553	-7.999981235676154
4	-0.000318763174830	-0.000318763346323
5	10.669430795525177	10.669430796641096
6	-0.013820286184988	-0.013820290914619
7	-5.647075631448992	-5.647075619959385
8	-0.075316015998390	-0.075316036589419
9	1.693606955875625	1.693606976803618
10	0.006032114972655	0.006032099645104
11	-0.374241706174742	-0.374241699881279
12	0.088040576569573	0.088040575462356

Problem 4G

Firstly, based on a post on piazza, we can assume that the QR factorization algorithm from part e produces the most accurate results. Looking at the table, it seems that parts c and d produced very similar results to part e. In addition, since parts c and d produced very similar results to part e, it can be explained that those algorithms are more stable than the ones used in parts a and b.

From the table, I noticed that the classical gram schmidt algorithm produced results that were vastly different from the other algorithms. The first 3 results were quite similar to the results from the other algorithms but from results 4 and onward, it is clear that the results are off compared to the other algorithm results. A possible explanation for this difference is due to how the algorithm works. In the classical gram schmidt procedure, the loss of orthogonality after each calculation is not accounted for. After repeated loops, round off errors accumulate which affect the orthogonality of the vectors it calculates.

Problem 5

First, we will prove that if A^TA has linearly independent columns, A^TA is invertible. To begin, if a matrix has linearly independent columns, that means it has a null space of $\vec{0}$. From the given conditions in the problem statement, we know that A is an mxn matrix. This means that A^TA results in a square matrix. Since A^TA is a square matrix and it has a null space of 0, this means that A^TA is invertible.

Next, we will prove that if A^TA is invertible, A^TA has linearly independent columns.

First, refer to the definition of nullspace

$$N(A) = \left\{\vec{x} : A\vec{x} = \vec{0}\right\} \tag{1}$$

Here, it is given that A^TA is invertible so we can refer to properties of invertible matrices. From these properties, we know that A^TA has a nullspace of 0 therefore $\vec{x} = \vec{0}$. If the nullspace of A^TA is the zero vector, then that means the matrix is of full rank. A matrix having full rank indicates that the matrix does not have any zero rows in its row reduced form. From this, we know that the nullity of the matrix is 0, therefore we know all the rows are independent. From the 'Rank = Column Space theorem', because the rows are independent, the columns must also be independent. Thus the proof is complete.

Problem 6A

First, refer to the formula for computing the psuedoinverse

$$A^{+} = (A^{T}A)^{-1}A^{T} \tag{1}$$

$$= \left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 3 & 3.0002 \\ 3.0002 & 3.00040002 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix}$$

$$= \begin{bmatrix} 150020001 & -150010000 \\ -150010000 & 150000000 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix}$$

$$= \begin{bmatrix} 10001 & -5000 & -5000 \\ -10000 & 5000 & 5000 \end{bmatrix}$$

Next, solve for the projector

$$P = AA^+ \tag{2}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} 10001 & -5000 & -5000 \\ -10000 & 5000 & 5000 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Problem 6B

To solve for the least squares solutions, refer to the normal equations

$$A^T A\hat{x} = A^T \overrightarrow{b} \tag{1}$$

Next, we can rearrange the normal equations formula to solve for x using the pseudoinverse as follows

$$\hat{x} = (A^T A)^{-1} A^T \vec{b} \tag{2}$$

$$= A^+ \vec{b}$$

$$= \begin{bmatrix} 10001 & -5000 & -5000 \\ -10000 & 5000 & 5000 \end{bmatrix} \begin{bmatrix} 2 \\ 0.0001 \\ 4.0001 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

From solving the least squares problem, both the intercept and slope of the least squares line are 1. Therefore the least squares line is as follows: y = x + 1.

Problem 6C

According to MATLAB, the condition number of A is 4.242923541610648e+04