Homework 2

● Graded

Student

Kevin Vuong

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Total Points

24 / 25 pts

Question 1

Question #1

6 / 6 pts

✔ **− 0 pts** Correct

− 1 pt #1b incorrect

− 0.5 pts #1c partially incorrect

− 6 pts Click here to replace this description.

Question 2

Question #2

4 / 4 pts

✔ **− 0 pts** Correct

− 1 pt #2b incorrect

− 1 pt No python code

Question 3

Question #3

3.5 / 4 pts

− 0 pts Correct

✔ **− 0.5 pts** #3a incorrect

− 1 pt #3d missing

− 4 pts Click here to replace this description.

Question 4

Question #4

1.5 / 2 pts

− 0 pts Correct

✔ **− 0.5 pts** #4b incorrect

− 1 pt No link to python code

Question 5

Question #5

3 / 3 pts

✔ **− 0 pts** Correct

− 1 pt #5 incorrect

− 3 pts Click here to replace this description.

Question 6

Question #6

6 / 6 pts

✔ **− 0 pts** Correct

− 2 pts #6b incomplete

− 1 pt No code for part c

− 1 pt #6c incorrect

− 2 pts #6c incomplete

Questions assigned to the following page: 1, 2, 3, and 4

Homework 2

Instructions: Solutions must be typed using LaTeX. All work and any accompanying Python code should be uploaded to Gradescope by 11:59PM on Friday, February 2, 2024.

1. (6 pts) Consider the system

$$\begin{cases} x - y + 2z &= a \\ 2x + 4y - 3z &= b \\ 4x + 2y + z &= c \end{cases}$$

Find the values of a, b, and c for which the system will have

(a) no solution
(b) exactly one solution
(c) infinitely many solutions

let $A = \left[\begin{array}{ccc|c} x & -y & 2z & a \\ 2x & 4y & -3z & b \\ 4x & 2y & z & c \end{array}\right]$

if we perform rref(A) (Subtract $2\times Row1$ from Row 2, $4\times Row1$ from Row 3, and then subtract $1\times$(New) Row 2 from Row 3), we get. $A = \left[\begin{array}{ccc|c} x & -y & 2z & a \\ 0 & 6y & -7z & b-2a \\ 0 & 0 & 0 & c-2a-b \end{array}\right]$

(a) A will have no solution when the bottom row is not all 0's, i.e. $c \neq 2a + b$

(b) A cannot have exactly one solution. $rank(A) = 2 < 3 = n = m$. This implies A must have either no solutions or infinite solutions.

(c) A will have infinitely many solutions when the bottom row is all 0's i.e. $c = 2a + b$

2. (4 pts) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 2 & 6 & 13 \end{bmatrix}$.

(a) Use Python to find A^{-1}. Please check this Colab for answers to part a and b

(b) Use A^{-1} to solve the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

3. (4 pts) Let $A, B \in \mathbb{R}^{3\times 3}$ with $\det(A) = 2$ and $\det(B) = 3$. Evaluate each of the following:

(a) $\det(-2A^{-1})$ (b) $\det(AB^T)$ (c) $\det(A^{-1}B^T)$ (d) $\det((AB)^{-1})$

(a). By the rules of Determinant linearity (i.e. multiplying a row/column by a scalar n multiplies the determinant by n) and the fact that the dimension of our matrices is $3x3$, we have

$$\det(-2A^{-1}) = (-2)^3 * \frac{1}{\det(A)} = \frac{-8}{6} = -\frac{4}{3}$$

(b).

$$\det(AB^T) = \det(A) * \det(B^T) = \det(A) * \det(B) = 6$$

(c).

$$\det(A^{-1}B^T) = \det(A^{-1}) * \det(B^T) = \frac{1}{\det(A)} * \det(B) = \frac{3}{2}$$

(d).

$$\det((AB)^{-1}) = \det(B^{-1}A^{-1}) = \det(B^{-1}) * \det(A^{-1}) = \frac{1}{\det(B)} * \frac{1}{\det(A)} = \frac{1}{6}$$

4. (2 pts) Use Python to find the determinant of each of the following:

Questions assigned to the following page: 4, 5, and 6

(a) $\begin{bmatrix} 6 & 9 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 7 & 11 \\ 3 & 4 & 10 \\ 5 & 6 & 16 \end{bmatrix}$

Please follow this Here

5. (3 pts) Find all vectors in $\mathbb{R}^2$ that are orthogonal to $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with respect to the dot product.

We want to find all vectors in the left nullspace of A ($N(A^T)$), i.e. all w such that $v^T w = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$

Trivially, we see that $v^T w = 0, \forall w$ such that $-3w_1 = w_2$.

$$\implies N(A^T) = \text{span}\{w\} = c \begin{bmatrix} -3 \\ 1 \end{bmatrix}, c \in \mathbb{R}$$

6. (6 pts) Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$.

(a) Verify that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

(b) Use Gram-Schmidt to find an orthonormal basis for $\mathbb{R}^3$.

(c) Use Python to check that the basis you obtained is orthonormal.

(a) we check for linear independence by showing $\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \neq 0$ through the big/permutation formula.

$$\det([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \det(\begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & -2 \\ -3 & 3 & 5 \end{bmatrix}) = a_{11}b_{22}c_{33} + a_{12}b_{23}c_{31} + a_{13}b_{21}c_{32} - a_{13}b_{22}c_{31} - a_{11}b_{23}c_{32} - a_{12}b_{21}c_{31}$$

$$= 0 + 6 + 0 - 0 - (-24) - 10 = 20$$

Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $\mathbb{R}^3$

(b). We will normalize at the end. We set our first orthogonal vector $u_1 = v_1$. We then set our next vector

$$u_2 = v_2 - \frac{u_1^T v_2}{||u_1||^2} u_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{-5}{29} \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} = \frac{1}{29} \begin{bmatrix} 49 \\ 10 \\ 72 \end{bmatrix}$$

Finally, we set our third vector

$$u_3 = v_3 - \frac{u_1^T v_3}{||u_1||^2} u_1 - -\frac{u_2^T v_3}{||u_2||^2} u_2 = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} - \frac{19}{29} \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} - \frac{(340/29)}{(265/29)} (\frac{1}{29} \begin{bmatrix} 49 \\ 10 \\ 72 \end{bmatrix}) = \frac{1}{53} \begin{bmatrix} 24 \\ -60 \\ -8 \end{bmatrix}$$

Thus, our orthogonal basis is

$$\{ \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{29} \begin{bmatrix} 49 \\ 10 \\ 72 \end{bmatrix}, \frac{1}{53} \begin{bmatrix} 24 \\ -60 \\ -8 \end{bmatrix} \}$$

We now create an orthnormalized basis by dividng each vector by its length

$$\{ \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} /||u_1||, \frac{1}{29} \begin{bmatrix} 49 \\ 10 \\ 72 \end{bmatrix} /||u_2||, \frac{1}{53} \begin{bmatrix} 24 \\ -60 \\ -8 \end{bmatrix} /||u_3||\}$$

Question assigned to the following page: 6

$$= \{ \frac{1}{\sqrt{29}} \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{265}} \begin{bmatrix} 49 \\ 2 \\ 72 \end{bmatrix}, \frac{1}{53} \begin{bmatrix} 24 \\ -60 \\ -8 \end{bmatrix} /1.74 \}$$

Please note that I rounded off the last norm to approximately 1.74, since I couldn't determine a rational-number divisor in NumPy. I display the full solution within the next solution's colab.

(c) Please check this Colab

[1]

[1] (Raw Link in case links don't work): https://colab.research.google.com/drive/1XpQFOrN7kJzPh20q1wfvm8lsnf4vPWX8?usp=sharing