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Dynamics of spring-block models: Tuning to criticality

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We have studied the homogeneous Burridge-Knopoff spring-block model with nonlinear friction introduced by Carlson and Langer [Phys. Rev. Lett. **62**, 2632 (1989); Phys. Rev. A **40**, 6470 (1989)]. There are several different velocity scales that define the model and divide the behavior of the system into distinct regimes. Over much of the parameter space defined by these velocities the system appears to be self-organized to what is reminiscent of a first-order transition. As the friction nonlinearity is varied, there appears to be a continuous (critical) transition to a regime where a global event is continually occurring.

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A variety of dynamical systems demonstrate scaling properties with power-law correlations. One example of such behavior is brittle fracture, which, in the context of earthquakes, has the Gutenberg-Richter law [1] describing the distribution of seismic moments $\rho(M_0)$ as a power law in the moment M_0. Because such scaling extends over many decades in M_0, earthquakes have been regarded as prototypical "self-organized critical" systems [2]. According to the tenets of self-organized criticality, there can be an analogy between the dynamics of a driven, spatially extended, steady-state system and the static correlations occurring at critical points.

The scaling properties of earthquake distribution functions have motivated the construction of many models that can produce such behavior [3]. Independent of their relevance to earthquakes, these models often have interesting dynamical behavior. One model that has received considerable attention is the spring-block model of Burridge and Knopoff [4] shown in the inset of Fig. 1(a). In a homogeneous version of this model, a one-dimensional array of N blocks, each of mass m, are coupled by springs of spring constant k_c to one another, and by springs of constant k_p to a rigid pulling rod that moves at constant velocity V. In equilibrium, when all the springs are unstretched, adjacent blocks are separated by a distance a. The blocks rest upon a stationary surface, which provides a frictional force that impedes the motion of the blocks. In the particular version considered by Carlson and Langer [5] the friction is a decreasing function of velocity, the same for all blocks; the equation of motion for the jth mass is

$$m\ddot{X}_j = k_c(X_{j+1} - 2X_j + X_{j-1}) + k_p(Vt - X_j) - \frac{F_0}{1+|\dot{X}_j/V_f|}\operatorname{sgn}(\dot{X}_j)\ , \quad \dot{X}_j \neq 0 \tag{1}$$

where X_j is the displacement of the jth block from its equilibrium position. The last term in Eq. (1) represents the frictional force where V_f is a characteristic velocity for the friction. In this homogeneous model, where no inherent disorder was present except for the randomness in the initial displacements of the blocks, the dynamics nonetheless displayed scaling properties [5]. By defining the seismic moment as the sum over all displacements during an event, $M_0 \equiv a\sum_j \delta X_j$, those authors showed that the model had a regime where the distribution of event moments scaled as $\rho(M_0) \propto M_0^{-b-1}$, with $b \approx 1$, consistent with the Gutenberg-Richter law observed in much seismic data [6,7].

Clearly, this model must have a variety of regimes since there are *five independent velocity scales* that can be defined. Two appear explicitly in Eq. (1): (i) the pulling velocity V and (ii) the characteristic friction velocity V_f. Two more can be defined in terms of the spring constants, the mass, and F_0: (iii) $V_0 = F_0/\sqrt{mk_p}$ and (iv) $V_l = (F_0/k_p)\sqrt{k_c/m}$. Here V_0 and V_l correspond to the maximum velocities of a single block held by a spring of constant k_p or k_c, respectively, when it has been displaced by the characteristic distance F_0/k_p in the absence of friction. The fifth velocity, the sound velocity $V_s = a\sqrt{k_c/m}$, depends on the equilibrium spacing of the blocks a, which does not appear explicitly in Eq. (1). Thus only the first four velocities determine the nature of the solutions to that equation, and, in general, we would expect different behavior depending on their relative magnitudes.

In this paper we investigate the behavior of a slowly driven system such that $V \ll V_0 \ll V_l$. We describe our simulations in two different regimes depending on the relative magnitude of V_f and V_0. We find in the case with $V_f < V_0$, first studied extensively in Ref. [5], that the dominant motion of the fault is through rapidly moving, finite-size, ruptures. By tuning the ratio V_f/V_0 through 1, we find that a transition takes place to a new regime where a system-spanning, slowly moving event is continuously occurring. This transition appears to be continuous (i.e., critical) in our simulations. We argue that the first regime, where cracks propagate for finite distances, appears analogous to a first-order transition and that the parameters of the model must be *tuned* in order to bring the system to a critical state. A growth in the correlation length as V_f was increased in a model with a different friction law had previously been observed [8].

Following Carlson and Langer [5], we introduce di-

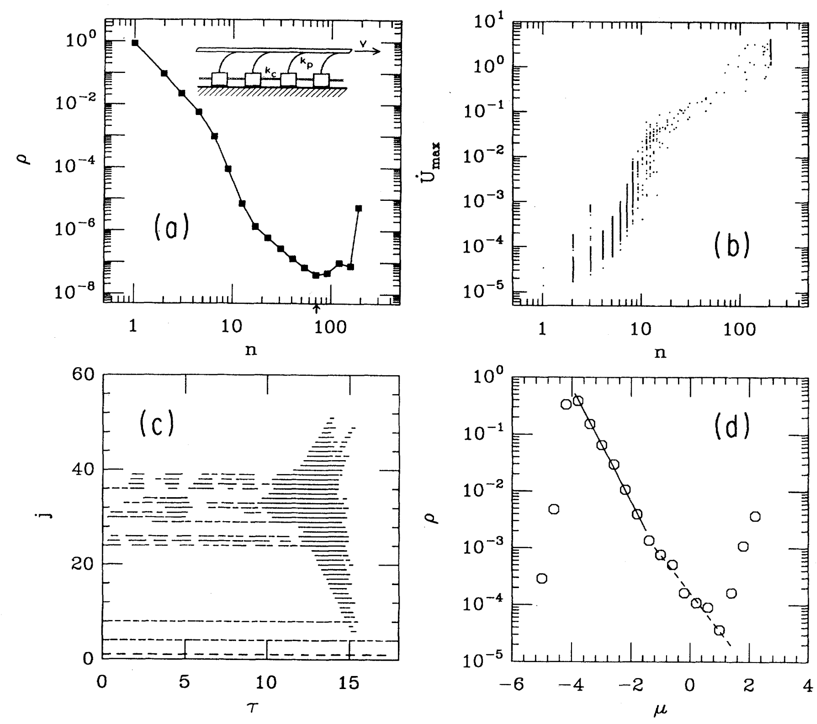

FIG. 1. The results of a simulation with 200 blocks, $\nu=10^{-3}$, $\nu_f=\frac{1}{6}$, and $\nu_l=10$. (a) The distribution $\rho(n)$ of events, appropriately binned, associated with n moving masses vs n. The arrow indicates the position of $\tilde{\xi}/a=71$. Inset: a schematic diagram of the Burridge-Knopoff model. (b) The maximum velocities attained by a mass in an event $\dot{U}_{max}$ vs n. (c) A projection of the first 60 block velocities $\dot{U}_j(t)$ onto the j-t plane. (The zero of time is arbitrary.) Note the two types of events: the small events with $n<12$ and the large V-shaped pulse. (d) The distribution of magnitudes $\rho(\mu)$. Lines indicating scaling with $b=1$ (solid) and $b=\frac{2}{3}$ (dashed) are shown for comparison.

mensionless variables

$$U_j=\frac{k_p}{F_0}X_j\ ,\quad \tau=\left[\frac{m}{k_p}\right]^{1/2}t\ , \tag{2}$$

so that Eq. (1) can be written in dimensionless form

$$\ddot{U}_j=\nu_l^2(U_{j+1}-2U_j+U_{j+1})-U_j+\nu\tau-\frac{\mathrm{sgn}(\dot{U}_j)}{1+|\dot{U}_j/\nu_f|}\ , \tag{3}$$

where $\nu_l=\sqrt{k_c/k_p}$, $\nu=V/V_0$, and $\nu_f=V_f/V_0$. [We depart slightly from the notation of Ref. [5] to emphasize explicitly the different velocity scales. In their notation $\nu_f=(2\alpha)^{-1}$ and $\nu_l\equiv l$.] Dots now represent derivatives with respect ot the scaled time τ. The four relevant velocities V, V_f, V_l, and V_0 have been transformed respectively into ν, ν_f, ν_l, and $\nu_0\equiv 1$. These dimensionless parameters completely determine the behavior of the model. In dimensionless units, the moment of an event is $\mu_0=a(k_p/F_0)\sum_j\delta U_j$, where again the sum is over the blocks displaced during the event. The magnitude of an event is defined as $\mu\equiv\log_{10}\mu_0$.

For the case studied by Carlson and Langer we are in the parameter regime where $\nu\ll\nu_f<\nu_0(\equiv 1)\ll\nu_l$. In Fig. 1 we show the results for a simulation of a chain with $N=200$ blocks, $\nu=0.001$, $\nu_f=\frac{1}{6}$, and $\nu_l=10$. The distribution $\rho(n)$ of events associated with n moving masses is shown in Fig. 1(a). We note three distinct regions in this plot. First we have a region of many small events ($n<n_c\approx 12$) followed by a power-law regime of intermediate-size events ($12<n<70$). Finally there is a peak at $N=200$, corresponding to events that involve all the blocks in the chain, which falls far off the scaling behavior. These three regions correspond to what was designated in Ref. [5] as "microscopic," "localized," and "delocalized" events, respectively. The upper end of the scaling regime is in good agreement with their predicted correlation length $\tilde{\xi}/a=4\nu_l\nu_f\ln(4\nu_l/\nu)=71$ for that crossover. That the largest events in this simulation involve all the blocks in the system is a finite-size effect: if we were to increase N or decrease ν_l we find, as did Carlson *et al.* [8], that these large ruptures do not reach the system boundaries.

The distinction between the first two regions can be seen more clearly in Fig. 1(b), where we plot the maximum velocities attained by a mass in an event, $\dot{U}_{max}$, versus n, the number of blocks moving in the event. Here we see a distinct change in slope and width at $n_c\approx 12$. The vast majority ($>99.9\%$) of the events lie in the region with $n<n_c$. Though numerous, these events contribute very little (less than 1%) to the forward motion of the chain. More than 99% of the motion is due to the few big ruptures with $\dot{U}_{max}>\nu_f$.

We can also look at the geometry of the individual events. In Fig. 1(c) we show a projection of a three-dimensional plot of the block velocities $\dot{U}_j(t)$ onto the j-t plane. In this graph the horizontal solid lines indicate the time intervals during which the corresponding block was in motion. In this representation, an event is a cluster of nearest-neighbor occupied sites (j,t) where by occupied we mean that the jth block was moving at time t. In that figure we can see clearly the two types of behavior: (i) the small motion, with a number of moving blocks less than n_c; and (ii) the ruptures [9] that appear V shaped in the plot. These latter events (which include both localized and delocalized pulses) travel at the speed of sound. The events with $n < n_c$ move more slowly than those involving a larger number of blocks and, as seen in Fig. 1(b), have a different slope $d\dot{U}_{\max}/dn$. Furthermore, they also have a qualitatively different appearance: the small events tend to move in a somewhat coherent fashion [notice the almost periodic repetition of the small events in Fig. 1(c)]. Events with $n < n_c$ are not just smaller events of the same kind as seen in the ruptures; we suggest that they correspond to creep motion of the fault and serve to nucleate the larger ruptures when n reaches n_c.

In Fig. 1(d) we plot the distribution of magnitudes, $\rho(\mu)$ versus μ. We find, in agreement with Carlson and Langer, that the behavior for small μ is consistent with a power law, $\rho(\mu) \propto \mu^{-b}$ with $b \approx 1$, and that the very large ruptures have a much greater occurrence than is predicted by an extrapolation of this behavior. The low-magnitude behavior is dominated by the abundant events with $n < n_c$. Because of the relative scarcity of events in the intermediate region with $12 < n < 70$, it is difficult for us to establish whether those ruptures fall on the same scaling curve as the smaller events. Our best estimate indicates that these events have a smaller slope [10] ($b \approx \frac{2}{3}$) and thus form a separate scaling region.

The behavior of the model in this regime is reminiscent of what occurs at a first-order phase transition. As in such a transition there is *hysteresis* as the fault stores energy until a large characteristic event is *nucleated* when $n \approx \xi/a$. The fact that even the largest ruptures in our model are of finite size does not invalidate this analogy any more than does the finite size of dendrites preclude crystallization from being a first-order transition [11].

We will now describe what happens in the model as ν_f is increased until $\nu_f \approx \nu_0 \equiv 1$. In this limit we will see that there is a transition to another, qualitatively different type of behavior. In the limit where $\nu_f = \infty$ (i.e., the friction is constant and independent of the velocity), the motion for a single block ($N = 1$) can be solved analytically: $\dot{U} = \nu[1 - \cos\tau]$. Likewise, in a chain of N blocks we find in our simulations that no block ever stops moving [12]. We find that the blocks move in a coherent fashion with an approximate period of 2π, as predicted by the preceding equation. This type of behavior is very different from what we have shown for $\nu_f < 1$ (where the motion occurs in abrupt pulses) and there must be some sort of transition between these two regimes. In Fig. 2 we show a series of pictures of the motion of a system as ν_f is varied and all the other parameters are held fixed ($N = 200$, $\nu = 10^{-3}$, and $\nu_l = 2$). As mentioned above, for $\nu_f = \infty$ all the blocks are continuously moving. As ν_f is decreased, small regions of stationary blocks start to appear. When ν_f becomes less than 1.0 these stationary (event-free) regions begin to percolate across the entire system. The movement of the fault in this small ν_f region now occurs in the large, abrupt events.

We have investigated the nature of this transition by calculating the fraction of time Π that any given block participates in a global-sized event as a function of ν_f^{-1}. This is shown in Fig. 3. For low ν_f^{-1} this fraction approaches 1 and decays smoothly towards zero as ν_f^{-1} is increased. If we think of Π as an "order parameter," this transition appears to be a continuous one. The transition occurs near $\nu_f = 1.0$, that is, the point where $V_f = V_0$. For large values of ν_f^{-1}, the relaxation-oscillation behavior of the system is analogous to a first-order transition: there is large hysteresis as the blocks undergo a loading

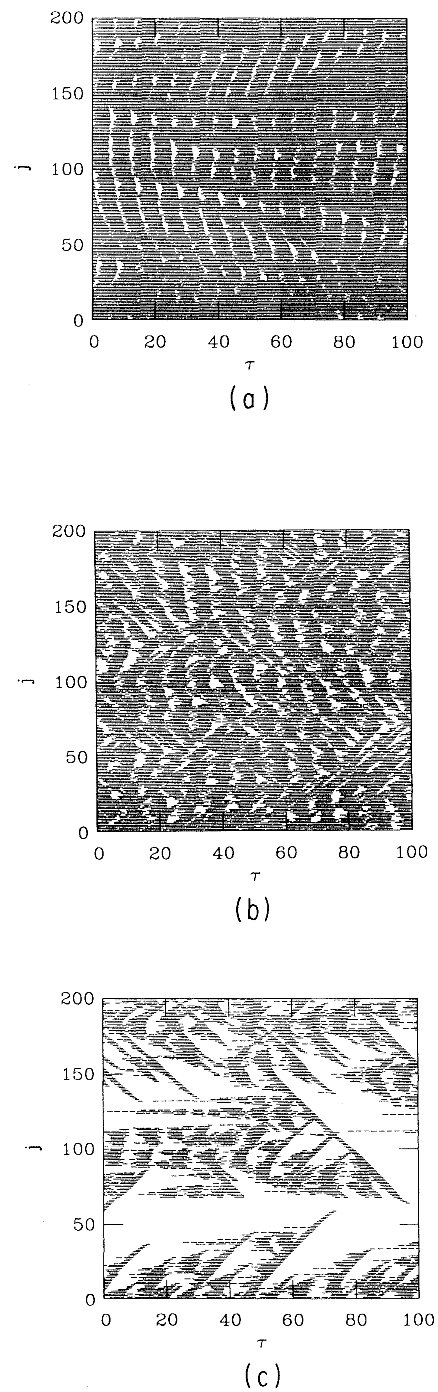

FIG. 2. A projection of the block velocities $\dot{U}_j(t)$ onto the j-t plane for a 200-block system with $\nu = 10^{-3}$ and $\nu_l = 2$. (a) $\nu_f = 5$, (b) $\nu_f = \frac{5}{3}$, and (c) $\nu_f = 1$. Note that as ν_f decreases the blank areas, indicating stationary blocks, become more extended and begin to percolate through the entire system.

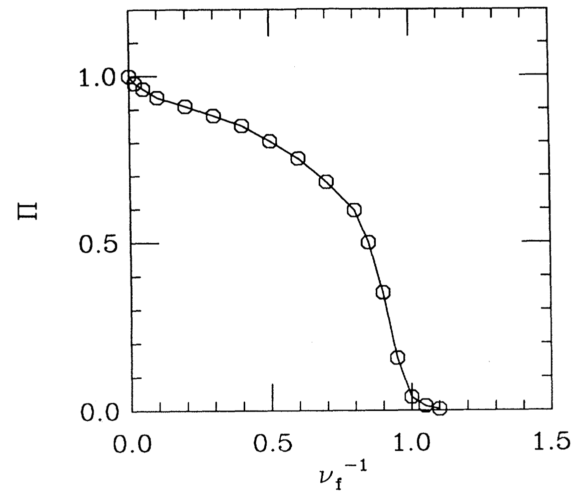

FIG. 3. The fraction of time Π that any given block participates in a global-sized event as a function of ν_f^{-1}. The results are for simulations of 200 blocks with $\nu_l=2$ and $\nu=10^{-3}$.

cycle and the large events produce almost all the motion of the "fault." As ν_f is increased towards 1.0, the loading-unloading hysteresis becomes less dramatic. If we push the analogy with static phase transitions further, this behavior is like a line of first-order transitions ending at a critical point. As in ordinary statistical mechanics, first-order transitions are more common than second-order ones.

It remains an open question whether the behavior of the Burridge-Knopoff model detailed above is a good description of real earthquake phenomena. Avalanches in granular systems [13] have behavior similar to what we have found here in the regime $\nu_f<\nu_0=1$. What appears to be different is the additional scaling region found in this model, which can occur all the way up to the beginning of the delocalized regime. However, this model clearly displays rich dynamics and the variety of behaviors that can be found by varying the four relevant velocity scales deserves further investigation in its own right. We conclude by pointing out that a continuous transition is observed when two of these velocity scales cross. It is tempting to think of this equality of two scales as analogous to a symmetry leading to a critical point in statistical mechanics.

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