EXAMPLE 7.2
Least-squares positioning from ranging in two dimensions

INPUTS:

True user position	$x_{pa}^{p} =$	1000	m
	$y_{pa}^{p} =$	100	m
Predicted user position	$\hat{x}_{pa}^{p-} =$	0	m
	$\hat{y}_{pa}^{p-} =$	0	m
Transmitter 1 position	$x_{p1`}^{p} =$	0	m
	$y_{p1`}^{p} =$	1000	m
Transmitter 2 position	$x_{p2`}^{p} =$	0	m
	$y_{p2`}^{p} =$	-1000	m
Transmitter 3 position	$x_{p3`}^{p} =$	2000	m
	$y_{p3`}^{p} =$	100	m

True and Measured Ranges

From (7.1), $r_{at} = \sqrt{\left(x_{pt}^{p} - x_{pa}^{p}\right)^2 + \left(y_{pt}^{p} - y_{pa}^{p}\right)^2} \qquad t \in 1,2,3$

In this example, there are no measurement errors, so:

$\tilde{r}_{a1} = r_{a1} =$	1345.362405	m
$\tilde{r}_{a2} = r_{a2} =$	1486.606875	m
$\tilde{r}_{a3} = r_{a3} =$	1000	m

First Iteration

Predicted position:

$\hat{x}_{pa}^{p-} =$	0	m
$\hat{y}_{pa}^{p-} =$	0	m

Calculate predicted ranges:

From (7.28), $r_{at} = \sqrt{\left(x_{pt}^{p} - x_{pa}^{p}\right)^2 + \left(y_{pt}^{p} - y_{pa}^{p}\right)^2} \qquad t \in 1,2,3$

$\hat{r}_{a1}^{-} =$	1000	m
$\hat{r}_{a2}^{-} =$	1000	m
$\hat{r}_{a3}^{-} =$	2002.498439	m

Calculate measurement matrix:

From (7.37),

$$\mathbf{H}_p = \begin{pmatrix} -\dfrac{x_{p1`}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a1}^{-}} & -\dfrac{y_{p1`}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a1}^{-}} \\ -\dfrac{x_{p2`}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a2}^{-}} & -\dfrac{y_{p2`}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a2}^{-}} \\ -\dfrac{x_{p3`}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a3}^{-}} & -\dfrac{y_{p3`}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a3}^{-}} \end{pmatrix}$$

$\mathbf{H}_p =$

0	-1
0	1
-0.998752339	-0.049937617

Update position estimate:

From (7.36), $$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} = \begin{pmatrix} \hat{x}_{pa}^{p-} \\ \hat{y}_{pa}^{p-} \end{pmatrix} + \left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^{-} \\ \tilde{r}_{a2} - \hat{r}_{a2}^{-} \\ \tilde{r}_{a3} - \hat{r}_{a3}^{-} \end{pmatrix}$$

$\tilde{r}_{a1} - \hat{r}_{a1}^{-} =$	345.3624047	m
$\tilde{r}_{a2} - \hat{r}_{a2}^{-} =$	486.6068747	m
$\tilde{r}_{a3} - \hat{r}_{a3}^{-} =$	-1002.498439	m

$\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p =$

0.997506234	0.049875312
0.049875312	2.002493766

$\left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} =$

0.025	-0.025	-1.00124922
-0.5	0.5	-3.46945E-18

$\left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^{-} \\ \tilde{r}_{a2} - \hat{r}_{a2}^{-} \\ \tilde{r}_{a3} - \hat{r}_{a3}^{-} \end{pmatrix} =$

1000.219669	m
70.62223501	m

$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} =$

1000.219669	m	Error	0.21967	m
70.62223501	m		-29.37776	m

Second Iteration

Predicted position:

$\hat{x}_{pa}^{p-} =$	1000.219669	m
$\hat{y}_{pa}^{p-} =$	70.62223501	m

Calculate predicted ranges:

From (7.28), $$r_{at} = \sqrt{\left(x_{pt}^{p} - x_{pa}^{p}\right)^2 + \left(y_{pt}^{p} - y_{pa}^{p}\right)^2} \qquad t \in 1,2,3$$

$\hat{r}_{a1}^{-} =$	1365.350657	m
$\hat{r}_{a2}^{-} =$	1465.152332	m
$\hat{r}_{a3}^{-} =$	1000.21186	m

Calculate measurement matrix:

From (7.37), $$\mathbf{H}_p = \begin{pmatrix} -\dfrac{x_{p1}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a1}^{-}} & -\dfrac{y_{p1}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a1}^{-}} \\ -\dfrac{x_{p2}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a2}^{-}} & -\dfrac{y_{p2}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a2}^{-}} \\ -\dfrac{x_{p3}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a3}^{-}} & -\dfrac{y_{p3}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a3}^{-}} \end{pmatrix}$$

$\mathbf{H}_p =$

0.73257347	-0.680687969
0.682672816	0.730724179
-0.999568563	-0.029371542

Update position estimate:

From (7.36), $$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} = \begin{pmatrix} \hat{x}_{pa}^{p-} \\ \hat{y}_{pa}^{p-} \end{pmatrix} + \left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^- \\ \tilde{r}_{a3} - \hat{r}_{a3}^- \end{pmatrix}$$

$\tilde{r}_{a1} - \hat{r}_{a1}^- =$	-19.98825185	m
$\tilde{r}_{a2} - \hat{r}_{a2}^- =$	21.45454292	m
$\tilde{r}_{a3} - \hat{r}_{a3}^- =$	-0.211859698	m

$\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p =$

2.001843376	0.029550456
0.029550456	0.998156624

$\left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} =$

0.376180457	0.33036	-0.499107808
-0.693081883	0.722293	-0.014649684

$\left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^- \\ \tilde{r}_{a3} - \hat{r}_{a3}^- \end{pmatrix} =$

-0.325728868	m
29.35307264	m

$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} =$

999.8939397	m	Error	-0.10606	m
99.97530765	m		-0.02469	m

Third Iteration

Predicted position:

$\hat{x}_{pa}^{p-} =$	999.8939397	m
$\hat{y}_{pa}^{p-} =$	99.97530765	m

Calculate predicted ranges:

From (7.28), $r_{at} = \sqrt{\left(x_{pt}^p - x_{pa}^p\right)^2 + \left(y_{pt}^p - y_{pa}^p\right)^2} \qquad t \in 1,2,3$

$\hat{r}_{a1}^- =$	1345.300092	m
$\hat{r}_{a2}^- =$	1486.517261	m
$\hat{r}_{a3}^- =$	1000.106061	m

Calculate measurement matrix:

From (7.37), $$\mathbf{H}_p = \begin{pmatrix} -\dfrac{x_{p1}^p - \hat{x}_{pa}^{p-}}{\hat{r}_{a1}^-} & -\dfrac{y_{p1}^p - \hat{y}_{pa}^{p-}}{\hat{r}_{a1}^-} \\ -\dfrac{x_{p2}^p - \hat{x}_{pa}^{p-}}{\hat{r}_{a2}^-} & -\dfrac{y_{p2}^p - \hat{y}_{pa}^{p-}}{\hat{r}_{a2}^-} \\ -\dfrac{x_{p3}^p - \hat{x}_{pa}^{p-}}{\hat{r}_{a3}^-} & -\dfrac{y_{p3}^p - \hat{y}_{pa}^{p-}}{\hat{r}_{a3}^-} \end{pmatrix}$$

$\mathbf{H}_p =$

0.743249737	-0.669014072
0.672641997	0.739968069
-1	-2.46897E-05

Update position estimate:

From (7.36),

$$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} = \begin{pmatrix} \hat{x}_{pa}^{p-} \\ \hat{y}_{pa}^{p-} \end{pmatrix} + \left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^{-} \\ \tilde{r}_{a2} - \hat{r}_{a2}^{-} \\ \tilde{r}_{a3} - \hat{r}_{a3}^{-} \end{pmatrix}$$

$\tilde{r}_{a1} - \hat{r}_{a1}^{-} =$	0.062312759	m
$\tilde{r}_{a2} - \hat{r}_{a2}^{-} =$	0.089613478	m
$\tilde{r}_{a3} - \hat{r}_{a3}^{-} =$	-0.106060649	m

$\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p =$	2.004867428	0.000513757
	0.000513757	0.995132572

$\left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} =$	0.370894961	0.335314	-0.498786157
	-0.672477859	0.743414	0.000232698

$\left(\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p\right)^{-1}\mathbf{H}_p^{\mathrm{T}} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^{-} \\ \tilde{r}_{a2} - \hat{r}_{a2}^{-} \\ \tilde{r}_{a3} - \hat{r}_{a3}^{-} \end{pmatrix} =$	0.106061723	m
	0.024691312	m

$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} =$	1000.000001	m	Error	0.00000	m
	99.99999896	m		0.00000	m