NDI and INDI briefing for Joshua Stokes

Dr Ir Thomas Lombaerts

SJSU SAN JOSÉ STATE UNIVERSITY

San José State University, February, 2022

Outline

- Introduction
- Recap Flight Mechanics
- Nonlinear Dynamic Inversion
- Incremental Nonlinear Dynamic Inversion

SJSU

Recap flight mechanics

Kinematic equations for linear and angular accelerations:

$$
\begin{aligned}
\dot{\mathbf{V}} &= \frac{1}{m}\left(\mathbf{F}_{\text{total}} - \omega \times m\mathbf{V}\right) \\
\dot{\omega} &= \mathbf{I}^{-1}\left(\mathbf{M}_{\text{total}} - \omega \times \mathbf{I}\omega\right)
\end{aligned}
\qquad
\begin{aligned}
\mathbf{F}_{\text{total}} &= \mathbf{F}_{\text{aero}} + \mathbf{F}_{\text{prop}} + \mathbf{F}_{\text{grav}} \\
\mathbf{M}_{\text{total}} &= \mathbf{M}_{\text{aero}} + \mathbf{M}_{\text{prop}}
\end{aligned}
$$

Aerodynamic contributions:

$$
\mathbf{F}_{\text{aero}} = \begin{bmatrix} X_{\text{aero}} \\ Y_{\text{aero}} \\ Z_{\text{aero}} \end{bmatrix} = \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} \bar{q} S
$$

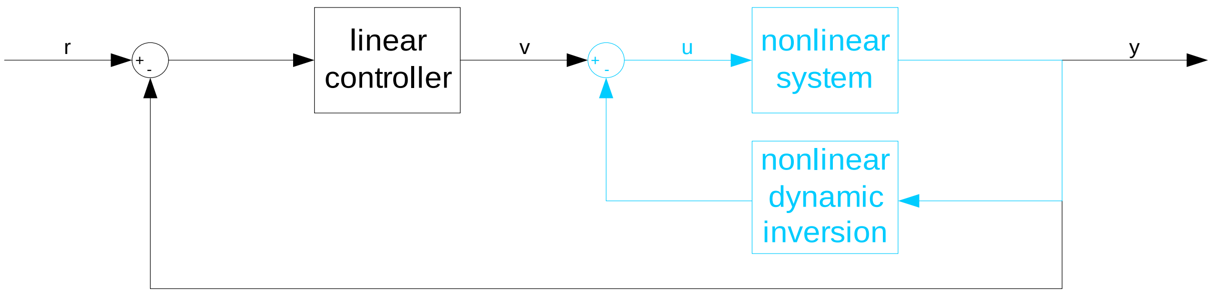
Depend on Lift, Drag and Sideforce

$$
\mathbf{M}_{\text{aero}} = \begin{bmatrix} L_{\text{aero}} \\ M_{\text{aero}} \\ N_{\text{aero}} \end{bmatrix} = \begin{bmatrix} bC_l \\ \bar{c}C_m \\ bC_n \end{bmatrix} \bar{q} S
\qquad
\begin{bmatrix} C_l \\ C_m \\ C_n \end{bmatrix}_{\text{aero}} = \overbrace{\begin{bmatrix} C_l(\beta, p, r) \\ C_m(\alpha, q, \delta_{\text{fl}}) \\ C_n(\beta, p, r) \end{bmatrix}}^{\textit{State dependent}} + \overbrace{\begin{bmatrix} C_{l_{\delta_a}} & 0 & C_{l_{\delta_r}} \\ 0 & C_{m_{\delta_e}} & 0 \\ C_{n_{\delta_a}} & 0 & C_{n_{\delta_r}} \end{bmatrix}}^{\textit{Control effectivity}} \cdot \overbrace{\begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \end{bmatrix}}^{\textit{Control surfaces}}
$$

The concept of Nonlinear Dynamic Inversion

Application of Nonlinear Dynamic Inversion:

Feedback path cancels all nonlinearities, linear closed loop system



Assumptions:

- effector blending is invertible
- reasonable accurate model information is available
  (not always the case)

The concept of NDI

Consider the nonlinear first order MIMO system dynamic model, which is assumed to be affine in the input:

$$\dot{\mathbf{x}} = \underbrace{\mathbf{f}(\mathbf{x},\mathbf{p})}_{\textit{Airframe model}} + \underbrace{\mathbf{G}(\mathbf{x},\mathbf{p})}_{\textit{Effector blending model}}\mathbf{u}$$

Solving for **u** by introducing a virtual outer loop control input vector ν :

$$\mathbf{u} = \mathbf{G}^{-1}(\hat{\mathbf{x}},\hat{\mathbf{p}})\ [\nu - \mathbf{f}(\hat{\mathbf{x}},\hat{\mathbf{p}})]$$

results in a closed-loop system with a linear and decoupled input-output relation: $\dot{\mathbf{x}} = \nu$, for which a linear controller can be designed with exponentially stable tracking dynamics.

SJSU

NDI for aircraft example

Dynamic equation of an aircraft:

$$\dot{\omega} = \mathbf{I}^{-1}\left(\mathbf{M}_{\text{total}} - \omega \times \mathbf{I}\omega\right)$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \mathbf{I}^{-1} \begin{bmatrix} L \\ M \\ N \end{bmatrix} - \mathbf{I}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \mathbf{I} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

Expanding leads to:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \frac{1}{2}\rho V^2 S \mathbf{I}^{-1} \left(\begin{bmatrix} bC_{l_{states}} \\ \bar{c}C_{m_{states}} \\ bC_{n_{states}} \end{bmatrix} + \begin{bmatrix} b\tilde{C}_{l_{\delta_a}} & 0 & b\tilde{C}_{l_{\delta_r}} \\ 0 & \bar{c}\tilde{C}_{m_{\delta_e}} & 0 \\ b\tilde{C}_{n_{\delta_a}} & 0 & b\tilde{C}_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \end{bmatrix} \right) + \\ -\mathbf{I}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \mathbf{I} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

SJSU

NDI for aircraft example

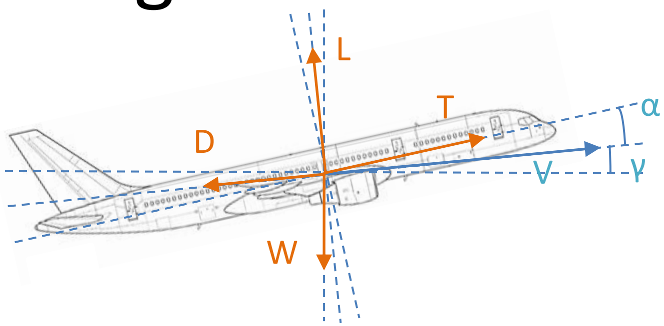
State inversion with airframe model:

$$\begin{bmatrix} L_{\text{req}} \\ M_{\text{req}} \\ N_{\text{req}} \\ F_{\text{v req}} \end{bmatrix} = \begin{bmatrix} I_{XX}\dot{p}_{\text{req}} - I_{XZ}\dot{r}_{\text{req}} - q\left(I_{XZ}p - I_{ZZ}r\right) - I_{YY}qr & -C_{l_{\text{aero}}}\bar{q}Sb \\ I_{YY}\dot{q}_{\text{req}} + r\left(I_{XX}q - I_{XZ}r\right) + p\left(I_{XZ}p - I_{ZZ}r\right) & -C_{m_{\text{aero}}}\bar{q}S\bar{c} \\ I_{ZZ}\dot{r}_{\text{req}} - I_{XZ}\dot{p}_{\text{req}} - q\left(I_{XX}p - I_{XZ}r\right) + I_{YY}pq & -C_{n_{\text{aero}}}\bar{q}Sb \\ m\left(\dot{V}_{\text{req}} + g\sin\gamma\right) & +C_D\bar{q}S \end{bmatrix}$$

Control allocation / effector blending:

$$\begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \\ T \end{bmatrix} = \mathbf{G}^{-1} \begin{bmatrix} L_{\text{req}} \\ M_{\text{req}} \\ N_{\text{req}} \\ F_{\text{v req}} \end{bmatrix}$$

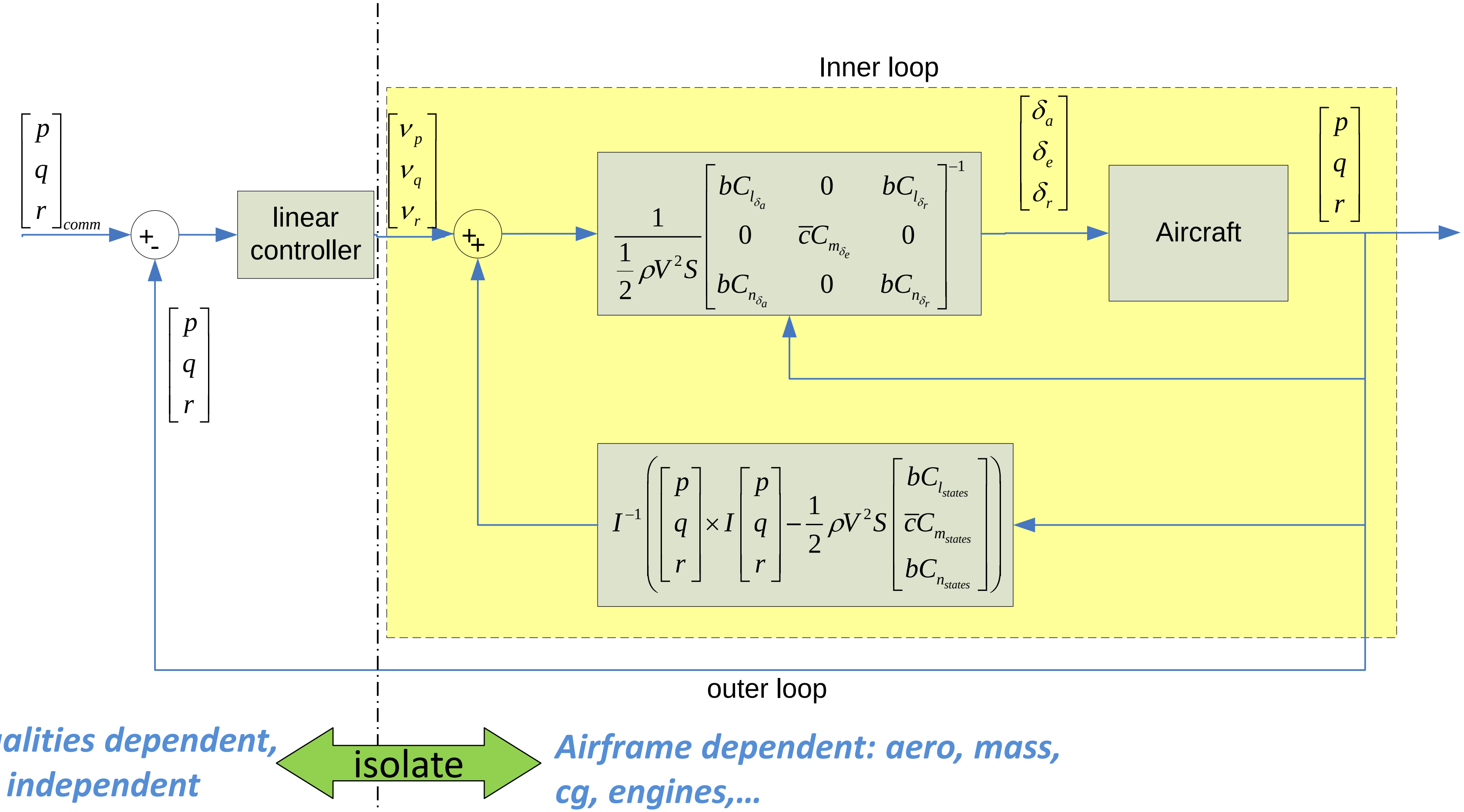
$$\mathbf{G} = \begin{bmatrix} \bar{q}SbC_{l_{\delta_a}} & 0 & \bar{q}SbC_{l_{\delta_r}} & 0 \\ 0 & \bar{q}S\bar{c}C_{m_{\delta_e}} & 0 & 0 \\ \bar{q}SbC_{n_{\delta_a}} & 0 & \bar{q}SbC_{n_{\delta_r}} & 0 \\ 0 & 0 & 0 & \cos\alpha \end{bmatrix}$$



$$m\dot{V} = T\cos\alpha - D - mg\sin\gamma$$

$$T_{\text{req}} = \frac{1}{\cos\alpha}\left(m\left(\dot{V}_{\text{req}} + g\sin\gamma\right) + D\right)$$

NDI for aircraft example



Flying qualities dependent, airframe independent

Airframe dependent: aero, mass, cg, engines,…

Recap NDI

Advantages:

- Physical interpretation of internal signals
- Decoupled steering channels (simplified steering)
- Control laws split: airframe dependent vs. independent part, no gain scheduling

Assumptions:

- effector blending is invertible
- reasonable accurate model information is available

  (not always the case)

NDI vs INDI

System description:
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x},\mathbf{p}\right) + \mathbf{G}\left(\mathbf{x},\mathbf{p}\right)\mathbf{u}$$

- Nonlinear Dynamic Inversion (NDI):

$$\mathbf{u} = \mathbf{G}^{-1}\left(\hat{\mathbf{x}},\hat{\mathbf{p}}\right)\left(\boldsymbol{\nu} - \mathbf{f}\left(\hat{\mathbf{x}},\hat{\mathbf{p}}\right)\right)$$

  assumptions:
  - effector blending is invertible
  - reasonable accurate model information is available

- Incremental Nonlinear Dynamic Inversion (INDI):

$$\Delta\mathbf{u} \simeq \mathbf{G}^{-1}\left(\hat{\mathbf{x}}_{\mathbf{0}},\hat{\mathbf{p}}\right)\left(\boldsymbol{\nu} - \hat{\dot{\mathbf{x}}}_{\mathbf{0}}\right) \qquad \mathbf{u} = \hat{\mathbf{u}}_{\mathbf{0}} + \Delta\mathbf{u}$$

  assumptions:
  - time scale separation: $\mathbf{x} - \mathbf{x}_0 << \mathbf{u} - \mathbf{u}_0$
  - synchronous filtering on state derivatives and previous inputs
  - less model information needed compared to NDI

SJSU

The concept of INDI

Consider the nonlinear first order MIMO system dynamic model, which is assumed to be affine in the input:

$$\dot{\mathbf{x}} = \underbrace{\mathbf{f}(\mathbf{x},\mathbf{p})}_{\textit{Airframe model}} + \underbrace{\mathbf{G}(\mathbf{x},\mathbf{p})}_{\textit{Effector blending model}}\mathbf{u}$$

Nonlinear system is approximated following a first-order Taylor series expansion around operating condition ‘0’:

$$\dot{\mathrm{x}} \approx \dot{\mathrm{x}}_0 + \frac{\partial}{\partial \mathrm{x}}\left[\mathrm{f}(\mathrm{x},\mathrm{p}) + \mathrm{G}(\mathrm{x},\mathrm{p})\mathrm{u}\right]\Big|_{\mathrm{x}=\mathrm{x}_0,\mathrm{u}=\mathrm{u}_0} \underbrace{(\mathrm{x}-\mathrm{x}_0)}_{\Delta \mathrm{x}} + \frac{\partial}{\partial \mathrm{u}}\left[\mathrm{f}(\mathrm{x},\mathrm{p}) + \mathrm{G}(\mathrm{x},\mathrm{p})\mathrm{u}\right]\Big|_{\mathrm{x}=\mathrm{x}_0,\mathrm{u}=\mathrm{u}_0} \underbrace{(\mathrm{u}-\mathrm{u}_0)}_{\Delta \mathrm{u}}$$

time-scale separation: the change in state is considered significantly slower than the change in control input for very small time increments and fast control action: $\mathbf{u}-\mathbf{u}_0 \gg \mathbf{x}-\mathbf{x}_0 \approx 0$

$$\dot{\mathbf{x}} \simeq \dot{\mathbf{x}}_0 + \mathbf{G}\left(\mathbf{x}_0,\mathbf{p}\right)\Delta\mathbf{u}$$

SJSU

The concept of INDI

Given: $\dot{\mathbf{x}} \simeq \dot{\mathbf{x}}_0 + \mathbf{G}(\mathbf{x}_0, \mathbf{p})\Delta\mathbf{u}$

And assuming output is equal to state $\mathbf{y} = \mathbf{x}$, inverting the equation and setting $\dot{\mathbf{x}} = \boldsymbol{\nu}$ results in:

$$\Delta\mathbf{u} \simeq \mathbf{G}^{-1}(\hat{\mathbf{x}}_0, \hat{\mathbf{p}})\left(\boldsymbol{\nu} - \hat{\dot{\mathbf{x}}}_0\right)$$

assumption: effector blending is invertible

Total input by adding current input to calculated increment:

$$\mathbf{u} = \hat{\mathbf{u}}_0 + \Delta\mathbf{u}$$

- Advantage: less model info needed compared to NDI (no **f**, only **G**)
- Additional feedback signals required: state derivatives and inputs.
- Note: synchronous filtering on state derivatives and previous inputs
- Controller should be discretized with sufficiently high sampling rate.

SJSU

INDI for aircraft example

Revisit dynamic equation of an aircraft:

$$\dot{\omega} = \mathbf{I}^{-1}\left(\mathbf{M}_{\text{total}} - \omega \times \mathbf{I}\omega\right)$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \mathbf{I}^{-1}\begin{bmatrix} L \\ M \\ N \end{bmatrix} - \mathbf{I}^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \mathbf{I}\begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

Expanding leads to:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \frac{1}{2}\rho V^2 S \mathbf{I}^{-1}\left(\begin{bmatrix} bC_{l_{states}} \\ \bar{c}C_{m_{states}} \\ bC_{n_{states}} \end{bmatrix} + \begin{bmatrix} b\tilde{C}_{l_{\delta_a}} & 0 & b\tilde{C}_{l_{\delta_r}} \\ 0 & \bar{c}\tilde{C}_{m_{\delta_e}} & 0 \\ b\tilde{C}_{n_{\delta_a}} & 0 & b\tilde{C}_{n_{\delta_r}} \end{bmatrix}\begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \end{bmatrix}\right) + \\ -\mathbf{I}^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \mathbf{I}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

SJSU

INDI for aircraft example

Rewrite as:

$$\dot{\boldsymbol{\omega}}_{\text{req}} = \mathbf{I}^{-1}\left[\mathbf{M}_{\text{aero}} + \mathbf{M}_{\text{CA}}\boldsymbol{\delta} - \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega}\right]$$

Taylor expansion around previous step ‘0’:

$$\dot{\boldsymbol{\omega}}_{\text{req}} \approx \mathbf{I}^{-1}\left[\mathbf{M}_{\text{aero}_0} + \mathbf{M}_{\text{CA}}\boldsymbol{\delta}_0 - \boldsymbol{\omega}_0 \times \mathbf{I}\boldsymbol{\omega}_0\right] + \frac{\partial}{\partial \boldsymbol{\omega}}\left[\mathbf{I}^{-1}\left[\mathbf{M}_{\text{aero}} - \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega}\right]\right]\Bigg|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0, \boldsymbol{\delta}=\boldsymbol{\delta}_0} \left(\boldsymbol{\omega} - \boldsymbol{\omega}_0\right) +$$

$$\frac{\partial}{\partial \boldsymbol{\delta}}\left[\mathbf{I}^{-1}\left[\mathbf{M}_{\text{CA}}\boldsymbol{\delta}\right]\right]\Bigg|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0, \boldsymbol{\delta}=\boldsymbol{\delta}_0} \left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right)$$

Previous step is defined as: $\dot{\boldsymbol{\omega}}_0 = \mathbf{I}^{-1}\left[\mathbf{M}_{\text{aero}_0} + \mathbf{M}_{\text{CA}}\boldsymbol{\delta}_0 - \boldsymbol{\omega}_0 \times \mathbf{I}\boldsymbol{\omega}_0\right]$

Combining and simplifying for assumption $\boldsymbol{\omega} - \boldsymbol{\omega}_0 << \boldsymbol{\delta} - \boldsymbol{\delta}_0$

for small time steps:

$$\dot{\boldsymbol{\omega}}_{\text{req}} \approx \dot{\boldsymbol{\omega}}_0 + \mathbf{I}^{-1}\mathbf{M}_{\text{CA}}\Delta\boldsymbol{\delta} \qquad \Rightarrow \qquad \Delta\boldsymbol{\delta} = \mathbf{M}_{\text{CA}}^{-1}\mathbf{I}\left(\dot{\boldsymbol{\omega}}_{\text{req}} - \dot{\boldsymbol{\omega}}_0\right) \qquad \boldsymbol{\delta} = \boldsymbol{\delta}_0 + \Delta\boldsymbol{\delta}$$

SJSU

INDI for aircraft example

State inversion with airframe model:

$$\begin{bmatrix} L_{\text{req}} \\ M_{\text{req}} \\ N_{\text{req}} \\ F_{\text{v req}} \end{bmatrix} = \begin{bmatrix} I_{XX}\left(\dot{p}_{\text{req}} - \hat{\dot{p}}_0\right) - I_{XZ}\left(\dot{r}_{\text{req}} - \hat{\dot{r}}_0\right) \\ I_{YY}\left(\dot{q}_{\text{req}} - \hat{\dot{q}}_0\right) \\ I_{ZZ}\left(\dot{r}_{\text{req}} - \hat{\dot{r}}_0\right) - I_{XZ}\left(\dot{p}_{\text{req}} - \hat{\dot{p}}_0\right) \\ m\left(\dot{V}_{\text{req}} - \hat{\dot{V}}_0\right) \end{bmatrix}$$

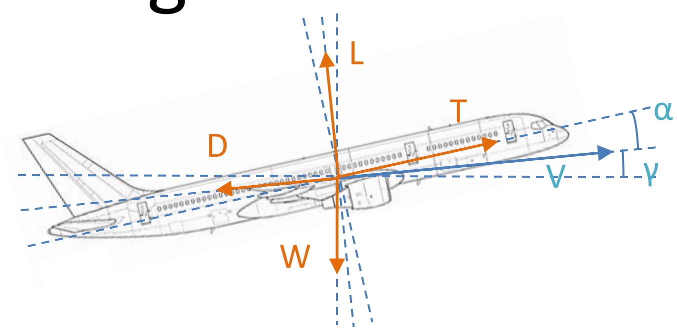
Instead of

$$\begin{bmatrix} L_{\text{req}} \\ M_{\text{req}} \\ N_{\text{req}} \\ F_{\text{v req}} \end{bmatrix} = \begin{bmatrix} I_{XX}\dot{p}_{\text{req}} - I_{XZ}\dot{r}_{\text{req}} - q\left(I_{XZ}p - I_{ZZ}r\right) - I_{YY}qr & -C_{l_{\text{aero}}}\bar{q}Sb \\ I_{YY}\dot{q}_{\text{req}} + r\left(I_{XX}q - I_{XZ}r\right) + p\left(I_{XZ}p - I_{ZZ}r\right) & -C_{m_{\text{aero}}}\bar{q}S\bar{c} \\ I_{ZZ}\dot{r}_{\text{req}} - I_{XZ}\dot{p}_{\text{req}} - q\left(I_{XX}p - I_{XZ}r\right) + I_{YY}pq & -C_{n_{\text{aero}}}\bar{q}Sb \\ m\left(\dot{V}_{\text{req}} + g\sin\gamma\right) & +C_D\bar{q}S \end{bmatrix}$$

Control allocation / effector blending:

$$\begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \\ T \end{bmatrix} = \mathbf{G}^{-1} \begin{bmatrix} L_{\text{req}} \\ M_{\text{req}} \\ N_{\text{req}} \\ F_{\text{v req}} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \bar{q}SbC_{l_{\delta_a}} & 0 & \bar{q}SbC_{l_{\delta_r}} & 0 \\ 0 & \bar{q}S\bar{c}C_{m_{\delta_e}} & 0 & 0 \\ \bar{q}SbC_{n_{\delta_a}} & 0 & \bar{q}SbC_{n_{\delta_r}} & 0 \\ 0 & 0 & 0 & \cos\alpha \end{bmatrix}$$



$$m\dot{V} = T\cos\alpha - D - mg\sin\gamma$$

$$T_{\text{req}} = \frac{m}{\cos\alpha}\left(\dot{V}_{\text{req}} - \hat{\dot{V}}_0\right)$$

SJSU