EXAMPLE 3.1 (Chapter 3 Example A)
Kalman filter estimating single-axis position and velocity

INPUTS:

Initial position		0	m
Initial velocity		2	m s^{-1}
Initial position uncertainty		1	m
Initial velocity uncertainty		0.5	m s^{-1}
Initial position-velocity covariance		0.1	m^2 s^{-1}
Acceleration PSD		0.2	m^2 s^{-3}
Time between epochs	τ_s =	0.5	s
Position measurement (including noise)		2	m
Measurement noise SD		1.5	m

INITIALIZATION

State vector estimate

From (3.12),

$$\mathbf{x}_A = \begin{pmatrix} r^i_{ib,x} \\ v^i_{ib,x} \end{pmatrix}$$

Thus,

$$\hat{\mathbf{x}}_0^+ = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{matrix} \text{m} \\ \text{m s}^{-1} \end{matrix}$$

Error covariance matrix

$$\mathbf{P}_0^+ = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 0.25 \end{pmatrix}$$

SYSTEM PROPAGATION PHASE

Step 1: Calculate transition matrix

From (3.12),

$$\Phi_A = \begin{pmatrix} 1 & \tau_s \\ 0 & 1 \end{pmatrix}$$

Thus,

$$\Phi_0 = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$$

Step 2: Calculate system noise covariance matrix

From (3.47),

$$\mathbf{Q}_A = \begin{pmatrix} \frac{1}{3} S_a \tau_s^3 & \frac{1}{2} S_a \tau_s^2 \\ \frac{1}{2} S_a \tau_s^2 & S_a \tau_s \end{pmatrix}$$

Thus,

$$\mathbf{Q}_0 = \begin{pmatrix} 0.008333333 & 0.025 \\ 0.025 & 0.1 \end{pmatrix}$$

Step 3: State vector time propagation

From (3.14), $\hat{\mathbf{x}}_k^- = \mathbf{\Phi}_{k-1} \hat{\mathbf{x}}_{k-1}^+$

Thus,

$$\hat{\mathbf{x}}_1^- = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{matrix} \text{m} \\ \text{m s}^{-1} \end{matrix} \qquad \mathbf{x}_A = \begin{pmatrix} r^i_{ib,x} \\ v^i_{ib,x} \end{pmatrix}$$

Step 4: Error covariance matrix time propagation

From (3.15), $\mathbf{P}_k^- = \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}^+\mathbf{\Phi}_{k-1}^\mathrm{T} + \mathbf{Q}_{k-1}$

Thus,

$$\mathbf{\Phi}_0\mathbf{P}_0^+\mathbf{\Phi}_0^\mathrm{T} = \begin{pmatrix} 1.1625 & 0.225 \\ 0.225 & 0.25 \end{pmatrix}$$

$$\mathbf{P}_1^- = \begin{pmatrix} 1.170833333 & 0.25 \\ 0.25 & 0.35 \end{pmatrix}$$

MEASUREMENT UPDATE PHASE

Step 5: Calculate Measurement Matrix

From (3.17), $\mathbf{H}_A = \begin{pmatrix} 1 & 0 \end{pmatrix}$

Thus,

$$\mathbf{H}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Step 6: Calculate Measurement Noise Covariance Matrix

For a scalar measurement, this is just the square of the measurement noise SD:

$$\mathbf{R}_1 = 2.25$$

Step 7: Calculate Kalman Gain Matrix

From (3.21), $\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}_k^\mathrm{T}\left(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^\mathrm{T} + \mathbf{R}_k\right)^{-1}$

$$\mathbf{P}_1^-\mathbf{H}_1^\mathrm{T} = \begin{pmatrix} 1.170833333 \\ 0.25 \end{pmatrix}$$

$$\mathbf{H}_1\mathbf{P}_1^-\mathbf{H}_1^\mathrm{T} = 1.170833333$$

$$\mathbf{H}_1\mathbf{P}_1^-\mathbf{H}_1^\mathrm{T} + \mathbf{R}_1 = 3.420833333$$

$$\left(\mathbf{H}_1\mathbf{P}_1^-\mathbf{H}_1^\mathrm{T} + \mathbf{R}_1\right)^{-1} = 0.292326431$$

$$\mathbf{K}_1 = \begin{pmatrix} 0.34226553 \\ 0.073081608 \end{pmatrix}$$

Step 8: Formulate Measurement

From (3.17), $z_A = r_{ib,x}^i + w_m$

$$\mathbf{z}_1 = 2 \text{ m}$$

Step 9: Update State Vector

From (3.24), $\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k\left(\mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-\right)$

$$\mathbf{z}_1 - \mathbf{H}_1\hat{\mathbf{x}}_1^- = 1$$

$$\mathbf{K}_1\left(\mathbf{z}_1 - \mathbf{H}_1\hat{\mathbf{x}}_1^-\right) = \begin{pmatrix} 0.34226553 \\ 0.073081608 \end{pmatrix}$$

$$\hat{\mathbf{x}}_1^+ = \begin{pmatrix} 1.34226553 \text{ m} \\ 2.073081608 \text{ m s}^{-1} \end{pmatrix} \qquad \mathbf{x}_A = \begin{pmatrix} r_{ib,x}^i \\ v_{ib,x}^i \end{pmatrix}$$

Step 10: Update Error Covariance Matrix

From (3.25), $\mathbf{P}_k^+ = \left(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k\right) \mathbf{P}_k^-$

$\mathbf{K}_1 \mathbf{H}_1 =$

0.34226553	0
0.073081608	0

$\left(\mathbf{I} - \mathbf{K}_1 \mathbf{H}_1\right) =$

0.65773447	0
-0.07308161	1

$\mathbf{P}_1^+ =$

0.770097442	0.164433618
0.164433618	0.331729598