EXAMPLE 9.1

Least-squares Positioning from GNSS Pseudo-range Measurements

Note: an ECI frame is used for simplicity

INPUTS:

True user position $\mathbf{r}^i_{ia}\left(t^s_{sa,a}\right) =$

	4245849	m
	-2451342	m
	4113840	m

True user clock offset $\delta\rho^a_c =$ 1000000 m

Predicted user position $\hat{\mathbf{r}}^{i-}_{ia}\left(t^s_{sa,a}\right) =$

	0	m
	0	m
	0	m

Predicted user clock offset $\delta\hat{\rho}^{a-}_c =$ 0 m

Satellite Positions $\mathbf{r}^i_{is}\left(t^s_{st,a}\right) =$

	x	y	z	
Satellite 1	21630742.37	-7872946.37	13290000	m
Satellite 2	9799722.428	-11678854.4	21773061.34	m
Satellite 3	15014045.82	2647381.37	21773061.34	m
Satellite 4	17020279.96	-20283979.8	2316599.642	m
Satellite 5	26076581.77	4598004.93	2316599.642	m

True and Measured Pseudo-ranges

From (8.50), $$\rho^s_{a,C} = \left|\mathbf{r}^i_{is}(t^s_{st,a}) - \mathbf{r}^i_{ia}(t^s_{sa,a})\right| + \delta\rho^a_c(t^s_{sa,a}) \qquad s = 1\ldots5$$

In this example, there are no measurement errors, so:

$\tilde{\rho}^1_{a,C} = \rho^1_{a,C} =$	21391915.65	m
$\tilde{\rho}^2_{a,C} = \rho^2_{a,C} =$	21684307.91	m
$\tilde{\rho}^3_{a,C} = \rho^3_{a,C} =$	22302561.84	m
$\tilde{\rho}^4_{a,C} = \rho^4_{a,C} =$	23009523.62	m
$\tilde{\rho}^5_{a,C} = \rho^5_{a,C} =$	24010959.53	m

First Iteration

Predicted position and receiever clock offset:

$\hat{\mathbf{r}}^{i-}_{ia}\left(t^s_{sa,a}\right) =$	0	m
	0	m
	0	m
$\delta\hat{\rho}^{a-}_c =$	0	m

Calculate predicted psuedo-ranges:

From (9.131), $$\hat{\rho}^{j-}_{a,C} = \sqrt{\left[\hat{\mathbf{r}}^i_{ij}\left(\tilde{t}^j_{st,a}\right) - \hat{\mathbf{r}}^{i-}_{ia}\left(\tilde{t}^j_{sa,a}\right)\right]^{\mathrm{T}}\left[\hat{\mathbf{r}}^i_{ij}\left(\tilde{t}^j_{st,a}\right) - \hat{\mathbf{r}}^{i-}_{ia}\left(\tilde{t}^j_{sa,a}\right)\right]} + \delta\hat{\rho}^{a-}_c \qquad j \in 1\ldots5$$

$\hat{\rho}^{1-}_{a,C} =$	26580000	m
$\hat{\rho}^{2-}_{a,C} =$	26580000	m
$\hat{\rho}^{3-}_{a,C} =$	26580000	m
$\hat{\rho}^{4-}_{a,C} =$	26580000	m
$\hat{\rho}^{5-}_{a,C} =$	26580000	m

Calculate measurement matrix:

From (9.133),

$$\mathbf{H}_G^i = \begin{pmatrix} -u_{a1,x}^i & -u_{a1,y}^i & -u_{a1,z}^i & 1 \\ -u_{a2,x}^i & -u_{a2,y}^i & -u_{a2,z}^i & 1 \\ -u_{a3,x}^i & -u_{a3,y}^i & -u_{a3,z}^i & 1 \\ -u_{a4,x}^i & -u_{a4,y}^i & -u_{a4,z}^i & 1 \\ -u_{a5,x}^i & -u_{a5,y}^i & -u_{a5,z}^i & 1 \end{pmatrix}_{\mathbf{r}_{ia}^i = \hat{\mathbf{r}}_{ia}^{i-}}$$

$\mathbf{H}_G^i =$

-0.813797681	0.296198133	-0.5	1
-0.368687826	0.439385042	-0.81915204	1
-0.564862521	-0.099600503	-0.81915204	1
-0.640341609	0.763129413	-0.08715574	1
-0.981060262	-0.172987394	-0.08715574	1

Update position estimate:

From (9.135),

$$\begin{pmatrix} \hat{\mathbf{r}}_{ia}^{i+} \\ \delta\hat{\rho}_c^{a+} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}}_{ia}^{i-} \\ \delta\hat{\rho}_c^{a-} \end{pmatrix} + \left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} \begin{pmatrix} \tilde{\rho}_{a,C}^1 - \hat{\rho}_{a,C}^{1-} \\ \tilde{\rho}_{a,C}^2 - \hat{\rho}_{a,C}^{2-} \\ \vdots \\ \tilde{\rho}_{a,C}^m - \hat{\rho}_{a,C}^{m-} \end{pmatrix}$$

$\tilde{\rho}_{a,C}^1 - \hat{\rho}_{a,C}^{1-} =$	-5188084.353	m
$\tilde{\rho}_{a,C}^2 - \hat{\rho}_{a,C}^{2-} =$	-4895692.093	m
$\tilde{\rho}_{a,C}^3 - \hat{\rho}_{a,C}^{3-} =$	-4277438.159	m
$\tilde{\rho}_{a,C}^4 - \hat{\rho}_{a,C}^{4-} =$	-3570476.383	m
$\tilde{\rho}_{a,C}^5 - \hat{\rho}_{a,C}^{5-} =$	-2569040.475	m

$\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i =$

2.489783662	-0.665733136	1.312933	-3.3687499
-0.665733136	0.903003948	-0.47786853	1.22612469
1.312933001	-0.477868532	1.60721239	-2.312615574
-3.3687499	1.22612469	-2.31261557	5

$\left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} =$

-4.480751953	1.41202741	1.115106548	1.234657	0.718961
1.630860338	-0.05200909	-0.867792454	0.352902	-1.06396
-2.384158316	-0.01073361	-0.010733614	1.202813	1.20281
-4.321562495	1.15914282	1.159142821	1.501638	1.50164

$$\left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} \begin{pmatrix} \tilde{\rho}_{a,C}^1 - \hat{\rho}_{a,C}^{1-} \\ \tilde{\rho}_{a,C}^2 - \hat{\rho}_{a,C}^{2-} \\ \vdots \\ \tilde{\rho}_{a,C}^m - \hat{\rho}_{a,C}^{m-} \end{pmatrix} =$$

5308514.89
-3021161.84
5082986
2568328.25

$\begin{pmatrix} \hat{\mathbf{r}}_{ia}^{i+} \\ \delta\hat{\rho}_c^{a+} \end{pmatrix} =$

5308514.886	m	Error	1062665.886	m
-3021161.836	m		-569819.8356	m
5082986.002	m		969146.0019	m
2568328.248	m		1568328.248	m

Second Iteration

Predicted position and receiever clock offset:

$\hat{\mathbf{r}}_{ia}^{i-}\left(t_{sa,a}^{s}\right) =$	5308514.886	m
	-3021161.836	m
	5082986.002	m
$\delta\hat{\rho}_{c}^{a-} =$	2568328.248	m

Calculate predicted psuedo-ranges:

From (9.131), $\hat{\rho}_{a,C}^{j-} = \sqrt{\left[\hat{\mathbf{r}}_{ij}^{i}\left(\tilde{t}_{st,a}^{j}\right) - \hat{\mathbf{r}}_{ia}^{i-}\left(\tilde{t}_{sa,a}^{j}\right)\right]^{\mathrm{T}}\left[\hat{\mathbf{r}}_{ij}^{i}\left(\tilde{t}_{st,a}^{j}\right) - \hat{\mathbf{r}}_{ia}^{i-}\left(\tilde{t}_{sa,a}^{j}\right)\right]} + \delta\hat{\rho}_{c}^{a-} \qquad j \in 1 \ldots 5$

$\hat{\rho}_{a,C}^{1-} =$	21470973.62	m
$\hat{\rho}_{a,C}^{2-} =$	21899267.2	m
$\hat{\rho}_{a,C}^{3-} =$	22690165.33	m
$\hat{\rho}_{a,C}^{4-} =$	23611693.44	m
$\hat{\rho}_{a,C}^{5-} =$	24862210.77	m

Calculate measurement matrix:

From (9.133),

$$\mathbf{H}_{G}^{i} = \left.\begin{pmatrix} -u_{a1,x}^{i} & -u_{a1,y}^{i} & -u_{a1,z}^{i} & 1 \\ -u_{a2,x}^{i} & -u_{a2,y}^{i} & -u_{a2,z}^{i} & 1 \\ -u_{a3,x}^{i} & -u_{a3,y}^{i} & -u_{a3,z}^{i} & 1 \\ -u_{a4,x}^{i} & -u_{a4,y}^{i} & -u_{a4,z}^{i} & 1 \\ -u_{a5,x}^{i} & -u_{a5,y}^{i} & -u_{a5,z}^{i} & 1 \end{pmatrix}\right|_{\mathbf{r}_{ia}^{i} = \hat{\mathbf{r}}_{ia}^{i-}}$$

$\mathbf{H}_{G}^{i} =$	-0.86348906	0.256672251	-0.43417278	1
	-0.232332612	0.447867152	-0.86338669	1
	-0.482338213	-0.281711018	-0.82945087	1
	-0.55655381	0.820344931	0.13146122	1
	-0.931558999	-0.341760425	0.12408724	1

Update position estimate:

From (9.135),

$$\begin{pmatrix} \hat{\mathbf{r}}_{ia}^{i+} \\ \delta\hat{\rho}_{c}^{a+} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}}_{ia}^{i-} \\ \delta\hat{\rho}_{c}^{a-} \end{pmatrix} + \left(\mathbf{H}_{G}^{i\,\mathrm{T}}\mathbf{H}_{G}^{i}\right)^{-1}\mathbf{H}_{G}^{i\,\mathrm{T}} \begin{pmatrix} \tilde{\rho}_{a,C}^{1} - \hat{\rho}_{a,C}^{1-} \\ \tilde{\rho}_{a,C}^{2} - \hat{\rho}_{a,C}^{2-} \\ \vdots \\ \tilde{\rho}_{a,C}^{m} - \hat{\rho}_{a,C}^{m-} \end{pmatrix}$$

$\tilde{\rho}_{a,C}^{1} - \hat{\rho}_{a,C}^{1-} =$	-79057.97032	m
$\tilde{\rho}_{a,C}^{2} - \hat{\rho}_{a,C}^{2-} =$	-214959.2896	m
$\tilde{\rho}_{a,C}^{3} - \hat{\rho}_{a,C}^{3-} =$	-387603.4873	m
$\tilde{\rho}_{a,C}^{4} - \hat{\rho}_{a,C}^{4-} =$	-602169.8186	m
$\tilde{\rho}_{a,C}^{5} - \hat{\rho}_{a,C}^{5-} =$	-851251.2414	m

$\mathbf{H}_{G}^{i\,\mathrm{T}}\mathbf{H}_{G}^{i} =$	2.209796263	-0.328003934	0.78681235	-3.066272694
	-0.328003934	1.135592721	-0.19902176	0.901412891
	0.786812352	-0.199021757	1.65461102	-1.87146188
	-3.066272694	0.901412891	-1.87146188	5

$$\left(\mathbf{H}_G^{i\,\mathrm{T}}\mathbf{H}_G^i\right)^{-1}\mathbf{H}_G^{i\,\mathrm{T}} =$$

-2.64977875	0.86808937	0.65808753	0.740664	0.382938
0.859790281	0.10277255	-0.606871337	0.37928	-0.73497
-1.189635947	-0.19918253	-0.140128122	0.755191	0.77376
-2.025265719	0.63927914	0.660534606	0.8685	0.85695

$$\left(\mathbf{H}_G^{i\,\mathrm{T}}\mathbf{H}_G^i\right)^{-1}\mathbf{H}_G^{i\,\mathrm{T}}\begin{pmatrix}\tilde{\rho}_{a,C}^{1}-\hat{\rho}_{a,C}^{1-}\\ \tilde{\rho}_{a,C}^{2}-\hat{\rho}_{a,C}^{2-}\\ \vdots \\ \tilde{\rho}_{a,C}^{m}-\hat{\rho}_{a,C}^{m-}\end{pmatrix} =$$

-1004176.7
542414.753
-922233.121
-1485797.04

$$\begin{pmatrix}\hat{\mathbf{r}}_{ia}^{i+}\\ \delta\hat{\rho}_{c}^{a+}\end{pmatrix} =$$

			Error		
4304338.189	m			58489.18921	m
-2478747.082	m			-27405.08248	m
4160752.88	m			46912.88044	m
1082531.213	m			82531.21337	m

Third Iteration

Predicted position and receiever clock offset:

$\hat{\mathbf{r}}_{ia}^{i-}\left(t_{sa,a}^{s}\right)=$	4304338.189	m
	-2478747.082	m
	4160752.88	m
$\delta\hat{\rho}_{c}^{a-} =$	1082531.213	m

Calculate predicted psuedo-ranges:

From (9.131), $\hat{\rho}_{a,C}^{j-} = \sqrt{\left[\hat{\mathbf{r}}_{ij}^{i}\left(\tilde{t}_{st,a}^{j}\right)-\hat{\mathbf{r}}_{ia}^{i-}\left(\tilde{t}_{sa,a}^{j}\right)\right]^{\mathrm{T}}\left[\hat{\mathbf{r}}_{ij}^{i}\left(\tilde{t}_{st,a}^{j}\right)-\hat{\mathbf{r}}_{ia}^{i-}\left(\tilde{t}_{sa,a}^{j}\right)\right]}+\delta\hat{\rho}_{c}^{a-}$ $\quad j \in 1 \ldots 5$

$\hat{\rho}_{a,C}^{1-} =$	21396192.15	m
$\hat{\rho}_{a,C}^{2-} =$	21698899.27	m
$\hat{\rho}_{a,C}^{3-} =$	22323257.18	m
$\hat{\rho}_{a,C}^{4-} =$	23039816.75	m
$\hat{\rho}_{a,C}^{5-} =$	24050158.66	m

Calculate measurement matrix:

From (9.133),

$$\mathbf{H}_G^i = \begin{pmatrix} -u_{a1,x}^{i} & -u_{a1,y}^{i} & -u_{a1,z}^{i} & 1\\ -u_{a2,x}^{i} & -u_{a2,y}^{i} & -u_{a2,z}^{i} & 1\\ -u_{a3,x}^{i} & -u_{a3,y}^{i} & -u_{a3,z}^{i} & 1\\ -u_{a4,x}^{i} & -u_{a4,y}^{i} & -u_{a4,z}^{i} & 1\\ -u_{a5,x}^{i} & -u_{a5,y}^{i} & -u_{a5,z}^{i} & 1 \end{pmatrix}_{\mathbf{r}_{ia}^{i}=\hat{\mathbf{r}}_{ia}^{i-}}$$

$$\mathbf{H}_G^i =$$

-0.852943457	0.265545403	-0.44941417	1
-0.26655443	0.446252575	-0.85428764	1
-0.50420629	-0.241334899	-0.82917639	1
-0.579121757	0.810903182	0.08398822	1
-0.947953533	-0.308118548	0.08029359	1

Update position estimate:

From (9.135),

$$\begin{pmatrix} \hat{\mathbf{r}}_{ia}^{i+} \\ \delta\hat{\rho}_c^{a+} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}}_{ia}^{i-} \\ \delta\hat{\rho}_c^{a-} \end{pmatrix} + \left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} \begin{pmatrix} \tilde{\rho}_{a,C}^1 - \hat{\rho}_{a,C}^{1-} \\ \tilde{\rho}_{a,C}^2 - \hat{\rho}_{a,C}^{2-} \\ \vdots \\ \tilde{\rho}_{a,C}^m - \hat{\rho}_{a,C}^{m-} \end{pmatrix}$$

$\tilde{\rho}_{a,C}^1 - \hat{\rho}_{a,C}^{1-} =$	-4276.505192	m
$\tilde{\rho}_{a,C}^2 - \hat{\rho}_{a,C}^{2-} =$	-14591.366	m
$\tilde{\rho}_{a,C}^3 - \hat{\rho}_{a,C}^{3-} =$	-20695.33971	m
$\tilde{\rho}_{a,C}^4 - \hat{\rho}_{a,C}^{4-} =$	-30293.13205	m
$\tilde{\rho}_{a,C}^5 - \hat{\rho}_{a,C}^{5-} =$	-39199.13869	m

$\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i =$

2.286785697	-0.401292851	0.90436099	-3.150779466
-0.401292851	1.080399265	-0.25709236	0.973247714
0.904360989	-0.25709236	1.63281504	-1.968596399
-3.150779466	0.973247714	-1.9685964	5

$\left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} =$

-2.941366777	0.95828998	0.731731595	0.818199	0.433147
0.972036778	0.07885046	-0.64562993	0.380701	-0.78596
-1.360594883	-0.1729784	-0.125990227	0.821636	0.83793
-2.37841856	0.72041894	0.737171766	0.964983	0.95584

$$\left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} \begin{pmatrix} \tilde{\rho}_{a,C}^1 - \hat{\rho}_{a,C}^{1-} \\ \tilde{\rho}_{a,C}^2 - \hat{\rho}_{a,C}^{2-} \\ \vdots \\ \tilde{\rho}_{a,C}^m - \hat{\rho}_{a,C}^{m-} \end{pmatrix} =$$

-58312.1947
27330.3365
-46785.9741
-82297.247

$\begin{pmatrix} \hat{\mathbf{r}}_{ia}^{i+} \\ \delta\hat{\rho}_c^{a+} \end{pmatrix} =$

			Error	
4246025.994	m		176.9944686	m
-2451416.746	m		-74.74602809	m
4113966.906	m		126.9063111	m
1000233.966	m		233.9663531	m

Fourth Iteration

Predicted position and receiever clock offset:

$\hat{\mathbf{r}}_{ia}^{i-}\left(t_{sa,a}^s\right) =$	4246025.994	m
	-2451416.746	m
	4113966.906	m
$\delta\hat{\rho}_c^{a-} =$	1000233.966	m

Calculate predicted psuedo-ranges:

From (9.131), $\hat{\rho}_{a,C}^{j-} = \sqrt{\left[\hat{\mathbf{r}}_{ij}^i\left(\tilde{t}_{st,a}^j\right) - \hat{\mathbf{r}}_{ia}^{i-}\left(\tilde{t}_{sa,a}^j\right)\right]^{\mathrm{T}}\left[\hat{\mathbf{r}}_{ij}^i\left(\tilde{t}_{st,a}^j\right) - \hat{\mathbf{r}}_{ia}^{i-}\left(\tilde{t}_{sa,a}^j\right)\right]} + \delta\hat{\rho}_c^{a-}$ $\quad j \in 1 \ldots 5$

$\hat{\rho}_{a,C}^{1-} =$	21391921.74	m
$\hat{\rho}_{a,C}^{2-} =$	21684352.66	m
$\hat{\rho}_{a,C}^{3-} =$	22302619.03	m
$\hat{\rho}_{a,C}^{4-} =$	23009604.66	m
$\hat{\rho}_{a,C}^{5-} =$	24011058.39	m

Calculate measurement matrix:

From (9.133),

$$\mathbf{H}_G^i = \begin{pmatrix} -u^i_{a1,x} & -u^i_{a1,y} & -u^i_{a1,z} & 1 \\ -u^i_{a2,x} & -u^i_{a2,y} & -u^i_{a2,z} & 1 \\ -u^i_{a3,x} & -u^i_{a3,y} & -u^i_{a3,z} & 1 \\ -u^i_{a4,x} & -u^i_{a4,y} & -u^i_{a4,z} & 1 \\ -u^i_{a5,x} & -u^i_{a5,y} & -u^i_{a5,z} & 1 \end{pmatrix}_{\mathbf{r}^i_{ia} = \hat{\mathbf{r}}^{i-}_{ia}}$$

$\mathbf{H}_G^i =$

-0.852539357	0.265869588	-0.4499889	1
-0.268500511	0.446112199	-0.85375136	1
-0.505484235	-0.239353392	-0.82897264	1
-0.580400691	0.810225939	0.08166373	1
-0.948708111	-0.306352417	0.07810964	1

Update position estimate:

From (9.135),

$$\begin{pmatrix} \hat{\mathbf{r}}^{i+}_{ia} \\ \delta\hat{\rho}^{a+}_{c} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}}^{i-}_{ia} \\ \delta\hat{\rho}^{a-}_{c} \end{pmatrix} + \left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} \begin{pmatrix} \tilde{\rho}^1_{a,C} - \hat{\rho}^{1-}_{a,C} \\ \tilde{\rho}^2_{a,C} - \hat{\rho}^{2-}_{a,C} \\ \vdots \\ \tilde{\rho}^m_{a,C} - \hat{\rho}^{m-}_{a,C} \end{pmatrix}$$

$\tilde{\rho}^1_{a,C} - \hat{\rho}^{1-}_{a,C} =$	-6.092449263	m
$\tilde{\rho}^2_{a,C} - \hat{\rho}^{2-}_{a,C} =$	-44.75128129	m
$\tilde{\rho}^3_{a,C} - \hat{\rho}^{3-}_{a,C} =$	-57.18678384	m
$\tilde{\rho}^4_{a,C} - \hat{\rho}^{4-}_{a,C} =$	-81.04043965	m
$\tilde{\rho}^5_{a,C} - \hat{\rho}^{5-}_{a,C} =$	-98.8607426	m

$\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i =$

2.291342234	-0.405072947	0.91039759	-3.155632906
-0.405072947	1.077310654	-0.25985285	0.976501917
0.910397588	-0.259852852	1.63134711	-1.972939528
-3.155632906	0.976501917	-1.97293953	5

$\left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} =$

-2.958467492	0.96332612	0.735986842	0.8228	0.436354
0.978317303	0.07768256	-0.647883956	0.380752	-0.78887
-1.370196138	-0.17167202	-0.12511923	0.825393	0.84159
-2.398896042	0.72506958	0.74166231	0.97062	0.96154

$\left(\mathbf{H}_G^{i\,\mathrm{T}} \mathbf{H}_G^i\right)^{-1} \mathbf{H}_G^{i\,\mathrm{T}} \begin{pmatrix} \tilde{\rho}^1_{a,C} - \hat{\rho}^{1-}_{a,C} \\ \tilde{\rho}^2_{a,C} - \hat{\rho}^{2-}_{a,C} \\ \vdots \\ \tilde{\rho}^m_{a,C} - \hat{\rho}^{m-}_{a,C} \end{pmatrix} =$

-176.992887
74.745416
-126.905292
-233.964357

$\begin{pmatrix} \hat{\mathbf{r}}^{i+}_{ia} \\ \delta\hat{\rho}^{a+}_{c} \end{pmatrix} =$

4245849.002	m	Error	0.001581227	m
-2451342.001	m		-0.000612094	m
4113840.001	m		0.001019315	m
1000000.002	m		0.00199621	m