APPENDIX C

Position Representations, Transformations, and Conversions

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This appendix presents additional information on position representations, datum transformations, and coordinate conversions, building on Sections 2.3.2 and 2.4.2–2.4.5. Section C.1 discusses datums and describes the transformation of coordinates between them. Section C.2 presents some additional Cartesian-to-curvilinear position coordinate conversion methods. Section C.3 introduces the normal vector representation of curvilinear position. Finally, Section C.4 describes the properties of the transverse Mercator projection and the conversion of latitude and longitude to and from transverse Mercator coordinates.

C.1 Datums

The datum is introduced in Section 2.4.1. There are three main classes of datum: global, regional, and historical. A global datum may be used for positioning anywhere on the planet. Examples include ITRF, WGS84, GTRF, PZ-90.02, and CGCS 2000. Global datums are used for GNSS positioning and inertial navigation, but are not suitable for mapping. This is because points on the surface of the Earth move with respect to the datum over time due to the motion of the tectonic plates and other effects.

A regional datum avoids this problem by moving with the tectonic plates so that points on the surface of the Earth are fixed with respect to the datum. This enables it to be used for mapping. The area over which a regional datum is valid is bounded by the relevant tectonic plate. Examples of regional datums include the North American Datum of 1983 (NAD83), used in the United States and Canada, the European Terrestrial Reference Frame 1989 (ETRF89), the Geodetic Datum of Australia, and New Zealand Geodetic Datum 2000. Many of these regional datums are defined as coincident with a global datum at a particular time. For example, ETRF89 coincided with WGS84 at the beginning of 1989 [1, 2]. Both regional and global datums are realized using satellite measurements and use the GRS80 or WGS84 reference ellipsoid.

Historical datums were realized using terrestrial surveying techniques and use local-fit ellipsoids. Examples include the North American Datum of 1927 (NAD27), the European Datum 1950 (ED50), and Ordnance Survey Great Britain 1936 (OSGB36). Due to limitations in the original surveying methodology, these datums exhibit distortions (i.e., spatially varying scale-factor errors). However, they are retained to provide continuity of mapping data. In many countries, the national historical datum is being superseded by the appropriate regional datum.

A position solution may be transformed from one datum to another in Cartesian, curvilinear, or projected form. Only Cartesian position transformation is presented here as it is the simplest approach. Both curvilinear and projected positions may be transformed through conversion to their Cartesian counterparts, noting that different ellipsoid parameters may be required for the conversions to and from Cartesian position. Methods for transforming curvilinear and projected positions directly are described in [3].

For datums with parallel axes and a common scaling, such as all of the global datums, positions may be transformed simply by applying a linear translation. Thus,

$$\begin{aligned}\mathbf{r}_{Eb}^{E} &= \mathbf{r}_{eb}^{e} + \mathbf{r}_{Ee}^{E} \\ &= \mathbf{r}_{eb}^{e} - \mathbf{r}_{eE}^{e}\end{aligned}, \tag{C.1}$$

where e and E denote the two ECEF coordinate frames and $\mathbf{r}_{eE}^{e} = -\mathbf{r}_{Ee}^{E}$ is the relative position of their origins. This is known as a 3-parameter coordinate transformation [3].

Not all datums are parallel so a coordinate transformation may also require rotation of the axes. Furthermore, it may be necessary to apply a scale-factor correction where a historical datum is involved due to variations in the definition of the meter (or foot). In these cases a 7-parameter transformation is applied [3]:

$$\begin{aligned}\mathbf{r}_{Eb}^{E} &= \mu_e^E \mathbf{C}_e^E \mathbf{r}_{eb}^{e} + \mathbf{r}_{Ee}^{E} \\ &= \mu_e^E \mathbf{C}_e^E \left(\mathbf{r}_{eb}^{e} - \mathbf{r}_{eE}^{e}\right)\end{aligned}, \tag{C.2}$$

where μ_e^E is the scale factor ($\mu_e^E \mu_E^e = 1$). Note that, for the 7-parameter transformation, $\mathbf{r}_{eE}^{e} \neq -\mathbf{r}_{Ee}^{E}$. The coordinate transformation matrix, $\mathbf{C}_e^E$, is obtained from Euler angles using (2.22) or (2.26). However, there are two different sign conventions in use. The coordinate frame convention provides $\boldsymbol{\psi}_{eE}$ for a transformation from reference frame e to reference frame E, whereas the position vector convention, which is more common, provides $\boldsymbol{\psi}_{Ee}$ [3].

Where a coordinate transformation involves a historical datum, the transformation parameters will be spatially variant due to the limitations of the original surveying methodology. An accurate transformation must account for these distortions in the datum. There are a number of different methods. However, to ensure standardization, the method recommended by the relevant national mapping agency should always be used [3].

C.2 Cartesian to Curvilinear Coordinate Conversion Methods

This section describes methods of obtaining geodetic latitude and height from Cartesian position, building on the introduction in Section 2.4.3. Iterative methods are described first followed by the Heikkinen and Borkowski closed-form exact solutions. Example 2.2 on CD includes all of the methods and may be edited using Microsoft Excel.

C.2.1 Iterative methods

Four different iterative methods for obtaining latitude and height are described here (noting that obtaining longitude is straightforward). In each case, k is used to denote the iteration. The first method, based on (2.113), iterates the latitude and height together. The initial latitude, $L_{b,0}$, is the last known latitude or geocentric latitude.

The transverse radius of curvature is calculated first:

$$R_E(L_{b,k-1}) = \frac{R_0}{\sqrt{1 - e^2 \sin^2 L_{b,k-1}}}, \tag{C.3}$$

followed by the height:

$$h_{b,k-1} = \frac{\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}}}{\cos L_{b,k-1}} - R_E(L_{b,k-1}), \tag{C.4}$$

and finally the geodetic latitude:

$$\tan L_{b,k} = \frac{z_{eb}^{e}\left[R_E(L_{b,k-1}) + h_{b,k-1}\right]}{\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}}\left[\left(1-e^2\right)R_E(L_{b,k-1}) + h_{b,k-1}\right]}. \tag{C.5}$$

The iteration continues until $|L_{b,k} - L_{b,k-1}|$ and $|h_{b,k} - h_{b,k-1}|$ are below their respective precision thresholds. In polar regions, (C.4) and (C.5) are replaced by

$$h_{b,k-1} = \frac{z_{eb}^{e}}{\sin L_{b,k-1}} - \left(1-e^2\right)R_E(L_{b,k-1}) \tag{C.6}$$

and

$$\tan\left(\frac{\pi}{2} - L_{b,k}\right) = \frac{\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}}\left[\left(1-e^2\right)R_E(L_{b,k-1}) + h_{b,k-1}\right]}{z_{eb}^{e}\left[R_E(L_{b,k-1}) + h_{b,k-1}\right]}, \tag{C.7}$$

respectively.

The other three methods iterate the latitude calculation only until $|L_{b,k} - L_{b,k-1}|$ is below its precision threshold. The final height solution is then obtained using

$$h_b = \frac{\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}}}{\cos L_b} - \frac{R_0}{\sqrt{1 - e^2\sin^2 L_b}} \tag{C.8}$$

or, in polar regions,

$$h_b = \frac{z_{eb}^{e}}{\sin L_b} - \frac{\left(1-e^2\right)R_0}{\sqrt{1 - e^2\sin^2 L_b}}. \tag{C.9}$$

The second iterative method is based on the approximate closed-form solution of (2.116) and (2.117) [4]. The reduced (or parametric) latitude is first calculated using

$$\tan\zeta_{b,0} = \frac{z_{eb}^{e}}{\sqrt{1-e^2}\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}}} \tag{C.10}$$

on initialization and

$$\tan\zeta_{b,k-1} = \sqrt{1-e^2}\tan L_{b,k-1} \tag{C.11}$$

on subsequent iterations. The latitude solution is then updated using

$$\tan L_{b,k} = \frac{z_{eb}^{e}\sqrt{1-e^2} + e^2 R_0 \sin^3\zeta_{b,k-1}}{\sqrt{1-e^2}\left(\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}} - e^2 R_0\cos^3\zeta_{b,k-1}\right)}. \tag{C.12}$$

The third iterative method [5] initializes the latitude using

$$\tan L_{b,0} = \frac{z_{eb}^{e}}{\sqrt{1-e^2}\sqrt{x_{eb}^{e\,2} + y_{eb}^{e\,2}}} \tag{C.13}$$

and then iterates it using

$$\tan L_{b,k} = \frac{z_{eb}^{e} + e^2 R_E\left(L_{b,k-1}\right)\sin L_{b,k-1}}{\sqrt{x_{eb}^{e\ 2} + y_{eb}^{e\ 2}}}, \tag{C.14}$$

where $R_E(L_{b,k-1})$ is given by (C.3).

The fourth method [6] initializes the reduced latitude using (C.10) and then iterates it using

$$\zeta_{b,k} = \zeta_{b,k-1} - \frac{2\sin\left(\zeta_{b,k-1} - A\right) - B\sin 2\zeta_{b,k-1}}{2\cos\left(\zeta_{b,k-1} - A\right) - B\cos 2\zeta_{b,k-1}}, \tag{C.15}$$

where

$$A = \arctan\left(\frac{z_{eb}^{e}\sqrt{1-e^2}}{\sqrt{x_{eb}^{e\ 2} + y_{eb}^{e\ 2}}}\right), \qquad B = \frac{e^2 R_0}{\sqrt{x_{eb}^{e\ 2} + y_{eb}^{e\ 2} + \left(1-e^2\right)z_{eb}^{e\ 2}}}. \tag{C.16}$$

Once the reduced latitude has converged, the geodetic latitude is obtained using

$$L_b = \arctan\left(\frac{\tan\zeta_b}{\sqrt{1-e^2}}\right). \tag{C.17}$$

C.2.2 Heikkinen Closed-Form Exact Solution

The geodetic latitude and height may be obtained from Cartesian position using the following steps in sequence [7]:

$$\beta_{eb}^{e} = \sqrt{x_{eb}^{e\ 2} + y_{eb}^{e\ 2}}, \tag{C.18}$$

$$F = 54\left(1-e^2\right)R_0^2 z_{eb}^{e\ 2}, \tag{C.19}$$

$$G = \beta_{eb}^{e\ 2} + \left(1-e^2\right)z_{eb}^{e\ 2} - e^4 R_0^2, \tag{C.20}$$

$$C = \frac{e^4 F \beta_{eb}^{e\ 2}}{G^3}, \tag{C.21}$$

$$S = \left(1 + C + \sqrt{C^2 + 2C}\right)^{1/3}, \tag{C.22}$$

$$P = \frac{F}{3\left(S + \frac{1}{S} + 1\right)^2 G^2}, \tag{C.23}$$

$$Q = \sqrt{1 + 2e^4 P}, \tag{C.24}$$

$$T = \sqrt{\frac{R_0^2}{2}\left(1 + \frac{1}{Q}\right) - \frac{P\left(1-e^2\right)z_{eb}^{e\ 2}}{Q(1+Q)} - \frac{P\beta_{eb}^{e\ 2}}{2}} - \frac{Pe^2\beta_{eb}^{e}}{1+Q}, \tag{C.25}$$

$$V = \sqrt{\left(\beta_{eb}^{e} - e^2 T\right)^2 + \left(1-e^2\right)z_{eb}^{e\ 2}}, \tag{C.26}$$

$$L_b = \arctan\left[\left(1 + \frac{e^2 R_0}{V}\right)\frac{z_{eb}^e}{\beta_{eb}^e}\right], \tag{C.27}$$

$$h_b = \left[1 - \frac{\left(1-e^2\right)R_0}{V}\right]\sqrt{\left(\beta_{eb}^e - e^2 T\right)^2 + z_{eb}^{e\,2}}\,. \tag{C.28}$$

Note that the intermediate terms, C, F, G, P, Q, S, T, and V, have no meaning outside this calculation.

C.2.3 Borkowski Closed-Form Exact Solution

The geodetic latitude and height may be obtained from Cartesian position using the following steps in sequence [6]:

$$E = \frac{\sqrt{1-e^2}\left|z_{eb}^b\right| - e^2 R_0}{\beta_{eb}^b}, \tag{C.29}$$

where β_{eb}^e is given by (C.18),

$$F = \frac{\sqrt{1-e^2}\left|z_{eb}^b\right| + e^2 R_0}{\beta_{eb}^b}, \tag{C.30}$$

$$P = \frac{4}{3}\left(EF + 1\right), \tag{C.31}$$

$$Q = 2\left(E^2 - F^2\right), \tag{C.32}$$

$$D = P^3 + Q^2, \tag{C.33}$$

$$V = \left(D^{1/2} - Q\right)^{1/3} - \left(D^{1/2} + Q\right)^{1/3}, \tag{C.34}$$

$$G = \frac{1}{2}\left(\sqrt{E^2 + V} + E\right), \tag{C.35}$$

$$T = \sqrt{G^2 + \frac{F - VG}{2G - E}} - G, \tag{C.36}$$

$$L_b = \mathrm{sign}(z_{eb}^e)\arctan\left(\frac{1-T^2}{2T\sqrt{1-e^2}}\right), \tag{C.37}$$

$$h_b = \left(\beta_{eb}^e - R_0 T\right)\cos L_b + \left(z_{eb}^e - \mathrm{sign}(z_{eb}^e) R_0\sqrt{1-e^2}\right)\sin L_b\,. \tag{C.38}$$

Note that the intermediate terms, D, E, F, G, P, Q, T, and V, have no meaning outside this calculation.

C.3 Normal Vector

The normal vector, $\mathbf{n}_{eb}^e$, is the line-of-sight unit vector that describes the normal to the ellipsoid. It may be used as an alternative to latitude and longitude for expressing horizontal curvilinear position [8]. Unlike longitude, the normal vector is not singular at the poles. Its relationship to latitude and longitude is analogous to the relationship of quaternion attitude to Euler attitude. By introducing an additional parameter, the singularity is avoided. Thus, although the normal vector has three components, as with any unit vector, only two of them are independent.

Transformation from and to latitude and longitude is straightforward:

$$\mathbf{n}_{eb}^{e} = \begin{pmatrix} \cos L_b \cos \lambda_b \\ \cos L_b \sin \lambda_b \\ \sin L_b \end{pmatrix}, \tag{C.39}$$

$$\begin{aligned} L_b &= \arcsin\left(n_{eb,z}^{e}\right) \\ \lambda_b &= \arctan_2\left(n_{eb,y}^{e}, n_{eb,x}^{e}\right) \end{aligned}. \tag{C.40}$$

The meridian and transverse radii of curvature are readily expressed in terms of the z-component of the normal vector:

$$R_N\left(n_{eb,z}^{e}\right) = \frac{R_0\left(1-e^2\right)}{\left(1-e^2 n_{eb,z}^{e\ 2}\right)^{3/2}}, \qquad R_E\left(n_{eb,z}^{e}\right) = \frac{R_0}{\sqrt{1-e^2 n_{eb,z}^{e\ 2}}}. \tag{C.41}$$

The relative orientation of commonly realized Earth and local navigation frames may be expressed in terms of the normal vector using

$$\begin{aligned} \mathbf{C}_e^n &= \begin{pmatrix} -\dfrac{n_{eb,x}^{e} n_{eb,z}^{e}}{\sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}}} & -\dfrac{n_{eb,y}^{e} n_{eb,z}^{e}}{\sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}}} & \sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}} \\ -\dfrac{n_{eb,y}^{e}}{\sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}}} & \dfrac{n_{eb,x}^{e}}{\sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}}} & 0 \\ -n_{eb,x}^{e} & -n_{eb,y}^{e} & -n_{eb,z}^{e} \end{pmatrix} \\ \mathbf{C}_n^e &= \begin{pmatrix} -\dfrac{n_{eb,x}^{e} n_{eb,z}^{e}}{\sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}}} & -\dfrac{n_{eb,y}^{e}}{\sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}}} & -n_{eb,x}^{e} \\ -\dfrac{n_{eb,y}^{e} n_{eb,z}^{e}}{\sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}}} & \dfrac{n_{eb,x}^{e}}{\sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}}} & -n_{eb,y}^{e} \\ \sqrt{n_{eb,x}^{e\ 2} + n_{eb,y}^{e\ 2}} & 0 & -n_{eb,z}^{e} \end{pmatrix} \end{aligned}. \tag{C.42}$$

As with latitude and longitude, the time derivative of the normal vector may be expressed as a function of the velocity and the radii of curvature, enabling it to be integrated directly from velocity without having to use Cartesian position as an intermediary. The relationship is [8]

$$\begin{aligned} \dot{\mathbf{n}}_{eb}^{e} &= \mathbf{n}_{eb}^{e} \wedge \left[\mathbf{C}_n^e \begin{pmatrix} \left(R_N(n_{eb,z}^{e}) + h_b\right)^{-1} & 0 & 0 \\ 0 & \left(R_E(n_{eb,z}^{e}) + h_b\right)^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_{eb}^{n} \right] \wedge \mathbf{n}_{eb}^{e} \\ &= \mathbf{n}_{eb}^{e} \wedge \left[\mathbf{C}_n^e \begin{pmatrix} \left(R_N(n_{eb,z}^{e}) + h_b\right)^{-1} & 0 & 0 \\ 0 & \left(R_E(n_{eb,z}^{e}) + h_b\right)^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{C}_e^n \mathbf{v}_{eb}^{e} \right] \wedge \mathbf{n}_{eb}^{e} \end{aligned}, \tag{C.43}$$

$$\begin{aligned} \dot{h}_b &= \mathbf{n}_{eb}^{e\,\mathrm{T}} \mathbf{v}_{eb}^e \\ &= \mathbf{n}_{eb}^{e\,\mathrm{T}} \mathbf{C}_n^e \mathbf{v}_{eb}^n \end{aligned} . \tag{C.44}$$

Conversion of the normal vector and height to Cartesian ECEF position is straightforward:

$$\begin{aligned} x_{eb}^e &= \left(R_E(n_{eb,z}^e) + h_b\right) n_{eb,x}^e \\ y_{eb}^e &= \left(R_E(n_{eb,z}^e) + h_b\right) n_{eb,y}^e \\ z_{eb}^e &= \left[\left(1 - e^2\right) R_E(n_{eb,z}^e) + h_b\right] n_{eb,z}^e \end{aligned} . \tag{C.45}$$

The reverse conversion is more complex. The geodetic latitude (which is not subject to a singularity at the poles) and the height may be obtained using one of the methods described in Sections 2.4.3 and C.2. The normal vector may then be determined using

$$\mathbf{n}_{eb}^e = \begin{pmatrix} x_{eb}^e / \left[R_E(L_b) + h_b\right] \\ y_{eb}^e / \left[R_E(L_b) + h_b\right] \\ \sin L_b \end{pmatrix}, \tag{C.46}$$

which avoids computation of the longitude. A closed-form exact method that determines the normal vector and height directly is described in [8].

C.4 Transverse Mercator Projection

A transverse Mercator projection is a conformal transverse cylindrical projection (see Figure 2.24). It has an origin latitude and longitude, L_0 and λ_0, and a focal point on the opposite side of both the ellipsoid and the projection cylinder at latitude $-L_0$ and longitude $\lambda_0 \pm \pi$. For any point along the origin meridian or the origin parallel, the x-axis of the projection is aligned with east and the y-axis of the projection is aligned with north. Elsewhere, the axes of the projected coordinates are rotated with respect to north and east.

Where no overall scaling factor is applied, the scale is unity along the origin meridian, so distances along the ellipsoid match the change in projected coordinates. Elsewhere the scaling factor increases with distance from the origin meridian. As a transverse Mercator projection is conformal, the scale is the same in all directions at a given location. In many implementations, an overall scaling factor (< 1) is applied in order to reduce the average scale factor error across the coverage area of the projection. In a scaled transverse Mercator projection, there are two values of the x-axis projected coordinate, equidistant from the origin meridian, at which the scale is unity.

Although not strictly correct, the x-axis projected coordinate is commonly known as an *easting*, E, and the y-axis coordinate is known as a *northing*, N. Thus,

$$E_b \equiv x_{pb}^p, \qquad N_b \equiv y_{pb}^p . \tag{C.47}$$

A transverse Mercator projection is defined by five parameters: L_0 and λ_0, the scale factor at the origin, k_0, and the easting and northing at the origin, E_0 and N_0. In most projections, the convention is adopted that the easting and northing are always positive. To minimize distortions, the origin should be close to the center of the projection's coverage area. Therefore, E_0 and N_0 are normally nonzero and are sometimes known as the false easting and false northing.

Sections C.4.1 and C.4.2 describe the coordinate conversion from curvilinear to projected and projected to curvilinear, respectively. Section C.4.3 describes some common transverse Mercator projections.

C.4.1 Curvilinear to Projected Coordinate Conversion

The easting and northing of a body b are obtained from the latitude and longitude using [1]

$$\begin{aligned} E_b &= E_0 + E_1(\lambda_b - \lambda_0) + E_2(\lambda_b - \lambda_0)^3 + E_3(\lambda_b - \lambda_0)^5 \\ N_b &= N_0 + N_1 + N_2(\lambda_b - \lambda_0)^2 + N_3(\lambda_b - \lambda_0)^4 + N_4(\lambda_b - \lambda_0)^6 \end{aligned}, \tag{C.48}$$

where

$$N_1 = k_0 R_P \left[\begin{aligned} &\left(1 + n + \frac{5}{4}n^2 + \frac{5}{4}n^3\right)(L_b - L_0) - \left(3n + 3n^2 + \frac{21}{8}n^3\right)\sin(L_b - L_0)\cos(L_b + L_0) \\ &+ \frac{15}{8}(n^2 + n^3)\sin 2(L_b - L_0)\cos 2(L_b + L_0) - \frac{35}{24}n^3 \sin 3(L_b - L_0)\cos 3(L_b + L_0) \end{aligned} \right] \tag{C.49}$$

and

$$\begin{aligned} N_2 &= \frac{k_0 R_E}{2}\sin L_b \cos L_b \\ N_3 &= \frac{k_0 R_E}{24}\sin L_b \cos^3 L_b\left(5 - \tan^2 L_b + 9\eta^2\right) \\ N_4 &= \frac{k_0 R_E}{720}\sin L_b \cos^5 L_b\left(61 - 58\tan^2 L_b + \tan^4 L_b\right) \\ E_1 &= k_0 R_E \cos L_b \\ E_2 &= \frac{k_0 R_E}{6}\cos^3 L_b\left(\frac{R_E}{R_N} - \tan^2 L_b\right) \\ E_3 &= \frac{k_0 R_E}{120}\cos^5 L_b\left(5 - 18\tan^2 L_b + \tan^4 L_b + 14\eta^2 - 58\eta^2\tan^2 L_b\right) \end{aligned}, \tag{C.50}$$

where

$$n = \frac{R_0 - R_P}{R_0 + R_P}, \qquad \eta^2 = \frac{e^2\left(1 - \sin^2 L_b\right)}{1 - e^2}. \tag{C.51}$$

C.4.2 Projected to Curvilinear Coordinate Conversion

To obtain the latitude and longitude from the easting and northing, it is first necessary to obtain L'_b iteratively [1]. This is initialized using

$$L'_{b,0} = L_0 + \frac{N_b - N_0}{k_0 R_0} \tag{C.52}$$

and then iterated using

$$L'_{b,k} = L'_{b,k-1} + \frac{N_b - N_0 - N_1\left(L'_{b,k-1}\right)}{k_0 R_0}, \tag{C.53}$$

with N_1 calculated using (C.49) with $L_b = L'_b$. The iteration continues until

$$N_b - N_0 - N_1\left(L'_{b,k-1}\right) < 10^{-5}\,\text{m}. \tag{C.54}$$

The latitude and longitude are then obtained using [1]

$$\begin{aligned} L_b &= L'_b - N_5(E_b - E_0)^2 + N_6(E_b - E_0)^4 - N_7(E_b - E_0)^6 \\ \lambda_b &= \lambda_0 + E_4(E_b - E_0) - E_5(E_b - E_0)^3 + E_6(E_b - E_0)^5 - E_7(E_b - E_0)^7 \end{aligned}, \tag{C.55}$$

where

$$
\begin{aligned}
N_5 &= \frac{\tan L_b'}{2k_0^2 R_N R_E} \\
N_6 &= \frac{\tan L_b'}{24k_0^4 R_N R_E^3}\left(5 + 3\tan^2 L_b' + \eta^2 - 9\eta^2 \tan^2 L_b'\right) \\
N_7 &= \frac{\tan L_b'}{720k_0^6 R_N R_E^5}\left(61 + 90\tan^2 L_b' + 45\tan^4 L_b'\right) \\
E_4 &= \frac{1}{k_0 R_E \cos L_b'} \\
E_5 &= \frac{1}{6k_0^3 R_E^3 \cos L_b'}\left(\frac{R_E}{R_N} + 2\tan^2 L_b'\right) \\
E_6 &= \frac{1}{120k_0^5 R_E^5 \cos L_b'}\left(5 + 28\tan^2 L_b' + 24\tan^4 L_b'\right) \\
E_7 &= \frac{1}{5040k_0^7 R_E^7 \cos L_b'}\left(61 + 662\tan^2 L_b' + 1320\tan^4 L' + 720\tan^6 L_{bb}'\right)
\end{aligned}
\quad . \qquad \text{(C.56)}
$$

C.4.3 Common Implementations

The Universal Transverse Mercator system divides the world into 60 numbered longitudinal zones, each zone spanning 6° with the longitude origin in the middle of the zone. Each longitudinal zone is further divided into northern and southern hemispherical zones. For the n^{th} zone, the origin latitude and longitude are $L_0 = 0°$, $\lambda_0 = (6n - 183)°$. The scale factor at the origin is $k_0 = 0.9996$. The easting origin is 500 km, while the northing origin is 0 in the northern hemisphere and 10,000 km in the southern hemisphere. Different countries use UTM in association with different datums.

The Military Grid Reference System (MGRS) is a derivative of UTM that further divides each zone into latitudinal bands. These are designated by the letters C to X, omitting I and O. Bands C to W span 8° of latitude and band X spans 12°. Band C is the most southerly, extending from –80° to –72°, and band X is the most northerly, extending from 72° to 84°. Each band has its own northing origin. MGRS also incorporates a stereographic planar projection for each pole. The United States National Grid (USNG) is based on MGRS and uses a mixture of the NAD27 and WGS84 datums.

The Gauss-Krueger system is similar to the UTM system, but typically divides the world into 3°-wide longitudinal zones. The origin latitude and longitude of the n^{th} zone are $L_0 = 0°$, $\lambda_0 = (3n - 1.5)°$, and the scale factor at the origin is unity. There is also a variant with 6°-wide zones for which $\lambda_0 = (6n - 3)°$. The Gauss-Krueger system is commonly used in Northern, Central and Eastern Europe; Russia; and China.

The U.K. National Grid is defined by $L_0 = 49°$, $\lambda_0 = -2°$, $k_0 = 0.9996012717$, $E_0 = 400$ km, $N_0 = -100$ km [1]. Details of the U.S. State Plane system may be found in [9].

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