Higher-Order Ordinary Differential Equations

Initial Value Problems

(Existence of Unique Solution)

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

- Let $a_n(x)$, $a_{n-1}(x)$, …, $a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the above initial-value problem exists on the interval and is unique.

$$3y''' + 5y'' - y' + 7y = 0, \quad y(1) = 0, \; y'(1) = 0, \; y''(1) = 0 \quad \Rightarrow \quad y = 0$$

$$y'' - 4y = 12x, \quad y(0) = 4, \; y'(0) = 1 \quad \Rightarrow \quad y = 3e^{2x} + e^{-2x} - 3x$$

Boundary-Value Problem

- A problem such as

$$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$

$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$$

  is called a **boundary-value problem**.
- A boundary value problem can have many, one or no solutions.

$$x'' + 16x = 0 \quad \Rightarrow \quad x = c_1 \cos 4t + c_2 \sin 4t$$

$$x(0) = 0, \; x(\pi/2) = 0 \quad \Rightarrow \quad \text{Has infinite number of solutions.}$$

$$x(0) = 0, \; x(\pi/8) = 0 \quad \Rightarrow \quad \text{Has only one solution.}$$

$$x(0) = 0, \; x(\pi/2) = 1 \quad \Rightarrow \quad \text{Has no solution.}$$

Differential Operator

- Differentiation is often denoted by the capital letter D such as

$$\frac{dy}{dx} = Dy \qquad \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = D^2 y$$

The symbol D is called a differential operator.

- Polynomial expressions involving D are also differential operators;

$$\left(5x^3D^3 - 6x^2D^2 + 4xD + 9\right)y = 5x^3y''' - 6x^2y'' + 4xy' + 9y$$

- An ***n*th-order differential operator** is defined as

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots + a(x)D + a_0(x).$$

- L has a linear property;

$$L[\alpha f(x) + \beta g(x)] = \alpha L(f(x)) + \beta L(g(x)).$$

Superposition Principle – Homogeneous Equations

- Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order linear differential equation on an interval I. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_k y_k(x),$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants, is also a solution of this equation on the interval I.

- A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exists constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent.**

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

$$1 \qquad 1 \qquad -1 \qquad 1$$

Fundamental Set of Solutions – Homogeneous Equations

- Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least n-1 derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

is called the **Wronskian** of the functions.

- Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous nth-order linear differential equation on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

- Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous nth-order linear differential equation on an interval I in said to be a **fundamental set of solutions** on the interval.

General Solution – Homogeneous Equation

- There exists a fundamental set of solutions for the homogeneous linear nth-order differential equation on an interval I.

- Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous nth-order linear differential equation on an interval I. Then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants.

$$y'' - 9y = 0 \quad \Rightarrow \quad y_1 = e^{3x} \text{ and } y_2 = e^{-3x} \text{ are two solutions.}$$

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

$$y = c_1 e^{3x} + c_2 e^{-3x} \text{ is the general solution.}$$

Nonhomogeneous Equations

- Any function y_p free of arbitrary parameters that satisfies a nonhomogeneous linear differential equation is called a **particular solution** or **particular integral** of the equation.
- Let y_p be any particular solution of the nonhomogeneous nth-order linear differential equation on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation on I. Then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) + y_p$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants.
- Suppose y_{pi} denotes a particular solution of the differential equation

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = g_i(x)$$

where $i = 1, 2, \ldots, k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \ldots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = g_1(x) + g_2(x) + \ldots + g_k(x).$$

Reduction of Order

- The method of reduction of order is used to find the second solution of a 2nd order linear ODE if the first solution is known.
- Let's assume that $y_1(x)$ is a solution of the second-order ODE

$$y''+P(x)y'+Q(x)y=0$$

- We assume that the second solution of this ODE is $y_2(x)=u(x)\,y_1(x)$.
- Substituting into the equation gives

$$u''y_1+u'(2y_1'+Py_1)+u(y_1''+Py_1'+Qy_1)=0$$

- Note that the coefficient of u is zero. Assuming $w=u'$ gives

$$w'y_1+w(2y_1'+Py_1)=0$$

- This is a separable first-order ODE for w.
- Example: $y_1=x$ is a solution of $(x^2-x)y''-xy'+y=0$. Find the second solution.

Homogeneous Linear Equations with Constant Coefficients

- Consider the homogeneous second-order linear equation

$$ay''+by'+cy=0$$

and assume a solution of the form $y=e^{mx}$.
- This results in the **auxiliary equation** for this differential equation.

$$am^2+bm+c=0.$$

- The general solution of this equation depends on the roots of its auxiliary equation.
  - If the auxiliary equation has two **unequal real roots** $m_1\neq m_2$,

$$y=c_1e^{m_1x}+c_2e^{m_2x}.$$

  - If the auxiliary equation has **equal real roots** $m_1=m_2$,

$$y=c_1e^{m_1x}+c_2xe^{m_1x}.$$

  - If $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$ are **conjugate roots**,

$$y=C_1e^{(\alpha+i\beta)x}+C_2e^{(\alpha-i\beta)x}=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x).$$

Homogeneous Linear Equations with Constant Coefficients

- To solve an nth-order linear differential equation with constant coefficients

$$a_ny^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_1y'+a_0y=0$$

we must solve an nth-order auxiliary equation

$$a_nm^n+a_{n-1}m^{n-1}+\ldots+a_1m+a_0=0.$$

- The auxiliary equation may have p unequal real, q equal real, and r pairs of conjugate roots where $n=p+q+2r$.

$$y'''+3y''-4y=0$$
$$m^3+3m^2-4=0$$
$$(m-1)(m^2+4m+4)=0$$
$$(m-1)(m+2)^2=0$$
$$m_1=1,\quad m_2=m_3=-2$$
$$y=c_1e^x+c_2e^{-2x}+c_3xe^{-2x}$$

$$y''''+2y''+y=0$$
$$m^4+2m^2+1=0$$
$$(m^2+1)^2=0$$
$$m_1=m_2=i,\quad m_3=m_4=-i$$
$$y=C_1e^{ix}+C_2e^{-ix}+C_3xe^{ix}+C_4xe^{-ix}$$
$$y=c_1\cos x+c_2\sin x+c_3x\cos x+c_4x\sin x$$

Method of Undetermined Coefficients

- This method is used to find particular solution y_p of a nonhomogeneous linear differential equation with constant coefficients.
- The underlying idea in this method is an educated guess about the form of the particular solution.

$g(x)$	Form of y_p
4	A
$5x+7$	$Ax+B$
$3x^2-2$	Ax^2+Bx+C
x^3-x+1	Ax^3+Bx^2+Cx+D
$\sin 4x$	$A\cos 4x+B\sin 4x$
$\cos 4x$	$A\cos 4x+B\sin 4x$
e^{5x}	Ae^{5x}
$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(Ex^2+Fx+G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$

Method of Undetermined Coefficients (Examples)

$$y''+4y'-2y=2x^2-3x+6$$

$$y''-y'+y=2\sin 3x$$

$$y''-2y'-3y=4x-5+6xe^{2x}$$

$$y''-9y'+14y=3x^2-5\sin 2x+7xe^{6x}$$

Method of Undetermined Coefficients (A Glitch)

- The particular solution of the differential equation

$$y''-5y'+4y=8e^x$$

can not be of the form $y_p=Ae^x$ since e^x is a solution of the homogeneous equation.

- The appropriate particular solution has the form $y_p=Axe^x$.
- If any particular solution y_p contains terms that duplicate terms in the solution of the homogenous equation, then that y_p must be multiplied by x^n where n is the smallest positive integer that eliminates that duplication.

$$y''+y=4x+10\sin x$$

$$y''-6y'+9y=6x^2+2-12e^{3x}$$

$$y^{(4)}+y'''=1-x^2e^{-x}$$

Method of Variation of Parameters

- Consider the second-order linear differential equation

$$a_2(x)y''+a_1(x)y'+a_0(x)y=g(x).$$

- First, this equation is converted to the standard form

$$y''+P(x)y'+Q(x)y=f(x).$$

- Then, we seek a particular solution in the form

$$y_p=u_1(x)y_1(x)+u_2(x)y_2(x),$$

where y_1 and y_2 are solutions of the homogeneous equation. Inserting y_p in the equation gives

$$\frac{d}{dx}\left[y_1u_1'+y_2u_2'\right]+P(x)\left[y_1u_1'+y_2u_2'\right]+y_1'u_1'+y_2'u_2'=f(x).$$

- u_1 and u_2 are obtained by the solution of the following two equations.

$$y_1u_1'+y_2u_2'=0$$
$$y_1'u_1'+y_2'u_2'=f(x)$$

Method of Variation of Parameters (Examples)

$$y''-4y'+4y=(x+1)e^{2x}$$

$$m^2-4m+4=0$$

$$m_1=m_2=2$$

$$y_1=e^{2x} \qquad y_2=xe^{2x}$$

$$y_p=u_1e^{2x}+u_2xe^{2x}$$

$$\begin{cases} y_1u_1'+y_2u_2'=0 \\ y_1'u_1'+y_2'u_2'=(x+1)e^{2x} \end{cases}$$

$$y''+9y=\frac{1}{4}\csc 3x$$

$$m^2+9=0$$

$$m_1=3i, \qquad m_2=-3i$$

$$y_1=\cos 3x, \qquad y_2=\sin 3x$$

$$y_p=u_1\cos 3x+u_2\sin 3x$$

$$\begin{cases} y_1u_1'+y_2u_2'=0 \\ y_1'u_1'+y_2'u_2'=\frac{1}{4}\csc 3x \end{cases}$$

Cauchy-Euler Equation

- The linear differential equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \ldots + a_1 x y' + a_0 y = g(x)$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants is known as **Cauchy-Euler equation**.

- The solution to this equation is obtained by substituting $y=x^m$ in the equation.

- Consider the second-order homogeneous Cauchy-Euler equation;

$$ax^2 y'' + bxy' + cy = 0.$$

- Inserting $y=x^m$ in this equation results in the auxiliary equation

$$am^2 + (b-a)m + c = 0.$$

- The solution of Cauchy-Euler equation depends on the roots of the auxiliary equation.

Second-Order Cauchy-Euler Equation

- If the auxiliary equation has two distinct real roots $m_1 \neq m_2$, the solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

- If the auxiliary equation has two repeated real roots $m_1 = m_2$, the solution is

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x.$$

- If the auxiliary equation has a pair of conjugate complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, the solution is

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right].$$

- Examples:

$$x^2 y'' - 2xy' - 4y = 0$$

$$4x^2 y'' + 8xy' + y = 0 \qquad 4x^2 y'' + 17y = 0$$

Nonlinear Second-Order Differential Equations

- **Method of Reduction of Order**: Nonlinear second order differential equations $F(x,y',y'')=0$ and $F(y,y',y'')=0$ can be reduced to a first order equation for $u=y'$.

$$y'' = 2x(y')^2 \quad \Rightarrow \quad u' = 2xu^2$$

$$yy'' = (y')^2 \quad \Rightarrow \quad yu\frac{du}{dy} = u^2$$

- **Use of Taylor Series**: It is assumed that the solution of an initial-value problem can be expressed as a Taylor series expansion around x_0.

$$y(x) = y(x_0) + \sum_{1}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n$$

Example:

$$y'' = x + y - y^2 \qquad y(0) = -1, \qquad y'(0) = 1$$

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots$$

$$y''(0) = 0 - 1 - 1 = -2$$

$$y'''(x) = 1 + y' - 2yy' \quad \Rightarrow \quad y'''(0) = 1 + 1 - 2(-1)(1) = 4$$

$$y(x) = -1 + x - x^2 + \frac{2}{3}x^3 + \ldots$$

Spring-Mass System – Free Undamped Motion

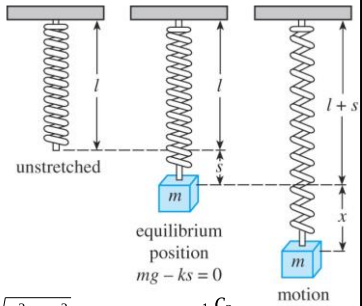
$$F = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\omega^2 = \frac{k}{m} \quad \Rightarrow \quad \frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$x = c_1 \cos\omega t + c_2 \sin\omega t$$



$$\text{or} \quad x = A\cos(\omega t - \phi) \quad \text{where} \quad A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \phi = \tan^{-1}\frac{c_2}{c_1},$$

$$\text{or} \quad x = A\sin(\omega t + \phi) \quad \text{where} \quad A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \phi = \tan^{-1}\frac{c_1}{c_2}.$$

Spring-Mass System – Free Undamped Motion

$$\frac{d^2x}{dt^2}+64x=0$$

$$x(0)=\frac{2}{3},\quad x'(0)=-\frac{4}{3}$$

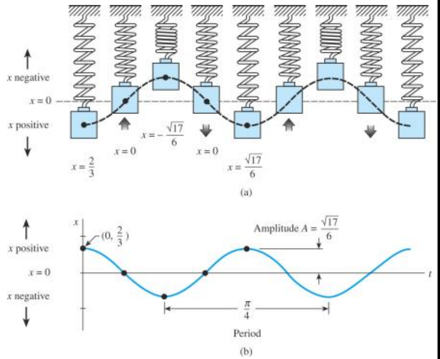
$$x(t)=\frac{2}{3}\cos 8t-\frac{1}{6}\sin 8t$$

or

$$x(t)=\frac{\sqrt{17}}{6}\sin(8t+1.816)$$

or

$$x(t)=\frac{\sqrt{17}}{6}\cos(8t+0.245)$$



Spring-Mass System – Free Damped Motion

$$F=ma$$

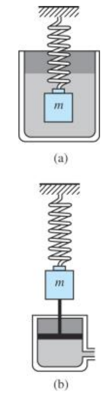
$$-kx-\beta\frac{dx}{dt}=m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=0$$

$$\omega^2=\frac{k}{m}\ ,\quad 2\lambda=\frac{\beta}{m}\quad\Rightarrow\quad\frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=0$$

$$m^2+2\lambda m+\omega^2=0$$

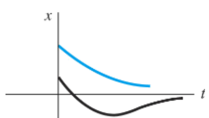
$$m_1=-\lambda+\sqrt{\lambda^2-\omega^2},\quad m_2=-\lambda-\sqrt{\lambda^2-\omega^2}$$



Spring-Mass System – Free Damped Motion

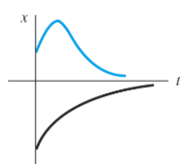
- Overdamped Motion ($\lambda^2-\omega^2>0$). In this case both m_1 and m_2 are negative.

$$x(t)=c_1e^{m_1t}+c_2e^{m_2t}$$



- Critically damped motion ($\lambda^2-\omega^2=0$)

$$x(t)=e^{-\lambda t}(c_1+c_2t)$$



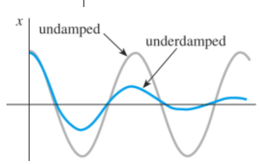
- Underdamped Motion ($\lambda^2-\omega^2<0$). If define the frequency of motion as $\omega^*=\sqrt{\omega^2-\lambda^2}$

$$x(t)=e^{-\lambda t}(c_1\cos\omega^*t+c_2\sin\omega^*t)$$

or

$$x(t)=Ae^{-\lambda t}\sin(\omega^*t+\phi)$$

where $A=\sqrt{c_1^2+c_2^2}$ and $\phi=\tan^{-1}\dfrac{c_1}{c_2}$.



Spring-Mass System – Forced Undamped Motion

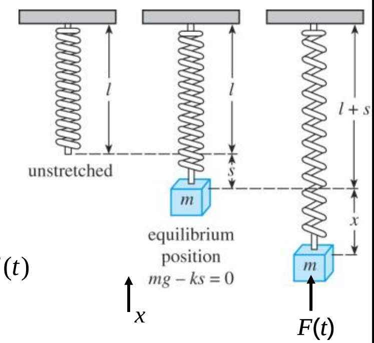
$$F=ma$$

$$-kx+F(t)=m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2}+kx=F(t)$$

$$\omega^2=\frac{k}{m}\quad\Rightarrow\quad\frac{d^2x}{dt^2}+\omega^2x=F(t)$$

$$x=c_1\cos\omega t+c_2\sin\omega t+x_p(t)$$



Spring-Mass System – Forced Undamped Motion

Consider the forced undamped motion given by the equation

$$\frac{d^2x}{dt^2}+\omega^2 x = F_0 \sin\omega_0 t$$

where F_0 and ω_0 are amplitude and frequnecy of the force function.

The general solution to this equation is

$$x = c_1\cos\omega t + c_2 \sin\omega \quad t + \frac{F_0}{\omega^2-\omega_0^2}\sin\omega_0 t$$

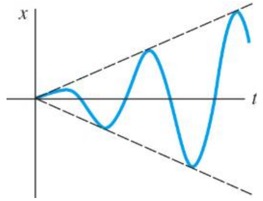
or

$$x = A\sin(\omega t+\phi)+\frac{F_0}{\omega^2-\omega_0^2}\sin\omega_0 t$$

Note that at resonance where $\omega_0=\omega$, $x\to\infty$.

The correct solution in this case is

$$x = A\sin(\omega t+\phi)-\frac{F_0}{2\omega}t\cos\omega t$$



Spring-Mass System – Forced Damped Motion

$$F = ma$$

$$-kx-\beta\frac{dx}{dt}+F(t)=m\frac{d^2x}{dt^2}$$

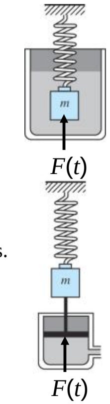
$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=F(t)$$

$$\omega^2=\frac{k}{m}\ ,\quad 2\lambda=\frac{\beta}{m}\quad\Rightarrow\quad \frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=F(t)$$

$$x(t)=x_h(t)+x_p(t)$$

Note that the homogeneous solution $x_h(t)$ dies off as time increases.

Therefore, $x_p(t)$ is called the steady-state solution.



Spring-Mass System – Forced Damped Motion

Consider the forced damped motion given by

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=F_0\cos\omega_0 t$$

The steady-state solution to this equation is

$$x_p(t)=A\cos(\omega_0 t-\theta)$$

where both A and θ are functions of ω_0;

$$A(\omega_0)=\frac{F_0}{\sqrt{m^2(\omega^2-\omega_0^2)^2+\omega_0^2\beta^2}}\quad\text{and}\quad \theta=\tan^{-1}\frac{\omega_0\beta}{m(\omega^2-\omega_0^2)}$$

The maximum value of $A(\omega_0)$ is achieved at a value of ω_0 where $\frac{dA}{d\omega_0}=0$.

It can be shown that if $\beta^2<2mk$, then the value of ω_0 that results in higehst $A(\omega_0)$ is

$$\omega_{0,\max}=\sqrt{\omega^2-\frac{\beta^2}{2m^2}}$$

and the corresponding value of A is

$$A_{\max}(\omega_{0,\max})=\frac{2mF_0}{\beta\sqrt{4m^2\omega^2-\beta^2}}.$$

Modeling Series RLC Circuit

The differential equation governing an RLC circuit with the voltage $E(t)=E_0\sin\omega_0 t$ is

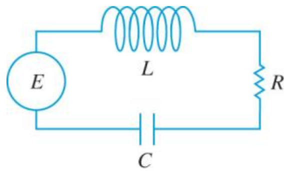
$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=E_0\omega_0\cos\omega_0 t$$

The steady-state solution to this equation is

$$i_p(t)=I_0\sin(\omega_0 t-\theta)$$

where both I_0 and θ are functions of ω_0;

$$I_0(\omega_0)=\frac{E_0}{\sqrt{R^2+S^2}}\quad\text{and}\quad\theta=\tan^{-1}\frac{S}{R}$$



$S=\omega_0 L-\frac{1}{\omega_0 C}$ and $\sqrt{R^2+S^2}$ are called the "reactance" and "impedance" of the circuit.

Note that $\frac{E_0}{I_0}=\sqrt{R^2+S^2}$.

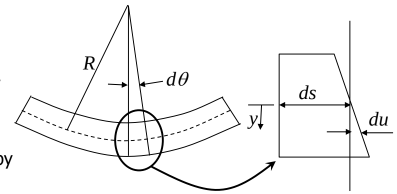
That is why $\sqrt{R^2+S^2}$ is also called the "apparent resistance" of the circuit.

Deflection of a Beam – Boundary Value Problem

- Consider a section of a deflected beam. Let's assume that ds is the measured on the neutral axis and y is measured from the neutral axis.

$$\frac{du}{ds} = \frac{y d\theta}{ds} = \frac{y}{R}$$

Since $\varepsilon = \dfrac{du}{ds}$ and $\sigma = \varepsilon E, \quad \sigma = \dfrac{E}{R} y$



- On the other hand, the radius of deflection for the beam is given by

$$R = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Bigg/ \left(\frac{d^2y}{dx^2}\right)$$

- For small deflection of the beam, $dy/dx << 1$ and $R = 1 \Big/ \left(\dfrac{d^2y}{dx^2}\right)$.
- On the other hand the bending moment at a point is given by

$$M(x) = \int (\sigma dA) y = \int \sigma y dA = \int \frac{E}{R} y^2 dA = \frac{E}{R} I$$

- Therefore, the beam deflection equation becomes

$$M(x) = EI \frac{d^2y}{dx^2}$$

Deflection of a Beam – Boundary Value Problem

- Since shear force $Q(x) = dM(x)/dx$ and load distribution $w(x) = dQ(x)/dx$, the beam deflection equation can be written in any of these forms:

$$M(x) = EI \frac{d^2y}{dx^2}$$

$$Q(x) = EI \frac{d^3y}{dx^3}$$

$$w(x) = EI \frac{d^4y}{dx^4}$$

- The boundary conditions depend on how the ends of the beam are supported.
  - Embedded end

    $$y = 0 \quad \text{and} \quad y' = 0$$

  - Free end

    $$y'' = 0 \quad \text{and} \quad y''' = 0$$

  - Simply supported or hinged end

    $$y = 0 \quad \text{and} \quad y'' = 0$$

(a) Embedded at both ends

(b) Cantilever beam: embedded at the left end, free at the right end

(c) Simply supported at both ends

General Two-Point Boundary-Value Problems

- A general two-point boundary-value problem involves a second-order differential equation and boundary conditions as shown below.

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad a < x < b$$

$$A_1 y(a) + B_1 y'(a) = C_1$$

$$A_2 y(b) + B_2 y'(b) = C_2$$

- If $g(x) = 0$, $C_1 = 0$ and $C_2 = 0$, the boundary-value problem is homogeneous. Otherwise, it is called nonhomogeneous.
- The trivial solution of any homogeneous boundary-value problem is zero.
- The coefficients of many homogeneous boundary-value problems may depend on a constant parameter λ. The value of λ that results in a non-trivial solution is called **eigenvalue** and the corresponding solution is called **eigenfunction** of the boundary-value problem.

Eigenvalues and Eigenfunctions

- Consider the homogeneous boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

We consider three cases; $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$.

- **Case 1**:

For $\lambda = 0$ the solution is $y = c_1 x + c_2$.

Applying the boundary conditions give $c_1 = c_2 = 0$ and therefore, the only solution is the trivial solution $y = 0$.

- **Case 2**:

For $\lambda < 0$, it is convenient to write $\lambda = -\alpha^2$ where $\alpha > 0$.

The solution in this case is $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$.

Applying the boundary conditions give $c_1 = c_2 = 0$ and therefore, the only solution is the trivial solution $y = 0$.

Eigenvalues and Eigenfunctions

- **Case 3**: For $\lambda > 0$, it is convenient to write $\lambda = \alpha^2$ where $\alpha > 0$.

  The solution in this case is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$.

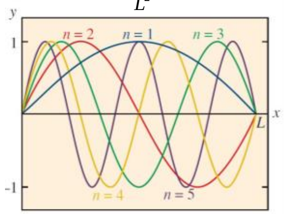
  Applying the first boundary conditions give $c_1 = 0$.

  The second boundary condition gives

  $$c_2 \sin \alpha L = 0$$

  $c_2 = 0$ results in the trivial solution $y = 0$. However, nontrivial solution is obtained if

  $$\sin \alpha L = 0 \quad \Rightarrow \quad \alpha L = n\pi \quad \Rightarrow \quad \alpha = \frac{n\pi}{L} \quad \Rightarrow \quad \lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{L^2}$$



- The numbers λ_n, $n = 1, 2, 3, \ldots$ for which the boundary value problem has non-trivial solutions are called **eigenvalues**.
- The corresponding non-trivial solutions $y_n = \sin(n\pi x/L)$ are called **eigenfunctions**.

Buckling Beam/Column

$$\sum M_x = 0$$

$$EIy'' + Py = 0$$

$$y'' + \frac{P}{EI} y = 0$$

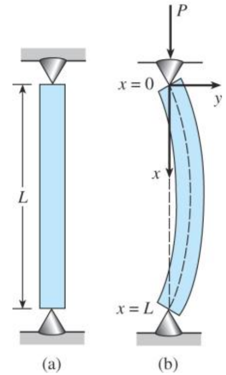
$$y(0) = 0, \quad y(L) = 0$$

$$y = c_1 \cos \sqrt{\frac{P}{EI}} x + c_2 \sin \sqrt{\frac{P}{EI}} x$$

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y(L) = 0 \Rightarrow c_2 \sin \sqrt{\frac{P}{EI}} L = 0$$

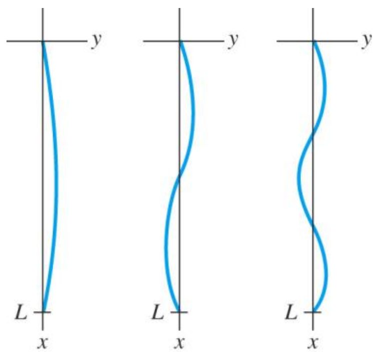
$$\sqrt{\frac{P}{EI}} L = n\pi \quad \Rightarrow \quad P_n = \frac{n^2\pi^2 EI}{L^2}, \quad n = 1,2,3,\ldots$$



Critical Loads

Buckling Beam/Column

$$y_n = \sin \frac{n\pi}{L} x, \quad n = 1,2,3,\ldots$$



n=1 n=2 n=3