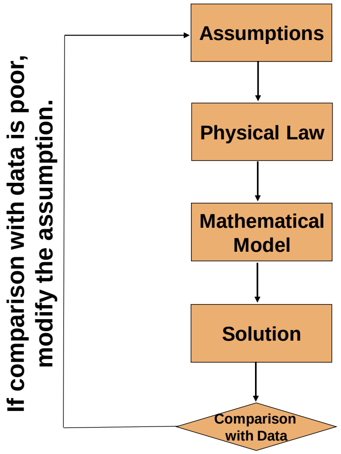
Introduction

Mathematical Modeling Process



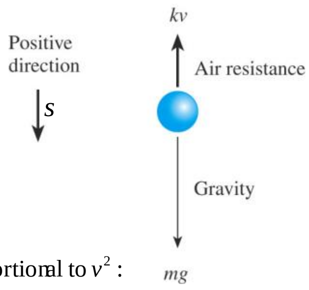
Falling Bodies with Air Resistance

$$F = ma$$

$$mg - kv = m\frac{dv}{dt}$$

$$v = \frac{ds}{dt} \quad \Rightarrow \quad mg - k\frac{ds}{dt} = m\frac{d^2s}{dt^2}$$

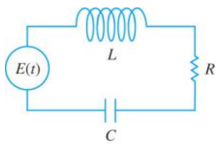
$$\frac{d^2s}{dt^2} + \frac{k}{m}\frac{ds}{dt} - g = 0$$



At higher speeds air resistance is proportional to v^2 :

$$\frac{d^2s}{dt^2} + \frac{k}{m}\left(\frac{ds}{dt}\right)^2 - g = 0$$

Series Circuits

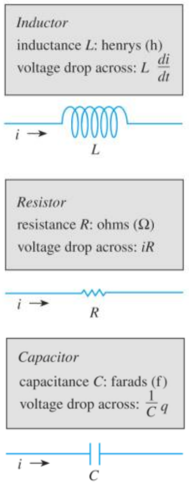


- **Kirchhoff's Second Law**: The applied voltage on a circuit is equal to sum of the voltage drops in the circuit.

$$L\frac{di}{dt} + iR + \frac{q}{C} = E(t)$$

$$i = \frac{dq}{dt} \quad \Rightarrow \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)$$

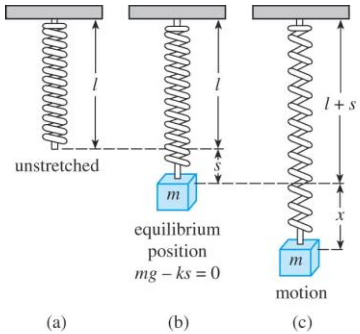


Spring-Mass System – Free Motion

$$F = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

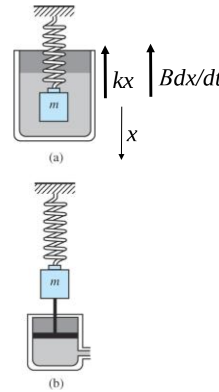


Spring-Mass System – Damped Motion

$$F = ma$$

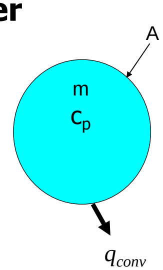
$$-kx - \beta\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$



Transient Heat Transfer

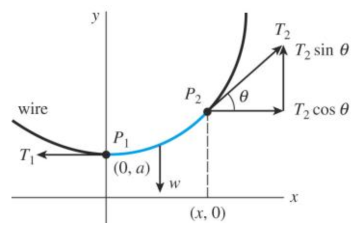
- Consider a system with mass m, specific heat c_p, surface area A, and temperature T.
- The system is placed in air with temperature T_∞ and heat transfer coefficient h.
- The energy balance for this system is

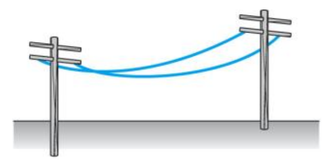


$$\dot{E}_{in} - \dot{E}_{out} = \frac{dE}{dt} \quad \Rightarrow \quad -q_{conv} = \frac{dE}{dt}$$

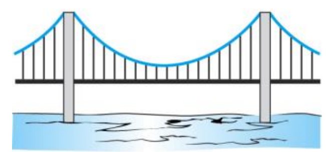
$$-hA(T - T_\infty) = mc_p\frac{dT}{dt}$$

Suspended Cables





(a) telephone wires



(b) suspension bridge

$$\begin{aligned} T_1 &= T_2\cos\theta \\ W &= T_2\sin\theta \end{aligned} \quad \Rightarrow \quad \tan\theta = \frac{W}{T_1}$$

$$\tan\theta = \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{W}{T_1}$$

Note that T_1 is constant.

Suspended Cables

- For a telephone wire, $W = \rho s$ where ρ is the linear weight of the wire and s is the length of the segment P_1P_2.

$$\frac{dy}{dx} = \frac{\rho s}{T_1} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\rho}{T_1}\frac{ds}{dx}$$

$$\text{Since} \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

$$\frac{d^2y}{dx^2} = \frac{\rho}{T_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

- For a suspension with a horizontal roadway, $W = wx$ where w is weight per unit length of the roadway.

$$\frac{dy}{dx} = \frac{w}{T_1}x$$

Definition

- **Differential Equation**

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables.

$$\frac{dy}{dx} = 0.2xy$$

$$\frac{dT}{dx} + 5T^2 = e^x$$

$$\frac{d^3y}{dx^3} + x\frac{dy}{dx} + 6y = 0$$

$$y''' + xy' + 6y = 0$$

$$\frac{du}{dt} + \frac{dv}{dt} + 2x + y = 0$$

$$\dot{y} + \dot{x} + 2x + y = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

$$T_{xx} = T_{tt} - 2T_t$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Classification by Type

- **Ordinary Differential Equation**

A differential equation containing only ordinary derivatives of one or more dependent variables with respect to a single independent variable.

- **Partial Differential Equation**

A differential equation containing partial derivatives of one or more dependent variables.

$$\frac{dy}{dx} = 0.2xy$$

$$\frac{du}{dx} + 5u^2 = e^x$$

$$\frac{d^3y}{dx^3} + x\frac{dy}{dx} + 6y = 0$$

$$\frac{dy}{dt} + \frac{dx}{dt} + 2x + y = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Classification by Order

- The order of a differential equation is the order of the highest derivative in the equation.

$$\frac{dy}{dx} = 0.2xy$$

$$\frac{dy}{dx} + 5y^2 = e^x$$

$$y''' + xy' + 6y = 0$$

$$\frac{d^2y}{dt^2} + \frac{dx}{dt} + 2x + y = 0$$

$$u_{xx} + u_{yy} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Classification by Linearity

- A differential equation is linear if

  1. The dependent variable and all its derivatives are of the first power.

  2. The coefficients of the dependent variable and all its derivatives are at most functions of the independent variables.

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$\frac{dy}{dx}=0.2xy \qquad xu_{xx}+yu_{yy}=0$$

$$\frac{dy}{dx}+5y^2=e^x \qquad \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial t}$$

$$y'''+xy'+6y=0$$

$$\frac{d^2y}{dt^2}+y\frac{dx}{dt}+2x+y=0 \qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=0$$

Solution of a Differential Equation

- The solution of an nth-order differential equation is any function, defined on an interval I and possesses at least n continuous derivatives in that internal, which when substituted in the differential equation reduces that into an identity.

$$\frac{dy}{dx}=xy^{1/2} \quad \Rightarrow \quad y=x^4/16$$

$$y''-2y'+y=0 \quad \Rightarrow \quad y=xe^x$$

Explicit and Implicit Solutions

- **Explicit Solution**

  A solution in which the dependent variable is expressed solely in terms of the independent variable and constants.

$$\frac{dy}{dx}=xy^{1/2} \quad \Rightarrow \quad y=x^4/16$$

$$y''-2y'+y=0 \quad \Rightarrow \quad y=xe^x$$

- **Implicit Solution**

  A solution in which the relationship between dependent and independent variables is expressed by an implicit equation $G(x,y)=0$.

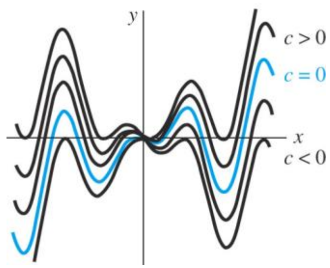
$$\frac{dy}{dx}=-\frac{x}{y} \quad \Rightarrow \quad x^2+y^2=16$$

Families of Solutions

- The solution of an nth-order differential equation includes n parameters c_1, c_2, …, c_n with infinite number of choices for those parameters.
- This set of solutions is called an ***n*-parameter family of solutions**.

$$xy'-y=x^2\sin x$$

$$y=cx-x\cos x$$



Initial Value Problems

- On an interval I containing x_0, the problem

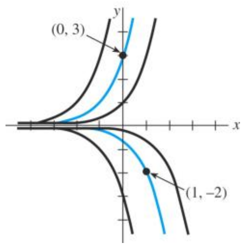
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1},$$

where $y_0, y_1, \ldots, y_n$ are arbitrarily specified real constants, is called an **initial-value problem**.

$$\left.\begin{aligned} y'&=y \\ y(0)&=3 \end{aligned}\right\} \Rightarrow \quad y=3e^{x}$$

$$\left.\begin{aligned} y'&=y \\ y(1)&=-2 \end{aligned}\right\} \Rightarrow \quad y=-2e^{x-1}$$



Chapter Exercise

- Determine the order and linearity

$$x\frac{d^3y}{dx^3}-\left(\frac{dy}{dx}\right)^4+y=0 \qquad \frac{d^2u}{dr^2}+\frac{du}{dr}+u=\cos(u+r)$$

- Verify Solutions

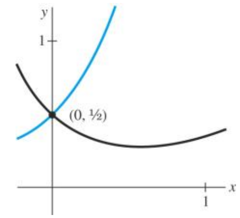
$$y''-6y'+13y=0 \quad\Rightarrow\quad y=e^{3x}\cos 2x$$

$$y'=25+y^2 \quad\Rightarrow\quad y=5\tan 5x$$

- Which curve is the solution to

$$y'=x-2y$$

$$y(0)=\frac{1}{2}$$



Chapter Exercise

Determine which functions meet each set of the boundary conditions.

$y(1)=1, y'(1)=-2$

$y(-1)=0, y'(-1)=-4$

$y(1)=1, y'(1)=2$

$y(0)=-1, y'(0)=2$

$y(0)=-1, y'(0)=0$

$y(0)=-4, y'(0)=-2$