First-Order Ordinary Differential Equations

Separable Variables

- A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be separable or to have separable variables.

$$\frac{dy}{dx} = g(x)h(y) \quad \Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$p(y) = \frac{1}{h(y)} \quad \Rightarrow \quad p(y)dy = g(x)dx$$

$$\int p(y)dy = \int g(x)dx + c$$

Separable Variables (Examples)

$$(e^{2y} - y)\cos x \frac{dy}{dx} = e^{y}\sin 2x, \quad y(0) = 0$$

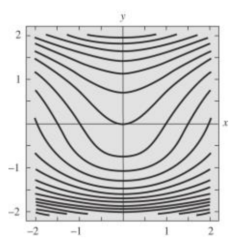
$$\frac{(e^{2y} - y)}{e^{y}}dy = \frac{\sin 2x}{\cos x}dx$$

$$\int (e^{y} - ye^{-y})dy = 2\int \sin x dx$$

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + c$$

$$y(0) = 0 \quad \Rightarrow \quad c = 4$$

$$e^{y} + ye^{-y} + e^{-y} = 4 - 2\cos x$$



$$(1+x)dy - ydx = 0$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x} + c$$

$$\ln|y| = \ln|1+x| + c$$

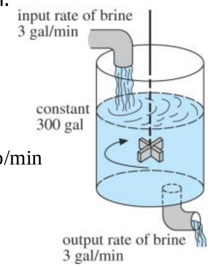
$$\ln\left|\frac{y}{1+x}\right| = c$$

$$\frac{y}{1+x} = e^{c} = c_1$$

$$y = c_1(1+x)$$

Mixing Problem

- Let's assume $y(t)$ is the amount of salt in the tank at time t.
- The concentration of the salt in inflow is 2 lb/gal.



$$\frac{dy}{dt} = \text{Input Rate of Salt - Output Rate of Salt}$$

$$\text{Input Rate of Salt} = (2\text{ lb/gal})(3\text{ gal/min}) = 6\text{ lb/min}$$

$$\text{Output Rate of Salt} = (\frac{y}{300}\text{ lb/gal})(3\text{ gal/min}) = \frac{y}{100}\text{ lb/min}$$

$$\frac{dy}{dt} = 6 - \frac{y}{100} \quad \Rightarrow \quad \frac{100dy}{600 - y} = dt$$

Cooling of an Object

- Consider an object, such as a potato or a cake, which is removed from an oven with initial temperature of 300°F. Three minutes later its temperature is 200°F. How long will it take for the object to reach within one degree of the room temperature of 70°F.

$$-hA(T-T_\infty)=mc_p\frac{dT}{dt}\quad\Rightarrow\quad\frac{dT}{T-T_\infty}=-\frac{hA}{mc_p}dt$$

$$\text{Let's take }\frac{hA}{mc_p}=b,\text{ then}\quad\frac{dT}{T-70}=-bdt$$

$$\ln|T-70|=-bt+c\quad\text{or}\quad T-70=c_1e^{-bt}$$

$$T(0)=300\quad\Rightarrow\quad c_1=230\quad\Rightarrow\quad T=70+230e^{-bt}$$

$$T(3)=200\quad\Rightarrow\quad b=0.19$$

$$T=70+230e^{-0.19t}$$

$$T(t)=71°F\quad\Rightarrow\quad t=28.62\,\text{min}$$

Linear Equations

- A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

is said to be a linear equation in the dependent variable y.

- If $g(x)=0$, the equation is said to be homogeneous.
- To solve this equation, it is converted to the standard form

$$\frac{dy}{dx}+P(x)y=f(x).$$

- Then, it is multiplied by the integrating factor $e^{\int P(x)dx}$.

$$e^{\int P(x)dx}\frac{dy}{dx}+P(x)e^{\int P(x)dx}y=e^{\int P(x)dx}f(x)\quad\Rightarrow\quad\frac{d}{dx}\left[e^{\int P(x)dx}y\right]=e^{\int P(x)dx}f(x)$$

- Integrate both sides to find $y(x)$.

Linear Equations (Examples)

$$x\frac{dy}{dx}-4y=x^6e^x$$

$$\frac{dy}{dx}-\frac{4}{x}y=x^5e^x$$

$$P(x)=-\frac{4}{x}$$

$$e^{\int P(x)dx}=e^{\int-(4/x)dx}=e^{-4\ln x}=e^{\ln x^{-4}}=x^{-4}$$

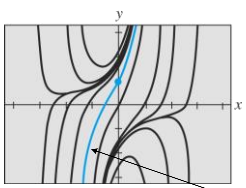
$$x^{-4}\frac{dy}{dx}-x^{-4}\frac{4}{x}y=x^{-4}x^5e^x$$

$$x^{-4}\frac{dy}{dx}-4x^{-5}y=xe^x$$

$$\frac{d}{dx}\left[x^{-4}y\right]=xe^x$$

$$x^{-4}y=xe^x-e^x+c$$

$$y=x^5e^x-x^4e^x+cx^4$$



$$\frac{dy}{dx}-2xy=2,\quad y(0)=1$$

$$P(x)=-2x$$

$$e^{\int P(x)dx}=e^{\int-2xdx}=e^{-x^2}$$

$$e^{-x^2}\frac{dy}{dx}-2xye^{-x^2}=2e^{-x^2}$$

$$\frac{d}{dx}\left[e^{-x^2}y\right]=2e^{-x^2}$$

$$e^{-x^2}y=\int_0^x2e^{-t^2}dt+c$$

$$y=2e^{x^2}\int_0^xe^{-t^2}dt+ce^{x^2}$$

$$y(0)=1\quad\Rightarrow\quad c=1$$

$$y=2e^{x^2}\int_0^xe^{-t^2}dt+e^{x^2}$$

$$y=e^{x^2}(1+\sqrt{\pi}\,\mathrm{erf}(x))$$

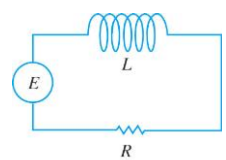
RL and RC Circuits

$$L\frac{di}{dt}+Ri=E(t)\quad\Rightarrow\quad\frac{di}{dt}+\frac{R}{L}i=\frac{1}{L}E(t)$$

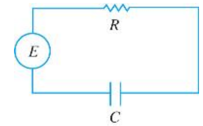
$$\frac{d}{dt}\left[i\cdot e^{\frac{R}{L}t}\right]=e^{\frac{R}{L}t}\frac{1}{L}E(t)$$

If E(t) is constant with time,

$$i(t)=\frac{E}{R}+c_1e^{-\frac{R}{L}t}$$



$$Ri+\frac{1}{C}q=E(t)\quad\text{or}\quad R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$



Exact Equations

- A first-order differential equation $M(x,y)dx+N(x,y)dy=0$ is an **exact differential equation** in a region R of the xy-plane if the left side corresponds to the exact differential of some function $f(x,y)$ in that region; i.e. $M(x,y)=\partial f/\partial x$ and $N(x,y)=\partial f/\partial y$.
- Examples of exact differentials are state equations such as ideal gas equation $P(\rho,T)=\rho RT$.
- A necessary and sufficient condition that $M(x,y)dx+N(x,y)dy$ be an exact differential is $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$
- Solution method is as follows

$$\frac{\partial f}{\partial x}=M(x,y) \quad \Rightarrow \quad f(x,y)=\int M(x,y)dx+g(y)$$

$$\frac{\partial f}{\partial y}=N(x,y)=\frac{\partial}{\partial y}\int M(x,y)dx+g'(y)$$

$$g'(y)=N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx \quad \Rightarrow \quad g(y)=\int N(x,y)dy-\int\frac{\partial}{\partial y}\int M(x,y)dxdy$$

Exact Equations (Examples)

$$2xydx+(x^2-1)dy=0$$

$$M(x,y)=2xy \quad \text{and} \quad N(x,y)=x^2-1$$

$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x}=M(x,y)=2xy$$

$$\Rightarrow \quad f(x,y)=x^2y+g(y)$$

$$\frac{\partial f}{\partial y}=N(x,y)$$

$$\Rightarrow \quad x^2+g'(y)=x^2-1$$

$$g(y)=-y$$

$$f(x,y)=x^2y-y$$

$$x^2y-y=c$$

$$y=\frac{c}{x^2-1}$$

$$(e^{2y}-y\cos xy)dx+(2xe^{2y}-x\cos xy+2y)dy=0$$

$$\frac{\partial M}{\partial y}=2e^{2y}+xy\sin xy-\cos xy=\frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial y}=N(x,y)=2xe^{2y}-x\cos xy+2y$$

$$\Rightarrow \quad f(x,y)=\int(2xe^{2y}-x\cos xy+2y)dy+h(x)$$

$$f(x,y)=xe^{2y}-\sin xy+y^2+h(x)$$

$$\frac{\partial f}{\partial x}=M(x,y)$$

$$\Rightarrow \quad e^{2y}-y\cos xy+h'(x)=e^{2y}-y\cos xy$$

$$h'(x)=0 \quad \Rightarrow \quad h(x)=c$$

$$xe^{2y}-\sin xy+y^2+c=0$$

Solution by Substitution

- Sometimes the first step in solving a differential equation is transforming it into another differential equation by **substitution**.
- If a function f possesses the property

  $$f(tx,ty)=t^{\alpha}f(x,y),$$

  then f is called a **homogeneous function** of order α.
- A first order differential equation

  $$P(x,y)dx+Q(x,y)dy=0$$

  is a **homogeneous** equation if both coefficients P and Q are homogeneous functions of the same degree.
- A homogeneous differential equation

  $$P(x,y)dx+Q(x,y)dy=0$$

  can be transformed to a separable differential equation by either of the substitutions

  $$y=ux \text{ or } x=vy.$$

Solution by Substitution

$$(x^2+y^2)dx+(x^2-xy)dy=0$$

$$\left.\begin{array}{l} y=ux \\ dy=udx+xdu \end{array}\right\} \quad \Rightarrow \quad (x^2+u^2x^2)dx+(x^2-ux^2)(udx+xdu)=0$$

$$x^2(1+u)dx+x^3(1-u)du=0$$

$$\frac{1-u}{1+u}du+\frac{dx}{x}=0$$

$$\left(-1+\frac{2}{1+u}\right)du+\frac{dx}{x}=0$$

$$-u+2\ln|1+u|+\ln|x|=c$$

$$\ln\left|x(1+u)^2\right|=c+u$$

$$x(1+u)^2=e^{c+u}=e^ce^u=c_1e^u$$

$$x(1+\frac{y}{x})^2=c_1e^{y/x}$$

$$(x+y)^2=c_1xe^{y/x}$$

Solution by Substitution

- The differential equation

$$\frac{dy}{dx}+P(x)y=f(x)y^n,$$

where n is any real number, is called **Bernoulli's equation**.

- The substitution $u=y^{1-n}$ reduces any Bernoulli's equation to a linear equation.

$$x\frac{dy}{dx}+y=x^2y^2 \qquad \frac{dy}{dx}+\frac{1}{x}y=xy^2$$

$$n=2 \quad\Rightarrow\quad u=y^{-1} \quad\Rightarrow\quad y=u^{-1} \quad\Rightarrow\quad \frac{dy}{dx}=-u^{-2}\frac{du}{dx}$$

$$-xu^{-2}\frac{du}{dx}+u^{-1}=x^2u^{-2} \quad\Rightarrow\quad \frac{du}{dx}-\frac{1}{x}u=-x$$

Solution by Substitution

- The differential equation

$$\frac{dy}{dx}=f(Ax+By+C)$$

can always be reduced to a separable equation by the substitution $u=Ax+By+C$.

$$\frac{dy}{dx}=(-2x+y)^2-7$$

$$u=-2x+y \quad\Rightarrow\quad \frac{du}{dx}=-2+\frac{dy}{dx}$$

$$2+\frac{du}{dx}=u^2-7 \quad\Rightarrow\quad \frac{du}{dx}=u^2-9$$

$$\frac{du}{u^2-9}=dx \quad\Rightarrow\quad \frac{du}{(u-3)(u+3)}=dx \quad\Rightarrow\quad \left[\frac{1}{u-3}-\frac{1}{u+3}\right]du=6dx$$

Chapter Exercises

$$\frac{dy}{dt}+2y=1, \qquad y(0)=2.5$$

$$\frac{dN}{dt}+N=Nte^{t+2}$$

$$x^2y'+x(x+2)y=e^x$$

$$\left(1-\frac{3}{y}+x\right)\frac{dy}{dx}+y=\frac{3}{x}-1$$

$$t^2\frac{dy}{dt}+y^2=ty$$

$$xy^2\frac{dy}{dx}=y^3-x^3, \quad y(1)=2$$

$$\frac{dy}{dx}=1+e^{y-x+5}$$