APPENDIX I

Additional Topics on Integration and Alignment

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This appendix contains alternative implementations and some advanced topics on the subjects of INS/GNSS integration, INS alignment, and multisensor integration, building on Chapters 14, 15, and 16. Section I.1 presents higher-order ECEF-frame and local-navigation-frame INS state transition matrices. Section I.2 shows how local-navigation-frame INS/GNSS and multisensor integration and quasi-stationary fine alignment may be implemented with Cartesian position error states. Section I.3 describes wander-azimuth implementations of INS/GNSS integration, INS alignment, and Doppler radar and sonar integration. Section I.4 describes a local-tangent-plane-frame implementation of INS/GNSS integration. Section I.5 describes INS/GNSS and multisensor integration and transfer alignment with body-frame-resolved attitude error states. Section I.6 shows how an INS/GNSS or multisensor integration algorithm and a transfer alignment algorithm may be augmented to estimate the time synchronization error. Section I.7 describes how an INS/GNSS integration algorithm may be extended to estimate GNSS wavelength ambiguities. Finally, Section I.8 presents additional information on the integration of dead reckoning, attitude, and height, while Section I.9 presents additional information on the integration of terrestrial radio navigation and environmental feature matching.

I.1 Higher-Order INS State Transition Matrices

This section presents third-order approximations of the transition matrices for ECEF-frame and local-navigation-frame implementations of INS error-state propagation. These provide more accurate state propagation over longer time intervals. The third-order transition matrix for the ECI-frame implementation is given by (14.38). Note that the attitude, velocity, position rows and columns are applicable to any type of dead-reckoning system integrated as the reference in an error-state integration architecture.

Where the Kalman-filter-estimated attitude, velocity, and position errors are referenced to and resolved in an ECEF frame, the system matrix for a state vector defined by (14.39) is given by (14.48) and (14.49) and the transition matrix to first order in $\mathbf{F}\tau_s$ is given by (14.50). Where terms of up to third order in $\mathbf{F}\tau_s$ are included and the assumptions $\left|F_{23,ij}^e\right|\tau_s^2 << 1$ for all i and j and $\omega_{ie}^2\tau_s^2 << 1$ are made, the transition matrix becomes

$$\mathbf{\Phi}_{INS}^e \approx \begin{bmatrix} \mathbf{\Phi}_{11}^e & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{15}^e \\ \mathbf{\Phi}_{21}^e & \mathbf{\Phi}_{22}^e & \mathbf{F}_{23}^e\tau_s & \mathbf{\Phi}_{24}^e & \mathbf{\Phi}_{25}^e \\ \mathbf{\Phi}_{31}^e & \mathbf{\Phi}_{32}^e & \mathbf{I}_3 & \mathbf{\Phi}_{34}^e & \frac{1}{6}\mathbf{F}_{21}^e\hat{\mathbf{C}}_b^e\tau_s^3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, \tag{I.1}$$

where

$$\begin{aligned}
\mathbf{\Phi}_{11}^e &= \mathbf{I}_3 - \mathbf{\Omega}_{ie}^e\tau_s \\
\mathbf{\Phi}_{15}^e &= \hat{\mathbf{C}}_b^e\tau_s - \tfrac{1}{2}\mathbf{\Omega}_{ie}^e\hat{\mathbf{C}}_b^e\tau_s^2 \\
\mathbf{\Phi}_{21}^e &= \mathbf{F}_{21}^e\tau_s - \tfrac{1}{2}\mathbf{F}_{21}^e\mathbf{\Omega}_{ie}^e\tau_s^2 - \mathbf{\Omega}_{ie}^e\mathbf{F}_{21}^e\tau_s^2 \\
\mathbf{\Phi}_{22}^e &= \mathbf{I}_3 - 2\mathbf{\Omega}_{ie}^e\tau_s \\
\mathbf{\Phi}_{24}^e &= \hat{\mathbf{C}}_b^e\tau_s - \mathbf{\Omega}_{ie}^e\hat{\mathbf{C}}_b^e\tau_s^2 \\
\mathbf{\Phi}_{25}^e &= \tfrac{1}{2}\mathbf{F}_{21}^e\hat{\mathbf{C}}_b^e\tau_s^2 - \tfrac{1}{6}\mathbf{F}_{21}^e\mathbf{\Omega}_{ie}^e\hat{\mathbf{C}}_b^e\tau_s^3 - \tfrac{1}{3}\mathbf{\Omega}_{ie}^e\mathbf{F}_{21}^e\hat{\mathbf{C}}_b^e\tau_s^3 \\
\mathbf{\Phi}_{31}^e &= \tfrac{1}{2}\mathbf{F}_{21}^e\tau_s^2 - \tfrac{1}{6}\mathbf{F}_{21}^e\mathbf{\Omega}_{ie}^e\tau_s^3 - \tfrac{1}{3}\mathbf{\Omega}_{ie}^e\mathbf{F}_{21}^e\tau_s^3 \\
\mathbf{\Phi}_{32}^e &= \mathbf{I}_3\tau_s - \mathbf{\Omega}_{ie}^e\tau_s^2 \\
\mathbf{\Phi}_{34}^e &= \tfrac{1}{2}\hat{\mathbf{C}}_b^e\tau_s^2 - \tfrac{1}{3}\mathbf{\Omega}_{ie}^e\hat{\mathbf{C}}_b^e\tau_s^3
\end{aligned}. \tag{I.2}$$

Where the Kalman-filter-estimated attitude, velocity, and position errors are referenced to and resolved in a local navigation frame, the system matrix for a state vector defined by (14.51) is given by (14.63–71) and the transition matrix to first order in $\mathbf{F}\tau_s$ is given by (14.72). The following assumptions are made about the magnitudes of the system matrix components:

$$\begin{aligned} \left|F^n_{12,i,j}\right|\tau_s^2 &<< 1/\left|\mathbf{f}_{ib}\right| \\ \left|\left(\mathbf{F}^n_{13}\mathbf{T}^p_{r(n)}\right)_{ij}\right|\tau_s^2 &<< 1/\left|\mathbf{f}_{ib}\right| \\ \left|\left(\mathbf{F}^n_{22}\mathbf{F}^n_{22}\right)_{ij}\right|\tau_s^2 &<< 1 \qquad i,j \in 1,2,3 \\ \left|\left(\mathbf{F}^n_{23}\mathbf{T}^p_{r(n)}\right)_{ij}\right|\tau_s^2 &<< 1 \\ \left|\left(\mathbf{F}^n_{33}\mathbf{F}^n_{33}\right)_{ij}\right|\tau_s^2 &<< 1 \end{aligned} \tag{I.3}$$

The transition matrix including significant terms of up to third order in $\mathbf{F}\tau_s$ is thus

$$\mathbf{\Phi}^e_{INS} \approx \begin{bmatrix} \mathbf{\Phi}^n_{11} & \mathbf{\Phi}^n_{12} & \mathbf{\Phi}^n_{13} & \mathbf{\Phi}^n_{14} & \mathbf{\Phi}^n_{15} \\ \mathbf{\Phi}^n_{21} & \mathbf{\Phi}^n_{22} & \mathbf{\Phi}^n_{23} & \mathbf{\Phi}^n_{24} & \mathbf{\Phi}^n_{25} \\ \mathbf{\Phi}^n_{31} & \mathbf{\Phi}^n_{32} & \mathbf{\Phi}^n_{33} & \mathbf{\Phi}^n_{34} & \mathbf{\Phi}^n_{35} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, \tag{I.4}$$

where

$$\begin{aligned} \mathbf{\Phi}^n_{11} &= \mathbf{I}_3 + \mathbf{F}^n_{11}\tau_s + \tfrac{1}{2}\mathbf{F}^n_{11}\mathbf{F}^n_{11}\tau_s^2 + \tfrac{1}{6}\mathbf{F}^n_{11}\mathbf{F}^n_{11}\mathbf{F}^n_{11}\tau_s^3 \\ \mathbf{\Phi}^n_{12} &= \begin{array}{l} \mathbf{F}^n_{12}\tau_s + \tfrac{1}{2}\left(\mathbf{F}^n_{11}\mathbf{F}^n_{12} + \mathbf{F}^n_{12}\mathbf{F}^n_{22} + \mathbf{F}^n_{13}\mathbf{F}^n_{32}\right)\tau_s^2 \\ + \tfrac{1}{6}\left[\mathbf{F}^n_{11}\left(\mathbf{F}^n_{11}\mathbf{F}^n_{12} + \mathbf{F}^n_{12}\mathbf{F}^n_{22} + \mathbf{F}^n_{13}\mathbf{F}^n_{32}\right) + \mathbf{F}^n_{13}\left(\mathbf{F}^n_{32}\mathbf{F}^n_{22} + \mathbf{F}^n_{33}\mathbf{F}^n_{32}\right)\right]\tau_s^3 \end{array} \\ \mathbf{\Phi}^n_{13} &= \mathbf{F}^n_{13}\tau_s + \tfrac{1}{2}\left(\mathbf{F}^n_{11}\mathbf{F}^n_{13} + \mathbf{F}^n_{12}\mathbf{F}^n_{23} + \mathbf{F}^n_{13}\mathbf{F}^n_{33}\right)\tau_s^2 + \tfrac{1}{6}\left(\mathbf{F}^n_{11}\mathbf{F}^n_{11}\mathbf{F}^n_{13} + \mathbf{F}^n_{11}\mathbf{F}^n_{13}\mathbf{F}^n_{33}\right)\tau_s^3 \\ \mathbf{\Phi}^n_{14} &= \tfrac{1}{2}\mathbf{F}^n_{12}\hat{\mathbf{C}}^n_b\tau_s^2 + \tfrac{1}{6}\left(\mathbf{F}^n_{11}\mathbf{F}^n_{12} + \mathbf{F}^n_{12}\mathbf{F}^n_{22} + \mathbf{F}^n_{13}\mathbf{F}^n_{32}\right)\hat{\mathbf{C}}^n_b\tau_s^3 \\ \mathbf{\Phi}^n_{15} &= \hat{\mathbf{C}}^n_b\tau_s + \tfrac{1}{2}\mathbf{F}^n_{11}\hat{\mathbf{C}}^n_b\tau_s^2 + \tfrac{1}{6}\mathbf{F}^n_{11}\mathbf{F}^n_{11}\hat{\mathbf{C}}^n_b\tau_s^3 \end{aligned}, \tag{I.5}$$

$$\begin{aligned} \mathbf{\Phi}^n_{21} &= \mathbf{F}^n_{21}\tau_s + \tfrac{1}{2}\left(\mathbf{F}^n_{21}\mathbf{F}^n_{11} + \mathbf{F}^n_{22}\mathbf{F}^n_{21}\right)\tau_s^2 + \tfrac{1}{6}\left(\mathbf{F}^n_{21}\mathbf{F}^n_{11}\mathbf{F}^n_{11} + \mathbf{F}^n_{22}\mathbf{F}^n_{21}\mathbf{F}^n_{11}\right)\tau_s^3 \\ \mathbf{\Phi}^n_{22} &= \mathbf{I}_3 + \mathbf{F}^n_{22}\tau_s \\ \mathbf{\Phi}^n_{23} &= \mathbf{F}^n_{23}\tau_s + \tfrac{1}{2}\left(\mathbf{F}^n_{21}\mathbf{F}^n_{13} + \mathbf{F}^n_{22}\mathbf{F}^n_{23} + \mathbf{F}^n_{23}\mathbf{F}^n_{33}\right)\tau_s^2 + \tfrac{1}{6}\left(\mathbf{F}^n_{21}\mathbf{F}^n_{11}\mathbf{F}^n_{13} + \mathbf{F}^n_{21}\mathbf{F}^n_{13}\mathbf{F}^n_{33} + \mathbf{F}^n_{22}\mathbf{F}^n_{21}\mathbf{F}^n_{13}\right)\tau_s^3 \\ \mathbf{\Phi}^n_{24} &= \hat{\mathbf{C}}^n_b\tau_s + \tfrac{1}{2}\mathbf{F}^n_{22}\hat{\mathbf{C}}^n_b\tau_s^2 \\ \mathbf{\Phi}^n_{25} &= \tfrac{1}{2}\mathbf{F}^n_{21}\hat{\mathbf{C}}^n_b\tau_s^2 + \tfrac{1}{6}\left(\mathbf{F}^n_{21}\mathbf{F}^n_{11} + \mathbf{F}^n_{22}\mathbf{F}^n_{21}\right)\hat{\mathbf{C}}^n_b\tau_s^3 \end{aligned}, \tag{I.6}$$

$$\begin{aligned} \mathbf{\Phi}^n_{31} &= \tfrac{1}{2}\mathbf{F}^n_{32}\mathbf{F}^n_{21}\tau_s^2 + \tfrac{1}{6}\left(\mathbf{F}^n_{32}\mathbf{F}^n_{21}\mathbf{F}^n_{11} + \mathbf{F}^n_{32}\mathbf{F}^n_{22}\mathbf{F}^n_{21}\right)\tau_s^3 \\ \mathbf{\Phi}^n_{32} &= \mathbf{F}^n_{32}\tau_s + \tfrac{1}{2}\left(\mathbf{F}^n_{32}\mathbf{F}^n_{22} + \mathbf{F}^n_{33}\mathbf{F}^n_{32}\right)\tau_s^2 + \tfrac{1}{6}\mathbf{F}^n_{33}\mathbf{F}^n_{32}\mathbf{F}^n_{22}\tau_s^3 \\ \mathbf{\Phi}^n_{33} &= \mathbf{I}_3 + \mathbf{F}^n_{33}\tau_s \\ \mathbf{\Phi}^n_{34} &= \tfrac{1}{2}\mathbf{F}^n_{32}\hat{\mathbf{C}}^n_b\tau_s^2 + \tfrac{1}{6}\left(\mathbf{F}^n_{32}\mathbf{F}^n_{22} + \mathbf{F}^n_{33}\mathbf{F}^n_{32}\right)\hat{\mathbf{C}}^n_b\tau_s^3 \\ \mathbf{\Phi}^n_{35} &= \tfrac{1}{6}\mathbf{F}^n_{32}\mathbf{F}^n_{21}\hat{\mathbf{C}}^n_b\tau_s^3 \end{aligned}. \tag{I.7}$$

I.2 Local-Navigation-Frame Cartesian Position Error

This section describes an alternative local-navigation-frame implementation of INS/GNSS and multisensor integration and quasi-stationary fine alignment, in which a Cartesian position error, resolved about local-navigation-frame axes, is estimated instead of latitude, longitude, and height errors. This simplifies the measurement model, removing the need for position measurements to be rescaled in order to prevent numerical errors.

Section I.2.1 defines the error states for this implementation, including how the inertial position solution is corrected. Sections I.2.2 and I.2.3 describe typical system models for INS and dead reckoning, respectively. Sections I.2.4 and I.2.5 present measurement models for loosely coupled and tightly coupled INS/GNSS integration, respectively. These may be applied to the integration of GNSS with dead-reckoning sensors by replacing the bias states with the appropriate instrument error states, for which the measurement matrix columns are zero. Finally, Section I.2.6 describes the quasi-stationary fine alignment measurement models and Section I.2.7 presents terrestrial radio navigation and environmental feature-matching measurement models. Note that the dead-reckoning aiding-sensor and total-state measurement models presented in Section 16.2 are unaffected by the position error implementation.

As for the examples presented in Sections 14.2 and 14.3, it is assumed that the inertial error states estimated by the Kalman filter (or EKF) are attitude error, velocity error, position error, accelerometer biases, and gyro biases. Note that the transfer alignment examples presented in Section 15.1 are unaffected because position error states are not estimated.

I.2.1 State Definitions

For a Cartesian local-navigation-frame implementation of INS/GNSS integration and quasi-stationary fine alignment, the INS error state vector is defined as

$$\mathbf{x}_{INS}^{nC} = \begin{pmatrix} \delta\boldsymbol{\psi}_{nb}^{n} \\ \delta\mathbf{v}_{eb}^{n} \\ \delta\mathbf{r}_{eb}^{n} \\ \mathbf{b}_{a} \\ \mathbf{b}_{g} \end{pmatrix}. \tag{I.8}$$

The Cartesian and curvilinear position error states transform as

$$\delta\mathbf{p}_{b} \approx \mathbf{T}_{r(n)}^{p}\delta\mathbf{r}_{eb}^{n}, \qquad \delta\mathbf{r}_{eb}^{n} \approx \mathbf{T}_{p}^{r(n)}\delta\mathbf{p}_{b} \tag{I.9}$$

where the transformation matrices, $\mathbf{T}_{r(n)}^{p}$ and $\mathbf{T}_{p}^{r(n)}$, are [repeating (2.119) and (2.120)]

$$\mathbf{T}_{r(n)}^{p} = \frac{\partial\mathbf{p}_{b}}{\partial\mathbf{r}_{eb}^{n}} = \begin{pmatrix} \dfrac{1}{R_N(L_b)+h_b} & 0 & 0 \\ 0 & \dfrac{1}{\left(R_E(L_b)+h_b\right)\cos L_b} & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\mathbf{T}_{p}^{r(n)} = \frac{\partial\mathbf{r}_{eb}^{n}}{\partial\mathbf{p}_{b}} = \begin{pmatrix} R_N(L_b)+h_b & 0 & 0 \\ 0 & \left(R_E(L_b)+h_b\right)\cos L_b & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The inertial latitude, longitude, and height solution is corrected using

$$\begin{pmatrix} \hat{L}_b \\ \hat{\lambda}_b \\ \hat{h}_b \end{pmatrix} = \begin{pmatrix} \tilde{L}_b \\ \tilde{\lambda}_b \\ \tilde{h}_b \end{pmatrix} - \hat{\mathbf{T}}^{p}_{r(n)} \delta\hat{\mathbf{r}}^{n}_{eb} \tag{I.10}$$

for open-loop correction and

$$\begin{pmatrix} \hat{L}_b(+) \\ \hat{\lambda}_b(+) \\ \hat{h}_b(+) \end{pmatrix} = \begin{pmatrix} \hat{L}_b(-) \\ \hat{\lambda}_b(-) \\ \hat{h}_b(-) \end{pmatrix} - \hat{\mathbf{T}}^{p}_{r(n)} \delta\hat{\mathbf{r}}^{n}_{eb} \tag{I.11}$$

for closed-loop correction.

The attitude error, velocity error, and position error states in (I.8) are also applicable to the integration of odometry, PDR using step detection, and Doppler radar/sonar as the reference navigation system. The remaining states are replaced by scale factor error states for odometry and Doppler radar/sonar, and by model coefficient and boresight error states for PDR, as described in Section 16.2.

I.2.2 INS System Model

The INS system model derivation presented in Section 14.2.4 is applicable with the curvilinear position error states replaced with the corresponding Cartesian states. Consequently, the system matrix can be obtained directly from its curvilinear counterpart using transformation matrices:

$$\mathbf{F}^{nC}_{INS} = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \hat{\mathbf{T}}^{r(n)}_{p} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix} \mathbf{F}^{n}_{INS} \begin{pmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \hat{\mathbf{T}}^{p}_{r(n)} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix}. \tag{I.12}$$

Thus, from (14.63),

$$\mathbf{F}^{nC}_{INS} = \begin{pmatrix} \mathbf{F}^{n}_{11} & \mathbf{F}^{n}_{12} & \mathbf{F}^{nC}_{13} & \mathbf{0}_3 & \hat{\mathbf{C}}^{n}_{b} \\ \mathbf{F}^{n}_{21} & \mathbf{F}^{n}_{22} & \mathbf{F}^{nC}_{23} & \hat{\mathbf{C}}^{n}_{b} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{F}^{nC}_{33} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}, \tag{I.13}$$

where $\mathbf{F}^{n}_{11}$, $\mathbf{F}^{n}_{12}$, $\mathbf{F}^{n}_{21}$, and $\mathbf{F}^{n}_{22}$ are as given by (14.64), (14.65), (14.67), and (14.68), respectively, while, from (14.66), (14.69), and (14.71),

$$\mathbf{F}^{nC}_{13} = \begin{bmatrix} \dfrac{\omega_{ie} \sin \hat{L}_b}{R_N(\hat{L}_b) + \hat{h}_b} & 0 & \dfrac{-\hat{v}^{n}_{eb,E}}{\left(R_E(\hat{L}_b) + \hat{h}_b\right)^2} \\ 0 & 0 & \dfrac{\hat{v}^{n}_{eb,N}}{\left(R_N(\hat{L}_b) + \hat{h}_b\right)^2} \\ \dfrac{\omega_{ie} \cos \hat{L}_b}{R_N(\hat{L}_b) + \hat{h}_b} + \dfrac{\hat{v}^{n}_{eb,E}}{\left(R_N(\hat{L}_b) + \hat{h}_b\right)\left(R_E(\hat{L}_b) + \hat{h}_b\right) \cos^2 \hat{L}_b} & 0 & \dfrac{\hat{v}^{n}_{eb,E} \tan \hat{L}_b}{\left(R_E(\hat{L}_b) + \hat{h}_b\right)^2} \end{bmatrix}, \tag{I.14}$$

$$\mathbf{F}_{23}^{nC} = \begin{bmatrix} \left(-\dfrac{(\hat{v}_{eb,E}^{n})^2 \sec^2 \hat{L}_b}{\left(R_N(\hat{L}_b)+\hat{h}_b\right)\left(R_E(\hat{L}_b)+\hat{h}_b\right)} \atop -2\hat{v}_{eb,E}^{n}\omega_{ie}\cos\hat{L}_b \Big/ \left(R_N(\hat{L}_b)+\hat{h}_b\right) \right) & 0 & -\dfrac{(\hat{v}_{eb,E}^{n})^2 \tan\hat{L}_b}{\left(R_E(\hat{L}_b)+\hat{h}_b\right)^2} + \dfrac{\hat{v}_{eb,N}^{n}\hat{v}_{eb,D}^{n}}{\left(R_N(\hat{L}_b)+\hat{h}_b\right)^2} \\ \left(\dfrac{\hat{v}_{eb,N}^{n}\hat{v}_{eb,E}^{n}\sec^2\hat{L}_b}{\left(R_N(\hat{L}_b)+\hat{h}_b\right)\left(R_E(\hat{L}_b)+\hat{h}_b\right)} + \dfrac{2\left(\omega_{ie}\hat{v}_{eb,N}^{n}\cos\hat{L}_b - \hat{v}_{eb,D}^{n}\sin\hat{L}_b\right)}{R_N(\hat{L}_b)+\hat{h}_b} \right) & 0 & \dfrac{\hat{v}_{eb,N}^{n}\hat{v}_{eb,E}^{n}\tan\hat{L}_b + \hat{v}_{eb,E}^{n}\hat{v}_{eb,D}^{n}}{\left(R_E(\hat{L}_b)+\hat{h}_b\right)^2} \\ \dfrac{2\hat{v}_{eb,E}^{n}\omega_{ie}\sin\hat{L}_b}{R_N(\hat{L}_b)+\hat{h}_b} & 0 & -\dfrac{(\hat{v}_{eb,E}^{n})^2}{\left(R_E(\hat{L}_b)+\hat{h}_b\right)^2} - \dfrac{(\hat{v}_{eb,N}^{n})^2}{\left(R_N(\hat{L}_b)+\hat{h}_b\right)^2} + \dfrac{2g_0(\hat{L}_b)}{r_{eS}^{e}(\hat{L}_b)} \end{bmatrix}, \tag{I.15}$$

$$\mathbf{F}_{33}^{nC} = \begin{bmatrix} 0 & 0 & \dfrac{\hat{v}_{eb,N}^{n}}{R_N(\hat{L}_b)+\hat{h}_b} \\ \dfrac{\hat{v}_{eb,E}^{n}\tan\hat{L}_b}{R_N(\hat{L}_b)+\hat{h}_b} & 0 & \dfrac{\hat{v}_{eb,E}^{n}}{R_E(\hat{L}_b)+\hat{h}_b} \\ 0 & 0 & 0 \end{bmatrix}. \tag{I.16}$$

The transition matrix, limited to first order in $\mathbf{F}\tau_s$, is

$$\mathbf{\Phi}_{INS}^{nC} \approx \begin{pmatrix} \mathbf{I}_3 + \mathbf{F}_{11}^{n}\tau_s & \mathbf{F}_{12}^{n}\tau_s & \mathbf{F}_{13}^{nC}\tau_s & \mathbf{0}_3 & \hat{\mathbf{C}}_b^n\tau_s \\ \mathbf{F}_{21}^{n}\tau_s & \mathbf{I}_3 + \mathbf{F}_{22}^{n}\tau_s & \mathbf{F}_{23}^{nC}\tau_s & \hat{\mathbf{C}}_b^n\tau_s & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3\tau_s & \mathbf{I}_3 + \mathbf{F}_{33}^{nC}\tau_s & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix}. \tag{I.17}$$

I.2.3 Dead-Reckoning System Models

The odometry system model presented in Section 16.2.3 is applicable, except for the position error state dynamics, which are given by

$$\delta\dot{\mathbf{r}}_{eb}^{n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_{eb}^{n} s_{or}, \tag{I.18}$$

where s_{or} is the rear-wheel scale-factor error.

The system model presented in Section 16.2.4 for PDR using step detection is applicable, except for the position error state dynamics, which are given by

$$\delta\mathbf{r}_{eb,k}^{n-} = \begin{matrix} \delta\mathbf{r}_{eb,k-1}^{n+} + \begin{pmatrix} \cos(\hat{\psi}_{nb} + \hat{\psi}_{bh}) \\ \sin(\hat{\psi}_{nb} + \hat{\psi}_{bh}) \\ 0 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{1}{\tau_P} & \sigma_f^2 & \hat{\theta}_{nb} \end{pmatrix} \delta\mathbf{c}_{P,k-1}^{-} \\ + \begin{pmatrix} -\sin(\hat{\psi}_{nb} + \hat{\psi}_{bh}) \\ \cos(\hat{\psi}_{nb} + \hat{\psi}_{bh}) \\ 0 \end{pmatrix} \Delta\hat{r}_P \left(\delta\psi_{nb,k-1}^{-} + \delta\psi_{bh,k-1}^{-}\right) \end{matrix}, \tag{I.19}$$

where $\hat{\psi}_{bh}$, τ_P, σ_f^2, $\delta\mathbf{c}_P^-$, and $\Delta\hat{r}_P$ are as defined in Section 16.2.4.

The Doppler radar and sonar system model presented in Section 16.2.5 is applicable, except for the position error state dynamics, which are given by

$$\delta\dot{\mathbf{r}}_{eb}^{n} = s_D \mathbf{v}_{eb}^{n} + \mathbf{v}_{es}^{n}, \tag{I.20}$$

where s_D is the scale-factor error and $\mathbf{v}_{es}^{n}$ is the surface velocity.

I.2.4 Loosely Coupled GNSS Measurement Model

In this implementation, the position measurement innovation is expressed in terms of Cartesian, not curvilinear position. The velocity measurement innovation is the same as for the conventional local-navigation-frame implementation. Thus (14.103) is replaced by

$$\delta\mathbf{z}_{G,k}^{nC-} = \begin{pmatrix} \hat{\mathbf{T}}_p^{r(n)}\left(\hat{\mathbf{p}}_{aG} - \hat{\mathbf{p}}_b\right) - \hat{\mathbf{C}}_b^n \mathbf{l}_{ba}^b \\ \hat{\mathbf{v}}_{eaG}^n - \hat{\mathbf{v}}_{eb}^n - \hat{\mathbf{C}}_b^n\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right) + \hat{\boldsymbol{\Omega}}_{ie}^n \hat{\mathbf{C}}_b^n \mathbf{l}_{ba}^b \end{pmatrix}_k. \tag{I.21}$$

Following the measurement model derivation in Section 14.3.1, the measurement matrix is

$$\mathbf{H}_{G,k}^{nC} = \begin{pmatrix} \mathbf{H}_{r1}^{nC} & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{H}_{v1}^{n} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{v5}^{n} \end{pmatrix}_k, \tag{I.22}$$

where

$$\mathbf{H}_{r1}^{nC} \approx \left[\left(\hat{\mathbf{C}}_b^n \mathbf{l}_{ba}^b\right)\wedge\right] \tag{I.23}$$

and $\mathbf{H}_{v1}^{n}$ and $\mathbf{H}_{v5}^{n}$ are as defined by (14.114).

Where coupling of the attitude errors and gyro biases into the measurements through the lever arm terms is neglected, the measurement matrix simplifies to

$$\mathbf{H}_{G,k}^{nC} \approx \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}_k. \tag{I.24}$$

I.2.5 Tightly Coupled GNSS Measurement Model

For tightly coupled integration, the measurement innovations are the same as for the conventional local-navigation-frame implementation. However, the measurement matrix simplifies to

$$\mathbf{H}_{G,k}^{nC} \approx \left.\begin{pmatrix} \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{a1}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 \\ \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{a2}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{am}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 \\ \hdashline \mathbf{0}_{1,3} & \mathbf{u}_{a1}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \\ \mathbf{0}_{1,3} & \mathbf{u}_{a2}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{1,3} & \mathbf{u}_{am}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \end{pmatrix}\right|_{\mathbf{x}=\hat{\mathbf{x}}_k}, \tag{I.25}$$

where, again, the coupling of the attitude errors and gyro biases into the measurements through the lever arm terms is neglected.

I.2.6 Quasi-Stationary Fine Alignment Measurement Model

For implementation with Cartesian position error states, the measurement innovation for the quasi-stationary fine alignment algorithm presented in Section 15.2.2 must be transformed from curvilinear to Cartesian position. Thus, (15.26) is replaced by

$$\delta\mathbf{z}_{Q,k}^{nC-} = \hat{\mathbf{T}}_p^{r(n)}\left(\hat{\mathbf{p}}_b(t_0) - \hat{\mathbf{p}}_b(t)\right). \tag{I.26}$$

The measurement matrix, given by (15.30), is unchanged.

I.2.7 Terrestrial Radio Navigation and Feature Matching Measurement Models

Loosely coupled integration of terrestrial radio navigation and environmental feature-matching position fixes is described in Section 16.3.1. For the local-navigation-frame implementation with Cartesian position error, the measurement innovation is

$$\delta\mathbf{z}_{R,k}^{nC-} = \left[\hat{\mathbf{T}}_p^{r(n)}\left(\tilde{\mathbf{p}}_{aR} - \hat{\mathbf{p}}_b\right) - \hat{\mathbf{C}}_b^n \mathbf{l}_{ba}^b - \hat{\mathbf{b}}_R^{nC}\right]_k^-, \tag{I.27}$$

where $\hat{\mathbf{b}}_R^{nC}$ is the estimated Cartesian position solution bias, resolved about local-navigation-frame axes.

Defining the state vector as

$$\mathbf{x}^{nC} = \begin{pmatrix} \mathbf{r}_{eb}^n \\ \vdots \\ \mathbf{b}_R^{nC} \end{pmatrix} \tag{I.28}$$

for total-state integration or

$$\mathbf{x}^{nC} = \begin{pmatrix} \delta\mathbf{r}_{eb}^n \\ \vdots \\ \mathbf{b}_R^{nC} \end{pmatrix} \tag{I.29}$$

for error-state integration, the measurement matrix is

$$\mathbf{H}_{R,k}^{nC} = \left(k_R\mathbf{I} \quad \mathbf{0} \quad \mathbf{I}\right), \tag{I.30}$$

where k_R is 1 for total-state integration and −1 for error-state integration.

Tightly coupled integration of terrestrial radio navigation and environmental feature-matching range measurements is described in Section 16.3.2. For the local-navigation-frame implementation with Cartesian position error, the state vector may be defined as

$$\mathbf{x}^{nC} = \begin{pmatrix} \mathbf{r}_{eb}^{n} \\ \vdots \\ \mathbf{b}_{r} \\ \delta\rho_{c}^{a} \\ \delta\dot{\rho}_{c}^{a} \end{pmatrix} \tag{I.31}$$

for total-state integration or

$$\mathbf{x}^{nC} = \begin{pmatrix} \delta\mathbf{r}_{eb}^{n} \\ \vdots \\ \mathbf{b}_{r} \\ \delta\rho_{c}^{a} \\ \delta\dot{\rho}_{c}^{a} \end{pmatrix} \tag{I.32}$$

where the clock offset and drift states are included for pseudo-range measurements and excluded for two-way ranging.

The measurement matrix for pseudo-range measurements is approximately

$$\mathbf{H}_{R,k}^{nC} \approx \begin{pmatrix} -k_R \mathbf{u}_{a1}^{n\,\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{1\,\mathrm{T}} & 1 & \mathbf{0} \\ -k_R \mathbf{u}_{a2}^{n\,\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{2\,\mathrm{T}} & 1 & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -k_R \mathbf{u}_{am}^{n\,\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{m\,\mathrm{T}} & 1 & \mathbf{0} \end{pmatrix}_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}_k^{\gamma -}}, \tag{I.33}$$

where [repeating (16.86)]

$$\mathbf{h}_b^{j\,\mathrm{T}} = \begin{pmatrix} \delta_{1j} & \delta_{2j} & \cdots & \delta_{nj} \end{pmatrix}.$$

For measurements of the TDOA across transmitters, the measurement matrix is approximately

$$\mathbf{H}_{R,k}^{nC} \approx \begin{pmatrix} k_R \left(\mathbf{u}_{ar}^{n} - \mathbf{u}_{a1}^{n}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{1\,\mathrm{T}} \\ k_R \left(\mathbf{u}_{ar}^{n} - \mathbf{u}_{a2}^{n}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{2\,\mathrm{T}} \\ \vdots & \vdots & \vdots \\ k_R \left(\mathbf{u}_{ar}^{n} - \mathbf{u}_{am}^{n}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{m\,\mathrm{T}} \end{pmatrix}_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}_k^{\gamma -}}. \tag{I.34}$$

I.3 Wander-Azimuth Implementation

This section describes how the algorithms presented in Chapters 14 and 15 may be modified to estimate attitude, velocity, and position errors resolved about the axes of a wander-azimuth frame. A wander-azimuth frame is a modified local navigation frame that does not exhibit singularities at the Earth's poles. Inertial navigation equations have traditionally been implemented in a wander-azimuth frame where there is a need for operation in polar regions. See Sections 2.1.6 and 5.4.5 for more information. Note that ECI- and ECEF-frame implementations of inertial navigation, INS/GNSS integration, and alignment are also suitable for polar use, while it is not necessary for the inertial navigation and integration algorithms to be implemented using the same coordinate frames.

Section I.3.1 defines the error states for a wander-azimuth implementation, including how they are used to correct the inertial navigation solution. Section I.3.2 describes a typical system model. Sections I.3.3 and I.3.4 present measurement models for loosely coupled and tightly coupled INS/GNSS integration, respectively. As for the examples presented in Sections 14.2 and 14.3, it is assumed that the inertial error states estimated by the integration Kalman filter (or EKF) are attitude error, velocity error, position error, accelerometer biases, and gyro biases. Section I.3.5 discusses wander-azimuth-frame transfer alignment, Section I.3.6 describes quasi-stationary fine alignment, and Section I.3.7 discusses zero updates and motion constraints. Finally, Section I.3.8 presents a measurement model for Doppler radar and sonar integrated as an aiding sensor. Other dead-reckoning sensors and terrestrial radio navigation are rarely used in polar regions.

I.3.1 State Definitions

For the wander-azimuth implementations presented here, the INS error state vector is defined as

$$\mathbf{x}_{INS}^{w} = \begin{pmatrix} \delta\boldsymbol{\psi}_{wb}^{w} \\ \delta\mathbf{v}_{eb}^{w} \\ \delta\boldsymbol{\psi}_{ew}^{w} \\ \delta h_b \\ \mathbf{b}_a \\ \mathbf{b}_g \end{pmatrix}, \tag{I.35}$$

where the small-angle attitude error, $\delta\boldsymbol{\psi}_{wb}^{w}$, velocity error, $\delta\mathbf{v}_{eb}^{w}$, and, height error, δh_b, are defined by (5.111), (5.107), and (5.108), respectively, while $\delta\boldsymbol{\psi}_{ew}^{w}$ is the wander azimuth position error, defined from (5.110) and (5.111) by

$$\begin{aligned} \left[\delta\boldsymbol{\psi}_{ew}^{w} \wedge\right] &\approx \mathbf{I}_3 - \delta\mathbf{C}_e^w \\ &= \mathbf{I}_3 - \tilde{\mathbf{C}}_e^w \mathbf{C}_w^e \end{aligned}. \tag{I.36}$$

Note that $\mathbf{C}_e^w$ is a function of the latitude, longitude, and wander azimuth (see Section 5.4.5). The third component of $\delta\boldsymbol{\psi}_{ew}^{w}$ is equal to the wander azimuth error, $\delta\psi_{nw}$, and the remaining components are functions of the latitude and longitude errors.

The INS attitude, velocity, and height errors are corrected using, respectively, (14.6), (14.2), and (14.4) for open-loop correction and (14.7), (14.8), and (14.10) for closed-loop correction. The horizontal position error is corrected using

$$\hat{\mathbf{C}}_e^w \approx \left(\mathbf{I}_3 + \left[\delta\hat{\boldsymbol{\psi}}_{ew}^{w} \wedge\right]\right)\tilde{\mathbf{C}}_e^w \tag{I.37}$$

for open-loop correction and

$$\hat{\mathbf{C}}_e^w(+) \approx \left(\mathbf{I}_3 + \left[\delta\hat{\boldsymbol{\psi}}_{ew}^{w} \wedge\right]\right)\hat{\mathbf{C}}_e^w(-) \tag{I.38}$$

for closed-loop correction.

I.3.2 INS System Model

As stated in Section 5.4.5, the attitude propagation equations incorporate a transport-rate term in addition to the Earth-rate and gyro-measurement terms. Following its ECI-frame counterpart from (14.16) to (14.22), the attitude error derivative in a wander-azimuth frame is

$$\delta\dot{\boldsymbol{\psi}}_{wb}^{w} \approx \hat{\mathbf{C}}_b^w\left(\tilde{\boldsymbol{\omega}}_{wb}^{b} - \boldsymbol{\omega}_{wb}^{b}\right). \tag{I.39}$$

Expanding this into gyro-measurement, Earth-rate, and transport-rate terms,

$$\delta\dot{\boldsymbol{\psi}}_{wb}^{w} \approx \hat{\mathbf{C}}_b^w \delta\boldsymbol{\omega}_{ib}^{b} - \hat{\mathbf{C}}_b^w\left(\tilde{\boldsymbol{\omega}}_{ie}^{b} - \boldsymbol{\omega}_{ie}^{b}\right) - \hat{\mathbf{C}}_b^w\left(\tilde{\boldsymbol{\omega}}_{ew}^{b} - \boldsymbol{\omega}_{ew}^{b}\right). \tag{I.40}$$

Expanding the Earth-rate and transport-rate terms, neglecting products of error states,

$$\begin{aligned}\hat{\mathbf{C}}_b^w\left(\tilde{\boldsymbol{\omega}}_{ie}^b-\boldsymbol{\omega}_{ie}^b\right)+\hat{\mathbf{C}}_b^w\left(\tilde{\boldsymbol{\omega}}_{ew}^b-\boldsymbol{\omega}_{ew}^b\right) &\approx \hat{\mathbf{C}}_b^w\left(\tilde{\mathbf{C}}_w^b-\mathbf{C}_w^b\right)\left(\hat{\boldsymbol{\omega}}_{ie}^w+\hat{\boldsymbol{\omega}}_{ew}^w\right)+\left(\tilde{\boldsymbol{\omega}}_{ie}^w-\boldsymbol{\omega}_{ie}^w\right)+\left(\tilde{\boldsymbol{\omega}}_{ew}^w-\boldsymbol{\omega}_{ew}^w\right) \\ &\approx \boldsymbol{\Omega}_{iw}^w\delta\boldsymbol{\psi}_{wb}^w+\left(\tilde{\boldsymbol{\omega}}_{ie}^w-\boldsymbol{\omega}_{ie}^w\right)+\left(\tilde{\boldsymbol{\omega}}_{ew}^w-\boldsymbol{\omega}_{ew}^w\right)\end{aligned}. \tag{I.41}$$

Considering the Earth-rate term,

$$\tilde{\boldsymbol{\omega}}_{ie}^w-\boldsymbol{\omega}_{ie}^w=\left(\tilde{\mathbf{C}}_e^w-\mathbf{C}_e^w\right)\begin{pmatrix}0\\0\\\omega_{ie}\end{pmatrix}=\left(\tilde{\mathbf{C}}_e^w\mathbf{C}_w^e-\mathbf{I}_3\right)\mathbf{C}_e^w\begin{pmatrix}0\\0\\\omega_{ie}\end{pmatrix}. \tag{I.42}$$

Substituting in (I.36) and taking the expectation,

$$\begin{aligned}\mathrm{E}\left(\tilde{\boldsymbol{\omega}}_{ie}^w-\boldsymbol{\omega}_{ie}^w\right) &= -\left[\delta\boldsymbol{\psi}_{ew}^w\wedge\right]\hat{\mathbf{C}}_e^w\begin{pmatrix}0\\0\\\omega_{ie}\end{pmatrix} \\ &= \left[\hat{\mathbf{C}}_e^w\begin{pmatrix}0\\0\\\omega_{ie}\end{pmatrix}\right]\wedge\delta\boldsymbol{\psi}_{ew}^w, \\ &= \omega_{ie}\left[\hat{\mathbf{c}}_{e,:,3}^w\wedge\right]\delta\boldsymbol{\psi}_{ew}^w\end{aligned} \tag{I.43}$$

where $\hat{\mathbf{c}}_{e,:,3}^w$ is the third column of $\hat{\mathbf{C}}_e^w$.

Considering the transport term, from (5.59) and (5.63), neglecting products of error states and the variation of the radii or curvature with latitude,

$$\mathrm{E}\left(\tilde{\boldsymbol{\omega}}_{ew}^w-\boldsymbol{\omega}_{ew}^w\right)\approx\mathbf{W}_2\delta\mathbf{v}_{eb}^w+\mathbf{w}_3\delta\psi_{ew,3}^w+\mathbf{w}_4\delta h_b, \tag{I.44}$$

where

$$\mathbf{W}_2=\begin{bmatrix}\left(-\dfrac{\cos\hat{\psi}_{nw}\sin\hat{\psi}_{nw}}{R_N(\hat{C}_{e,3,3}^w)+\hat{h}_b}+\dfrac{\cos\hat{\psi}_{nw}\sin\hat{\psi}_{nw}}{R_E(\hat{C}_{e,3,3}^w)+\hat{h}_b}\right) & \left(\dfrac{\sin^2\hat{\psi}_{nw}}{R_N(\hat{C}_{e,3,3}^w)+\hat{h}_b}+\dfrac{\cos^2\hat{\psi}_{nw}}{R_E(\hat{C}_{e,3,3}^w)+\hat{h}_b}\right) & 0 \\ \left(-\dfrac{\cos^2\hat{\psi}_{nw}}{R_N(\hat{C}_{e,3,3}^w)+\hat{h}_b}-\dfrac{\sin^2\hat{\psi}_{nw}}{R_E(\hat{C}_{e,3,3}^w)+\hat{h}_b}\right) & \left(\dfrac{\cos\hat{\psi}_{nw}\sin\hat{\psi}_{nw}}{R_N(\hat{C}_{e,3,3}^w)+\hat{h}_b}-\dfrac{\cos\hat{\psi}_{nw}\sin\hat{\psi}_{nw}}{R_E(\hat{C}_{e,3,3}^w)+\hat{h}_b}\right) & 0 \\ 0 & 0 & 0\end{bmatrix}, \tag{I.45}$$

$$\mathbf{w}_3=\begin{pmatrix}\dfrac{-\hat{v}_{eb,x}^w\cos 2\hat{\psi}_{nw}+\hat{v}_{eb,y}^w\sin 2\hat{\psi}_{nw}}{R_N(\hat{C}_{e,3,3}^w)+\hat{h}_b}+\dfrac{\hat{v}_{eb,x}^w\cos 2\hat{\psi}_{nw}-\hat{v}_{eb,y}^w\sin 2\hat{\psi}_{nw}}{R_E(\hat{C}_{e,3,3}^w)+\hat{h}_b} \\ \dfrac{\hat{v}_{eb,x}^w\sin 2\hat{\psi}_{nw}+\hat{v}_{eb,y}^w\cos 2\hat{\psi}_{nw}}{R_N(\hat{C}_{e,3,3}^w)+\hat{h}_b}-\dfrac{\hat{v}_{eb,x}^w\sin 2\hat{\psi}_{nw}+\hat{v}_{eb,y}^w\cos 2\hat{\psi}_{nw}}{R_E(\hat{C}_{e,3,3}^w)+\hat{h}_b} \\ 0\end{pmatrix}, \tag{I.46}$$

$$\mathbf{w}_4 = \begin{pmatrix} \dfrac{\hat{v}^w_{eb,x}\cos\hat{\psi}_{nw}\sin\hat{\psi}_{nw} - \hat{v}^w_{eb,y}\sin^2\hat{\psi}_{nw}}{\left(R_N(\hat{C}^w_{e,3,3}) + \hat{h}_b\right)^2} - \dfrac{\hat{v}^w_{eb,x}\cos\hat{\psi}_{nw}\sin\hat{\psi}_{nw} + \hat{v}^w_{eb,y}\cos^2\hat{\psi}_{nw}}{\left(R_E(\hat{C}^w_{e,3,3}) + \hat{h}_b\right)^2} \\ \dfrac{\hat{v}^w_{eb,x}\cos^2\hat{\psi}_{nw} - \hat{v}^w_{eb,y}\cos\hat{\psi}_{nw}\sin\hat{\psi}_{nw}}{\left(R_N(\hat{C}^w_{e,3,3}) + \hat{h}_b\right)^2} + \dfrac{\hat{v}^w_{eb,x}\sin^2\hat{\psi}_{nw} + \hat{v}^w_{eb,y}\cos\hat{\psi}_{nw}\sin\hat{\psi}_{nw}}{\left(R_E(\hat{C}^w_{e,3,3}) + \hat{h}_b\right)^2} \\ 0 \end{pmatrix}, \quad (I.47)$$

noting that $\delta\psi^w_{ew,3} = \delta\psi_{nw}$ and that the meridian and transverse radii of curvature, respectively, R_N and R_E, can be computed directly from $C^w_{e,3,3}$ using (5.64).

Close to the poles, it is difficult to compute the wander angle accurately (see Section 5.4.5), while the meridian and transverse radii of curvature are the same at the poles. Therefore, the following approximations should be used near the poles (e.g., where $|C^w_{e,3,3}| > 0.99995$):

$$\begin{aligned} \mathbf{W}_2 &\approx \frac{1}{\left(R_0\big/\sqrt{1-e^2}\right) + \hat{h}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \mathbf{w}_3 &\approx \mathbf{0}_{3\times 1} \\ \mathbf{w}_4 &\approx \frac{1}{\left[\left(R_0\big/\sqrt{1-e^2}\right) + \hat{h}\right]^2} \begin{pmatrix} -\hat{v}^w_{eb,y} \\ \hat{v}^w_{eb,x} \\ 0 \end{pmatrix} \end{aligned}. \quad (I.48)$$

The time derivative of the velocity error takes the same form as for a local-navigation-frame implementation, as shown in Section 14.2.4. Thus, by analogy with (14.61), its expectation is

$$\begin{aligned} \mathrm{E}\left(\delta\dot{\mathbf{v}}^w_{eb}\right) \approx & -\left(\hat{\mathbf{C}}^w_b\hat{\mathbf{f}}^b_{ib}\right)\wedge\delta\boldsymbol{\psi}^w_{wb} - \left(\hat{\boldsymbol{\Omega}}^w_{ew} + 2\hat{\boldsymbol{\Omega}}^w_{ie}\right)\delta\mathbf{v}^w_{eb} + \hat{\mathbf{v}}^w_{eb}\wedge\mathrm{E}\left(\tilde{\boldsymbol{\omega}}^w_{ew} - \boldsymbol{\omega}^w_{ew}\right) \\ & + 2\hat{\mathbf{v}}^w_{eb}\wedge\mathrm{E}\left(\tilde{\boldsymbol{\omega}}^w_{ie} - \boldsymbol{\omega}^w_{ie}\right) - 2\frac{g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)}\hat{\mathbf{u}}^w_D\delta h_b + \hat{\mathbf{C}}^w_b\mathbf{b}_a \end{aligned}, \quad (I.49)$$

where $\hat{\mathbf{u}}^w_D$ is the down unit vector of the wander azimuth frame, which is the same as for its counterpart local navigation frame.

By analogy with the derivation of $\delta\dot{\boldsymbol{\psi}}^i_{ib}$ in Section 14.2.2.1, the time derivative of the wander azimuth position error is given by

$$\mathrm{E}\left(\delta\dot{\boldsymbol{\psi}}^w_{ew}\right) = \mathrm{E}\left(\tilde{\boldsymbol{\omega}}^w_{ew} - \boldsymbol{\omega}^w_{ew}\right), \quad (I.50)$$

where the approximation $\delta\boldsymbol{\psi}^w_{ew}\delta\dot{\boldsymbol{\psi}}^w_{ew} \approx 0$ is made.

As for a local-navigation-frame implementation, the height time derivative is

$$\delta\dot{h}_b \approx -\delta v^w_{eb,z}. \quad (I.51)$$

Substituting (I.40) to (I.51), and (14.34) into (3.26), the system matrix is

$$\mathbf{F}_{INS}^{w} = \begin{pmatrix} \mathbf{F}_{11}^{w} & \mathbf{F}_{12}^{w} & \mathbf{F}_{13}^{w} & \mathbf{f}_{14}^{w} & \mathbf{0}_3 & \hat{\mathbf{C}}_b^n \\ \mathbf{F}_{21}^{w} & \mathbf{F}_{22}^{w} & \mathbf{F}_{23}^{w} & \mathbf{f}_{24}^{w} & \hat{\mathbf{C}}_b^n & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{F}_{32}^{w} & \mathbf{F}_{33}^{w} & \mathbf{f}_{34}^{w} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_{1\times3} & \mathbf{F}_{42}^{w} & \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}, \tag{I.52}$$

where

$$\mathbf{F}_{11}^{w} = -\left[\hat{\boldsymbol{\omega}}_{iw}^{w} \wedge\right], \tag{I.53}$$

$$\mathbf{F}_{12}^{w} = -\mathbf{W}_2, \tag{I.54}$$

$$\mathbf{F}_{13}^{w} = -\left[\omega_{ie}\left[\hat{\mathbf{c}}_{e,:,3}^{w} \wedge\right] + \begin{pmatrix} \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{w}_3 \end{pmatrix}\right], \tag{I.55}$$

$$\mathbf{f}_{14}^{w} = -\mathbf{w}_4, \tag{I.56}$$

$$\mathbf{F}_{21}^{w} = -\left[\left(\hat{\mathbf{C}}_b^w \hat{\mathbf{f}}_{ib}^b\right) \wedge\right], \tag{I.57}$$

$$\mathbf{F}_{22}^{w} = -\left(\hat{\boldsymbol{\Omega}}_{ew}^{w} + 2\hat{\boldsymbol{\Omega}}_{ie}^{w}\right) + \left[\hat{\mathbf{v}}_{eb}^{w} \wedge\right]\mathbf{W}_2, \tag{I.58}$$

$$\mathbf{F}_{23}^{n} = \left[\hat{\mathbf{v}}_{eb}^{w} \wedge\right]\left\{2\omega_{ie}\left[\hat{\mathbf{c}}_{e,:,3}^{w} \wedge\right] + \begin{pmatrix} \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{w}_3 \end{pmatrix}\right\}, \tag{I.59}$$

$$\mathbf{f}_{24}^{w} = -2\frac{g_0(\hat{L}_b)}{r_{eS}^{e}(\hat{L}_b)}\hat{\mathbf{u}}_D^w + \left[\hat{\mathbf{v}}_{eb}^{w} \wedge\right]\mathbf{w}_4, \tag{I.60}$$

$$\mathbf{F}_{32}^{w} = \mathbf{W}_2, \qquad \mathbf{F}_{33}^{w} = \begin{pmatrix} \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{w}_3 \end{pmatrix}, \qquad \mathbf{f}_{34}^{w} = \mathbf{w}_4, \tag{I.61}$$

$$\mathbf{F}_{42}^{w} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}. \tag{I.62}$$

The transition matrix, limited to first order in $\mathbf{F}\tau_s$, is

$$\boldsymbol{\Phi}_{INS}^{w} = \begin{pmatrix} \mathbf{I}_3 + \mathbf{F}_{11}^{w}\tau_s & \mathbf{F}_{12}^{w}\tau_s & \mathbf{F}_{13}^{w}\tau_s & \mathbf{f}_{14}^{w}\tau_s & \mathbf{0}_3 & \hat{\mathbf{C}}_b^n\tau_s \\ \mathbf{F}_{21}^{w}\tau_s & \mathbf{I}_3 + \mathbf{F}_{22}^{w}\tau_s & \mathbf{F}_{23}^{w}\tau_s & \mathbf{f}_{24}^{w}\tau_s & \hat{\mathbf{C}}_b^n\tau_s & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{F}_{32}^{w}\tau_s & \mathbf{I}_3 + \mathbf{F}_{33}^{w}\tau_s & \mathbf{f}_{34}^{w}\tau_s & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_{1\times3} & \mathbf{F}_{42}^{w}\tau_s & \mathbf{0}_{1\times3} & 1 & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix}. \tag{I.63}$$

I.3.3 Loosely Coupled INS/GNSS Measurement Model

GNSS user equipment does not output position and velocity in a wander-azimuth frame, whilst using local-navigation-frame GNSS outputs defeats the purpose of the wander-azimuth implementation. Therefore, the measurements will be resolved in ECEF-frame axes.

ECEF-frame-resolved Cartesian position can be obtained from the third row of $\mathbf{C}_e^w$, which is independent of the wander angle and is not subject to a polar singularity. From (2.112) and (5.60),

$$\hat{\mathbf{r}}_{eb}^{e} = -\begin{pmatrix} \left(R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right)\hat{C}_{e,3,1}^{w} \\ \left(R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right)\hat{C}_{e,3,2}^{w} \\ \left[\left(1-e^2\right)R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right]\hat{C}_{e,3,3}^{w} \end{pmatrix}. \tag{I.64}$$

From (14.102), the measurement innovation is

$$\delta\mathbf{z}_{G,k}^{ew-} = \begin{pmatrix} \hat{\mathbf{r}}_{eaG}^{e} - \hat{\mathbf{r}}_{eb}^{e} - \hat{\mathbf{C}}_{w}^{e}\hat{\mathbf{C}}_{b}^{w}\mathbf{l}_{ba}^{b} \\ \hat{\mathbf{v}}_{eaG}^{e} - \hat{\mathbf{C}}_{w}^{e}\hat{\mathbf{v}}_{eb}^{w} - \hat{\mathbf{C}}_{w}^{e}\hat{\mathbf{C}}_{b}^{w}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right) + \boldsymbol{\Omega}_{ie}^{e}\hat{\mathbf{C}}_{w}^{e}\hat{\mathbf{C}}_{b}^{w}\mathbf{l}_{ba}^{b} \end{pmatrix}_k \tag{I.65}$$

Neglecting the coupling errors of the attitude errors and gyro biases into the measurements through the lever arm, the measurement matrix may be approximated to

$$\mathbf{H}_{G,k}^{ew} \approx \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{r3}^{ew} & \mathbf{h}_{r4}^{ew} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & -\hat{\mathbf{C}}_{w}^{e} & \mathbf{0}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}_k \tag{I.66}$$

where

$$\begin{aligned} \mathbf{H}_{r3}^{ew} &= -\left.\frac{\partial \mathbf{r}_{eb}^{e}}{\partial \boldsymbol{\psi}_{ew}^{w}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_k^-} = -\left.\frac{\partial \mathbf{r}_{eb}^{e}}{\partial \mathbf{C}_{e,3,:}^{w}}\frac{\partial \mathbf{C}_{e,3,:}^{w}}{\partial \boldsymbol{\psi}_{ew}^{w}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_k^-} \\ &= \begin{pmatrix} -\left(R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right)\hat{C}_{e,2,1}^{w} & \left(R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right)\hat{C}_{e,1,1}^{w} & 0 \\ -\left(R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right)\hat{C}_{e,2,2}^{w} & \left(R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right)\hat{C}_{e,1,2}^{w} & 0 \\ -\left[\left(1-e^2\right)R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right]\hat{C}_{e,2,3}^{w} & \left[\left(1-e^2\right)R_E(\hat{C}_{e,3,3}^{w}) + \hat{h}_b\right]\hat{C}_{e,1,3}^{w} & 0 \end{pmatrix}_k, \end{aligned} \tag{I.67}$$

$$\mathbf{h}_{r4}^{ew} = \begin{pmatrix} C_{e,3,1}^{w} \\ C_{e,3,2}^{w} \\ C_{e,3,3}^{w} \end{pmatrix}. \tag{I.68}$$

I.3.4 Tightly Coupled INS/GNSS Measurement Model

The ECEF-frame antenna position and velocity may be obtained from the wander-azimuth-frame inertial navigation solution using

$$\begin{aligned} \hat{\mathbf{r}}_{ea}^{e} &= \hat{\mathbf{r}}_{eb}^{e} + \hat{\mathbf{C}}_{w}^{e}\hat{\mathbf{C}}_{b}^{w}\mathbf{l}_{ba}^{b} \\ \hat{\mathbf{v}}_{ea}^{e} &= \hat{\mathbf{C}}_{w}^{e}\hat{\mathbf{v}}_{eb}^{w} + \hat{\mathbf{C}}_{w}^{e}\hat{\mathbf{C}}_{b}^{w}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right) + \boldsymbol{\Omega}_{ie}^{e}\hat{\mathbf{C}}_{w}^{e}\hat{\mathbf{C}}_{b}^{w}\mathbf{l}_{ba}^{b} \end{aligned}, \tag{I.69}$$

where $\hat{\mathbf{r}}_{eb}^{e}$ is as given by (I.64). The measurement innovations are then calculated using (14.119), where the estimated pseudo-range and pseudo-range-rate for the j^{th} satellite, $\hat{\rho}_C^j$ and $\hat{\dot{\rho}}_C^j$, are given by (9.164) or (9.165).

The state vector for tightly coupled integration is defined as

$$\mathbf{x}^{w} = \begin{pmatrix} \mathbf{x}_{INS}^{w} \\ \delta\rho_c^a \\ \delta\dot{\rho}_c^a \end{pmatrix}, \tag{I.70}$$

where $\mathbf{x}_{INS}^{w}$ is as defined by (I.35), $\delta\rho_c^a$ is the receiver clock offset, and $\delta\dot{\rho}_c^a$ is the clock drift. Neglecting the coupling of the attitude errors and gyro biases into the measurements through the lever arm, the measurement matrix may be approximated to

$$\mathbf{H}_{G,k}^{ew} \approx \left.\left(\begin{array}{cccccccc} \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & -\mathbf{u}_{a1}^{e\ \mathrm{T}}\mathbf{H}_{r3}^{ew} & -\mathbf{u}_{a1}^{e\ \mathrm{T}}\mathbf{h}_{r4}^{ew} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 0 \\ \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & -\mathbf{u}_{a2}^{e\ \mathrm{T}}\mathbf{H}_{r3}^{ew} & -\mathbf{u}_{a2}^{e\ \mathrm{T}}\mathbf{h}_{r4}^{ew} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & -\mathbf{u}_{am}^{e\ \mathrm{T}}\mathbf{H}_{r3}^{ew} & -\mathbf{u}_{am}^{e\ \mathrm{T}}\mathbf{h}_{r4}^{ew} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 0 \\ \hline \mathbf{0}_{1,3} & \mathbf{u}_{a1}^{e\ \mathrm{T}}\hat{\mathbf{C}}_{w}^{e} & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \\ \mathbf{0}_{1,3} & \mathbf{u}_{a2}^{e\ \mathrm{T}}\hat{\mathbf{C}}_{w}^{e} & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{1,3} & \mathbf{u}_{am}^{e\ \mathrm{T}}\hat{\mathbf{C}}_{w}^{e} & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \end{array}\right)\right|_{\mathbf{x}=\hat{\mathbf{x}}_k}, \qquad \text{(I.71)}$$

where $\mathbf{u}_{a1}^{e}$ is the line-of-sight unit vector from the user antenna to the j^{th} satellite resolved about an ECEF frame, and $\mathbf{H}_{r3}^{ew}$ and $\mathbf{h}_{r4}^{ew}$ are as defined by (I.67) and (I.68), respectively.

I.3.5 Transfer Alignment

In the transfer alignment algorithms presented in Section 15.1, position errors are not estimated as states. Therefore, transfer alignment may be implemented with a wander-azimuth-resolved attitude and velocity error using (15.3), (15.8) and (15.11) with n replaced by w; (15.4), (15.5), (15.9), (15.13) and (15.15–15.17) with $\{\beta,\gamma\} = \{e,w\}$; and (15.10) and (15.14) unchanged.

I.3.6 Quasi-Stationary Fine Alignment

Where the navigation system is stationary, both the ECEF frame to wander-azimuth frame coordinate transformation matrix, $\mathbf{C}_{e}^{w}$, and the height are constant. The position displacement measurement innovation can thus be expressed as

$$\delta\mathbf{z}_{Q,k}^{w-} = \begin{pmatrix} \delta\mathbf{z}_{Q1:3,k}^{w-} \\ \hat{h}_b(t_0) - \hat{h}_b(t) \end{pmatrix}, \qquad \mathbf{I}_3 + \left[\delta\mathbf{z}_{Q1:3,k}^{w-} \wedge\right] = \hat{\mathbf{C}}_{e}^{w}(t)\hat{\mathbf{C}}_{w}^{e}(t_0). \qquad \text{(I.72)}$$

Assuming the state vector defined by (I.35), the measurement matrix is

$$\mathbf{H}_{Q,k}^{w} = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & -1 & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \end{pmatrix}. \qquad \text{(I.73)}$$

I.3.7 Zero Updates and Motion Constraints

Where zero updates and/or motion constraints are implemented with a wander-azimuth-frame Kalman filter or EKF, only minor modifications to the algorithms presented in Sections 15.3 and 15.4.1 are required. For ZVUs, the measurement innovation, (15.31), is simply resolved about a wander-azimuth frame. Assuming the state vector defined by (I.35), the measurement matrix is

$$\mathbf{H}_{ZV,k}^{w} = \begin{pmatrix} \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}. \qquad \text{(I.74)}$$

For ZARUs, the measurement innovation, (15.35), is unchanged and the measurement matrix is

$$\mathbf{H}_{ZA,k}^{w} = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_3 & -\mathbf{I}_3 \end{pmatrix}. \qquad \text{(I.75)}$$

For vehicle velocity motion constraints, (15.37–39) are implemented with $\{\beta,\gamma\} = \{e,w\}$ and the measurement matrix is

$$\mathbf{H}^{w}_{VC,k} \approx \begin{pmatrix} \mathbf{0}_{2\times3} & -\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hat{\mathbf{C}}^{b}_{w} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} \end{pmatrix}_{k} . \tag{I.76}$$

I.3.8 Doppler Radar and Sonar Measurement Model

Section 16.2.5 describes the integration of Doppler radar and sonar measurements as an aiding sensor. This is also applicable to 3D visual odometry. For a wander azimuth implementation, if the state vector is defined as

$$\mathbf{x}^{w} = \begin{pmatrix} \delta\boldsymbol{\psi}^{w}_{wb} \\ \delta\mathbf{v}^{w}_{eb} \\ \vdots \\ s_{D} \\ \mathbf{v}^{w}_{es} \end{pmatrix}, \tag{I.77}$$

where s_D is the scale-factor error and $\mathbf{v}^{w}_{es}$ is the surface velocity, the measurement matrix is

$$\mathbf{H}^{w}_{D,k} = \left(-\hat{\mathbf{C}}^{b}_{w}\left[\left(\hat{\mathbf{v}}^{w}_{eb} - \hat{\mathbf{v}}^{w}_{es}\right)\wedge\right] \quad -\hat{\mathbf{C}}^{b}_{w} \quad \mathbf{0} \quad \hat{\mathbf{C}}^{b}_{w}\hat{\mathbf{v}}^{w}_{eb} \quad \hat{\mathbf{C}}^{b}_{w} \right)_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}^{\gamma -}_{k}} . \tag{I.78}$$

I.4 Local Tangent-Plane Implementation

This section describes INS/GNSS integration implemented with the position, velocity, and attitude errors resolved about a local tangent-plane frame. This is suited to navigation within a localized area, particularly where INS and GNSS are also integrated with a short-range position-fixing system. Note that the local-tangent-plane-frame inertial navigation equations are presented in Section E.7 of Appendix E, also on CD.

Section I.4.1 defines the state vector and describes a typical INS system model. Sections I.4.2 and I.4.3 then present measurement models for loosely coupled and tightly coupled INS/GNSS integration, respectively. Note that local-tangent-plane system and measurement models for a number of dead-reckoning, attitude measurement, terrestrial radio navigation, and environmental feature-matching systems are included in Chapter 16.

As for the examples presented in Sections 14.2 and 14.3, it is assumed that the inertial error states estimated by the Kalman filter (or EKF) are attitude error, velocity error, position error, accelerometer biases, and gyro biases.

I.4.1 State Definitions and INS System Model

For a local-tangent-plane-frame implementation of INS/GNSS integration, the INS error state vector is defined as

$$\mathbf{x}^{l}_{INS} = \begin{pmatrix} \delta\boldsymbol{\psi}^{l}_{lb} \\ \delta\mathbf{v}^{l}_{lb} \\ \delta\mathbf{r}^{l}_{lb} \\ \mathbf{b}_{a} \\ \mathbf{b}_{g} \end{pmatrix}, \tag{I.79}$$

where the attitude, velocity and position errors are as defined in Section 5.7 and corrected as described in Section 14.1.1 with $\alpha = b$, $\beta = l$, and $\psi = l$.

As the local tangent-plane frame is Earth-fixed, the system model is derived in a similar way to the ECEF frame implementation, described in Section 14.2.3. Therefore, the system matrix may be written as

$$\mathbf{F}_{INS}^{l} = \begin{pmatrix} -\boldsymbol{\Omega}_{il}^{l} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \hat{\mathbf{C}}_b^l \\ \mathbf{F}_{21}^{l} & -2\boldsymbol{\Omega}_{il}^{l} & \mathbf{F}_{23}^{l} & \hat{\mathbf{C}}_b^l & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}, \tag{I.80}$$

where $\boldsymbol{\Omega}_{il}^{l}$ is given by (E.47) and

$$\mathbf{F}_{21}^{l} = \left[-\left(\hat{\mathbf{C}}_b^l \hat{\mathbf{f}}_{ib}^b\right)\wedge\right], \qquad \mathbf{F}_{23}^{l} = -\hat{\mathbf{C}}_b^l \frac{2\hat{\boldsymbol{\gamma}}_{ib}^e}{r_{eS}^e(\hat{L}_b)} \frac{\hat{\mathbf{r}}_{eb}^{e\,\mathrm{T}}}{\left|\hat{\mathbf{r}}_{eb}^e\right|} \hat{\mathbf{C}}_l^b, \tag{I.81}$$

where $\mathbf{r}_{eb}^e$ is obtained from $\mathbf{r}_{lb}^l$ using (2.160). Note that $\mathbf{C}_e^l$ is constant because both frames are Earth fixed.

The transition matrix, limited to first order in $\mathbf{F}\tau_s$, is

$$\boldsymbol{\Phi}_{INS}^{l} \approx \begin{bmatrix} \mathbf{I}_3 - \boldsymbol{\Omega}_{il}^{l}\tau_s & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \hat{\mathbf{C}}_b^l\tau_s \\ \mathbf{F}_{21}^{l}\tau_s & \mathbf{I}_3 - 2\boldsymbol{\Omega}_{il}^{l}\tau_s & \mathbf{F}_{23}^{l}\tau_s & \hat{\mathbf{C}}_b^l\tau_s & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3\tau_s & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}. \tag{I.82}$$

I.4.2 Loosely Coupled GNSS Measurement Model

The loosely coupled GNSS measurement innovation in the local-tangent-plane implementation is

$$\delta\mathbf{z}_{G,k}^{l-} = \begin{pmatrix} \hat{\mathbf{r}}_{laG}^{l} - \hat{\mathbf{r}}_{lb}^{l} - \hat{\mathbf{C}}_b^l \mathbf{l}_{ba}^b \\ \hat{\mathbf{v}}_{laG}^{l} - \hat{\mathbf{v}}_{lb}^{l} - \hat{\mathbf{C}}_b^l\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right) + \boldsymbol{\Omega}_{il}^l \hat{\mathbf{C}}_b^l \mathbf{l}_{ba}^b \end{pmatrix}_k^{-} \tag{I.83}$$

where, from (2.159) and (2.160), the GNSS position and velocity are converted to a local-tangent-plane representation using

$$\begin{aligned} \hat{\mathbf{r}}_{laG}^{l} &= \mathbf{C}_e^l\left(\hat{\mathbf{r}}_{eaG}^{e} - \mathbf{r}_{el}^{e}\right) \\ \hat{\mathbf{v}}_{laG}^{l} &= \mathbf{C}_e^l \hat{\mathbf{v}}_{eaG}^{e} \end{aligned}, \tag{I.84}$$

Following the measurement model derivation in Section 14.3.1, the measurement matrix is

$$\mathbf{H}_{G,k}^{l} = \begin{pmatrix} \mathbf{H}_{r1}^{l} & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{H}_{v1}^{l} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{v5}^{l} \end{pmatrix}_k, \tag{I.85}$$

where

$$\begin{aligned} \mathbf{H}_{r1}^{l} &\approx \left[\left(\hat{\mathbf{C}}_b^l \mathbf{l}_{ba}^b\right)\wedge\right] \\ \mathbf{H}_{v1}^{l} &\approx \left[\left\{\hat{\mathbf{C}}_b^l\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right) - \boldsymbol{\Omega}_{il}^l \hat{\mathbf{C}}_b^l \mathbf{l}_{ba}^b\right\}\wedge\right] \\ \mathbf{H}_{v5}^{l} &\approx \hat{\mathbf{C}}_b^l\left[\mathbf{l}_{ba}^b \wedge\right] \end{aligned}. \tag{I.86}$$

Where coupling of the attitude errors and gyro biases into the measurements through the lever arm terms is neglected, the measurement matrix simplifies to

$$\mathbf{H}_{G,k}^{l} \approx \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}_k . \tag{I.87}$$

I.4.3 Tightly Coupled GNSS Measurement Model

For tightly coupled integration, the measurement innovations are as given by (14.119). It is recommended that the predicted pseudo-ranges and pseudo-range rates are calculated using (9.164) or (9.165) where the estimated antenna position and velocity are given by

$$\begin{aligned} \hat{\mathbf{r}}_{ea}^{e} &= \mathbf{r}_{el}^{e} + \mathbf{C}_{l}^{e}\left(\hat{\mathbf{r}}_{lb}^{l} + \hat{\mathbf{C}}_{b}^{l}\mathbf{l}_{ba}^{b}\right) \\ \hat{\mathbf{v}}_{ea}^{e} &= \mathbf{C}_{l}^{e}\left[\hat{\mathbf{v}}_{lb}^{l} + \hat{\mathbf{C}}_{b}^{l}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] + \boldsymbol{\Omega}_{ie}^{e}\mathbf{C}_{l}^{e}\hat{\mathbf{C}}_{b}^{l}\mathbf{l}_{ba}^{b} \end{aligned} . \tag{I.88}$$

The approximate measurement matrix, neglecting the dependence of the measurement innovations on the attitude error and of the pseudo-range-rate measurements on the position and gyro errors, is

$$\mathbf{H}_{G,k}^{l} \approx \left(\begin{array}{cccccccc} \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{a1}^{l\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 \\ \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{a2}^{l\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{am}^{l\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 \\ \hdashline \mathbf{0}_{1,3} & \mathbf{u}_{a1}^{l\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \\ \mathbf{0}_{1,3} & \mathbf{u}_{a2}^{l\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{1,3} & \mathbf{u}_{am}^{l\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 \end{array}\right)_{\mathbf{x}=\hat{\mathbf{x}}_k} , \tag{I.89}$$

where, the lines of sight may be calculated in the ECEF frame using $\hat{\mathbf{r}}_{ea}^{e}$ and $\hat{\mathbf{v}}_{ea}^{e}$, and transformed using

$$\mathbf{u}_{aj}^{l} = \mathbf{C}_{e}^{l}\mathbf{u}_{aj}^{e} . \tag{I.90}$$

I.5 Body-Frame-Resolved Attitude Error

This section describes alternative implementations of INS/GNSS and multisensor integration and transfer alignment, in which the attitude error is resolved about the body-frame axes. Section I.5.1 defines the error states for this implementation, including how the inertial attitude solution is corrected. Section I.5.2 describes a typical INS system model and discusses AHRS and odometry system models. Section I.5.3 discusses INS/GNSS measurement models. These may be applied to the integration of GNSS with dead-reckoning sensors by replacing the bias states with the appropriate instrument error states, for which the measurement matrix columns are zero. As for the examples presented in Sections 14.2 and 14.3, it is assumed that the inertial error states estimated by the integration Kalman filter (or EKF) are attitude error, velocity error, position error, accelerometer biases and gyro biases. Section I.5.4 discusses transfer alignment. Finally Section I.5.5 describes dead-reckoning measurement models for aiding-sensor and total-state integration. Note that the quasi-stationary alignment, zero update, and motion constraint measurement models are unaffected by the resolving axes of the attitude error states, as are the terrestrial radio navigation measurement models presented in Section 16.3.

By analogy with (2.58), the heading error may be estimated from the body-frame-resolved attitude error using

$$\delta\psi_{nb} \approx \begin{pmatrix} 0 & \dfrac{\sin\hat{\phi}_{nb}}{\cos\hat{\theta}_{nb}} & \dfrac{\cos\hat{\phi}_{nb}}{\cos\hat{\theta}_{nb}} \end{pmatrix} \delta\hat{\boldsymbol{\psi}}^{b}_{\gamma b}, \qquad \gamma \in i, e, n, l\,. \tag{I.91}$$

I.5.1 State Definitions

From (5.111), where the small-angle approximation is assumed, the body-frame-resolved attitude error is defined as

$$\left[\delta\boldsymbol{\psi}^{b}_{\gamma b}\wedge\right] \approx \mathbf{I}_3 - \tilde{\mathbf{C}}^{b}_{\gamma}\mathbf{C}^{\gamma}_{b} \qquad \gamma \in i, e, n, l\,. \tag{I.92}$$

The INS error state vectors defined by (14.15), (14.39), (14.51), and (I.79) are modified as follows for estimation of the attitude error resolved about the body frame

$$\mathbf{x}^{ib}_{INS} = \begin{pmatrix} \delta\boldsymbol{\psi}^{b}_{ib} \\ \delta\mathbf{v}^{i}_{ib} \\ \delta\mathbf{r}^{i}_{ib} \\ \mathbf{b}_a \\ \mathbf{b}_g \end{pmatrix}, \quad \mathbf{x}^{eb}_{INS} = \begin{pmatrix} \delta\boldsymbol{\psi}^{b}_{eb} \\ \delta\mathbf{v}^{e}_{eb} \\ \delta\mathbf{r}^{e}_{eb} \\ \mathbf{b}_a \\ \mathbf{b}_g \end{pmatrix}, \quad \mathbf{x}^{nb}_{INS} = \begin{pmatrix} \delta\boldsymbol{\psi}^{b}_{nb} \\ \delta\mathbf{v}^{n}_{eb} \\ \delta\mathbf{p}_b \\ \mathbf{b}_a \\ \mathbf{b}_g \end{pmatrix}, \quad \mathbf{x}^{lb}_{INS} = \begin{pmatrix} \delta\boldsymbol{\psi}^{b}_{lb} \\ \delta\mathbf{v}^{l}_{lb} \\ \delta\mathbf{r}^{l}_{lb} \\ \mathbf{b}_a \\ \mathbf{b}_g \end{pmatrix}. \tag{I.93}$$

The inertial attitude solution is corrected using

$$\hat{\mathbf{C}}^{\gamma}_{b} \approx \tilde{\mathbf{C}}^{\gamma}_{b}\left(\mathbf{I}_3 - \left[\delta\hat{\boldsymbol{\psi}}^{b}_{\gamma b}\wedge\right]\right) \qquad \gamma \in i, e, n, l \tag{I.94}$$

for open-loop correction and

$$\hat{\mathbf{C}}^{\gamma}_{b}(+) \approx \hat{\mathbf{C}}^{\gamma}_{b}(-)\left(\mathbf{I}_3 - \left[\delta\hat{\boldsymbol{\psi}}^{b}_{\gamma b}\wedge\right]\right) \qquad \gamma \in i, e, n, l \tag{I.95}$$

for closed-loop correction.

The attitude error, velocity error, and position error states in (I.93) are also applicable to the integration of odometry, PDR using step detection, and Doppler radar/sonar as the reference navigation system. The remaining states are replaced by scale factor error states for odometry and Doppler radar/sonar, and by model coefficient and boresight error states for PDR, as descried in Section 16.2.

I.5.2 System Models

From (5.111),

$$\tilde{\mathbf{C}}^{\gamma}_{b}\mathbf{C}^{b}_{\gamma} \approx \left[\delta\boldsymbol{\psi}^{\gamma}_{\gamma b}\wedge\right] + \mathbf{I}_3\,. \tag{I.96}$$

From (2.17)

$$\tilde{\mathbf{C}}^{b}_{\gamma}\mathbf{C}^{\gamma}_{b} = \mathbf{C}^{b}_{\gamma}\left(\tilde{\mathbf{C}}^{\gamma}_{b}\mathbf{C}^{b}_{\gamma}\right)^{\mathrm{T}}\mathbf{C}^{\gamma}_{b}\,. \tag{I.97}$$

Substituting in (I.92) and (I.96),

$$\begin{aligned} \mathbf{I}_3 - \left[\delta\boldsymbol{\psi}^{b}_{\gamma b}\wedge\right] &\approx \mathbf{C}^{b}_{\gamma}\left(\mathbf{I}_3 + \left[\delta\boldsymbol{\psi}^{\gamma}_{\gamma b}\wedge\right]\right)^{\mathrm{T}}\mathbf{C}^{\gamma}_{b} \\ &\approx \mathbf{C}^{b}_{\gamma}\left(\mathbf{I}_3 - \left[\delta\boldsymbol{\psi}^{\gamma}_{\gamma b}\wedge\right]\right)\mathbf{C}^{\gamma}_{b} \\ \Rightarrow & \\ \left[\delta\boldsymbol{\psi}^{b}_{\gamma b}\wedge\right] &\approx \mathbf{C}^{b}_{\gamma}\left[\delta\boldsymbol{\psi}^{\gamma}_{\gamma b}\wedge\right]\mathbf{C}^{\gamma}_{b} \end{aligned}. \tag{I.98}$$

Taking the components of the skew-symmetric matrices and rearranging,

$$\begin{aligned} \delta\boldsymbol{\psi}^{b}_{\gamma b} &\approx \mathbf{C}^{b}_{\gamma}\delta\boldsymbol{\psi}^{\gamma}_{\gamma b} \\ \delta\boldsymbol{\psi}^{\gamma}_{\gamma b} &\approx \mathbf{C}^{\gamma}_{b}\delta\boldsymbol{\psi}^{b}_{\gamma b} \end{aligned}. \tag{I.99}$$

Differentiating with respect to time,

$$\delta\dot{\boldsymbol{\psi}}_{\gamma b}^{b} \approx \mathbf{C}_{\gamma}^{b}\delta\dot{\boldsymbol{\psi}}_{\gamma b}^{\gamma} + \dot{\mathbf{C}}_{\gamma}^{b}\delta\boldsymbol{\psi}_{\gamma b}^{\gamma} . \quad (I.100)$$

Applying (2.56) and then (I.99),

$$\begin{aligned} \delta\dot{\boldsymbol{\psi}}_{\gamma b}^{b} &\approx \mathbf{C}_{\gamma}^{b}\delta\dot{\boldsymbol{\psi}}_{\gamma b}^{\gamma} - \boldsymbol{\Omega}_{\gamma b}^{b}\mathbf{C}_{\gamma}^{b}\delta\boldsymbol{\psi}_{\gamma b}^{\gamma} \\ &= \mathbf{C}_{\gamma}^{b}\delta\dot{\boldsymbol{\psi}}_{\gamma b}^{\gamma} - \boldsymbol{\Omega}_{\gamma b}^{b}\delta\boldsymbol{\psi}_{\gamma b}^{b} \end{aligned} . \quad (I.101)$$

Using (I.99) and (I.101), the INS system matrices may be expressed in terms of their counterparts in Section 14.2. Thus,

$$\mathbf{F}_{INS}^{\gamma b} = \begin{pmatrix} \hat{\mathbf{C}}_{\gamma}^{b} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix} \mathbf{F}_{INS}^{\gamma} \begin{pmatrix} \hat{\mathbf{C}}_{b}^{\gamma} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{pmatrix} - \begin{pmatrix} \hat{\boldsymbol{\Omega}}_{\gamma b}^{b} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix} . \quad \gamma \in i,e,n,l \quad (I.102)$$

Applying this to the system matrix for an ECI-frame implementation, given by (14.35),

$$\mathbf{F}_{INS}^{ib} = \begin{pmatrix} -\hat{\boldsymbol{\Omega}}_{ib}^{b} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{F}_{21}^{ib} & \mathbf{0}_3 & \mathbf{F}_{23}^{i} & \hat{\mathbf{C}}_{b}^{i} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix} , \quad (I.103)$$

where

$$\mathbf{F}_{21}^{ib} = -\hat{\mathbf{C}}_{b}^{i}\left[\hat{\mathbf{f}}_{ib}^{b} \wedge\right] . \quad (I.104)$$

and $\mathbf{F}_{23}^{i}$ is given by (14.36). The transition matrix, limited to first order in $\mathbf{F}\tau_s$, is

$$\boldsymbol{\Phi}_{INS}^{ib} \approx \begin{bmatrix} \mathbf{I}_3 - \hat{\boldsymbol{\Omega}}_{ib}^{b}\tau_s & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\tau_s \\ \mathbf{F}_{21}^{ib}\tau_s & \mathbf{I}_3 & \mathbf{F}_{23}^{i}\tau_s & \hat{\mathbf{C}}_{b}^{i}\tau_s & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3\tau_s & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} . \quad (I.105)$$

Applying (I.102) to the system matrix for an ECEF-frame implementation, given by (14.48),

$$\mathbf{F}_{INS}^{eb} = \begin{pmatrix} -\hat{\boldsymbol{\Omega}}_{ib}^{b} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{F}_{21}^{eb} & -2\boldsymbol{\Omega}_{ie}^{e} & \mathbf{F}_{23}^{e} & \hat{\mathbf{C}}_{b}^{e} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix} , \quad (I.106)$$

where

$$\mathbf{F}_{21}^{eb} = -\hat{\mathbf{C}}_{b}^{e}\left[\hat{\mathbf{f}}_{ib}^{b} \wedge\right] . \quad (I.107)$$

and $\mathbf{F}_{23}^{e}$ is given by (14.49). The transition matrix, limited to first order in $\mathbf{F}\tau_s$, is

$$\mathbf{\Phi}_{INS}^{eb} \approx \begin{bmatrix} \mathbf{I}_3 - \hat{\mathbf{\Omega}}_{ib}^{b}\tau_s & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\tau_s \\ \mathbf{F}_{21}^{eb}\tau_s & \mathbf{I}_3 - 2\mathbf{\Omega}_{ie}^{e}\tau_s & \mathbf{F}_{23}^{e}\tau_s & \hat{\mathbf{C}}_b^{e}\tau_s & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3\tau_s & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}. \tag{I.108}$$

Applying (I.102) to the system matrix for a local-navigation-frame implementation, given by (14.63),

$$\mathbf{F}_{INS}^{nb} = \begin{pmatrix} -\hat{\mathbf{\Omega}}_{ib}^{b} & \mathbf{F}_{12}^{nb} & \mathbf{F}_{13}^{nb} & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{F}_{21}^{nb} & \mathbf{F}_{22}^{n} & \mathbf{F}_{23}^{n} & \hat{\mathbf{C}}_b^{n} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{F}_{32}^{n} & \mathbf{F}_{33}^{n} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}, \tag{I.109}$$

where

$$\mathbf{F}_{12}^{nb} = \hat{\mathbf{C}}_n^{b} \begin{bmatrix} 0 & \dfrac{-1}{R_E(\hat{L}_b) + \hat{h}_b} & 0 \\ \dfrac{1}{R_N(\hat{L}_b) + \hat{h}_b} & 0 & 0 \\ 0 & \dfrac{\tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} & 0 \end{bmatrix}, \tag{I.110}$$

$$\mathbf{F}_{13}^{n} = \hat{\mathbf{C}}_n^{b} \begin{bmatrix} \omega_{ie} \sin \hat{L}_b & 0 & \dfrac{\hat{v}_{eb,E}^{n}}{\left(R_E(\hat{L}_b) + \hat{h}_b\right)^2} \\ 0 & 0 & \dfrac{-\hat{v}_{eb,N}^{n}}{\left(R_N(\hat{L}_b) + \hat{h}_b\right)^2} \\ \omega_{ie} \cos \hat{L}_b + \dfrac{\hat{v}_{eb,E}^{n}}{\left(R_E(\hat{L}_b) + \hat{h}_b\right) \cos^2 \hat{L}_b} & 0 & \dfrac{-\hat{v}_{eb,E}^{n} \tan \hat{L}_b}{\left(R_E(\hat{L}_b) + \hat{h}_b\right)^2} \end{bmatrix}, \tag{I.111}$$

$$\mathbf{F}_{21}^{nb} = -\hat{\mathbf{C}}_b^{n}\left[\hat{\mathbf{f}}_{ib}^{b} \wedge\right], \tag{I.112}$$

and $\mathbf{F}_{22}^{n}$, $\mathbf{F}_{23}^{n}$, $\mathbf{F}_{32}^{n}$, and $\mathbf{F}_{33}^{n}$ are given by (14.68–71). The transition matrix, limited to first order in $\mathbf{F}\tau_s$, is

$$\mathbf{\Phi}_{INS}^{nb} \approx \begin{bmatrix} \mathbf{I}_3 - \hat{\mathbf{\Omega}}_{ib}^{b}\tau_s & \mathbf{F}_{12}^{nb}\tau_s & \mathbf{F}_{13}^{nb}\tau_s & \mathbf{0}_3 & \mathbf{I}_3\tau_s \\ \mathbf{F}_{21}^{nb}\tau_s & \mathbf{I}_3 + \mathbf{F}_{22}^{n}\tau_s & \mathbf{F}_{23}^{n}\tau_s & \hat{\mathbf{C}}_b^{n}\tau_s & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{F}_{32}^{n}\tau_s & \mathbf{F}_{33}^{n}\tau_s & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}. \tag{I.113}$$

Applying (I.102) to the system matrix for an local-tangent-plane-frame implementation, given by (I.80),

$$\mathbf{F}_{INS}^{lb} = \begin{pmatrix} -\hat{\mathbf{\Omega}}_{ib}^{b} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{F}_{21}^{lb} & -2\mathbf{\Omega}_{il}^{l} & \mathbf{F}_{23}^{l} & \hat{\mathbf{C}}_{b}^{l} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix}, \tag{I.114}$$

where

$$\mathbf{F}_{21}^{lb} = -\hat{\mathbf{C}}_{b}^{l}\left[\hat{\mathbf{f}}_{ib}^{b} \wedge\right]. \tag{I.115}$$

and $\mathbf{F}_{23}^{l}$ is given by (I.81). The transition matrix, limited to first order in $\mathbf{F}\tau_s$, is

$$\mathbf{\Phi}_{INS}^{lb} \approx \begin{bmatrix} \mathbf{I}_3 - \hat{\mathbf{\Omega}}_{ib}^{b}\tau_s & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\tau_s \\ \mathbf{F}_{21}^{lb}\tau_s & \mathbf{I}_3 - 2\mathbf{\Omega}_{il}^{l}\tau_s & \mathbf{F}_{23}^{l}\tau_s & \hat{\mathbf{C}}_{b}^{l}\tau_s & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3\tau_s & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}. \tag{I.116}$$

The preceding INS system model is also applicable to an AHRS with the position and velocity states omitted and magnetometer error states optionally added, noting that these do not affect attitude error and bias state propagation.

The odometry attitude error propagation equation becomes

$$\delta\dot{\boldsymbol{\psi}}_{\gamma b}^{b} \approx \mathbf{C}_{n1:3,3}^{b}\left(\dot{\psi}_{nb}s_{or} + \frac{v_{er}}{T_r}s_{\Delta or}\right), \qquad \gamma \in i,e,n, \tag{I.117}$$

I.5.3 GNSS Measurement Models

The attitude error resolving axes to do not affect the measurement innovations for INS/GNSS integration. They also do not affect the measurement matrix where the coupling of the attitude errors and gyro biases into the measurements through the lever arm is neglected. In Section 14.3, measurement matrices that account for this coupling are only presented for loosely coupled integration, so the same policy is adopted here.

The ECI-frame loosely-coupled measurement matrix is

$$\mathbf{H}_{G,k}^{ib} = \begin{pmatrix} \mathbf{H}_{r1}^{ib} & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{H}_{v1}^{ib} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{v5}^{i} \end{pmatrix}_k, \tag{I.118}$$

where

$$\begin{aligned} \mathbf{H}_{r1}^{ib} &\approx \hat{\mathbf{C}}_{b}^{i}\left[\mathbf{l}_{ba}^{b} \wedge\right] \\ \mathbf{H}_{v1}^{ib} &\approx \hat{\mathbf{C}}_{b}^{i}\left[\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\wedge\right], \\ \mathbf{H}_{v5}^{i} &\approx \hat{\mathbf{C}}_{b}^{i}\left[\mathbf{l}_{ba}^{b} \wedge\right] \end{aligned} \tag{I.119}$$

The ECEF-frame loosely-coupled measurement matrix is

$$\mathbf{H}_{G,k}^{eb} = \begin{pmatrix} \mathbf{H}_{r1}^{eb} & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{H}_{v1}^{eb} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{v5}^{e} \end{pmatrix}_k, \tag{I.120}$$

where

$$\begin{aligned}\mathbf{H}_{r1}^{eb} &\approx \hat{\mathbf{C}}_b^e\left[\mathbf{l}_{ba}^b \wedge\right]\\ \mathbf{H}_{v1}^{eb} &\approx \hat{\mathbf{C}}_b^e\left[\left\{\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right) - \hat{\mathbf{C}}_e^b \boldsymbol{\Omega}_{ie}^e \hat{\mathbf{C}}_b^e \mathbf{l}_{ba}^b\right\}\wedge\right].\\ \mathbf{H}_{v5}^{e} &\approx \hat{\mathbf{C}}_b^e\left[\mathbf{l}_{ba}^b \wedge\right]\end{aligned} \tag{I.121}$$

The local-navigation-frame loosely-coupled measurement matrix is

$$\mathbf{H}_{G,k}^{nb} = \begin{pmatrix} \mathbf{H}_{r1}^{nb} & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{H}_{v1}^{nb} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{v5}^{n} \end{pmatrix}_k, \tag{I.122}$$

where

$$\begin{aligned}\mathbf{H}_{r1}^{nb} &\approx \hat{\mathbf{T}}_{r(n)}^{p} \hat{\mathbf{C}}_b^n\left[\mathbf{l}_{ba}^b \wedge\right]\\ \mathbf{H}_{v1}^{nb} &\approx \hat{\mathbf{C}}_b^n\left[\left\{\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right) - \hat{\mathbf{C}}_n^b \hat{\boldsymbol{\Omega}}_{ie}^n \hat{\mathbf{C}}_b^n \mathbf{l}_{ba}^b\right\}\wedge\right].\\ \mathbf{H}_{v5}^{n} &\approx \hat{\mathbf{C}}_b^n\left[\mathbf{l}_{ba}^b \wedge\right]\end{aligned} \tag{I.123}$$

Finally, the local-tangent-plane-frame loosely-coupled measurement matrix is

$$\mathbf{H}_{G,k}^{lb} = \begin{pmatrix} \mathbf{H}_{r1}^{lb} & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{H}_{v1}^{lb} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{v5}^{l} \end{pmatrix}_k, \tag{I.124}$$

where

$$\begin{aligned}\mathbf{H}_{r1}^{lb} &\approx \hat{\mathbf{C}}_b^l\left[\mathbf{l}_{ba}^b \wedge\right]\\ \mathbf{H}_{v1}^{lb} &\approx \hat{\mathbf{C}}_b^l\left[\left\{\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right) - \hat{\mathbf{C}}_l^b \boldsymbol{\Omega}_{il}^l \hat{\mathbf{C}}_b^l \mathbf{l}_{ba}^b\right\}\wedge\right].\\ \mathbf{H}_{v5}^{l} &\approx \hat{\mathbf{C}}_b^l\left[\mathbf{l}_{ba}^b \wedge\right]\end{aligned} \tag{I.125}$$

I.5.4 Transfer Alignment

Where the attitude error is resolved about the body-frame axes, the transfer alignment measurement innovations presented in Section 15.1 are unchanged. For velocity and heading measurement matching, the state vector is defined as

$$\mathbf{x}^{\gamma b} = \begin{pmatrix} \delta\boldsymbol{\psi}_{\gamma b}^{b} \\ \delta\mathbf{v}_{\beta b}^{\gamma} \\ \mathbf{b}_a \\ \mathbf{b}_g \\ \vdots \end{pmatrix} \quad \{\beta,\gamma\} \in \{i,i\},\{e,e\},\{e,n\}. \tag{I.126}$$

The velocity-matching measurement matrix is

$$\mathbf{H}_{V,k}^{\gamma b} = \begin{pmatrix} \mathbf{H}_{v1}^{\gamma b} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{H}_{v5}^{\gamma b} & \mathbf{0} \end{pmatrix}_k \quad \gamma \in i,e,n, \tag{I.127}$$

where

$$\begin{aligned}\mathbf{H}_{v1}^{ib} &\approx -\hat{\mathbf{C}}_b^i\left[\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{rb}^b\right)\wedge\right],\\ \mathbf{H}_{v5}^{ib} &\approx -\hat{\mathbf{C}}_b^i\left[\mathbf{l}_{rb}^b \wedge\right]\end{aligned} \tag{I.128}$$

$$\begin{aligned}\mathbf{H}_{v1}^{eb} &\approx -\hat{\mathbf{C}}_b^e\left[\left\{\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{rb}^b\right) - \hat{\mathbf{C}}_e^b \boldsymbol{\Omega}_{ie}^e \hat{\mathbf{C}}_b^e \mathbf{l}_{rb}^b\right\}\wedge\right].\\ \mathbf{H}_{v5}^{eb} &\approx -\hat{\mathbf{C}}_b^e\left[\mathbf{l}_{rb}^b \wedge\right]\end{aligned} \tag{I.129}$$

$$\begin{aligned}\mathbf{H}_{v1}^{nb} &\approx -\hat{\mathbf{C}}_b^n\left[\left\{\hat{\mathbf{C}}_b^n\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{rb}^b\right) - \hat{\mathbf{C}}_n^b \hat{\boldsymbol{\Omega}}_{ie}^n \hat{\mathbf{C}}_b^n \mathbf{l}_{rb}^b\right\}\wedge\right].\\ \mathbf{H}_{v5}^{nb} &\approx -\hat{\mathbf{C}}_b^n\left[\mathbf{l}_{rb}^b \wedge\right]\end{aligned} \tag{I.130}$$

Where the coupling of the attitude errors and gyro biases into the measurements through the lever arm is neglected, (15.9) is used as the measurement matrix.

The heading-matching measurement matrix is

$$\mathbf{H}^{\gamma b}_{\psi,k} = \begin{pmatrix} -\mathbf{C}^{n}_{b3,1:3} & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & \mathbf{0} \end{pmatrix}_k \quad \gamma \in i, e, n \,. \tag{I.131}$$

Where full attitude matching is implemented, the state vector is defined as

$$\mathbf{x}^{\gamma b} = \begin{pmatrix} \delta\boldsymbol{\psi}^{b}_{\gamma b} \\ \delta\mathbf{v}^{\gamma}_{\beta b} \\ \mathbf{b}_a \\ \mathbf{b}_g \\ \boldsymbol{\psi}_{rb,s} \\ \boldsymbol{\eta} \\ \vdots \end{pmatrix} \quad \{\beta,\gamma\} \in \{i,i\},\{e,e\},\{e,n\}, \tag{I.132}$$

where $\boldsymbol{\psi}_{rb,s}$ is the static relative orientation and, $\boldsymbol{\eta} = \{\eta_{xy}, \eta_{xz}, \eta_{yx}, \eta_{yz}, \eta_{zx}, \eta_{zy}\}$ are the flexure coefficients. The attitude measurement matrix is

$$\mathbf{H}^{\gamma b}_{A,k} = \begin{pmatrix} -\hat{\mathbf{C}}^{\gamma}_{b} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \hat{\mathbf{C}}^{\gamma}_{b} & \mathbf{H}^{\gamma}_{a6} & \mathbf{0} \end{pmatrix}_k \quad \gamma \in i, e, n \,, \tag{I.133}$$

where $\mathbf{H}^{\gamma}_{a6}$ is defined by (15.17).

I.5.5 Dead-Reckoning Measurement Models

This subsection describes the integration of magnetic heading, accelerometer leveling, odometry, PDR using step detection, and Doppler radar and sonar as aiding-sensor or total-state measurements with the attitude error states resolved about the body frame. The odometry and Doppler radar models may also be used for visual odometry.

Integration of magnetic heading is described in Section 16.2.1.1. If the state vector is defined as

$$\mathbf{x}^{\gamma b} = \begin{pmatrix} \delta\boldsymbol{\psi}^{b}_{\gamma b} \\ \vdots \\ \mathbf{x}_M \end{pmatrix} \qquad \gamma \in i, e, n, l \,, \tag{I.134}$$

where $\delta\boldsymbol{\psi}^{b}_{\gamma b}$ is the overall attitude error, the measurement matrix is

$$\mathbf{H}^{\gamma b}_{M,k} = \left. \begin{pmatrix} -\dfrac{\partial \tilde{\psi}_{nb}}{\partial \delta\boldsymbol{\psi}^{b}_{\gamma b}} & \mathbf{0} & \dfrac{\partial \tilde{\psi}_{mb}}{\partial \mathbf{x}_M} \end{pmatrix} \right|_{\mathbf{x}^{\gamma b} = \hat{\mathbf{x}}^{\gamma b -}_{k}} . \tag{I.135}$$

Where the small angle approximation applies,

$$\left. \frac{\partial \tilde{\psi}_{nb}}{\partial \delta\boldsymbol{\psi}^{b}_{\gamma b}} \right|_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}^{\gamma -}_{k}} \approx \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \hat{\mathbf{C}}^{n}_{b} \,. \tag{I.136}$$

Integration of accelerometer leveling is described in Section 16.2.1.3. Defining the state vector as

$$\mathbf{x}^{\gamma b} = \begin{pmatrix} \delta\boldsymbol{\psi}^{b}_{\gamma b} \\ \mathbf{b}_a \\ \vdots \end{pmatrix} \qquad \gamma \in i, e, n, l \,, \tag{I.137}$$

the measurement matrix is

$$\mathbf{H}_{L,k}^{\gamma b} = \left. \begin{pmatrix} -\dfrac{\partial \phi_{nb}}{\partial \delta \boldsymbol{\psi}_{\gamma b}^{b}} & \dfrac{\partial \phi_{nbA}}{\partial \mathbf{f}_{ib}^{b}} & \mathbf{0} \\ -\dfrac{\partial \theta_{nb}}{\partial \delta \boldsymbol{\psi}_{\gamma b}^{b}} & \dfrac{\partial \theta_{nbA}}{\partial \mathbf{f}_{ib}^{b}} & \mathbf{0} \end{pmatrix} \right|_{\mathbf{x}^{\gamma b} = \hat{\mathbf{x}}_k^{\gamma b -}} . \tag{I.138}$$

Where the small angle approximation applies,

$$\left. \begin{pmatrix} \partial \phi_{nb} \big/ \partial \delta \boldsymbol{\psi}_{\gamma b}^{b} \\ \partial \theta_{nb} \big/ \partial \delta \boldsymbol{\psi}_{\gamma b}^{b} \end{pmatrix} \right|_{\mathbf{x}^{\gamma b} = \hat{\mathbf{x}}_k^{\gamma b -}} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{I.139}$$

Integration of odometry is described in Section 16.2.3. Defining the state vector as

$$\mathbf{x}^{\gamma b} = \begin{pmatrix} \delta \boldsymbol{\psi}_{\gamma b}^{b} \\ \delta \mathbf{v}_{\beta b}^{\gamma} \\ \vdots \\ \mathbf{b}_g \\ \vdots \\ s_{or} \\ s_{\Delta or} \end{pmatrix}, \tag{I.140}$$

where $\{\beta, \gamma\} \in \{i,i\},\{e,e\},\{e,n\},\{l,l\}$ and noting that the gyro bias is only estimated where the reference navigation system includes gyros, the measurement matrix is

$$\mathbf{H}_{O,k}^{\gamma b} \approx \left. \begin{pmatrix} \mathbf{H}_{O11}^{\gamma b} & \mathbf{H}_{O12}^{\gamma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{v}_{er} & \mathbf{0} \\ \mathbf{H}_{O21}^{\gamma b} & \mathbf{0} & \mathbf{0} & -\dfrac{\cos \hat{\theta}_{nb}}{\tau_o} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \hat{\mathbf{C}}_b^n & \mathbf{0} & \hat{\dot{\psi}}_{nb} & \dfrac{\hat{v}_{er}}{T_r} \\ \mathbf{H}_{O31}^{\gamma b} & \mathbf{H}_{O32}^{\gamma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \right|_{\mathbf{x}^{\gamma b} = \hat{\mathbf{x}}_k^{\gamma b -}} , \tag{I.141}$$

where

$$\begin{aligned} \mathbf{H}_{O11}^{\gamma b} &= -\frac{1}{\tau_o} \int_{t-\tau_o}^{t} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \hat{\mathbf{C}}_{\gamma}^{b}(t') \left[\hat{\mathbf{v}}_{\beta b}^{\gamma}(t') \wedge \right] \hat{\mathbf{C}}_{b}^{\gamma}(t') dt' \\ \mathbf{H}_{O21}^{\gamma b} &\approx \frac{1}{\tau_o^2} \left[\hat{\psi}_{nb}(t) - \hat{\psi}_{nb}(t-\tau_o) \right] \int_{t-\tau_o}^{t} \sin \hat{\theta}_{nb} \begin{pmatrix} 0 & \cos \hat{\phi}_{nb} & \sin \hat{\phi}_{nb} \end{pmatrix} dt' \\ \mathbf{H}_{O31}^{\gamma b} &= -\frac{1}{\tau_o} \int_{t-\tau_o}^{t} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \hat{\mathbf{C}}_{\gamma}^{b}(t') \left[\hat{\mathbf{v}}_{\beta b}^{\gamma}(t') \wedge \right] \hat{\mathbf{C}}_{b}^{\gamma}(t') dt' \end{aligned} \tag{I.142}$$

and $\mathbf{H}_{O12}^{\gamma}$ and $\mathbf{H}_{O32}^{\gamma}$ are given by (16.49).

Integration of PDR using step detection is described in Section 16.2.4. Defining the state vector as

$$\mathbf{x}^{\gamma b} = \begin{pmatrix} \delta \boldsymbol{\psi}_{\gamma b}^{\gamma b} \\ \delta \mathbf{v}_{\beta b}^{\gamma} \\ \vdots \\ \mathbf{c}_P \\ \psi_{bh} \end{pmatrix}, \tag{I.143}$$

where $\{\beta, \gamma\} \in \{i,i\},\{e,e\},\{e,n\},\{l,l\}$, the measurement matrix is

$$\mathbf{H}_{P,k}^{\gamma b} = \begin{pmatrix} \mathbf{H}_{P1}^{\gamma b} & \mathbf{H}_{P2}^{\gamma} & \mathbf{0} & \mathbf{H}_{P4}^{\gamma} & \mathbf{H}_{P5}^{\gamma} \end{pmatrix}_{\mathbf{x}^{\gamma b} = \hat{\mathbf{x}}_k^{\gamma b -}}, \tag{I.144}$$

where

$$\mathbf{H}_{P1}^{\gamma b} \approx \begin{pmatrix} -\sin(\hat{\psi}_{nb} + \hat{\psi}_{bh}) \\ \cos(\hat{\psi}_{nb} + \hat{\psi}_{bh}) \end{pmatrix} \Delta \hat{r}_P \begin{pmatrix} 0 & \dfrac{\sin \hat{\phi}_{nb}}{\cos \hat{\theta}_{nb}} & \dfrac{\cos \hat{\phi}_{nb}}{\cos \hat{\theta}_{nb}} \end{pmatrix} - \int_{t-\tau_P}^{t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \hat{\mathbf{C}}_{\gamma}^{n}(t') \left[\mathbf{v}_{\beta b}^{\gamma}(t') \wedge \right] \hat{\mathbf{C}}_{b}^{\gamma}(t') dt' \tag{I.145}$$

and $\mathbf{H}_{P2}^{\gamma}$, $\mathbf{H}_{P4}^{\gamma}$, and $\mathbf{H}_{P5}^{\gamma}$ are given by (16.61–63).

Integration of Doppler radar and sonar is described in Section 16.2.5. Defining the state vector as

$$\mathbf{x}^{\gamma b} = \begin{pmatrix} \delta \boldsymbol{\psi}_{\gamma b}^{b} \\ \delta \mathbf{v}_{eb}^{\gamma} \\ \vdots \\ s_D \\ \mathbf{v}_{es}^{\gamma} \end{pmatrix} \qquad \gamma \in e, n, \tag{I.146}$$

the measurement matrix is

$$\mathbf{H}_{D,k}^{\gamma b} = \begin{pmatrix} -\hat{\mathbf{C}}_{\gamma}^{b} \left[\left(\hat{\mathbf{v}}_{eb}^{\gamma} - \hat{\mathbf{v}}_{es}^{\gamma} \right) \wedge \right] \hat{\mathbf{C}}_{b}^{\gamma} & -\hat{\mathbf{C}}_{\gamma}^{b} & \mathbf{0} & \hat{\mathbf{C}}_{\gamma}^{b} \hat{\mathbf{v}}_{eb}^{\gamma} & \hat{\mathbf{C}}_{\gamma}^{b} \end{pmatrix}_{\mathbf{x}^{\gamma b} = \hat{\mathbf{x}}_k^{\gamma b -}}. \tag{I.147}$$

I.6 Estimation of the Time Synchronization Error

This section describes how the time synchronization error between two navigation systems, for example an IMU and GNSS user equipment, may be estimated as a Kalman filter state and incorporated within the measurement innovations to correct for its effects [1, 2]. Time synchronization is discussed for Kalman filters in general in Section 3.3.4 and for INS/GNSS integration in Section 14.3.

The time synchronization error, δt_s, may be modeled as a constant, a random walk, or a first-order Markov process, depending on its behavior. Sections I.6.1 and I.6.2 present measurement models for loosely coupled and tightly coupled INS/GNSS integration, respectively. Section I.6.3 presents some transfer alignment measurement models. Finally, Sections I.6.4 and I.6.5 present measurement models for integration of, respectively, dead reckoning and terrestrial radio navigation as aiding sensors.

I.6.1 Loosely Coupled GNSS Measurement Model

The time synchronization error, δt_s, is defined as the amount that must be added to the IMU time tags to synchronize them with the GNSS time tags. The state vector is

$$\mathbf{x}^{\gamma} = \begin{pmatrix} \mathbf{x}_{INS}^{\gamma} \\ \delta t_s \end{pmatrix} \qquad \gamma \in i, e, n, l, \tag{I.148}$$

where the inertial state vectors given by (14.15), (14.39), and (14.51) are assumed. The measurement innovations for the various coordinate frame implementations are

$$\delta \mathbf{z}_{G,k}^{i-} = \begin{pmatrix} \hat{\mathbf{r}}_{iaG}^{i}(t_G) - \hat{\mathbf{r}}_{ib}^{i}(t_G + \delta \hat{t}_s) - \hat{\mathbf{C}}_b^i(t_G + \delta \hat{t}_s)\, \mathbf{l}_{ba}^{b} \\ \hat{\mathbf{v}}_{iaG}^{i}(t_G) - \hat{\mathbf{v}}_{ib}^{i}(t_G + \delta \hat{t}_s) - \hat{\mathbf{C}}_b^i(t_G + \delta \hat{t}_s)\left(\hat{\boldsymbol{\omega}}_{ib}^{b}(t_G + \delta \hat{t}_s) \wedge \mathbf{l}_{ba}^{b}\right) \end{pmatrix}_k, \tag{I.149}$$

$$\delta \mathbf{z}_{G,k}^{e-} = \begin{pmatrix} \hat{\mathbf{r}}_{eaG}^{e}(t_G) - \hat{\mathbf{r}}_{eb}^{e}(t_G + \delta \hat{t}_s) - \hat{\mathbf{C}}_b^e(t_G + \delta \hat{t}_s)\, \mathbf{l}_{ba}^{b} \\ \hat{\mathbf{v}}_{eaG}^{e}(t_G) - \hat{\mathbf{v}}_{eb}^{e}(t_G + \delta \hat{t}_s) - \hat{\mathbf{C}}_b^e(t_G + \delta \hat{t}_s)\left(\hat{\boldsymbol{\omega}}_{ib}^{b}(t_G + \delta \hat{t}_s) \wedge \mathbf{l}_{ba}^{b}\right) + \boldsymbol{\Omega}_{ie}^{e} \hat{\mathbf{C}}_b^e(t_G + \delta \hat{t}_s)\, \mathbf{l}_{ba}^{b} \end{pmatrix}_k, \tag{I.150}$$

$$\delta\mathbf{z}_{G,k}^{n-} = \begin{pmatrix} \mathbf{S}_p\left(\hat{\mathbf{p}}_{aG}(t_G) - \hat{\mathbf{p}}_b\left(t_G + \delta\hat{t}_s\right) - \hat{\mathbf{T}}_{r(n)}^p\hat{\mathbf{C}}_b^n\left(t_G + \delta\hat{t}_s\right)\mathbf{l}_{ba}^b\right) \\ \hat{\mathbf{v}}_{eaG}^n(t_G) - \hat{\mathbf{v}}_{eb}^n\left(t_G + \delta\hat{t}_s\right) - \hat{\mathbf{C}}_b^n\left(t_G + \delta\hat{t}_s\right)\left(\hat{\boldsymbol{\omega}}_{ib}^b\left(t_G + \delta\hat{t}_s\right) \wedge \mathbf{l}_{ba}^b\right) + \hat{\boldsymbol{\Omega}}_{ie}^n\hat{\mathbf{C}}_b^n\left(t_G + \delta\hat{t}_s\right)\mathbf{l}_{ba}^b \end{pmatrix}_k, \tag{I.151}$$

$$\delta\mathbf{z}_{G,k}^{l-} = \begin{pmatrix} \hat{\mathbf{r}}_{laG}^l(t_G) - \hat{\mathbf{r}}_{lb}^l\left(t_G + \delta\hat{t}_s\right) - \hat{\mathbf{C}}_b^l\left(t_G + \delta\hat{t}_s\right)\mathbf{l}_{ba}^b \\ \hat{\mathbf{v}}_{laG}^l(t_G) - \hat{\mathbf{v}}_{lb}^l\left(t_G + \delta\hat{t}_s\right) - \hat{\mathbf{C}}_b^l\left(t_G + \delta\hat{t}_s\right)\left(\hat{\boldsymbol{\omega}}_{ib}^b\left(t_G + \delta\hat{t}_s\right) \wedge \mathbf{l}_{ba}^b\right) + \boldsymbol{\Omega}_{il}^l\hat{\mathbf{C}}_b^l\left(t_G + \delta\hat{t}_s\right)\mathbf{l}_{ba}^b \end{pmatrix}_k^-, \tag{I.152}$$

where t_G is the GNSS measurement time of validity as indicated by the GNSS user equipment. It is assumed that the inertial navigation solution at a given time is obtained by interpolating from a data store of the recent solutions. Note that this is an EKF implementation of the time synchronization error state. Thus the measurement innovations are assumed to be a linear function of the residual time synchronization error (actual minus estimated), but not necessarily of the whole time synchronization error.

Where the coupling of the attitude errors and gyro biases into the measurements through the lever arm terms is neglected, together with the angular acceleration, Earth rotation rate, and transport rate, the measurement matrices may be approximated to

$$\mathbf{H}_{G,k}^\gamma \approx \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \left[\hat{\mathbf{v}}_{\gamma b}^\gamma + \hat{\mathbf{C}}_b^\gamma\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right)\right] \\ \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \left[\hat{\mathbf{a}}_{\gamma b}^\gamma + \hat{\mathbf{C}}_b^\gamma\hat{\boldsymbol{\Omega}}_{ib}^b\hat{\boldsymbol{\Omega}}_{ib}^b\mathbf{l}_{ba}^b\right] \end{pmatrix}_k \qquad \gamma \in i,e,l, \tag{I.153}$$

$$\mathbf{H}_{G,k}^n \approx \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{S}_p & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{S}_p\hat{\mathbf{T}}_{r(n)}^p\left[\hat{\mathbf{v}}_{eb}^n + \hat{\mathbf{C}}_b^n\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right)\right] \\ \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \left[\hat{\mathbf{a}}_{eb}^n + \hat{\mathbf{C}}_b^n\hat{\boldsymbol{\Omega}}_{ib}^b\hat{\boldsymbol{\Omega}}_{ib}^b\mathbf{l}_{ba}^b\right] \end{pmatrix}, \tag{I.154}$$

where all terms are assumed to be calculated at time $t_G + \delta\hat{t}_s$.

I.6.2 Tightly Coupled GNSS Measurement Model

As for loosely coupled integration, the time synchronization error, δt_s, is defined as the amount that must be added to the IMU time tags to synchronize them with the GNSS time tags. The state vector is

$$\mathbf{x}^\gamma = \begin{pmatrix} \mathbf{x}_{INS}^\gamma \\ \delta\rho_c^a \\ \delta\dot{\rho}_c^a \\ \delta t_s \end{pmatrix} \qquad \gamma \in i,e,n,l, \tag{I.155}$$

where the inertial state vectors given by (14.15), (14.39), and (14.51) are assumed.

The INS-indicated user antenna position and velocity, $\hat{\mathbf{r}}_{ib}^i$ and $\hat{\mathbf{v}}_{ib}^i$, or $\hat{\mathbf{r}}_{eb}^e$ and $\hat{\mathbf{v}}_{eb}^e$, are calculated at time $t_G + \delta\hat{t}_s$. Otherwise, the measurement innovations are calculated as described in Section 14.3.

The time synchronization error columns of the measurement matrix may be calculated using the chain rule:

$$\frac{\partial\mathbf{z}_\rho}{\partial\delta t_s} = \frac{\partial\mathbf{z}_\rho}{\partial\mathbf{r}_{\beta b}^\gamma}\frac{\partial\mathbf{r}_{\beta b}^\gamma}{\partial\delta t_s}, \qquad \frac{\partial\mathbf{z}_r}{\partial\delta t_s} = \frac{\partial\mathbf{z}_r}{\partial\mathbf{v}_{\beta b}^\gamma}\frac{\partial\mathbf{v}_{\beta b}^\gamma}{\partial\delta t_s}. \tag{I.156}$$

Thus, where the coupling of the attitude errors and gyro biases into the measurements through the lever arm terms is neglected, together with the angular acceleration, Earth rotation rate, and transport rate, the measurement matrices may be approximated to

$$
\mathbf{H}_{G,k}^{\gamma} \approx \left(\begin{array}{cccccccc}
\mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{a1}^{\gamma\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 & -\mathbf{u}_{a1}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\
\mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{a2}^{\gamma\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 & -\mathbf{u}_{a2}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{u}_{am}^{\gamma\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 & -\mathbf{u}_{am}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\
\hline
\mathbf{0}_{1,3} & \mathbf{u}_{a1}^{\gamma\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 & -\mathbf{u}_{a1}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{a}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\hat{\boldsymbol{\Omega}}_{ib}^{b}\hat{\boldsymbol{\Omega}}_{ib}^{b}\mathbf{l}_{ba}^{b}\right] \\
\mathbf{0}_{1,3} & \mathbf{u}_{a2}^{\gamma\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 & -\mathbf{u}_{a2}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{a}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\hat{\boldsymbol{\Omega}}_{ib}^{b}\hat{\boldsymbol{\Omega}}_{ib}^{b}\mathbf{l}_{ba}^{b}\right] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0}_{1,3} & \mathbf{u}_{am}^{\gamma\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 & -\mathbf{u}_{am}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{a}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\hat{\boldsymbol{\Omega}}_{ib}^{b}\hat{\boldsymbol{\Omega}}_{ib}^{b}\mathbf{l}_{ba}^{b}\right]
\end{array} \right)_{\mathbf{x}=\hat{\mathbf{x}}_k} \qquad \gamma \in i,e,l, \tag{I.157}
$$

$$
\mathbf{H}_{G,k}^{n} \approx \left(\begin{array}{cccccccc}
\mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{h}_{\rho p}^{1\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 & -\mathbf{u}_{a1}^{n\,\mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\
\mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{h}_{\rho p}^{2\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 & -\mathbf{u}_{a2}^{n\,\mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{h}_{\rho p}^{\mathrm{m}\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 1 & 0 & -\mathbf{u}_{am}^{n\,\mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\
\hline
\mathbf{0}_{1,3} & \mathbf{u}_{a1}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 & -\mathbf{u}_{a1}^{n\,\mathrm{T}}\left[\hat{\mathbf{a}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\hat{\boldsymbol{\Omega}}_{ib}^{b}\hat{\boldsymbol{\Omega}}_{ib}^{b}\mathbf{l}_{ba}^{b}\right] \\
\mathbf{0}_{1,3} & \mathbf{u}_{a2}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 & -\mathbf{u}_{a2}^{n\,\mathrm{T}}\left[\hat{\mathbf{a}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\hat{\boldsymbol{\Omega}}_{ib}^{b}\hat{\boldsymbol{\Omega}}_{ib}^{b}\mathbf{l}_{ba}^{b}\right] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0}_{1,3} & \mathbf{u}_{am}^{n\,\mathrm{T}} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0 & 1 & -\mathbf{u}_{am}^{n\,\mathrm{T}}\left[\hat{\mathbf{a}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\hat{\boldsymbol{\Omega}}_{ib}^{b}\hat{\boldsymbol{\Omega}}_{ib}^{b}\mathbf{l}_{ba}^{b}\right]
\end{array} \right)_{\mathbf{x}=\hat{\mathbf{x}}_k} . \tag{I.158}
$$

where $\mathbf{h}_{\rho p}^{j}$ is given by (14.128).

I.6.3 Transfer Alignment Measurement Models

For transfer alignment, the time synchronization error, δt_s, is defined as the amount that must be added to the aligning-INS time tags to synchronize them with the reference-navigation-system time tags. For velocity-matching and heading-matching transfer alignment (Section 15.1), the state vector defined as

$$
\mathbf{x}^{\gamma} = \begin{pmatrix} \delta\boldsymbol{\psi}_{\gamma b}^{\gamma} \\ \delta\mathbf{v}_{\beta b}^{\gamma} \\ \mathbf{b}_a \\ \mathbf{b}_g \\ \delta t_s \end{pmatrix} \qquad \{\beta,\gamma\} \in \{i,i\},\{e,e\},\{e,n\}\cdot \tag{I.159}
$$

The velocity measurement innovations for the various coordinate frame implementations are

$$
\delta\mathbf{z}_{V,k}^{i-} = \left[\hat{\mathbf{v}}_{irR}^{i}(t_R) - \hat{\mathbf{v}}_{ib}^{i}\left(t_R + \delta\hat{t}_s\right) + \hat{\mathbf{C}}_b^{i}\left(t_R + \delta\hat{t}_s\right)\left(\hat{\boldsymbol{\omega}}_{ib}^{b}\left(t_R + \delta\hat{t}_s\right) \wedge \mathbf{l}_{rb}^{b}\right)\right]_k , \tag{I.160}
$$

$$
\delta\mathbf{z}_{V,k}^{e-} = \left[\hat{\mathbf{v}}_{erR}^{e}(t_R) - \hat{\mathbf{v}}_{eb}^{e}\left(t_R + \delta\hat{t}_s\right) + \hat{\mathbf{C}}_b^{e}\left(t_R + \delta\hat{t}_s\right)\left(\hat{\boldsymbol{\omega}}_{ib}^{b}\left(t_R + \delta\hat{t}_s\right) \wedge \mathbf{l}_{rb}^{b}\right) - \boldsymbol{\Omega}_{ie}^{e}\hat{\mathbf{C}}_b^{e}\left(t_R + \delta\hat{t}_s\right)\,\mathbf{l}_{rb}^{b}\right]_k , \tag{I.161}
$$

$$
\delta\mathbf{z}_{V,k}^{n-} = \left[\hat{\mathbf{v}}_{erR}^{n}(t_R) - \hat{\mathbf{v}}_{eb}^{n}\left(t_R + \delta\hat{t}_s\right) + \hat{\mathbf{C}}_b^{n}\left(t_R + \delta\hat{t}_s\right)\left(\hat{\boldsymbol{\omega}}_{ib}^{b}\left(t_R + \delta\hat{t}_s\right) \wedge \mathbf{l}_{rb}^{b}\right) - \hat{\boldsymbol{\Omega}}_{ie}^{n}\hat{\mathbf{C}}_b^{n}\left(t_R + \delta\hat{t}_s\right)\,\mathbf{l}_{rb}^{b}\right]_k , \tag{I.162}
$$

where t_R is the reference measurement time of validity as indicated by the reference navigation system. It is assumed that the aligning-INS navigation solution at a given time is obtained by interpolating from a data store of the recent solutions. Note that this is an EKF implementation of the time synchronization error state. Thus the measurement innovations are assumed to be a linear function of the residual time synchronization error (actual minus estimated), but not necessarily of the whole time synchronization error.

Where the coupling of the attitude errors and gyro biases into the measurements through the lever arm terms is neglected, together with the angular acceleration, Earth rotation rate, and transport rate, the measurement matrices may be approximated to

$$\mathbf{H}_{V,k}^{\gamma} \approx \begin{pmatrix} \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \left[\hat{\mathbf{a}}_{\gamma b}^{\gamma} - \hat{\mathbf{C}}_b^{\gamma}\hat{\mathbf{\Omega}}_{ib}^b\hat{\mathbf{\Omega}}_{ib}^b\mathbf{l}_{rb}^b\right] \end{pmatrix}_k \qquad \gamma \in i,e,n, \tag{I.163}$$

where all terms are assumed to be calculated at time $t_R + \delta\hat{t}_s$.

Similarly, the heading measurement innovation and measurement matrix are

$$\delta z_{\psi,k} = \hat{\psi}_{nr}(t_R) - \hat{\psi}_{nb}\left(t_R + \delta\hat{t}_s\right) + \psi_{rb}, \tag{I.164}$$

$$\mathbf{H}_{\psi,k}^{n} = \begin{pmatrix} (0 & 0 & -1) & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{C}_{b3,1:3}^{n}\boldsymbol{\omega}_{ib}^{b} \end{pmatrix}_k, \tag{I.165}$$

$$\mathbf{H}_{\psi,k}^{\gamma} = \begin{pmatrix} -\mathbf{C}_{\gamma3,1:3}^{n} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{C}_{\gamma3,1:3}^{n}\mathbf{C}_b^{\gamma}\boldsymbol{\omega}_{ib}^{b} \end{pmatrix}_k \quad \gamma \in i,e. \tag{I.166}$$

Where full attitude matching is implemented, the state vector is defined as

$$\mathbf{x}^{\gamma} = \begin{pmatrix} \delta\boldsymbol{\psi}_{\gamma b}^{\gamma} \\ \delta\mathbf{v}_{\beta b}^{\gamma} \\ \mathbf{b}_a \\ \mathbf{b}_g \\ \boldsymbol{\psi}_{rb,s} \\ \boldsymbol{\eta} \\ \delta t_s \end{pmatrix} \quad \{\beta,\gamma\} \in \{i,i\},\{e,e\},\{e,n\}, \tag{I.167}$$

where $\boldsymbol{\psi}_{rb,s}$ is the static relative orientation and, $\boldsymbol{\eta} = \{\eta_{xy}, \eta_{xz}, \eta_{yx}, \eta_{yz}, \eta_{zx}, \eta_{zy}\}$ are the flexure coefficients.

The full attitude-matching measurement innovation, $\delta\mathbf{z}_{A,k}^{\gamma-}$, is given by

$$\mathbf{I}_3 + \left[\delta\mathbf{z}_{A,k}^{\gamma-} \wedge\right] = \hat{\mathbf{C}}_r^{\gamma}(t_R)\hat{\mathbf{C}}_b^{r}\hat{\mathbf{C}}_{\gamma}^{b}\left(t_R + \delta\hat{t}_s\right) \quad \gamma \in i,e,n, \tag{I.168}$$

where $\hat{\mathbf{C}}_b^r$ is modeled by (15.14). The attitude measurement matrix is

$$\mathbf{H}_{A,k}^{\gamma} = \begin{pmatrix} -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \hat{\mathbf{C}}_b^{\gamma} & \mathbf{H}_{a6}^{\gamma} & \hat{\mathbf{C}}_b^{\gamma}\hat{\boldsymbol{\omega}}_{ib}^{b} \end{pmatrix}_k \quad \gamma \in i,e,n, \tag{I.169}$$

where $\mathbf{H}_{a6}^{\gamma}$ is defined by (15.17).

I.6.4 Dead-Reckoning Measurement Models

This subsection describes estimation of the time synchronization error where magnetic heading, generic heading, accelerometer leveling, barometric height, odometry, PDR using step detection, and Doppler radar and sonar are integrated as aiding-sensor or total-state measurements. The odometry and Doppler radar models may also be used for visual odometry. The time synchronization error, δt_s, is defined as the amount that must be added to the reference time tags to synchronize them with the dead-reckoning time tags.

The dead-reckoning measurement innovations are given in Section 16.2. Following the convention of the preceding examples, it is assumed that, for time synchronization, the reference parameters in the measurement innovations are advanced in time by $\delta\hat{t}_s$. For example, for a measurement, x, produced by sensor S, the measurement innovation would be

$$\delta z_{S,k}^{-} = \left[\tilde{x}_S(t_S) - \hat{x}(t_S + \delta\hat{t}_s)\right]_k , \tag{I.170}$$

where t_S is the sensor S measurement time of validity indicated by that sensor. Otherwise, the innovations are as presented in Section 16.2.

The state vectors are as those presented in Section 16.2 with the addition of the time synchronization error, δt_s, as the final state.

For the magnetic heading measurement described in Section 16.2.1.1, the measurement matrix becomes

$$\mathbf{H}_{M,k}^{\gamma} = \left. \begin{pmatrix} -\dfrac{\partial\tilde{\psi}_{nb}}{\partial\delta\boldsymbol{\psi}_{\gamma b}^{\gamma}} & \mathbf{0} & \dfrac{\partial\tilde{\psi}_{mb}}{\partial\mathbf{x}_M} & \dot{\hat{\psi}}_{nb} \end{pmatrix} \right|_{\mathbf{x}^{\gamma}=\hat{\mathbf{x}}_k^{\gamma-}} . \tag{I.171}$$

For the generic heading measurement described in Section 16.2.1.2, the measurement matrix becomes

$$\mathbf{H}_{\psi,k}^{\gamma} = \begin{pmatrix} -1 & 1 & \mathbf{0} & \dot{\hat{\psi}}_{nb} \end{pmatrix}. \tag{I.172}$$

For the accelerometer leveling measurement described in Section 16.2.1.3, the measurement matrix becomes

$$\mathbf{H}_{L,k}^{\gamma} = \left. \begin{pmatrix} -\dfrac{\partial\phi_{nb}}{\partial\delta\boldsymbol{\psi}_{\gamma b}^{\gamma}} & \dfrac{\partial\phi_{nbA}}{\partial\mathbf{f}_{ib}^{b}} & \mathbf{0} & \dot{\phi}_{nb} \\ -\dfrac{\partial\theta_{nb}}{\partial\delta\boldsymbol{\psi}_{\gamma b}^{\gamma}} & \dfrac{\partial\theta_{nbA}}{\partial\mathbf{f}_{ib}^{b}} & \mathbf{0} & \dot{\theta}_{nb} \end{pmatrix} \right|_{\mathbf{x}^{\gamma}=\hat{\mathbf{x}}_k^{\gamma-}} , \tag{I.173}$$

where the roll rate and pitch rate (and yaw rate) may be obtained from the angular rate using (2.58), noting that the Earth rate and transport rate may be neglected, where relevant.

For the barometric height measurement described in Section 16.2.2, the measurement matrix becomes

$$\mathbf{H}_{B,k} \approx \begin{pmatrix} -1 & \mathbf{0} & 1 & \hat{h}_{b,k}^{-} & -\hat{v}_{eb,D}^{n} \end{pmatrix}. \tag{I.174}$$

For the odometry measurements described in Section 16.2.3, the measurement matrix becomes

$$\mathbf{H}_{O,k}^{\gamma} \approx \left. \begin{pmatrix} \mathbf{H}_{O11}^{\gamma} & \mathbf{H}_{O12}^{\gamma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{v}_{er} & \mathbf{0} & \mathbf{H}_{O18}^{\gamma} \\ \mathbf{H}_{O21}^{\gamma} & \mathbf{0} & \mathbf{0} & -\dfrac{\cos\hat{\theta}_{nb}}{\tau_o}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\hat{\mathbf{C}}_b^n & \mathbf{0} & \dot{\hat{\psi}}_{nb} & \dfrac{\hat{v}_{er}}{T_r} & \mathbf{H}_{O28}^{\gamma} \\ \mathbf{H}_{O31}^{\gamma} & \mathbf{H}_{O32}^{\gamma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_{O38}^{\gamma} \end{pmatrix} \right|_{\mathbf{x}^{\gamma}=\hat{\mathbf{x}}_k^{\gamma-}} , \tag{I.175}$$

where

$$\begin{aligned} \mathbf{H}_{O18}^{\gamma} &\approx \frac{1}{\tau_o} \int_{t-\tau_o}^{t} \begin{pmatrix}1 & 0 & 0\end{pmatrix} \left(\hat{\mathbf{C}}_{\gamma}^{b}(t') \left[\hat{\mathbf{a}}_{\beta b}^{\gamma}(t') \wedge\right] - \hat{\boldsymbol{\Omega}}_{\gamma b}^{b} \hat{\mathbf{C}}_{\gamma}^{b}(t') \left[\hat{\mathbf{v}}_{\beta b}^{\gamma}(t') \wedge\right] \right) dt' \\ \mathbf{H}_{O28}^{\gamma} &\approx \frac{1}{\tau_o^2} \left[\hat{\psi}_{nb}(t) - \hat{\psi}_{nb}(t-\tau_o)\right] \int_{t-\tau_o}^{t} \cos\hat{\theta}_{nb} dt' \\ &\quad + \frac{1}{\tau_o^2} \left[\hat{\psi}_{nb}(t) - \hat{\psi}_{nb}(t-\tau_o)\right] \left[\cos\hat{\theta}_{nb}(t) - \cos\hat{\theta}_{nb}(t-\tau_o)\right] \\ \mathbf{H}_{O38}^{\gamma} &\approx \frac{1}{\tau_o} \int_{t-\tau_o}^{t} \begin{pmatrix}0 & 1 & 0\end{pmatrix} \left(\hat{\mathbf{C}}_{\gamma}^{b}(t') \left[\hat{\mathbf{a}}_{\beta b}^{\gamma}(t') \wedge\right] - \hat{\boldsymbol{\Omega}}_{\gamma b}^{b} \hat{\mathbf{C}}_{\gamma}^{b}(t') \left[\hat{\mathbf{v}}_{\beta b}^{\gamma}(t') \wedge\right] \right) dt' \end{aligned} . \tag{I.176}$$

For the measurement described in Section 16.2.4 for PDR using step detection, the measurement matrix becomes

$$\mathbf{H}_{P,k}^{\gamma} = \left(\mathbf{H}_{P1}^{\gamma} \quad \mathbf{H}_{P2}^{\gamma} \quad \mathbf{0} \quad \mathbf{H}_{P4}^{\gamma} \quad \mathbf{H}_{P5}^{\gamma} \quad \mathbf{H}_{P6}^{\gamma}\right)_{\mathbf{x}^{\gamma}=\hat{\mathbf{x}}_k^{\gamma-}}, \tag{I.177}$$

where

$$\mathbf{H}_{P6}^{\gamma} \approx \int_{t-\tau_P}^{t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \left(\hat{\mathbf{C}}_{\gamma}^{n}(t')\hat{a}_{\beta b}^{\gamma}(t') - \hat{\mathbf{\Omega}}_{\gamma n}^{n}\hat{\mathbf{C}}_{\gamma}^{n}(t')\hat{\mathbf{v}}_{\beta b}^{\gamma}(t')\right) dt' \ . \tag{I.178}$$

For the Doppler radar and sonar measurements described in Section 16.2.5, the measurement matrix becomes

$$\mathbf{H}_{D,k}^{\gamma} = \left(-\hat{\mathbf{C}}_{\gamma}^{b}\left[\left(\hat{\mathbf{v}}_{eb}^{\gamma} - \hat{\mathbf{v}}_{es}^{\gamma}\right)\wedge\right] \quad -\hat{\mathbf{C}}_{\gamma}^{b} \quad \mathbf{0} \quad \hat{\mathbf{C}}_{\gamma}^{b}\hat{\mathbf{v}}_{eb}^{\gamma} \quad \hat{\mathbf{C}}_{\gamma}^{b} \quad \left[\hat{\mathbf{C}}_{\gamma}^{b}\hat{\mathbf{a}}_{eb}^{\gamma} - \hat{\mathbf{\Omega}}_{\gamma b}^{b}\hat{\mathbf{C}}_{\gamma}^{b}\hat{\mathbf{v}}_{eb}^{\gamma}\right]\right)_{\mathbf{x}^{\gamma}=\hat{\mathbf{x}}_k^{\gamma-}}. \tag{I.179}$$

I.6.5 Terrestrial Radio Navigation and Feature Matching Measurement Models

This subsection describes estimation of the time synchronization for the integration of terrestrial radio navigation and environmental feature matching as aiding-sensor or total-state measurements. The time synchronization error, δt_s, is defined as the amount that must be added to the reference time tags to synchronize them with the radio time tags.

The radio-navigation and environmental feature-matching measurement innovations are given in Section 16.3. Following the convention of the preceding examples, it is assumed that, for time synchronization, the reference parameters in the measurement innovations are advanced in time by $\delta\hat{t}_s$. For example, for a measurement, x, produced by the radio-navigation or feature-matching system, the measurement innovation would be

$$\delta z_{R,k}^{-} = \left[\tilde{x}_R(t_R) - \hat{x}(t_R + \delta\hat{t}_s)\right]_k, \tag{I.180}$$

where t_R is the measurement time of validity indicated by that sensor. Otherwise, the innovations are as presented in Section 16.3.

The state vectors are as those presented in Section 16.3 with the addition of the time synchronization error, δt_s, as the final state.

For position measurement integration, the measurement matrix becomes

$$\mathbf{H}_{R,k}^{n} \approx \mathbf{S}_p\left(k_R\mathbf{I}_3 \quad \mathbf{0}_3 \quad \mathbf{I}_3 \quad \hat{\mathbf{T}}_{r(n)}^{p}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right]\right), \tag{I.181}$$

or

$$\mathbf{H}_{R,k}^{\gamma} \approx \left(k_R\mathbf{I}_3 \quad \mathbf{0}_3 \quad \mathbf{I}_3 \quad \left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right]\right)_k \qquad \gamma \in i, e, l, \tag{I.182}$$

omitting the final row for two-dimensional position fixes.

For ranging measurement integration, the measurement matrix is approximately

$$\mathbf{H}_{R,k}^{\gamma} \approx \begin{pmatrix} -k_R\mathbf{u}_{a1}^{\gamma\,\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{1\,\mathrm{T}} & 1 & \mathbf{0} & -\mathbf{u}_{a1}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\ -k_R\mathbf{u}_{a2}^{\gamma\,\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{2\,\mathrm{T}} & 1 & \mathbf{0} & -\mathbf{u}_{a2}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -k_R\mathbf{u}_{am}^{\gamma\,\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{m\,\mathrm{T}} & 1 & \mathbf{0} & -\mathbf{u}_{am}^{\gamma\,\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \end{pmatrix}_{\mathbf{x}^{\gamma}=\hat{\mathbf{x}}_k^{\gamma-}} \qquad \gamma \in i, e, l \tag{I.183}$$

or

$$\mathbf{H}_{R,k}^{n} \approx \begin{pmatrix} -k_R \mathbf{h}_{\rho p}^{1\ \mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{1\ \mathrm{T}} & 1 & \mathbf{0} & -\mathbf{u}_{a1}^{n\ \mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^n\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\ -k_R \mathbf{h}_{\rho p}^{2\ \mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{2\ \mathrm{T}} & 1 & \mathbf{0} & -\mathbf{u}_{a2}^{n\ \mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^n\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -k_R \mathbf{h}_{\rho p}^{m\ \mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{m\ \mathrm{T}} & 1 & \mathbf{0} & -\mathbf{u}_{am}^{n\ \mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^n\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \end{pmatrix}_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}_k^{\gamma -}} \tag{I.184}$$

for pseudo-range measurements, and

$$\mathbf{H}_{R,k}^{\gamma} \approx \begin{pmatrix} k_R\left(\mathbf{u}_{ar}^{\gamma} - \mathbf{u}_{a1}^{\gamma}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{1\ \mathrm{T}} & -\left(\mathbf{u}_{a1}^{\gamma} - \mathbf{u}_{ar}^{\gamma}\right)^{\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\ k_R\left(\mathbf{u}_{ar}^{\gamma} - \mathbf{u}_{a2}^{\gamma}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{2\mathrm{T}} & -\left(\mathbf{u}_{a2}^{\gamma} - \mathbf{u}_{ar}^{\gamma}\right)^{\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\ \vdots & \vdots & \vdots & \vdots \\ k_R\left(\mathbf{u}_{ar}^{\gamma} - \mathbf{u}_{am}^{\gamma}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{m\mathrm{T}} & -\left(\mathbf{u}_{am}^{\gamma} - \mathbf{u}_{ar}^{\gamma}\right)^{\mathrm{T}}\left[\hat{\mathbf{v}}_{\gamma b}^{\gamma} + \hat{\mathbf{C}}_b^{\gamma}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \end{pmatrix}_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}_k^{\gamma -}} \qquad \gamma \in i, e, l \tag{I.185}$$

or

$$\mathbf{H}_{R,k}^{n} \approx \begin{pmatrix} k_R\left(\mathbf{h}_{\rho p}^{r} - \mathbf{h}_{\rho p}^{1}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{1\ \mathrm{T}} & -\left(\mathbf{u}_{a1}^{n} - \mathbf{u}_{ar}^{n}\right)^{\mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\ k_R\left(\mathbf{h}_{\rho p}^{r} - \mathbf{h}_{\rho p}^{2}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{2\mathrm{T}} & -\left(\mathbf{u}_{a2}^{n} - \mathbf{u}_{ar}^{n}\right)^{\mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \\ \vdots & \vdots & \vdots & \vdots \\ k_R\left(\mathbf{h}_{\rho p}^{r} - \mathbf{h}_{\rho p}^{m}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{h}_b^{m\mathrm{T}} & -\left(\mathbf{u}_{am}^{n} - \mathbf{u}_{ar}^{n}\right)^{\mathrm{T}}\left[\hat{\mathbf{v}}_{eb}^{n} + \hat{\mathbf{C}}_b^{n}\left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b}\right)\right] \end{pmatrix}_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}_k^{\gamma -}} \tag{I.186}$$

for measurements of TDOA across transmitters.

I.7 Integer Wavelength Ambiguity Resolution

This section describes an EKF that performs both tightly-coupled integration of GNSS pseudo-range and ADR measurements with INS (see Section 14.4.2) and geometry-based integer wavelength ambiguity resolution (see Section 10.2.3 and Section G.9 of Appendix G, also on CD). Only the ECEF-frame implementation is shown and the reference station position is assumed to be known and constant. Both a positioning float filter and an attitude float filter are described

I.7.1 Positioning Float Filter

For the positioning float filter, the state vector is defined as

$$\mathbf{x}^{e} = \begin{pmatrix} \mathbf{x}_{INS}^{e} \\ \mathbf{x}_{N} \end{pmatrix} \qquad \mathbf{x}_{N} = \begin{pmatrix} \nabla\Delta N_{ra}^{t1,l} \\ \nabla\Delta N_{ra}^{t2,l} \\ \vdots \\ \nabla\Delta N_{ra}^{tm,l} \end{pmatrix}, \tag{I.187}$$

where $\mathbf{x}_{INS}^{e}$ is as defined by (14.39).

The transition matrix is

$$\boldsymbol{\Phi}_{k-1}^{e} = \begin{pmatrix} \boldsymbol{\Phi}_{INS}^{e} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m} \end{pmatrix}_{k-1}, \tag{I.188}$$

where $\boldsymbol{\Phi}_{INS}^{e}$ is given by (14.50). The system noise covariance matrix is

$$\mathbf{Q}_{k-1}^{e} = \begin{pmatrix} \mathbf{Q}_{INS}^{e} & \mathbf{0} \\ \mathbf{0} & Q_{N}\mathbf{I}_{m} \end{pmatrix}_{k-1}, \tag{I.189}$$

where $\mathbf{Q}_{INS}^{e}$ is given by (14.80) or (14.82). The wavelength ambiguities are assumed to be constant so, in theory $Q_N = 0$. In practice, Q_N should be set to a very small value to compensate for the effect of numerical rounding errors (see Section 3.3.3).

The measurement innovations comprise the differences between the measured and predicted double-differenced pseudo-ranges and ADRs. Thus [repeating (G.100)],

$$\delta\mathbf{z}_{F,k}^{-} = \begin{pmatrix} \nabla\Delta\tilde{\rho}_{ra,R}^{t1,l} - \nabla\Delta\hat{r}_{ra}^{t1-} \\ \nabla\Delta\tilde{\rho}_{ra,R}^{t2,l} - \nabla\Delta\hat{r}_{ra}^{t2-} \\ \vdots \\ \nabla\Delta\tilde{\rho}_{ra,R}^{tm,l} - \nabla\Delta\hat{r}_{ra}^{tm-} \\ \hdashline \nabla\Delta\tilde{\Phi}_{ra,R}^{t1,l} - \nabla\Delta\hat{N}_{ra}^{t1,l-}\lambda_{ca}^{l} - \nabla\Delta\hat{r}_{ra}^{t1-} \\ \nabla\Delta\tilde{\Phi}_{ra,R}^{t2,l} - \nabla\Delta\hat{N}_{ra}^{t2,l-}\lambda_{ca}^{l} - \nabla\Delta\hat{r}_{ra}^{t2-} \\ \vdots \\ \nabla\Delta\tilde{\Phi}_{ra,R}^{tm,l} - \nabla\Delta\hat{N}_{ra}^{tm,l-}\lambda_{ca}^{l} - \nabla\Delta\hat{r}_{ra}^{tm-} \end{pmatrix}_k,$$

where the predicted double-differenced range is given by

$$\nabla\Delta\hat{r}_{ra,k}^{ts-} = \left|\hat{\mathbf{r}}_{es}^{e}\left(\tilde{t}_{st,a}^{s,l}\right) - \hat{\mathbf{r}}_{ea,k}^{e-}\right| - \left|\hat{\mathbf{r}}_{et}^{e}\left(\tilde{t}_{st,a}^{t,l}\right) - \hat{\mathbf{r}}_{ea,k}^{e-}\right| - \left|\hat{\mathbf{r}}_{es}^{e}\left(\tilde{t}_{st,r}^{s,l}\right) - \mathbf{r}_{er}^{e}\right| + \left|\hat{\mathbf{r}}_{et}^{e}\left(\tilde{t}_{st,r}^{t,l}\right) - \mathbf{r}_{er}^{e}\right|. \quad \text{(I.190)}$$

The measurement matrix is

$$\mathbf{H}_{F,k}^{e} = \begin{pmatrix} \mathbf{H}_{\Phi,k}^{\Delta\gamma} & \mathbf{0} \\ \mathbf{H}_{\Phi,k}^{\Delta\gamma} & \mathbf{I}_m \end{pmatrix}, \quad \text{(I.191)}$$

where $\mathbf{H}_{\Phi,k}^{\Delta\gamma}$ is given by (14.148). The measurement noise covariance matrix is [repeating (G.104)]

$$\mathbf{R}_F = \begin{pmatrix} \mathbf{D}_G\mathbf{R}_{\rho}^{\nabla}\mathbf{D}_G^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_G\mathbf{R}_{\Phi}^{\nabla}\mathbf{D}_G^{\mathrm{T}} \end{pmatrix},$$

where $\mathbf{D}_G$ is the differencing matrix, given by (10.18), $\mathbf{R}_{\rho}^{\nabla}$ is the measurement noise covariance matrix for the pseudo-range measurements and $\mathbf{R}_{\Phi}^{\nabla}$ is the measurement noise covariance matrix for the ADR measurements, both differenced across receivers.

The ambiguities may be fixed using the LAMBDA method as described in Section G.9.4 of Appendix G. After the ambiguities have been fixed, the float filter should continue to operate in order to maintain calibration of the INS errors. The ambiguity-fixed state estimates are determined from their float counterparts using

$$\breve{\mathbf{x}}_{INS,k}^{e+} = \hat{\mathbf{x}}_{INS,k}^{e+} + \mathbf{P}_{IN,k}^{+}\left(\mathbf{P}_{N,k}^{+}\right)^{-1}\left(\breve{\mathbf{x}}_N - \hat{\mathbf{x}}_{N,k}^{+}\right), \quad \text{(I.192)}$$

where

$$\mathbf{P}_k^{+} = \begin{pmatrix} \mathbf{P}_{I,k}^{+} & \mathbf{P}_{IN,k}^{+} \\ \mathbf{P}_{IN,k}^{+\,\mathrm{T}} & \mathbf{P}_{N,k}^{+} \end{pmatrix}. \quad \text{(I.193)}$$

Either float estimates, fixed estimates, or a mixture of the two may be fed back to correct the inertial navigation solution. One option is to feedback the fixed position error estimates and the float versions of the other states. Note, however, that if an ambiguity-fixed state estimate is fed back, the corresponding float state should not be zeroed. Instead, its fixed counterpart should be subtracted from it, which effectively sets the fixed value (which was fed back) to zero.

I.7.2 Attitude Float Filter

It is assumed that the attitude float filter also performs conventional tightly-coupled INS/GNSS integration as described in Section 14.3.2. The state vector is defined as

$$\mathbf{x}^e = \begin{pmatrix} \mathbf{x}^e_{INS} \\ \mathbf{x}_{GNSS} \\ \mathbf{x}_N \end{pmatrix}, \tag{I.194}$$

where $\mathbf{x}_{GNSS}$ comprises the receiver clock offset and drift.

The transition matrix is

$$\mathbf{\Phi}^e_{k-1} = \begin{pmatrix} \mathbf{\Phi}^e_{INS} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_{GNSS} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_m \end{pmatrix}_{k-1}, \tag{I.195}$$

where $\mathbf{\Phi}_{GNSS}$ is given by (14.87). The system noise covariance matrix is

$$\mathbf{Q}^e_{k-1} = \begin{pmatrix} \mathbf{Q}^e_{INS} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{GNSS} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q_N \mathbf{I}_m \end{pmatrix}_{k-1}, \tag{I.196}$$

where $\mathbf{Q}_{GNSS}$ is given by (14.88) or (14.89). The wavelength ambiguities are assumed to be constant so, in theory $Q_N = 0$. In practice, Q_N should be set to a very small value to compensate for the effect of numerical rounding errors (see Section 3.3.3).

The measurement innovations for attitude estimation comprise the differences between the measured and predicted double-differenced pseudo-ranges and ADRs. Thus, from (G.100),

$$\delta\mathbf{z}^{e-}_{G\psi,k} = \begin{pmatrix} \delta\mathbf{z}^-_G \\ \hline \nabla\Delta\tilde{\rho}^{t1,l}_{aA,R} + \left(\mathbf{u}^e_{a1} - \mathbf{u}^e_{at}\right)^{\mathrm{T}} \hat{\mathbf{C}}^e_b \mathbf{r}^b_{aA} \\ \nabla\Delta\tilde{\rho}^{t2,l}_{aA,R} + \left(\mathbf{u}^e_{a2} - \mathbf{u}^e_{at}\right)^{\mathrm{T}} \hat{\mathbf{C}}^e_b \mathbf{r}^b_{aA} \\ \vdots \\ \nabla\Delta\tilde{\rho}^{tm,l}_{aA,R} + \left(\mathbf{u}^e_{am} - \mathbf{u}^e_{at}\right)^{\mathrm{T}} \hat{\mathbf{C}}^e_b \mathbf{r}^b_{aA} \\ \hline \nabla\Delta\tilde{\Phi}^{t1,l}_{aA,R} - \nabla\Delta\hat{N}^{t1,l}_{aA}\lambda^l_{ca} + \left(\mathbf{u}^e_{a1} - \mathbf{u}^e_{at}\right)^{\mathrm{T}} \hat{\mathbf{C}}^e_b \mathbf{r}^b_{aA} \\ \nabla\Delta\tilde{\Phi}^{t2,l}_{aA,R} - \nabla\Delta\hat{N}^{t2,l}_{aA}\lambda^l_{ca} + \left(\mathbf{u}^e_{a2} - \mathbf{u}^e_{at}\right)^{\mathrm{T}} \hat{\mathbf{C}}^e_b \mathbf{r}^b_{aA} \\ \vdots \\ \nabla\Delta\tilde{\Phi}^{tm,l}_{aA,R} - \nabla\Delta\hat{N}^{tm,l}_{aA}\lambda^l_{ca} + \left(\mathbf{u}^e_{am} - \mathbf{u}^e_{at}\right)^{\mathrm{T}} \hat{\mathbf{C}}^e_b \mathbf{r}^b_{aA} \end{pmatrix}_k, \tag{I.197}$$

where $\delta\mathbf{z}^-_G$ is the conventional GNSS measurement innovation, $\nabla\Delta\tilde{\rho}^{ts,l}_{aA,R}$ is the double-differenced pseudo-range; $\nabla\Delta\tilde{\Phi}^{ts,l}_{aA,R}$ is the double-differenced ADR, given by (14.154); and $\nabla\Delta\hat{N}^{ts,l}_{aA}$ is the float double-differenced integer wavelength ambiguity estimate.

The measurement matrix is

$$\mathbf{H}^e_{G\psi,k} = \begin{pmatrix} \mathbf{H}^e_{G,k} & & \mathbf{0} \\ \mathbf{H}^e_{\psi,k} & \mathbf{0} & \mathbf{0} \\ \mathbf{H}^e_{\psi,k} & \mathbf{0} & \mathbf{I}_m \end{pmatrix}, \tag{I.198}$$

where $\mathbf{H}^e_{G,k}$ is given by (14.126). and applies to both INS and GNSS clock states, while $\mathbf{H}^e_{\psi,k}$ is given by (14.155) and applies to the INS states only. The measurement noise covariance matrix is

$$\mathbf{R}_{G\psi} = \begin{pmatrix} \mathbf{R}_G & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_G \mathbf{R}^{\nabla}_{\rho} \mathbf{D}^{\mathrm{T}}_G & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_G \mathbf{R}^{\nabla}_{\Phi} \mathbf{D}^{\mathrm{T}}_G \end{pmatrix}, \tag{I.199}$$

where $\mathbf{R}_G$ is the pseudo-range and pseudo-range rate measurement noise covariance matrix and $\mathbf{R}^{\nabla}_{\rho}$ and $\mathbf{R}^{\nabla}_{\Phi}$ are the measurement noise covariance matrices for, respectively, the pseudo-range and ADR measurements differenced across receivers (but not satellites).

The ambiguities may be fixed using the LAMBDA method as described in Section G.9.4 of Appendix G. After the ambiguities have been fixed, the float filter should continue to operate in order to maintain calibration of the INS and receiver errors. The ambiguity-fixed state estimates are determined from their float counterparts using

$$\begin{pmatrix} \breve{\mathbf{x}}^{e+}_{INS,k} \\ \breve{\mathbf{x}}^{+}_{GNSS,k} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}}^{e+}_{INS,k} \\ \hat{\mathbf{x}}^{+}_{GNSS,k} \end{pmatrix} + \begin{pmatrix} \mathbf{P}^{+}_{IN,k} \\ \mathbf{P}^{+}_{GN,k} \end{pmatrix} \left(\mathbf{P}^{+}_{N,k}\right)^{-1} \left(\breve{\mathbf{x}}_N - \hat{\mathbf{x}}^{+}_{N,k}\right), \tag{I.200}$$

where

$$\mathbf{P}^{+}_{k} = \begin{pmatrix} \mathbf{P}^{+}_{I,k} & \mathbf{P}^{+}_{IG,k} & \mathbf{P}^{+}_{IN,k} \\ \mathbf{P}^{+\ \mathrm{T}}_{IG,k} & \mathbf{P}^{+}_{G,k} & \mathbf{P}^{+}_{GN,k} \\ \mathbf{P}^{+\ \mathrm{T}}_{IN,k} & \mathbf{P}^{+\ \mathrm{T}}_{GN,k} & \mathbf{P}^{+}_{N,k} \end{pmatrix}. \tag{I.201}$$

I.8 Dead Reckoning, Attitude, and Height

This section presents additional information on the integration of dead-reckoning, attitude, and height measuremenst. Section I.8.1 presents the derivation of the attitude error columns of the measurement matrix for the integration of odometry, PDR using step detection, Doppler radar Doppler sonar, and visual odometry as aiding measurements. Section I.8.2 describes boresight error estimation for integration of the same sensors.

I.8.1 Dead Reckoning Attitude Error Dependency

As described in Sections 16.2.3, 16.2.4, and 16.2.5, odometry, PDR, and Doppler radar and sonar aiding measurements exhibit a dependency on the navigation solution attitude error. Here, the attitude error columns of the measurement matrices are derived. Note that the odometry and Doppler radar models may also be used for visual odometry.

The odometry measurement innovation is given by (16.42–45). Considering first the dependency of $\hat{v}_{er}$ and $\hat{v}_x$ on the attitude error,

$$\left(\tilde{\mathbf{C}}^b_\gamma - \mathbf{C}^b_\gamma\right)\mathbf{v}^\gamma_{\beta b} = \left(\tilde{\mathbf{C}}^b_\gamma \mathbf{C}^\gamma_b - \mathbf{I}_3\right)\mathbf{C}^b_\gamma \mathbf{v}^\gamma_{\beta b}. \tag{I.202}$$

Applying (5.110), (5.111), and (I.99)

$$\begin{aligned} \left(\tilde{\mathbf{C}}^b_\gamma - \mathbf{C}^b_\gamma\right)\mathbf{v}^\gamma_{\beta b} &= \left(\delta\mathbf{C}^b_\gamma - \mathbf{I}_3\right)\mathbf{C}^b_\gamma \mathbf{v}^\gamma_{\beta b} \\ &= -\left[\delta\boldsymbol{\psi}^b_{\gamma b} \wedge\right]\mathbf{C}^b_\gamma \mathbf{v}^\gamma_{\beta b} \\ &= -\mathbf{C}^b_\gamma\left[\delta\boldsymbol{\psi}^\gamma_{\gamma b} \wedge\right]\mathbf{v}^\gamma_{\beta b} \\ &= \mathbf{C}^b_\gamma\left[\mathbf{v}^\gamma_{\beta b} \wedge\right]\delta\boldsymbol{\psi}^\gamma_{\gamma b} \end{aligned}. \tag{I.203}$$

Therefore,

$$\frac{\partial\left(\mathbf{C}_\gamma^b \mathbf{v}_{\beta b}^\gamma\right)}{\partial \delta\boldsymbol{\psi}_{\gamma b}^\gamma} = \mathbf{C}_\gamma^b\left[\mathbf{v}_{\beta b}^\gamma \wedge\right]. \tag{I.204}$$

Applying this to (16.43) and (16.45),

$$\frac{\partial \hat{v}_{er}}{\partial \delta\boldsymbol{\psi}_{\gamma b}^\gamma} = \frac{1}{\tau_o}\int_{t-\tau_o}^{t}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\hat{\mathbf{C}}_\gamma^b(t')\left[\hat{\mathbf{v}}_{\beta b}^\gamma(t')\wedge\right]dt'. \tag{I.205}$$

$$\frac{\partial \hat{v}_{c}}{\partial \delta\boldsymbol{\psi}_{\gamma b}^\gamma} = \frac{1}{\tau_o}\int_{t-\tau_o}^{t}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\hat{\mathbf{C}}_\gamma^b(t')\left[\hat{\mathbf{v}}_{\beta b}^\gamma(t')\wedge\right]dt'. \tag{I.206}$$

Moving on to the dependency of $\hat{\psi}_{nb}\overline{\cos\hat{\theta}_{nb}}$ on the attitude error, it is assumed that $\partial\hat{\psi}_{nb}(t)/\partial\delta\boldsymbol{\psi}_{\gamma b}^\gamma \approx \partial\hat{\psi}_{nb}(t-\tau_o)/\partial\delta\boldsymbol{\psi}_{\gamma b}^\gamma$. Therefore, from (16.44)

$$\frac{\partial\left(\hat{\psi}_{nb}\overline{\cos\hat{\theta}_{nb}}\right)}{\partial \delta\boldsymbol{\psi}_{\gamma b}^\gamma} \approx \frac{1}{\tau_o^2}\left[\hat{\psi}_{nb}(t)-\hat{\psi}_{nb}(t-\tau_o)\right]\int_{t-\tau_o}^{t}\frac{\partial\left(\cos\hat{\theta}_{nb}\right)}{\partial\hat{\theta}_{nb}}\frac{\partial\hat{\theta}_{nb}}{\partial\delta\boldsymbol{\psi}_{\gamma b}^b}\frac{\partial\delta\boldsymbol{\psi}_{\gamma b}^b}{\partial\delta\boldsymbol{\psi}_{\gamma b}^\gamma}dt'. \tag{I.207}$$

By analogy with (2.58),

$$\frac{\partial\hat{\theta}_{nb}}{\partial\delta\boldsymbol{\psi}_{\gamma b}^b} = \begin{pmatrix}0 & \cos\hat{\phi}_{nb} & \sin\hat{\phi}_{nb}\end{pmatrix}, \tag{I.208}$$

while, from (I.99),

$$\frac{\partial\delta\boldsymbol{\psi}_{\gamma b}^b}{\partial\delta\boldsymbol{\psi}_{\gamma b}^\gamma} = \mathbf{C}_\gamma^b. \tag{I.209}$$

Substituting these into (I.207),

$$\frac{\partial\left(\hat{\psi}_{nb}\overline{\cos\hat{\theta}_{nb}}\right)}{\partial \delta\boldsymbol{\psi}_{\gamma b}^\gamma} \approx -\frac{1}{\tau_o^2}\left[\hat{\psi}_{nb}(t)-\hat{\psi}_{nb}(t-\tau_o)\right]\int_{t-\tau_o}^{t}\sin\hat{\theta}_{nb}\begin{pmatrix}0 & \cos\hat{\phi}_{nb} & \sin\hat{\phi}_{nb}\end{pmatrix}\hat{\mathbf{C}}_\gamma^b(t')dt'. \tag{I.210}$$

Applying (3.90) to (16.42) and substituting in (I.205), (I.204), and (I.210) gives the measurement matrix of (16.48) and (16.49).

The measurement innovation for PDR using step detection is given by (16.55–57). By analogy with (I.202–204),

$$\frac{\partial\left(\mathbf{C}_\gamma^n \mathbf{v}_{\beta b}^\gamma\right)}{\partial \delta\boldsymbol{\psi}_{\gamma b}^\gamma} = \mathbf{C}_\gamma^n\left[\mathbf{v}_{\beta b}^\gamma \wedge\right]. \tag{I.211}$$

Thus,

$$\begin{pmatrix}\partial\Delta r_N'/\partial\delta\boldsymbol{\psi}_{\gamma b}^\gamma \\ \partial\Delta r_E'/\partial\delta\boldsymbol{\psi}_{\gamma b}^\gamma\end{pmatrix} = \int_{t-\tau_P}^{t}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\hat{\mathbf{C}}_\gamma^n(t')\left[\mathbf{v}_{\beta b}^\gamma(t')\wedge\right]dt'. \tag{I.212}$$

Now, from (16.54),

$$\begin{aligned}\frac{\partial\cos(\psi_{nb}+\psi_{bh})}{\partial\delta\boldsymbol{\psi}_{\gamma b}^\gamma} &= \frac{\partial\cos(\psi_{nb}+\psi_{bh})}{\partial\psi_{nb}}\frac{\partial\psi_{nb}}{\partial\delta\boldsymbol{\psi}_{\gamma b}^\gamma}\\ &= -\sin(\psi_{nb}+\psi_{bh})\begin{pmatrix}0 & \dfrac{\sin\hat{\phi}_{nb}}{\cos\hat{\theta}_{nb}} & \dfrac{\cos\hat{\phi}_{nb}}{\cos\hat{\theta}_{nb}}\end{pmatrix}\hat{\mathbf{C}}_\gamma^b\end{aligned} \tag{I.213}$$

and

$$\begin{aligned}\frac{\partial \sin(\psi_{nb}+\psi_{bh})}{\partial \delta\boldsymbol{\psi}_{\gamma b}^{\gamma}} &= \frac{\partial \sin(\psi_{nb}+\psi_{bh})}{\partial \psi_{nb}} \frac{\partial \psi_{nb}}{\partial \delta\boldsymbol{\psi}_{\gamma b}^{\gamma}} \\ &= \cos(\psi_{nb}+\psi_{bh})\begin{pmatrix} 0 & \frac{\sin\hat{\phi}_{nb}}{\cos\hat{\theta}_{nb}} & \frac{\cos\hat{\phi}_{nb}}{\cos\hat{\theta}_{nb}} \end{pmatrix}\hat{\mathbf{C}}_{\gamma}^{b}\end{aligned} . \tag{I.214}$$

Applying (3.90) to (16.55) and substituting in (I.212–214) gives the measurement matrix of (16.59) and (16.60).

Finally, the Doppler radar and sonar measurement innovation is given by (16.67). By analogy with (I.202–204),

$$\frac{\partial\left[\mathbf{C}_{\gamma}^{b}\left(\mathbf{v}_{eb}^{\gamma}-\mathbf{v}_{es}^{\gamma}\right)\right]}{\partial \delta\boldsymbol{\psi}_{\gamma b}^{\gamma}} = \mathbf{C}_{\gamma}^{b}\left[\left(\mathbf{v}_{eb}^{\gamma}-\mathbf{v}_{es}^{\gamma}\right)\wedge\right]. \tag{I.215}$$

Applying (3.90) to (16.67) and substituting in (I.215) gives the measurement matrix of (16.69).

I.8.2 Boresight Error Estimation

Where odometry, Doppler radar, Doppler sonar, or visual odometry is integrated as an aiding sensor, there may be a misalignment between the aiding and reference sensor body frames that must be estimated as additional Kalman filter states. This is often known as the boresight error and is already included in the measurement model for PDR using step detection presented in Section 16.2.4.

For odometry, the rear-wheel and body frames are nominally aligned, so the boresight error may be expressed as the relative orientation of the two frames. The small angle approximation is applied and only the yaw-axis component is considered as the other two components have little impact and are difficult to observe. The body-to-rear-wheel-frame coordinate transformation matrix may be thus be approximated as

$$\mathbf{C}_{b}^{r} \approx \begin{pmatrix} 1 & \psi_{br} & 0 \\ -\psi_{br} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{I.216}$$

The forward and cross-track speeds are assumed resolved in the rear-wheel frame, so become

$$\hat{v}_{er} \approx \frac{1}{\tau_o}\int_{t-\tau_o}^{t}\begin{pmatrix}1 & \hat{\psi}_{br} & 0\end{pmatrix}\left[\hat{\mathbf{C}}_{\gamma}^{b}(t')\hat{\mathbf{v}}_{\beta b}^{\gamma}(t')+\left(\boldsymbol{\omega}_{\beta b}^{b}(t')\wedge\mathbf{l}_{br}^{b}\right)\right]dt', \tag{I.217}$$

$$\hat{v}_{c} \approx \frac{1}{\tau_o}\int_{t-\tau_o}^{t}\begin{pmatrix}-\hat{\psi}_{br} & 1 & 0\end{pmatrix}\left[\hat{\mathbf{C}}_{\gamma}^{b}(t')\hat{\mathbf{v}}_{\beta b}^{\gamma}(t')+\left(\boldsymbol{\omega}_{\beta b}^{b}(t')\wedge\mathbf{l}_{br}^{b}\right)\right]dt'. \tag{I.218}$$

The boresight error is appended to the state vector, giving

$$\mathbf{x}^{\gamma} = \begin{pmatrix} \delta\boldsymbol{\psi}_{\gamma b}^{\gamma} \\ \delta\mathbf{v}_{\beta b}^{\gamma} \\ \vdots \\ \mathbf{b}_{g} \\ \vdots \\ s_{or} \\ s_{\Delta or} \\ \psi_{br} \end{pmatrix}, \tag{I.219}$$

The measurement matrix then becomes

$$\mathbf{H}_{O,k}^{\gamma} \approx \left. \begin{pmatrix} \mathbf{H}_{O11}^{\gamma} & \mathbf{H}_{O12}^{\gamma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{v}_{er} & \mathbf{0} & H_{O18}^{\gamma} \\ \mathbf{H}_{O21}^{\gamma} & \mathbf{0} & \mathbf{0} & -\dfrac{\cos\hat{\theta}_{nb}}{\tau_{o}}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\hat{\mathbf{C}}_{b}^{n} & \mathbf{0} & \hat{\dot{\psi}}_{nb} & \dfrac{\hat{v}_{er}}{T_{r}} & 0 \\ \mathbf{H}_{O31}^{\gamma} & \mathbf{H}_{O32}^{\gamma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & H_{O38}^{\gamma} \end{pmatrix} \right|_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}_{k}^{\gamma -}}, \tag{I.220}$$

where

$$\begin{aligned}
\mathbf{H}_{O11}^{\gamma} &= -\frac{1}{\tau_{o}} \int_{t-\tau_{o}}^{t} \begin{pmatrix} 1 & \hat{\psi}_{br} & 0 \end{pmatrix} \hat{\mathbf{C}}_{\gamma}^{b}(t') \left[\hat{\mathbf{v}}_{\beta b}^{\gamma}(t') \wedge \right] dt' \\
\mathbf{H}_{O12}^{\gamma} &= -\frac{1}{\tau_{o}} \int_{t-\tau_{o}}^{t} \begin{pmatrix} 1 & \hat{\psi}_{br} & 0 \end{pmatrix} \hat{\mathbf{C}}_{\gamma}^{b}(t') dt' \\
\mathbf{H}_{O21}^{\gamma} &\approx \frac{1}{\tau_{o}^{2}} \left[\hat{\psi}_{nb}(t) - \hat{\psi}_{nb}(t-\tau_{o}) \right] \int_{t-\tau_{o}}^{t} \sin\hat{\theta}_{nb} \begin{pmatrix} 0 & \cos\hat{\phi}_{nb} & \sin\hat{\phi}_{nb} \end{pmatrix} \hat{\mathbf{C}}_{\gamma}^{b}(t') dt' \\
\mathbf{H}_{O31}^{\gamma} &= -\frac{1}{\tau_{o}} \int_{t-\tau_{o}}^{t} \begin{pmatrix} -\hat{\psi}_{br} & 1 & 0 \end{pmatrix} \hat{\mathbf{C}}_{\gamma}^{b}(t') \left[\hat{\mathbf{v}}_{\beta b}^{\gamma}(t') \wedge \right] dt' \\
\mathbf{H}_{O32}^{\gamma} &= -\frac{1}{\tau_{o}} \int_{t-\tau_{o}}^{t} \begin{pmatrix} -\hat{\psi}_{br} & 1 & 0 \end{pmatrix} \hat{\mathbf{C}}_{\gamma}^{b}(t') dt' \\
H_{O18}^{\gamma} &= \frac{1}{\tau_{o}} \int_{t-\tau_{o}}^{t} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \left[\hat{\mathbf{C}}_{\gamma}^{b}(t') \hat{\mathbf{v}}_{\beta b}^{\gamma}(t') + \left(\boldsymbol{\omega}_{\beta b}^{b}(t') \wedge \mathbf{l}_{br}^{b} \right) \right] dt' \\
H_{O38}^{\gamma} &= -\frac{1}{\tau_{o}} \int_{t-\tau_{o}}^{t} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \left[\hat{\mathbf{C}}_{\gamma}^{b}(t') \hat{\mathbf{v}}_{\beta b}^{\gamma}(t') + \left(\boldsymbol{\omega}_{\beta b}^{b}(t') \wedge \mathbf{l}_{br}^{b} \right) \right] dt'
\end{aligned}, \tag{I.221}$$

noting that this is an EKF measurement model (for a basic Kalman filter implementation, $\hat{\psi}_{br} = 0$ must be assumed within the measurement matrix).

For Doppler radar and sonar, all three components of the boresight error must be considered. The Doppler sensor body frame is not necessarily aligned with the reference navigation system body frame. However, it is assumed that the small angle approximation may be applied to the difference between the actual and assumed alignment. The boresight error is expressed here as small angle error in the assumed attitude of the Doppler sensor body frame, *d*, with respect to the reference body frame, $\delta\boldsymbol{\psi}_{bd}^{d}$. The measurement innovation becomes

$$\delta\mathbf{z}_{D,k}^{-} = \left[\tilde{\mathbf{v}}_{ebD}^{b} \left(1 - \hat{s}_{D}\right) - \left(\mathbf{I}_{3} - \left[\delta\hat{\boldsymbol{\psi}}_{bd}^{d} \wedge \right] \right) \hat{\mathbf{C}}_{\gamma}^{b} \left(\hat{\mathbf{v}}_{eb}^{\gamma} - \hat{\mathbf{v}}_{es}^{\gamma} \right) \right]_{k} \qquad \gamma \in e, n. \tag{I.222}$$

The state vector becomes

$$\mathbf{x}^{\gamma} = \begin{pmatrix} \delta\boldsymbol{\psi}^{\gamma}_{\gamma b} \\ \delta\mathbf{v}^{\gamma}_{eb} \\ \vdots \\ s_D \\ \mathbf{v}^{\gamma}_{es} \\ \delta\boldsymbol{\psi}^{d}_{bd} \end{pmatrix} \qquad \gamma \in e, n \tag{I.223}$$

and the measurement matrix becomes

$$\mathbf{H}^{\gamma}_{D,k} = \left(-\hat{\mathbf{C}}^{b}_{\gamma}\left[\left(\hat{\mathbf{v}}^{\gamma}_{eb} - \hat{\mathbf{v}}^{\gamma}_{es}\right)\wedge\right] \quad -\hat{\mathbf{C}}^{b}_{\gamma} \quad \mathbf{0} \quad \hat{\mathbf{C}}^{b}_{\gamma}\hat{\mathbf{v}}^{\gamma}_{eb} \quad \hat{\mathbf{C}}^{b}_{\gamma} \quad \left[\left\{\hat{\mathbf{C}}^{b}_{\gamma}\left(\hat{\mathbf{v}}^{\gamma}_{eb} - \hat{\mathbf{v}}^{\gamma}_{es}\right)\right\}\wedge\right] \right)_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}^{\gamma -}_{k}}. \tag{I.224}$$

The rear-wheel odometry model may be used for 2D visual odometry and the Doppler radar model for 3D visual odometry.

I.9 Terrestrial Radio Navigation and Environmental Feature Matching

This section presents additional information on the integration of terrestrial radio navigation and environmental feature matching. Sections I.9.1 and I.9.2 describe the integration of velocity and range rate measurements, respectively. Section I.9.3 then describes the incorporation of bearing measurements when the heading is unknown.

I.9.1 Velocity Measurement Integration

Some terrestrial radio navigation systems output velocity as well as position. Depending on the requirements of the application and the degree of correlation between the velocity and position errors, it may be beneficial to input the velocity measurements to a sensor integration algorithm. Here, the extension of the loosely coupled measurement model of Section 16.3.1 to incorporate velocity measurements is described.

The following measurement innovation vectors append velocity to curvilinear and Cartesian position, respectively:

$$\delta\mathbf{z}^{n-}_{R,k} = \begin{pmatrix} \mathbf{S}_p\left[\tilde{\mathbf{p}}_{aR} - \hat{\mathbf{p}}_b - \hat{\mathbf{T}}^{p}_{r(n)}\hat{\mathbf{C}}^{n}_{b}\mathbf{l}^{b}_{ba} - \hat{\mathbf{b}}^{p}_{R}\right] \\ \hat{\mathbf{v}}^{n}_{eaR} - \hat{\mathbf{v}}^{n}_{eb} - \hat{\mathbf{C}}^{n}_{b}\left(\hat{\boldsymbol{\omega}}^{b}_{ib} \wedge \mathbf{l}^{b}_{ba}\right) + \hat{\boldsymbol{\Omega}}^{n}_{ie}\hat{\mathbf{C}}^{n}_{b}\mathbf{l}^{b}_{ba} \end{pmatrix}^{-}_{k}, \tag{I.225}$$

$$\delta\mathbf{z}^{\gamma -}_{R,k} = \begin{pmatrix} \tilde{\mathbf{r}}^{\gamma}_{\gamma aR} - \hat{\mathbf{r}}^{\gamma}_{\gamma b} - \hat{\mathbf{C}}^{\gamma}_{b}\mathbf{l}^{b}_{ba} - \hat{\mathbf{b}}^{\gamma}_{R} \\ \hat{\mathbf{v}}^{\gamma}_{\gamma aR} - \hat{\mathbf{v}}^{\gamma}_{\gamma b} - \hat{\mathbf{C}}^{\gamma}_{b}\left(\hat{\boldsymbol{\omega}}^{b}_{ib} \wedge \mathbf{l}^{b}_{ba}\right) + \boldsymbol{\Omega}^{\gamma}_{i\gamma}\hat{\mathbf{C}}^{\gamma}_{b}\mathbf{l}^{b}_{ba} \end{pmatrix}^{-}_{k}, \qquad \gamma \in i, e, l, \tag{I.226}$$

where the rotation of the reference frame with respect to an inertial frame may be neglected unless the lever arm is very large (i.e., $\boldsymbol{\Omega}^{n}_{ie}\mathbf{C}^{n}_{b}\mathbf{l}^{b}_{ba} \approx \mathbf{0}$, $\boldsymbol{\Omega}^{\gamma}_{i\gamma}\mathbf{C}^{\gamma}_{b}\mathbf{l}^{b}_{ba} \approx \mathbf{0}$). Many terrestrial radio navigation systems omit the height component, in which case, the curvilinear or local-tangent-plane-frame position may be two dimensional.

Defining the state vector as

$$\mathbf{x}^{n} = \begin{pmatrix} \mathbf{p}_b \\ \mathbf{v}^{n}_{eb} \\ \vdots \\ \mathbf{b}^{p}_{R} \end{pmatrix}, \qquad \mathbf{x}^{\gamma} = \begin{pmatrix} \mathbf{r}^{\gamma}_{\gamma b} \\ \mathbf{v}^{\gamma}_{\gamma b} \\ \vdots \\ \mathbf{b}^{\gamma}_{R} \end{pmatrix}, \qquad \gamma \in i, e, l \tag{I.227}$$

for total-state integration or

$$\mathbf{x}^n = \begin{pmatrix} \delta\mathbf{p}_b \\ \delta\mathbf{v}_{eb}^n \\ \vdots \\ \mathbf{b}_R^p \end{pmatrix}, \quad \mathbf{x}^\gamma = \begin{pmatrix} \delta\mathbf{r}_{\gamma b}^\gamma \\ \delta\mathbf{v}_{\gamma b}^\gamma \\ \vdots \\ \mathbf{b}_R^\gamma \end{pmatrix}, \quad \gamma \in i,e,l \tag{I.228}$$

for error-state integration, the measurement matrix is

$$\begin{array}{l} \mathbf{H}_{R,k}^n = \begin{pmatrix} k_R\mathbf{S}_p & \mathbf{0} & \mathbf{0} & \mathbf{S}_p \\ \mathbf{0} & k_R\mathbf{I} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \\ \mathbf{H}_{R,k}^\gamma = \begin{pmatrix} k_R\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & k_R\mathbf{I} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \gamma \in i,e,l \end{array}, \tag{I.229}$$

where k_R is 1 for total-state integration and -1 for error-state integration. The measurement noise covariance, $\mathbf{R}_R$, should account for any noise correlation between corresponding components of position and velocity.

I.9.2 Range Rate Measurement Integration

Some terrestrial radio navigation systems produce range rate or pseudo-range rate measurements as well as pseudo-range or range measurements. Here, the extension of the tightly coupled measurement model of Section 16.3.2 to incorporate range rate and pseudo-range rate measurements is described.

For pseudo-range rate measurements from passive ranging, the following rows are appended to the pseudo-range measurement innovation of (16.77):

$$\delta\mathbf{z}_{R,k}^- = \begin{pmatrix} \vdots \\ \hdashline \tilde{\dot{\rho}}_{a,C}^1 - \hat{\dot{\rho}}_{a,C}^{1-} \\ \tilde{\dot{\rho}}_{a,C}^2 - \hat{\dot{\rho}}_{a,C}^{2-} \\ \vdots \\ \tilde{\dot{\rho}}_{a,C}^m - \hat{\dot{\rho}}_{a,C}^{m-} \end{pmatrix}_k, \tag{I.230}$$

where m is the number of transmitters used, $\tilde{\dot{\rho}}_{a,C}^j$ is the measured pseudo-range rate for transmitter (or tracking channel) j, corrected for any predictable errors (e.g., ELoran ASFs), and $\hat{\dot{\rho}}_{a,C}^{j-}$ is the prior estimate thereof, calculated using

$$\hat{\dot{\rho}}_{a,C,k}^{j-} = \left[\hat{\mathbf{u}}_{aj}^{\gamma-\mathrm{T}}\left(\hat{\mathbf{v}}_{\gamma jR}^\gamma - \hat{\mathbf{v}}_{\gamma b}^\gamma - \hat{\mathbf{C}}_b^\gamma\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right) + \boldsymbol{\Omega}_{i\gamma}^\gamma\hat{\mathbf{C}}_b^\gamma\mathbf{l}_{ba}^b\right) + \delta\hat{\dot{\rho}}_c^a\right]_k^-, \quad \gamma \in i,e,l \tag{I.231}$$

or

$$\hat{\dot{\rho}}_{a,C,k}^{j-} = \left[\hat{\mathbf{u}}_{aj}^{n-\mathrm{T}}\left(\hat{\mathbf{v}}_{ejR}^n - \hat{\mathbf{v}}_{eb}^n - \hat{\mathbf{C}}_b^n\left(\hat{\boldsymbol{\omega}}_{ib}^b \wedge \mathbf{l}_{ba}^b\right) + \boldsymbol{\Omega}_{ie}^n\hat{\mathbf{C}}_b^n\mathbf{l}_{ba}^b\right) + \delta\hat{\dot{\rho}}_c^a\right]_k^- \tag{I.232}$$

where it is assumed that the range is short enough for the Sagnac effect to be neglected. For most terrestrial radio navigation systems, the transmitter will be stationary with respect to the Earth, so $\mathbf{v}_{ejR}^n = \mathbf{v}_{ejR}^e = \mathbf{v}_{ljR}^l = \mathbf{0}$. Also, the rotation of the reference frame with respect to an inertial frame may be neglected unless the lever arm is very large (i.e., $\boldsymbol{\Omega}_{ie}^n\mathbf{C}_b^n\mathbf{l}_{ba}^b \approx \mathbf{0}$, $\boldsymbol{\Omega}_{i\gamma}^\gamma\mathbf{C}_b^\gamma\mathbf{l}_{ba}^b \approx \mathbf{0}$).

For range rate measurements from two-way ranging, the following rows are appended to the range measurement innovation of (16.79):

$$\delta \mathbf{z}_{R,k}^{-} = \begin{pmatrix} \vdots \\ \hline \tilde{\dot{r}}_{a1,C} - \hat{\dot{r}}_{a1,C}^{-} \\ \tilde{\dot{r}}_{a2,C} - \hat{\dot{r}}_{a2,C}^{-} \\ \vdots \\ \tilde{\dot{r}}_{am,C} - \hat{\dot{r}}_{am,C}^{-} \end{pmatrix}_k , \qquad \text{(I.233)}$$

where $\tilde{\dot{r}}_{aj,C}$ is the corrected measured range rate for transmitter j and $\hat{\dot{r}}_{aj,C}^{-}$ is the prior estimate thereof, given by

$$\hat{\dot{r}}_{aj,C,k}^{-} = \left[\hat{\mathbf{u}}_{aj}^{\gamma -\mathrm{T}} \left(\hat{\mathbf{v}}_{\gamma jR}^{\gamma} - \hat{\mathbf{v}}_{\gamma b}^{\gamma} - \hat{\mathbf{C}}_b^{\gamma} \left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b} \right) + \boldsymbol{\Omega}_{i\gamma}^{\gamma} \hat{\mathbf{C}}_b^{\gamma} \mathbf{l}_{ba}^{b} \right) \right]_k^{-}, \qquad \gamma \in i, e, l \qquad \text{(I.234)}$$

or

$$\hat{\dot{r}}_{aj,C,k}^{-} = \left[\hat{\mathbf{u}}_{aj}^{n -\mathrm{T}} \left(\hat{\mathbf{v}}_{ejR}^{n} - \hat{\mathbf{v}}_{eb}^{n} - \hat{\mathbf{C}}_b^{n} \left(\hat{\boldsymbol{\omega}}_{ib}^{b} \wedge \mathbf{l}_{ba}^{b} \right) + \boldsymbol{\Omega}_{ie}^{n} \hat{\mathbf{C}}_b^{n} \mathbf{l}_{ba}^{b} \right) \right]_k^{-}. \qquad \text{(I.235)}$$

For pseudo-range and pseudo-range rate measurements, the state vector may be defined as

$$\mathbf{x}^{\gamma} = \begin{pmatrix} \mathbf{r}_{\gamma b}^{\gamma} \\ \mathbf{v}_{\gamma b}^{\gamma} \\ \vdots \\ \mathbf{b}_r \\ \delta\rho_c^a \\ \delta\dot{\rho}_c^a \end{pmatrix} \quad \gamma \in i, e, l, \quad \mathbf{x}^{n} = \begin{pmatrix} \mathbf{p}_b \\ \mathbf{v}_{eb}^{n} \\ \vdots \\ \mathbf{b}_r \\ \delta\rho_c^a \\ \delta\dot{\rho}_c^a \end{pmatrix} \qquad \text{(I.236)}$$

for total-state integration or

$$\mathbf{x}^{\gamma} = \begin{pmatrix} \delta\mathbf{r}_{\gamma b}^{\gamma} \\ \delta\mathbf{v}_{\gamma b}^{\gamma} \\ \vdots \\ \mathbf{b}_r \\ \delta\rho_c^a \\ \delta\dot{\rho}_c^a \end{pmatrix} \quad \gamma \in i, e, l, \quad \mathbf{x}^{n} = \begin{pmatrix} \delta\mathbf{p}_b \\ \delta\mathbf{v}_{eb}^{n} \\ \vdots \\ \mathbf{b}_r \\ \delta\rho_c^a \\ \delta\dot{\rho}_c^a \end{pmatrix} \qquad \text{(I.237)}$$

for error-state integration. These state vectors may be used for two-way ranging with the clock states omitted. The additional measurement matrix rows for the pseudo-range rate measurements are approximately

$$\mathbf{H}_{R,k}^{\gamma} \approx \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \mathbf{0} & -k_R \mathbf{u}_{a1}^{\gamma\,\mathrm{T}} & \mathbf{0} & \mathbf{0} & 0 & 1 \\ \mathbf{0} & -k_R \mathbf{u}_{a2}^{\gamma\,\mathrm{T}} & \mathbf{0} & \mathbf{0} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & -k_R \mathbf{u}_{am}^{\gamma\,\mathrm{T}} & \mathbf{0} & \mathbf{0} & 0 & 1 \end{pmatrix}_{\mathbf{x}^{\gamma} = \hat{\mathbf{x}}_k^{\gamma -}} \qquad \gamma \in i, e, n, l \qquad \text{(I.238)}$$

These measurement matrices may be used for two-way ranging if the clock state columns are omitted.

I.9.3 Bearing Measurement Integration with Unknown Heading

Bearing measurements made by radio direction finding and image matching determine the direction of a transmitter or feature with respect to the sensor body frame. However, the transmitter or feature positions will be resolved about the axes of an Earth-referenced coordinate frame. Where the host vehicle attitude and sensor boresight are known, bearings may be transformed from one frame to another enabling position information to be determined from the bearing measurements.

Where the sensor azimuth is unknown, but the roll and pitch are known, the bearing measurements may be expressed in a plane parallel to that in which the transmitter or feature positions within the horizontal plane are expressed. The need to rotate one plane to align its axes with the other may be eliminated by differencing the bearing measurements.

The measurement innovations thus become

$$\delta\mathbf{z}_{\psi,k}^{\Delta-} = \begin{pmatrix} \tilde{\psi}_{bu}^{a1} - \tilde{\psi}_{bu}^{ar} - \hat{\psi}_{lu}^{a1-} + \hat{\psi}_{lu}^{ar-} \\ \tilde{\psi}_{bu}^{a2} - \tilde{\psi}_{bu}^{ar} - \hat{\psi}_{lu}^{a2-} + \hat{\psi}_{lu}^{ar-} \\ \vdots \\ \tilde{\psi}_{bu}^{am} - \tilde{\psi}_{bu}^{ar} - \hat{\psi}_{lu}^{am-} + \hat{\psi}_{lu}^{ar-} \end{pmatrix}_k , \tag{I.239}$$

where r denotes the reference bearing. For a local-tangent-plane implementation a suitable state vector is

$$\mathbf{x}^l = \begin{pmatrix} \mathbf{r}_{lb}^l \\ \mathbf{b}_\psi \\ \vdots \end{pmatrix} \tag{I.240}$$

for total-state integration or

$$\mathbf{x}^l = \begin{pmatrix} \delta\mathbf{r}_{lb}^l \\ \mathbf{b}_\psi \\ \vdots \end{pmatrix} \tag{I.241}$$

for error-state integration, where the bias states are now expressed relative to the bias of the reference bearing. The measurement matrix is then approximately

$$\mathbf{H}_{\psi,k}^{\Delta l} \approx \begin{pmatrix} k_\psi\left(\mathbf{h}_{\psi r}^1 - \mathbf{h}_{\psi r}^r\right)^{\mathrm{T}} & \mathbf{h}_b^{1\,\mathrm{T}} & \mathbf{0} \\ k_\psi\left(\mathbf{h}_{\psi r}^2 - \mathbf{h}_{\psi r}^r\right)^{\mathrm{T}} & \mathbf{h}_b^{2\,\mathrm{T}} & \mathbf{0} \\ \vdots & \vdots & \vdots \\ k_\psi\left(\mathbf{h}_{\psi r}^m - \mathbf{h}_{\psi r}^r\right)^{\mathrm{T}} & \mathbf{h}_b^{m\,\mathrm{T}} & \mathbf{0} \end{pmatrix}, \tag{I.242}$$

where $\mathbf{h}_{\psi r}$ is given by (16.99) and (16.101), $\mathbf{h}_b$ is given by (16.86), and k_ψ is 1 for total-state integration and –1 for error-state integration.

References

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