EXAMPLE 7.1
Exact positioning from ranging in two dimensions

INPUTS:

True user position	$x^p_{pa} =$	1000	m
	$y^p_{pa} =$	100	m
Predicted user position	$\hat{x}^{p-}_{pa} =$	100	m
	$\hat{y}^{p-}_{pa} =$	0	m
Transmitter 1 position	$x^p_{p1`} =$	0	m
	$y^p_{p1`} =$	1000	m
Transmitter 2 position	$x^p_{p2`} =$	0	m
	$y^p_{p2`} =$	-1000	m

True and Measured Ranges

From (7.1),

$$r_{a1} = \sqrt{\left(x^p_{p1} - x^p_{pa}\right)^2 + \left(y^p_{p1} - y^p_{pa}\right)^2}$$
$$r_{a2} = \sqrt{\left(x^p_{p2} - x^p_{pa}\right)^2 + \left(y^p_{p2} - y^p_{pa}\right)^2}$$

In this example, there are no measurement errors, so:

$\tilde{r}_{a1} = r_{a1} =$ 1345.362405 m
$\tilde{r}_{a2} = r_{a2} =$ 1486.606875 m

First Iteration

Predicted position:

$\hat{x}^{p-}_{pa} =$ 100 m
$\hat{y}^{p-}_{pa} =$ 0 m

Calculate predicted ranges:

From (7.28), $$\hat{r}^-_{aj} = \sqrt{\left(x^p_{pj} - \hat{x}^{p-}_{pa}\right)^2 + \left(y^p_{pj} - \hat{y}^{p-}_{pa}\right)^2} \qquad j \in 1,2$$

$\hat{r}^-_{a1} =$ 1004.987562 m
$\hat{r}^-_{a2} =$ 1004.987562 m

Calculate measurement matrix:

From (7.30), $$\mathbf{H}_p = \begin{pmatrix} -\dfrac{x^p_{p1`} - \hat{x}^{p-}_{pa}}{\hat{r}^-_{a1}} & -\dfrac{y^p_{p1`} - \hat{y}^{p-}_{pa}}{\hat{r}^-_{a1}} \\ -\dfrac{x^p_{p2`} - \hat{x}^{p-}_{pa}}{\hat{r}^-_{a2}} & -\dfrac{y^p_{p2`} - \hat{y}^{p-}_{pa}}{\hat{r}^-_{a2}} \end{pmatrix}$$

$\mathbf{H}_p =$

0.099503719	-0.99503719
0.099503719	0.99503719

Update position estimate:

From (7.31), $$\begin{pmatrix} \hat{x}^{p+}_{pa} \\ \hat{y}^{p+}_{pa} \end{pmatrix} = \begin{pmatrix} \hat{x}^{p-}_{pa} \\ \hat{y}^{p-}_{pa} \end{pmatrix} + \mathbf{H}_p^{-1} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}^-_{a1} \\ \tilde{r}_{a2} - \hat{r}^-_{a2} \end{pmatrix}$$

$\tilde{r}_{a1} - \hat{r}^-_{a1} =$ 340.3748426 m
$\tilde{r}_{a2} - \hat{r}^-_{a2} =$ 481.6193126 m

$\mathbf{H}_p^{-1} =$

5.024937811	5.024937811
-0.502493781	0.502493781

$\mathbf{H}_p^{-1}\begin{pmatrix}\tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^-\end{pmatrix} =$

4130.469511	m
70.9744678	m

$\begin{pmatrix}\hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+}\end{pmatrix} =$

4230.469511	m	Error	3230.46951	m
70.9744678	m		-29.02553	m

Second Iteration

Predicted position:

$\hat{x}_{pa}^{p-} =$	4230.469511	m
$\hat{y}_{pa}^{p-} =$	70.9744678	m

Calculate predicted ranges:

From (7.28), $\hat{r}_{aj}^- = \sqrt{\left(x_{pj}^p - \hat{x}_{pa}^{p-}\right)^2 + \left(y_{pj}^p - \hat{y}_{pa}^{p-}\right)^2} \qquad j \in 1,2$

$\hat{r}_{a1}^- =$	4331.277031	m
$\hat{r}_{a2}^- =$	4363.926969	m

Calculate measurement matrix:

From (7.30), $\mathbf{H}_p = \begin{pmatrix} -\dfrac{x_{p1}^p - \hat{x}_{pa}^{p-}}{\hat{r}_{a1}^-} & -\dfrac{y_{p1}^p - \hat{y}_{pa}^{p-}}{\hat{r}_{a1}^-} \\ -\dfrac{x_{p2}^p - \hat{x}_{pa}^{p-}}{\hat{r}_{a2}^-} & -\dfrac{y_{p2}^p - \hat{y}_{pa}^{p-}}{\hat{r}_{a2}^-} \end{pmatrix}$

$\mathbf{H}_p =$

0.976725682	-0.214492291
0.969418036	0.245415305

Update position estimate:

From (7.31), $\begin{pmatrix}\hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+}\end{pmatrix} = \begin{pmatrix}\hat{x}_{pa}^{p-} \\ \hat{y}_{pa}^{p-}\end{pmatrix} + \mathbf{H}_p^{-1}\begin{pmatrix}\tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^-\end{pmatrix}$

$\tilde{r}_{a1} - \hat{r}_{a1}^- =$	-2985.914626	m
$\tilde{r}_{a2} - \hat{r}_{a2}^- =$	-2877.320094	m

$\mathbf{H}_p^{-1} =$

0.548247316	0.479166623
-2.165638516	2.181963485

$\mathbf{H}_p^{-1}\begin{pmatrix}\tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^-\end{pmatrix} =$

-3015.735433	m
188.2043399	m

$\begin{pmatrix}\hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+}\end{pmatrix} =$

1214.734078	m	Error	214.73408	m
259.1788077	m		159.17881	m

Third Iteration

Predicted position:

$\hat{x}_{pa}^{p-} =$	1214.734078	m
$\hat{y}_{pa}^{p-} =$	259.1788077	m

Calculate predicted ranges:

From (7.28), $$\hat{r}_{aj}^{-} = \sqrt{\left(x_{pj}^{p} - \hat{x}_{pa}^{p-}\right)^2 + \left(y_{pj}^{p} - \hat{y}_{pa}^{p-}\right)^2} \qquad j \in 1,2$$

$\hat{r}_{a1}^{-} =$ 1422.812327 m

$\hat{r}_{a2}^{-} =$ 1749.602855 m

Calculate measurement matrix:

From (7.30), $$\mathbf{H}_p = \begin{pmatrix} -\dfrac{x_{p1}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a1}^{-}} & -\dfrac{y_{p1}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a1}^{-}} \\ -\dfrac{x_{p2}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a2}^{-}} & -\dfrac{y_{p2}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a2}^{-}} \end{pmatrix}$$

$$\mathbf{H}_p = \begin{pmatrix} 0.85375566 & -0.520673864 \\ 0.694291321 & 0.719694075 \end{pmatrix}$$

Update position estimate:

From (7.31), $$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} = \begin{pmatrix} \hat{x}_{pa}^{p-} \\ \hat{y}_{pa}^{p-} \end{pmatrix} + \mathbf{H}_p^{-1} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^{-} \\ \tilde{r}_{a2} - \hat{r}_{a2}^{-} \end{pmatrix}$$

$\tilde{r}_{a1} - \hat{r}_{a1}^{-} =$ -77.44992257 m

$\tilde{r}_{a2} - \hat{r}_{a2}^{-} =$ -262.9959801 m

$$\mathbf{H}_p^{-1} = \begin{pmatrix} 0.737435115 & 0.533508896 \\ -0.711406164 & 0.874801427 \end{pmatrix}$$

$$\mathbf{H}_p^{-1} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^{-} \\ \tilde{r}_{a2} - \hat{r}_{a2}^{-} \end{pmatrix} = \begin{pmatrix} -197.4249875 \\ -174.9709065 \end{pmatrix} \text{ m}$$

$$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} = \begin{pmatrix} 1017.30909 \\ 84.20790124 \end{pmatrix} \text{ m}$$

Error: 17.30909 m, -15.79210 m

Fourth Iteration

Predicted position:

$\hat{x}_{pa}^{p-} =$ 1017.30909 m

$\hat{y}_{pa}^{p-} =$ 84.20790124 m

Calculate predicted ranges:

From (7.28), $$\hat{r}_{aj}^{-} = \sqrt{\left(x_{pj}^{p} - \hat{x}_{pa}^{p-}\right)^2 + \left(y_{pj}^{p} - \hat{y}_{pa}^{p-}\right)^2} \qquad j \in 1,2$$

$\hat{r}_{a1}^{-} =$ 1368.792517 m

$\hat{r}_{a2}^{-} =$ 1486.749662 m

Calculate measurement matrix:

From (7.30), $$\mathbf{H}_p = \begin{pmatrix} -\dfrac{x_{p1}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a1}^{-}} & -\dfrac{y_{p1}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a1}^{-}} \\ -\dfrac{x_{p2}^{p} - \hat{x}_{pa}^{p-}}{\hat{r}_{a2}^{-}} & -\dfrac{y_{p2}^{p} - \hat{y}_{pa}^{p-}}{\hat{r}_{a2}^{-}} \end{pmatrix}$$

$$\mathbf{H}_p = \begin{pmatrix} 0.743216432 & -0.669051071 \\ 0.684250426 & 0.729247115 \end{pmatrix}$$

Update position estimate:

From (7.31), $\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} = \begin{pmatrix} \hat{x}_{pa}^{p-} \\ \hat{y}_{pa}^{p-} \end{pmatrix} + \mathbf{H}_p^{-1} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^- \end{pmatrix}$

$\tilde{r}_{a1} - \hat{r}_{a1}^- =$	-23.43011182	m
$\tilde{r}_{a2} - \hat{r}_{a2}^- =$	-0.142787542	m

$\mathbf{H}_p^{-1} =$	0.729402536	0.669193663
	-0.684396258	0.743374831

$\mathbf{H}_p^{-1} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^- \end{pmatrix} =$	-17.18553551	m
	15.92933619	m

$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} =$	1000.123555	m	Error	0.12355	m
	100.1372374	m		0.13724	m

Fifth Iteration

Predicted position:

$\hat{x}_{pa}^{p-} =$	1000.123555	m
$\hat{y}_{pa}^{p-} =$	100.1372374	m

Calculate predicted ranges:

From (7.28), $\hat{r}_{aj}^- = \sqrt{\left(x_{pj}^p - \hat{x}_{pa}^{p-}\right)^2 + \left(y_{pj}^p - \hat{y}_{pa}^{p-}\right)^2} \qquad j \in 1,2$

$\hat{r}_{a1}^- =$	1345.362448	m
$\hat{r}_{a2}^- =$	1486.791534	m

Calculate measurement matrix:

From (7.30),
$$\mathbf{H}_p = \begin{pmatrix} -\dfrac{x_{p1}^p - \hat{x}_{pa}^{p-}}{\hat{r}_{a1}^-} & -\dfrac{y_{p1}^p - \hat{y}_{pa}^{p-}}{\hat{r}_{a1}^-} \\ -\dfrac{x_{p2}^p - \hat{x}_{pa}^{p-}}{\hat{r}_{a2}^-} & -\dfrac{y_{p2}^p - \hat{y}_{pa}^{p-}}{\hat{r}_{a2}^-} \end{pmatrix}$$

$\mathbf{H}_p =$	0.74338596	-0.668862702
	0.67267235	0.739940477

Update position estimate:

From (7.31), $\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} = \begin{pmatrix} \hat{x}_{pa}^{p-} \\ \hat{y}_{pa}^{p-} \end{pmatrix} + \mathbf{H}_p^{-1} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^- \end{pmatrix}$

$\tilde{r}_{a1} - \hat{r}_{a1}^- =$	-4.32012E-05	m
$\tilde{r}_{a2} - \hat{r}_{a2}^- =$	-0.184659409	m

$\mathbf{H}_p^{-1} =$	0.739950239	0.668871526
	-0.672681224	0.743395767

$\mathbf{H}_p^{-1} \begin{pmatrix} \tilde{r}_{a1} - \hat{r}_{a1}^- \\ \tilde{r}_{a2} - \hat{r}_{a2}^- \end{pmatrix} =$	-0.123545388	m
	-0.137245963	m

$\begin{pmatrix} \hat{x}_{pa}^{p+} \\ \hat{y}_{pa}^{p+} \end{pmatrix} =$	1000.000009	m	Error	0.00001	m
	99.99999148	m		-0.00001	m