4.4.1. Derivadas parciales e interpretación geométrica para el caso de dos variables independientes

Definición. Derivadas parciales.

Sea la función $f : S \subset \mathbb{R}^n \to \mathbb{R}$ una función definida en el conjunto abierto S de $\mathbb{R}^n$ y sea $\overline{x}_0 \in S$. Se define la derivada parcial de f con respecto a su $i-$ésima variable en el punto $\overline{x}_0$, denotada por $\dfrac{\partial f}{\partial x_i}(\overline{x}_0)$, por

$$\frac{\partial f}{\partial x_i}(\overline{x}_0) = \lim_{h \to 0} \frac{f(\overline{x}_0 + h\overline{e}_i) - f(\overline{x}_0)}{h}$$

siempre que el límite exista.

$$n = 2,\ f : S \subset \mathbb{R}^2 \to \mathbb{R}\ ; \quad z = f(x, y)$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

siempre que estos límites existan.

$$n=3,\ \ f:S\subset\mathbb{R}^3\to\mathbb{R}\ ; \qquad w=f\left(x,y,z\right)$$

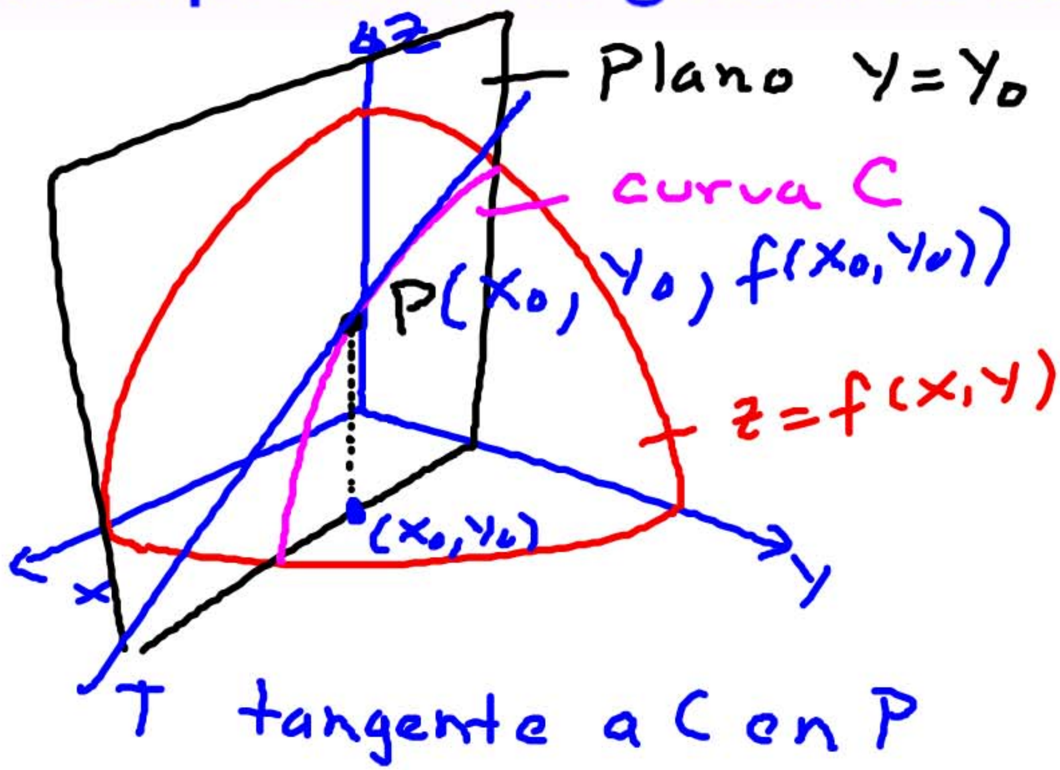
$$\frac{\partial f}{\partial x}\left(x_0,y_0,z_0\right)=\underset{h\to 0}{lím}\,\frac{f\left(x_0+h,y_0,z_0\right)-f\left(x_0,y_0,z_0\right)}{h}$$

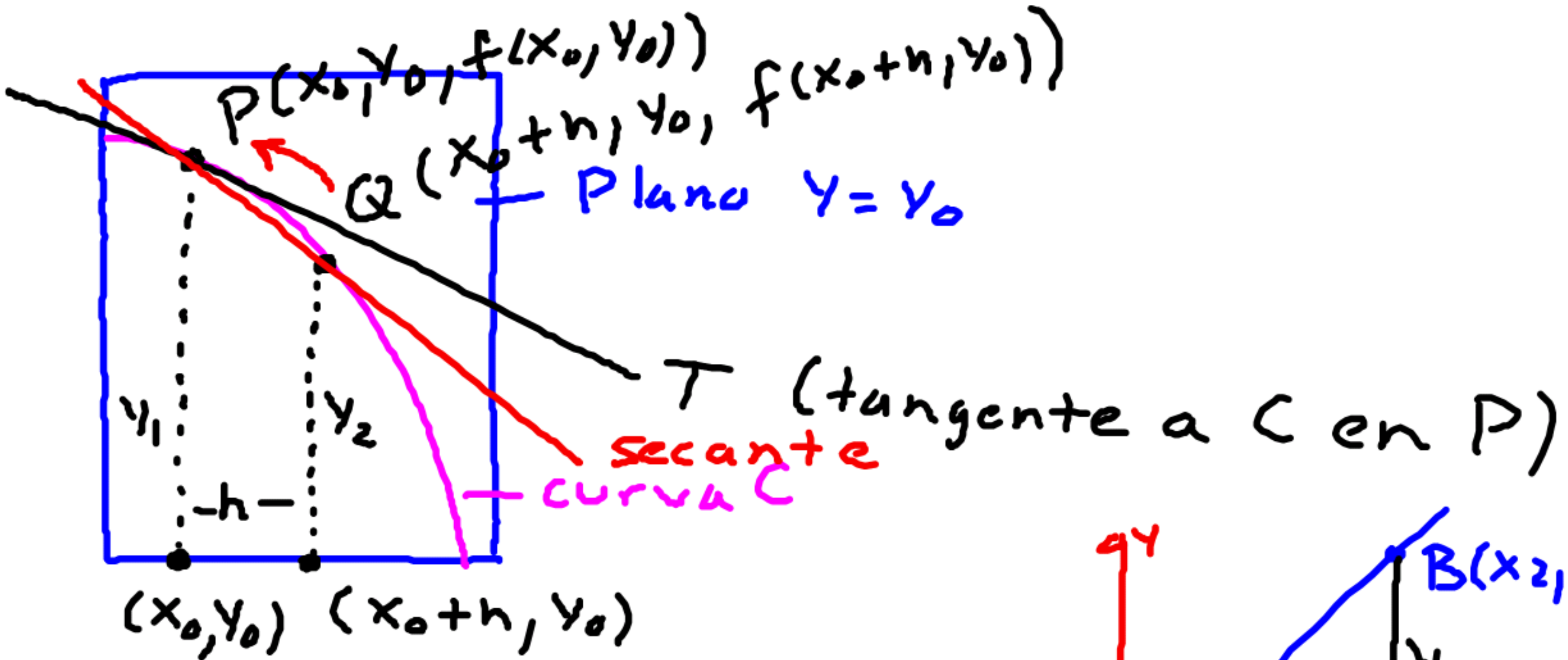
$$\frac{\partial f}{\partial y}\left(x_0,y_0,z_0\right)=\underset{h\to 0}{lím}\,\frac{f\left(x_0,y_0+h,z_0\right)-f\left(x_0,y_0,z_0\right)}{h}$$

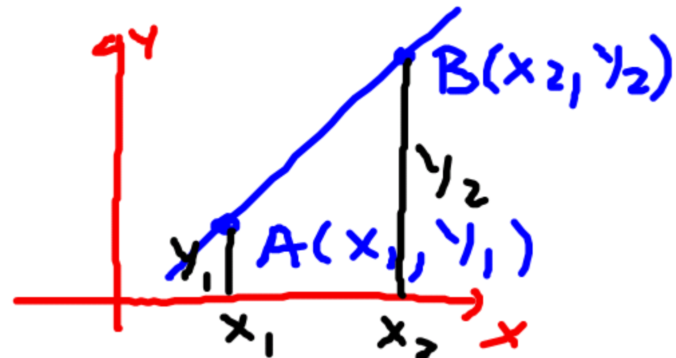
$$\frac{\partial f}{\partial z}\left(x_0,y_0,z_0\right)=\underset{h\to 0}{lím}\,\frac{f\left(x_0,y_0,z_0+h\right)-f\left(x_0,y_0,z_0\right)}{h}$$

siempre que estos límites existan.

Interpretación geométrica.







$$m_{\sec PQ} = ?$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\sec PQ} = \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$\lim_{Q \to P} m_{\sec PQ} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial x}(x_0, y_0)$$

$\frac{\partial f}{\partial x}(x_0, y_0)$ – es la pendiente de la recta tangente a C en el punto P en la dirección del eje x.

$\frac{\partial f}{\partial y}(x_0, y_0)$ – es la pendiente de la recta tangente a C en el punto P en la dirección del eje y.

Ejemplos:

1) $f(x,y) = xy + x^2y^2 + 2x^2y + 3y^3 + x^4$

$$\left.\frac{\partial f}{\partial x}\right|_{y=const} = y + 2xy^2 + 4xy + 4x^3$$

$$\left.\frac{\partial f}{\partial y}\right|_{x=const} = x + 2x^2y + 2x^2 + 9y^2$$

2) $f(x,y) = senh(xy)$

$$\left.\frac{\partial f}{\partial x}\right|_{y=const} = y\cosh(xy)$$

$$\left.\frac{\partial f}{\partial y}\right|_{x=const} = x\cosh(xy)$$

$$3)\ f(x,y)=x^{y}$$

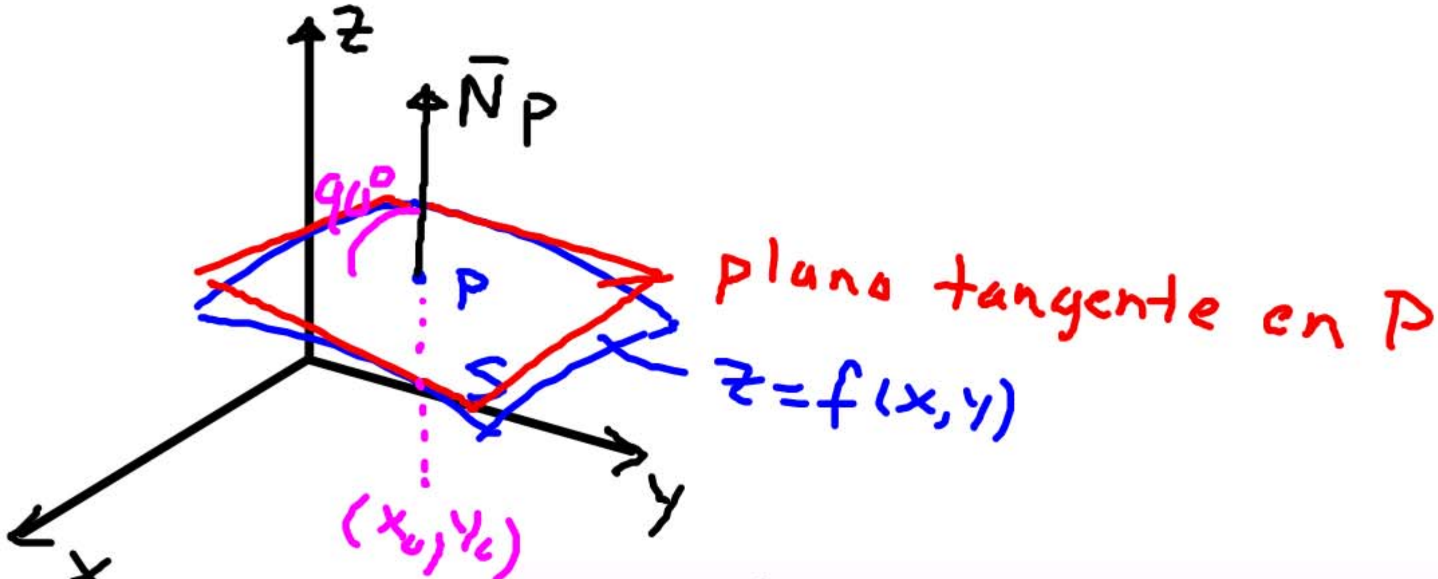
$$\left.\frac{\partial f}{\partial x}\right|_{y=\text{const}} = y\,x^{y-1}$$

$$\left.\frac{\partial f}{\partial y}\right|_{x=\text{const}} = x^{y}\ln x$$

$$(a^{x})' = a^{x}\ln a$$

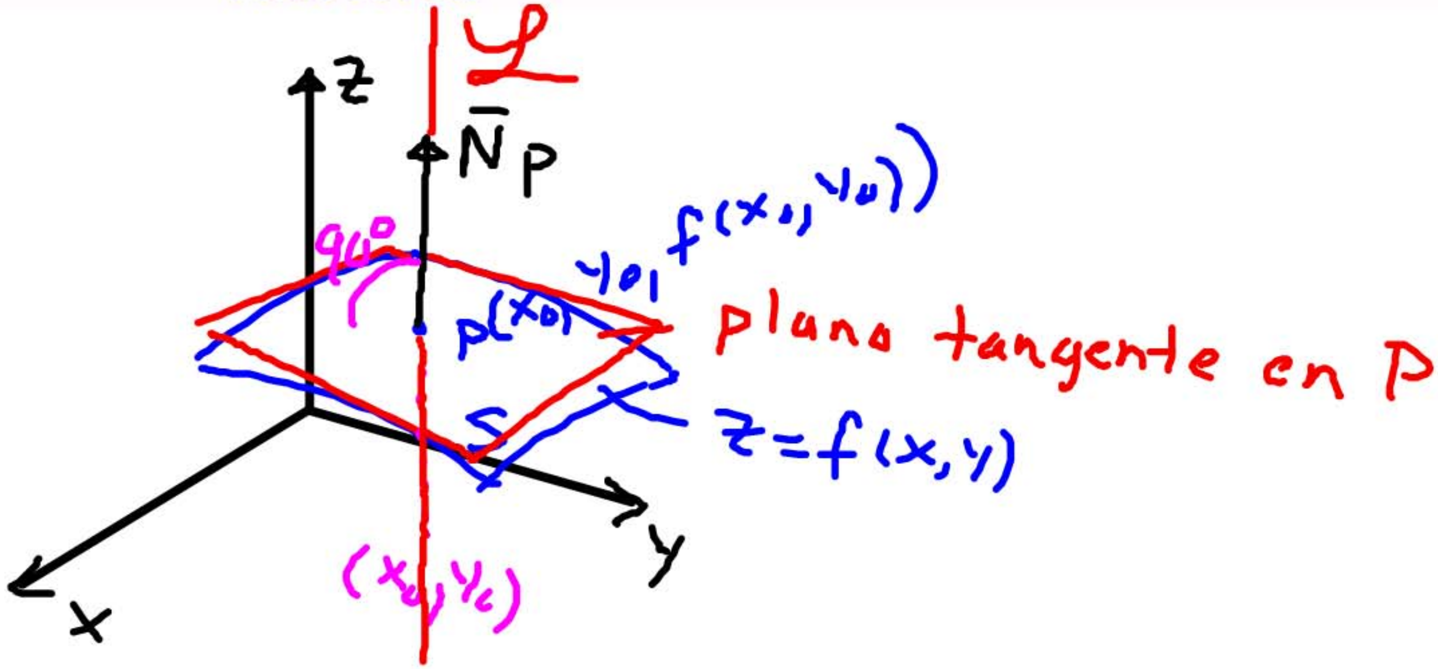
4.4.2. Vector normal a una superficie.

El vector normal a una superficie S en un punto P, será un vector perpendicular al plano tangente a la superficie en P.



$$\bar{N}_P = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1 \right)$$

4.4.3. Ecuaciones del plano tangente y de la recta normal



$[\bar{P} - \bar{P}_0] \cdot \bar{N} = 0$ Ecuación normal del plano.

$\bar{N} = \bar{N}_P$

$$\left[(x,y,z) - (x_0, y_0, f(x_0,y_0))\right] \cdot \left(-\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\right) = 0$$

$$\mathcal{L}: \begin{cases} x = x_0 - \frac{\partial f}{\partial x}(x_0, y_0)\, t \\ y = y_0 - \frac{\partial f}{\partial y}(x_0, y_0)\, t \\ z = f(x_0, y_0) + t \end{cases} ;\ t \in \mathbb{R}$$

Ecuaciones paramétricas de la recta normal.

4.5.1. derivadas parciales sucesivas.

$$z = f(x, y)$$

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

- en general son funciones de dos variables
- existen y son continuas

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$\Big\}$ mixtas

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x_0 + h, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)}{h}$$

$$\frac{\partial^2 f}{\partial y^2}(x_0, y_0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(x_0, y_0 + h) - \frac{\partial f}{\partial y}(x_0, y_0)}{h}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x_0, y_0 + h) - \frac{\partial f}{\partial x}(x_0, y_0)}{h}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(x_0 + h, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)}{h}$$

4.5.2. Teorema de derivadas parciales mixtas.

Teorema de Schwarz

Sea $f : S \subset \mathbb{R}^2 \to \mathbb{R}$ una función definida en el conjunto abierto S de $\mathbb{R}^2$. Si las derivadas

$\dfrac{\partial^2 f}{\partial x \partial y} : S \subset \mathbb{R}^2 \to \mathbb{R}$ y $\dfrac{\partial^2 f}{\partial y \partial x} : S \subset \mathbb{R}^2 \to \mathbb{R}$

(existen y) son funciones continuas en S,

entonces $\boxed{\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}}$ en cada punto de S.

Ejemplo:

$$f(x,y) = x\ senhy$$

$$\left.\frac{\partial f}{\partial x}\right|_{y=const} = senhy \qquad \left.\frac{\partial f}{\partial y}\right|_{x=const} = x\cosh y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left.\frac{\partial}{\partial x}(senhy)\right|_{y=const} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left.\frac{\partial}{\partial y}(x\cosh y)\right|_{x=const} = x\,senhy$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \left.\frac{\partial}{\partial y}(senhy)\right|_{x=const} = \cosh y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\cosh y\right)\Big|_{y=const} = \cosh y$$

clase_17_Cálculo Integral_Profesor Manny_19abril17_sem.17-2(33págs.)

4.6.1. Función diferenciable.

Definición

Sea $f: S \subset \mathbb{R}^n \to \mathbb{R}$ una función definida en el conjunto abierto S de $\mathbb{R}^n$ y sea $\bar{x}_0 \in \mathbb{R}^n$. Se dice que f es una función diferenciable en $\bar{x}_0$ si su incremento Δf puede expresarse como

$$\boxed{\Delta f = \sum_{i=1}^{n} A_i \Delta x_i + \sum_{i=1}^{n} \eta_i \Delta x_i} \qquad \eta_i(\Delta x_i)$$

donde las $\eta_i \to 0$ cuando las $\Delta x_i \to 0$ y las A_i son independientes de las Δx_i. La función f se dice que es diferenciable en una región R si es diferenciable en cada punto de R.

Ejemplo:

Probar que $f(x,y)=x^3+3y^2+5$ es diferenciable en todo punto del plano.

$$\Delta f=\sum_{i=1}^{2} A_i \Delta x_i+\sum_{i=1}^{2} \eta_i \Delta x_i$$

$$\Delta f=A_1\Delta x_1+A_2\Delta x_2+\eta_1\Delta x_1+\eta_2\Delta x_2$$

$$\underline{\Delta f=A_1\Delta x+A_2\Delta y+\eta_1\Delta x+\eta_2\Delta y}$$

$$\Delta f=f(x+\Delta x,y+\Delta y)-f(x,y)$$

$$\Delta f=(x+\Delta x)^3+3(y+\Delta y)^2+5-x^3-3y^2-5$$

$$\Delta f=\cancel{x^3}+3x^2\Delta x+3x\Delta x^2+\Delta x^3+\cancel{3y^2}+6y\Delta y+3\Delta y^2+\cancel{5}-\cancel{x^3}-\cancel{3y^2}-\cancel{5}$$

$$\Delta f = 3x^2\Delta x + 6y\Delta y + (3x\Delta x + \Delta x^2)\Delta x + (3\Delta y)\Delta y$$

$$\Delta f = A_1\Delta x + A_2\Delta y + \eta_1\Delta x + \eta_2\Delta y$$

$$A_1 = 3x^2 \,, \quad A_2 = 6y$$

$$\eta_1 = 3x\Delta x + \Delta x^2 \qquad \Delta x \to 0 \,, \ \eta_1 \to 0$$

$$\eta_2 = 3\Delta y \qquad \Delta y \to 0 \,, \ \eta_2 \to 0$$

La función es diferenciable en todo punto del plano xy.

Teorema

Si la función $f : S \subset \mathbb{R}^2 \to \mathbb{R}$ definida en el conjunto abierto S de $\mathbb{R}^2$ es diferenciable en el punto $(x_0, y_0) \in S$, entonces es continua en ese punto.

4.6.2. Diferencial total.

Definición

Sea $f : S \subset \mathbb{R}^n \to \mathbb{R}$ una función definida en el conjunto abierto S de $\mathbb{R}^n$. Su diferencial total se define como $df = \sum_{i=1}^{n} A_i \Delta x_i$ donde las A_i son constantes.

$$\Delta f = A_1 \Delta x_1 + A_2 \Delta x_2 + \cdots + A_n \Delta x_n + \eta_1 \Delta x_1 + \cdots + \eta_n \Delta x_n$$

$$\Delta x_i = 0 \quad \forall i \neq 1$$

$$\Delta f = A_1 \Delta x_1 + \eta_1 \Delta x_1 \quad \div \Delta x_1$$

$$\frac{\Delta f}{\Delta x_1} = A_1 + \eta_1$$

$$\underbrace{\lim_{\Delta x_1 \to 0} \frac{\Delta f}{\Delta x_1}}_{\frac{\partial f}{\partial x_1} = A_1} = \lim_{\Delta x_1 \to 0} A_1 + \eta_1$$

$$\Delta x_i = 0 \quad \forall i \neq 2$$

$$\Delta f = A_2 \Delta x_2 + \eta_2 \Delta x_2$$

$$\frac{\partial f}{\partial x_2} = A_2 \quad \ldots \quad \boxed{\frac{\partial f}{\partial x_i} = A_i}$$

C.D. $dx = \Delta x$

$$dx_i = \Delta x_i$$

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$ – diferencial total

La diferencial total para una función $f : S \subset \mathbb{R}^n \to \mathbb{R}$ puede ser calculada mediante

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Ejemplo:

Calcule la diferencial total de $f(x,y) = x^3 + 3y^2 + 5$

$$df = \sum_{i=1}^{2} \frac{\partial f}{\partial x_i} dx_i$$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$df = (3x^2)dx + (6y)dy$$ – diferencial total.

Comparación entre el incremento y la diferencial total.

La diferencia entre Δf y df es

$$\Delta f - df = \cancel{\sum_{i=1}^{n} A_i\Delta x_i} + \sum_{i=1}^{n} \eta_i \Delta x_i - \cancel{\sum_{i=1}^{n} A_i \Delta x_i}$$

$$\Delta f - df = \sum_{i=1}^{n} \eta_i \Delta x_i$$

$\Delta f \approx df$ — si los valores de las Δx_i son pequeños.

Ejemplo:

Determinar los valores de Δf y df para la función $f(x,y)=x^3+3y^2+5$ y los valores

$a)\ \ x=10,\ y=20;\ \ \Delta x=0.1,\ \ \Delta y=0.2$

$b)\ \ x=10,\ y=20;\ \ \Delta x=5,\ \ \Delta y=2$

$$\Delta f=3x^2\Delta x+6y\Delta y+\left(3x\Delta x+\Delta x^2\right)\Delta x+\left(3\Delta y\right)\Delta y$$

$$df=3x^2dx+6ydy$$

$$\Delta f\Big|_{a)} = 3(10)^2(0.1) + 6(20)(0.2) + \left(3 \bullet 10 \bullet (0.1) + (0.1)^2\right)(0.1) +$$
$$+3(0.2)(0.2)$$
$$\Delta f\Big|_{a)} = 54.421$$

$$df\Big|_{a)} = 3(10)^2(0.1) + 6(20)(0.2)$$
$$df\Big|_{a)} = 54$$

$$e_a = \text{Error absoluto} = \left|\Delta f - df\right| = \left|54.421 - 54\right| = 0.421$$
$$e_r = \text{Error relativo} = \frac{\left|\Delta f - df\right|}{\Delta f}100\% = \frac{0.421}{54.421}100\% = 0.7735\%$$

$$\Delta f\Big|_{b)} = 3(10)^2(5) + 6(20)(2) + \left(3\bullet 10\bullet(5) + (5)^2\right)(5) + 3(2)(2)$$

$$\Delta f\Big|_{b)} = 2627$$

$$df\Big|_{b)} = 3(10)^2(5) + 6(20)(2)$$

$$df\Big|_{b)} = 1740$$

$$e_a = |2627 - 1740| = 887$$

$$e_r = \frac{|2627 - 1740|}{2627}100\% = \frac{887}{2627}100\% = 33.76\%$$

Diferencial de orden superior.

$$z = f(x, y)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$ ← su diferencial total

$$d^2 f = d(df) = \frac{\partial}{\partial x}(df)\,dx + \frac{\partial}{\partial y}(df)\,dy$$

$$d^2 f = \left(\frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial x \partial y}dy \right)dx + \left(\frac{\partial^2 f}{\partial y \partial x}dx + \frac{\partial^2 f}{\partial y^2}dy \right)dy$$

$$d^2 f = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial x \partial y}dydx + \frac{\partial^2 f}{\partial y \partial x}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2$$

$$d^2 f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2$$

En forma simbólica

$$d^2 f = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^2 f$$

$$d^2 f = \left(\frac{\partial^2}{\partial x^2} dx^2 + 2 \frac{\partial^2}{\partial x \partial y} dx dy + \frac{\partial^2}{\partial y^2} dy^2 \right) f$$

$$d^2 f = \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2$$

La diferencial total de n-esimo orden de una función de dos variables es

$$d^n f = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n f$$

Para la función de un número mayor de variables es: $d^n f = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \cdots + \frac{\partial}{\partial x_n} dx_n \right)^n f$