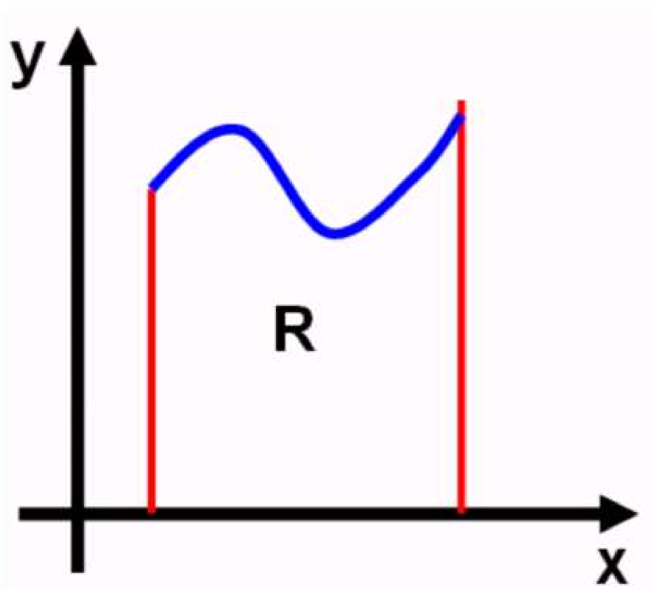
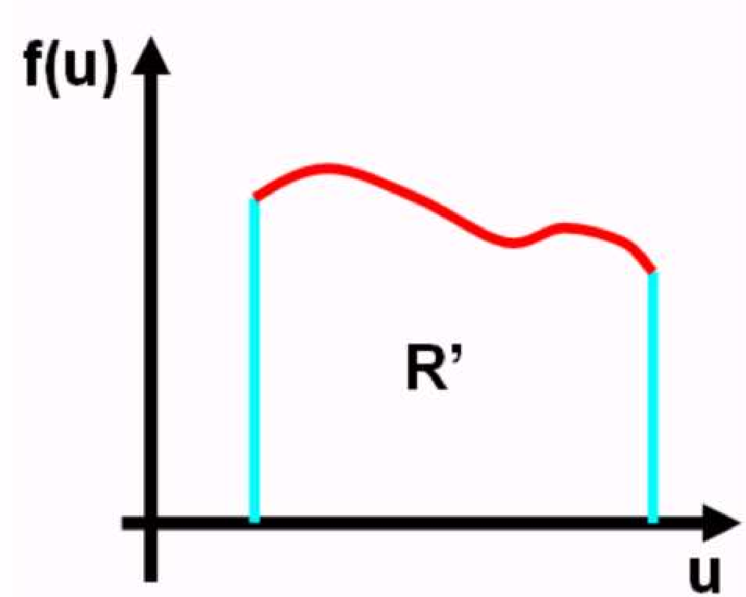
Teorema. Regla de sustitución para integrales

Suponga que g tiene una derivada continua en $[a,b]$, y sea f continua en el rango de g. Entonces

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Interpretación geométrica





$$A_R = A_{R'}$$

Ejemplos

Evalúe

1) $\int_0^2 \sqrt{2x^2+1}\, x dx$ $= \left[\begin{array}{ll} u = 2x^2+1 & x=2,\ u=9 \\ du = 4x dx & x=0,\ u=1 \end{array}\right] =$

$= \int_0^2 \sqrt{2x^2+1}\, \frac{4}{4} x dx = \frac{1}{4}\int_0^2 \sqrt{2x^2+1}\, 4x dx =$

$(9^{\frac{1}{2}})^3$

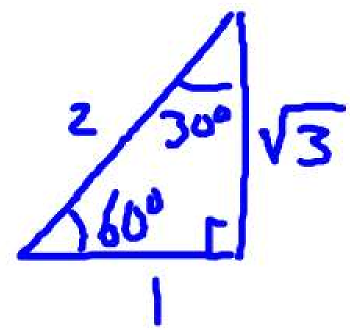
$= \frac{1}{4}\int_1^9 u^{\frac{1}{2}} du = \frac{1}{4}\frac{2}{3}u^{3/2}\Big]_1^9 = \frac{1}{6}\left[9^{3/2} - (1)^{3/2}\right] =$

$= \frac{1}{6}\left[27-1\right] = \frac{26}{6} = \underline{\frac{13}{3}}$

$$2)\quad \int_{\pi^2/9}^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \left[\begin{array}{ll} u=\sqrt{x} & x=\frac{\pi^2}{4}\rightarrow u=\frac{\pi}{2} \\ du=\frac{1}{2\sqrt{x}}\,dx & x=\frac{\pi^2}{9}\rightarrow u=\frac{\pi}{3} \end{array}\right] =$$

$$= \int_{\pi^2/9}^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,\frac{2}{2}\,dx = 2\int_{\pi^2/9}^{\pi^2/4} \cos\sqrt{x}\,\underbrace{\frac{dx}{2\sqrt{x}}}_{du} =$$

$$= 2\int_{\pi/3}^{\pi/2} \cos u\,du = 2\,\mathrm{sen}\,u\Big]_{\pi/3}^{\pi/2} =$$



$$= 2\left[\mathrm{sen}\,\frac{\pi}{2} - \mathrm{sen}\,\frac{\pi}{3}\right] = 2\left[1-\frac{\sqrt{3}}{2}\right] = \underline{2-\sqrt{3}}$$

Teorema. Teorema de simetría

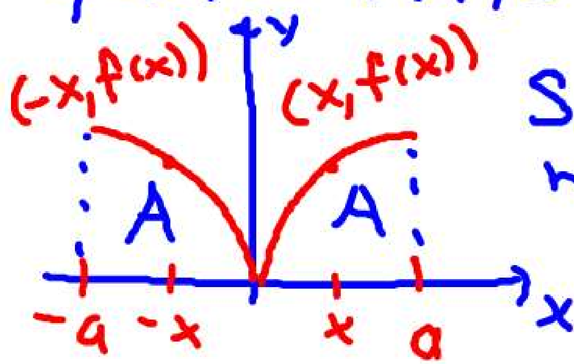
Si f es una función par, entonces

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Si f es una función impar, entonces

$$\int_{-a}^{a} f(x)\,dx = 0$$

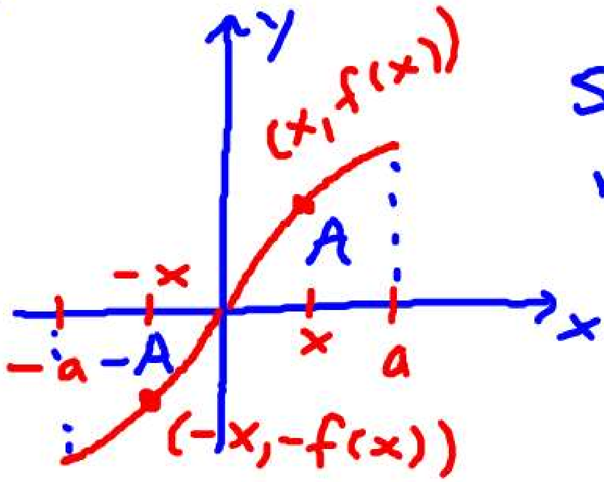
$f(x)$ – PAR $\qquad f(-x)=f(x)$



Simétrica respecto al eje y

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$f(x)$ - IMPAR $\qquad f(-x) = -f(x)$



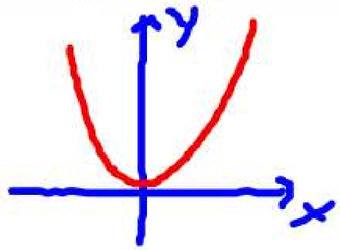
Simétrica respecto al origen.

$$\int_{-a}^{a} f(x)\,dx = A - A = 0$$

Ejemplos:

$$1)\ \int_{-1}^{1} x^2 dx = 2\int_0^1 x^2 dx = 2\left.\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

PAR



IMPAR

$$2)\ \int_{-1}^{1} x^3 dx = 0$$

2.6 Regla de L'Hôpital y sus aplicaciones a formas indeterminadas en límites de funciones.

$$\lim_{x\to 0}\frac{senx}{x} = \frac{0}{0}$$

$$\lim_{x\to 3}\frac{x^2-9}{x^2-x-6} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{senx}{x} = 1$$

$$\lim_{x\to 3}\frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x+2)} = \frac{6}{5}$$

Sean $f(x)$ y $g(x)$ dos funciones derivables en $x=a$ tales que $f(a)=g(a)=0$, y $g'(a)\neq 0$.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}=\frac{0}{0}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=$$

$$=\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}=\left[\begin{array}{ll} x-a=h & x\to a,\ h\to 0 \\ x=a+h & \end{array}\right]=$$

$$= \lim_{h \to 0} \frac{\frac{f(a+h)-f(a)}{h}}{\frac{g(a+h)-g(a)}{h}} = \frac{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}{\lim_{h \to 0} \frac{g(a+h)-g(a)}{h}} =$$

$$= \frac{f'(a)}{g'(a)}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Teorema. Regla de L'Hopital de la forma $\dfrac{0}{0}$

Suponga que $\lim\limits_{x\to u} f(x) = \lim\limits_{x\to u} g(x) = 0.$

Si el $\lim\limits_{x\to u} \dfrac{f'(x)}{g'(x)}$ existe en sentido finito ó

infinito (es decir, si su límite es el número finito

ó $-\infty$ ó bien $+\infty$), entonces

$$\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}$$

Aquí u puede sustituir a cualquiera de los símbolos

$a, a^-, a^+, -\infty,$ ó $+\infty.$

Ejemplos

$$\lim_{x\to 0}\frac{\overset{=0}{\operatorname{sen}x}}{\underset{\overset{\shortparallel}{0}}{x}} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

$$\lim_{x\to 3}\frac{\overset{=0}{x^2-9}}{\underset{\overset{\shortparallel}{0}}{x^2-x-6}} = \lim_{x\to 3}\frac{2x}{2x-1} = \frac{6}{5}$$

Teorema. Regla de L'Hopital de la forma $\frac{\infty}{\infty}$

Suponga que $\lim\limits_{x \to u} |f(x)| = \lim\limits_{x \to u} |g(x)| = \infty.$

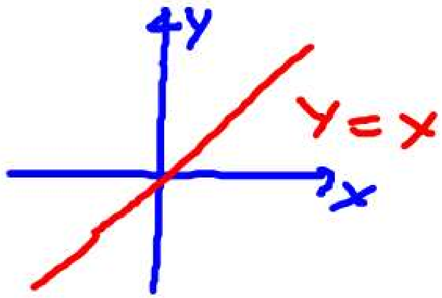
Si el $\lim\limits_{x \to u} \left[\dfrac{f'(x)}{g'(x)} \right]$ existe en sentido finito ó infinito, entonces

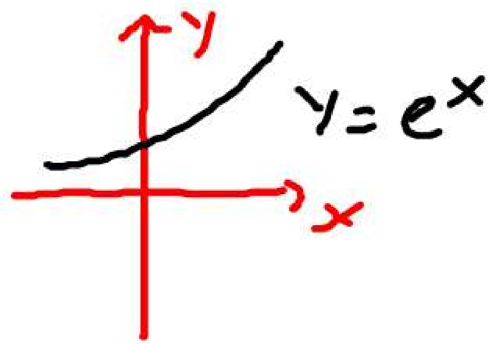
$$\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}$$

Aquí u puede sustituir a cualquiera de los símbolos $a, a^-, a^+, -\infty$, ó bien $+\infty$.

Ejemplo

$$\lim_{x\to+\infty}\frac{\overset{\infty}{x}}{\underset{\infty}{e^x}}=\lim_{x\to+\infty}\frac{1}{e^x}=0$$

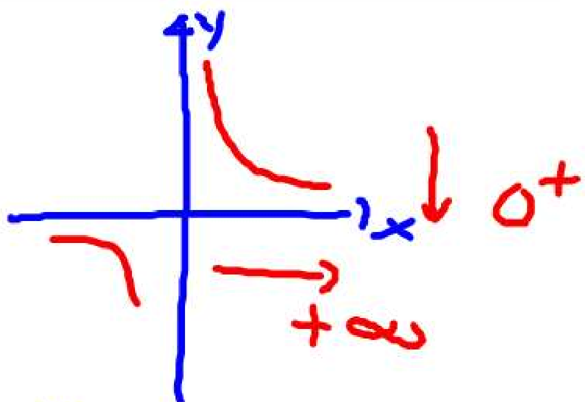




Formas indeterminadas $0\bullet\infty$ y $\infty-\infty$

Ejemplos:

$$\lim_{x\to\infty} \overset{\to\infty}{x}\,sen\left(\overset{\to 0^+}{\frac{1}{x}}\right) = \infty\cdot 0$$



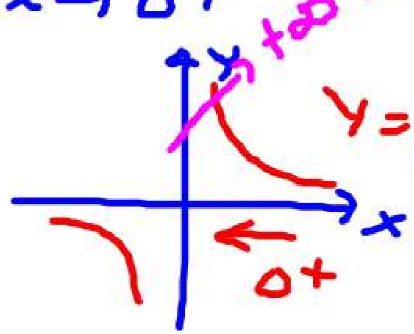
$$\lim_{x\to\infty} x\,sen\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\overset{0}{\overset{\shortparallel}{sen\left(\frac{1}{x}\right)}}}{\underset{0}{\underset{\shortparallel}{\frac{1}{x}}}} =$$

$$= \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = 1$$

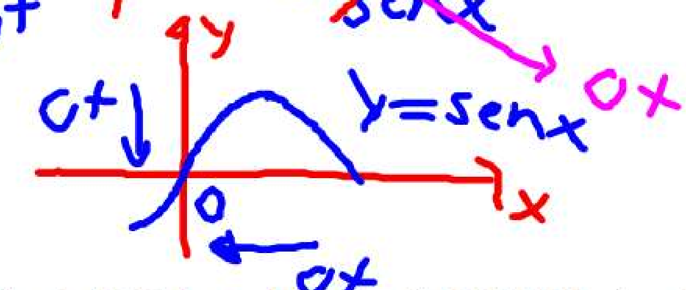
$$\lim_{x \to \infty} x \operatorname{sen}\left(\frac{1}{x}\right) = 1$$

$$\lim_{x\to0^+}\left[\frac{2}{x}-\left(\frac{3x+2}{\operatorname{sen}x}\right)\right] = \infty - \infty$$

$$\lim_{x\to0^+}\frac{2}{x} = \lim_{x\to0^+} 2\left(\overset{\to+\infty}{\frac{1}{x}}\right) = +\infty$$



$$\lim_{x\to0^+}\frac{3x+2}{\operatorname{sen}x} = \lim_{x\to0^+}\left(\overset{\to 2}{3x+2}\right)\overset{\to+\infty}{\frac{1}{\underset{\to 0^+}{\operatorname{sen}x}}} = +\infty$$



$$\lim_{x\to 0^+}\left[\frac{2}{x}-\left(\frac{3x+2}{senx}\right)\right]=\lim_{x\to 0^+}\frac{2senx-3x^2\overset{=0}{-}2x}{\underset{\overset{\shortparallel}{0}}{x\,senx}}=$$

$$=\lim_{x\to 0^+}\frac{2\cos x\overset{=0}{-}6x-2}{\underset{\overset{\shortparallel}{0}}{x\cos x+senx}}=$$

$$=\lim_{x\to 0^+}\frac{-2senx-6}{-xsenx+2\cos x}=-\frac{6}{2}=\underline{-3}$$

Formas indeterminadas $0^0, \infty^0, 1^\infty$

Ejemplo: $\lim\limits_{x \to 0^+} x^x = 0^0$

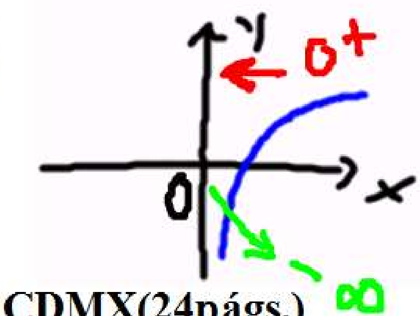
$$L = \lim_{x \to 0^+} x^x$$

$$\ln L = \ln\left[\lim_{x \to 0^+} x^x\right] \quad \text{Aplicando } \ln$$

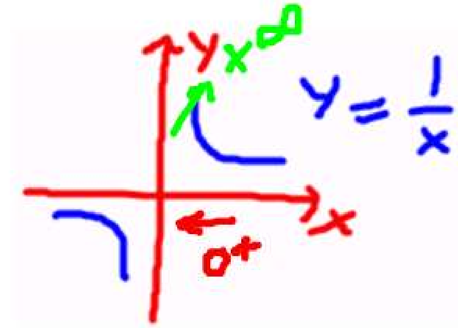
$$\ln L = \lim_{x \to 0^+} \ln x^x \quad \text{Continuidad de } \ln$$

$$\ln L = \lim_{x \to 0^+} x \ln x \quad 0 \cdot (-\infty)$$

$y = \ln x$



$$\ln L = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \qquad \frac{-\infty}{\infty}$$



$$\ln L = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

$$\ln L = 0 \quad \to \quad e^{\ln L} = e^0$$

$$L = 1$$

$$\lim_{x\to 0^+} x^x = 1$$

El número "e" como un límite.

Evaluar

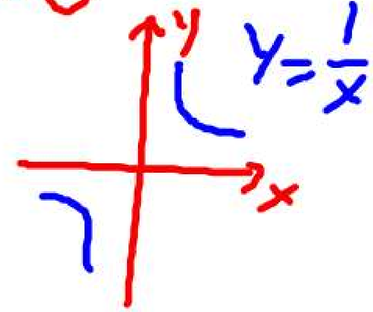
$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = 1^{\infty} \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = 1^{\infty}$$

$$L = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$

$$\ln L = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]$$

$$\ln L = \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^{x}$$

$$\ln L = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \qquad \infty\cdot 0$$



$$\ln L = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \frac{0}{0}$$

$$(\ln u)' = \frac{u'}{u}$$

$$\ln L = \lim_{x\to\infty} \frac{\frac{-\cancel{\frac{1}{x^2}}}{1+\frac{1}{x}}}{-\cancel{\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

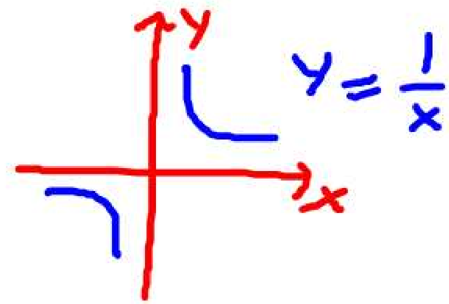
$$\ln L = 1 \rightarrow \underline{L = e}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = 1^{\infty}$$

$$L = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$\ln L = \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}$$



$$\ln L = \lim_{x\to 0}\frac{1}{x}\ln(1+x) \qquad \infty\cdot 0$$

$$\ln L = \lim_{x\to 0}\frac{\ln(1+x)}{x} \qquad \frac{0}{0}$$

$$\ln L = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$$

$$\ln L = 1$$

$$\underline{L = e}$$

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$