Integración por sustitución trigonométrica.

Integrandos que contienen $\sqrt{a^2-u^2}, \sqrt{a^2+u^2}, \sqrt{u^2-a^2}$.

Sustitución trigonométrica $(a>0)$.

1) $\sqrt{a^2-u^2} \quad \Rightarrow \quad u = a\,sen\phi \quad -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$

2) $\sqrt{a^2+u^2} \quad \Rightarrow \quad u = a\tan\phi \quad -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$

3) $\sqrt{u^2-a^2} \quad \Rightarrow \quad u = a\sec\phi \quad 0 \le \phi < \frac{\pi}{2}$ y $\pi \le \phi < \frac{3\pi}{2}$

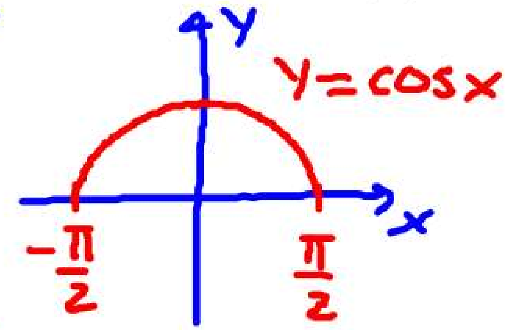
Sea ha restringido ϕ de modo que estas sustituciones sean invertibles (es decir, que seno, tangente y secante tengan inversas)

Efecto de estas sustituciones.

$$\sqrt{a^2-u^2} \;\Rightarrow\; u = a\,sen\phi$$

$$\sqrt{a^2-u^2}=\sqrt{a^2-a^2sen^2\phi}=\sqrt{a^2(1-sen^2\phi)}=$$

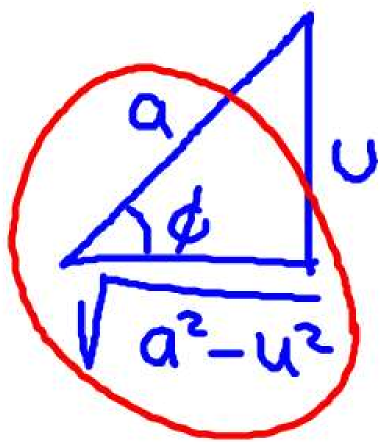
$$=\sqrt{a^2}\sqrt{\cos^2\phi}=a\left|\cos\phi\right|=$$



$$=$$

$$= a\cos\phi$$

$$sen\phi = \frac{u}{a}$$



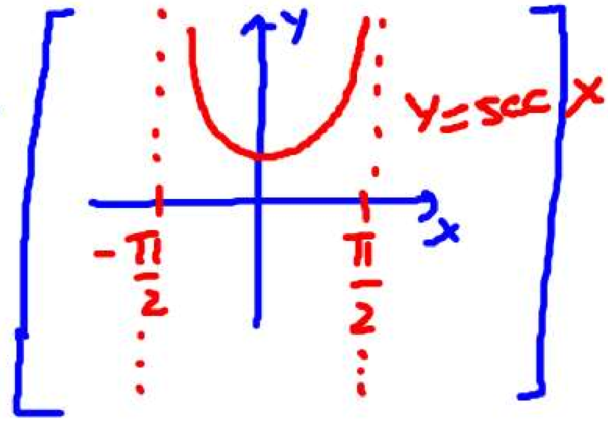
$$\cos\phi = \frac{\sqrt{a^2-u^2}}{a}$$

$$\sqrt{a^2-u^2} = a\cos\phi$$

$$\sqrt{a^2+u^2} \quad \Rightarrow \quad u = a\tan\phi$$

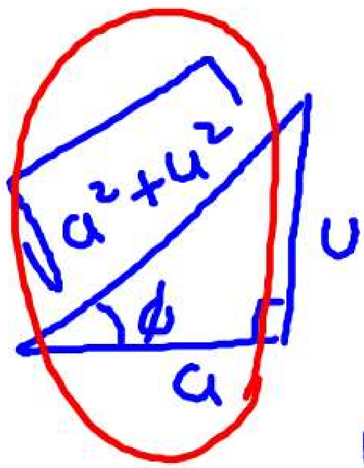
$$\sqrt{a^2+u^2}=\sqrt{a^2+a^2\tan^2\phi}=\sqrt{a^2(1+\tan^2\phi)}=$$

$$=\sqrt{a^2}\sqrt{\sec^2\phi}=a\left|\sec\phi\right|=$$



$$=a\sec\phi$$

$$\tan\phi=\frac{u}{a}$$



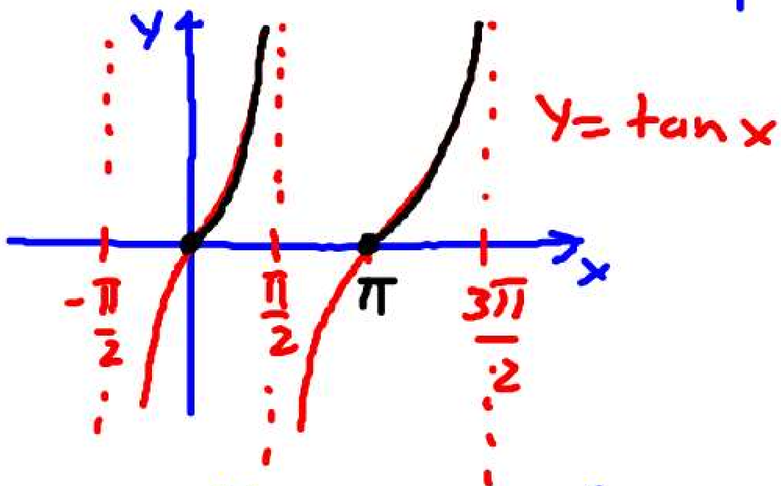
$$\sec\phi=\frac{\sqrt{a^2+u^2}}{a}$$

$$\sqrt{a^2+u^2}=a\sec\phi$$

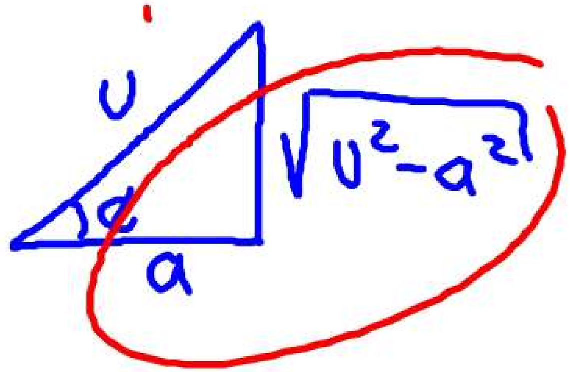
$$\sqrt{u^2 - a^2} \Rightarrow u = a\sec\phi$$

$$\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\phi - a^2} = \sqrt{a^2(\sec^2\phi - 1)} =$$

$$= \sqrt{a^2}\sqrt{\tan^2\phi} = a\left|\tan\phi\right| = a\tan\phi$$



$$\sec\phi = \frac{u}{a}$$



$$\tan\phi = \frac{\sqrt{u^2-a^2}}{a}$$

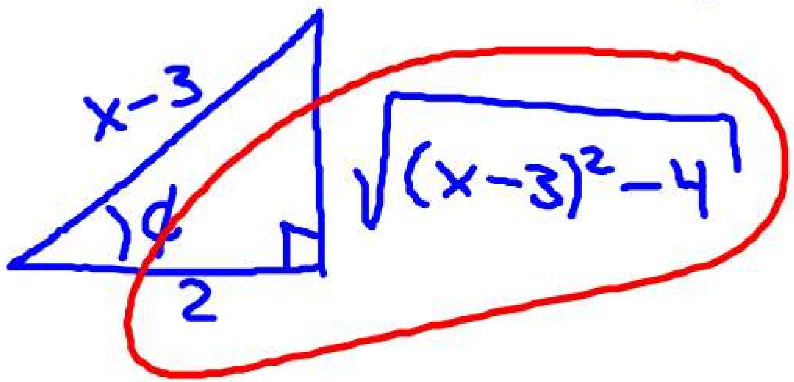
$$\sqrt{u^2-a^2} = a\tan\phi$$

Ejemplos:

$$\int \frac{x}{\sqrt{x^2-6x+5}}\,dx = \left[\; x^2-6x+5 = (x-3)^2-4 \;\right] =$$

$$= \int \frac{x}{\sqrt{(x-3)^2-4}}\,dx = \left[\; \sqrt{u^2-a^2} \longrightarrow u = a\sec\phi \quad \boxed{x-3 = 2\sec\phi} \quad x = 3+2\sec\phi \right.$$

$$dx = 2\sec\phi\tan\phi\,d\phi$$



$$\left. \sec\phi = \frac{x-3}{2} \qquad \tan\phi = \frac{\sqrt{(x-3)^2-4}}{2} \qquad \sqrt{(x-3)^2-4} = 2\tan\phi \;\right] =$$

$$= \int \frac{3 + 2\sec\phi}{\cancel{2}\,\cancel{\tan\phi}}\,\cancel{2}\sec\phi\,\cancel{\tan\phi}\,d\phi =$$

$$= 3\int \sec\phi\,d\phi + 2\int \sec^2\phi\,d\phi =$$

$$= 3\ln\left|\sec\phi + \tan\phi\right| + 2\tan\phi + C_1 =$$

$$= 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{(x-3)^2-4}}{2}\right| + \cancel{2}\,\frac{\sqrt{(x-3)^2-4}}{\cancel{2}} + C_1 =$$

$$= 3\ln\left|\frac{x-3+\sqrt{x^2-6x+5}}{2}\right| + \sqrt{x^2-6x+5} + C_1$$

$$= 3\ln\frac{\left|x-3+\sqrt{x^2-6x+5}\right|}{|2|} + \sqrt{x^2-6x+5} + C_1 =$$

$$= 3\ln\left|x-3+\sqrt{x^2-6x+5}\right| - 3\ln|2| +$$

$$\sqrt{x^2-6x+5} + C_1 =$$

$$= 3\ln\left|x-3+\sqrt{x^2-6x+5}\right| + \sqrt{x^2-6x+5} + \underbrace{C_1 - 3\ln 2}_{=\,C} =$$

$$= 3\ln\left|x-3+\sqrt{x^2-6x+5}\right| + \sqrt{x^2-6x+5} + C$$

$$\int \frac{1}{\left(x^2+1\right)^2}dx = \int \frac{1}{(x^2+1)^{\frac{4}{2}}}\,dx = \int \frac{1}{\left[(x^2+1)^{\frac{1}{2}}\right]^4}\,dx =$$

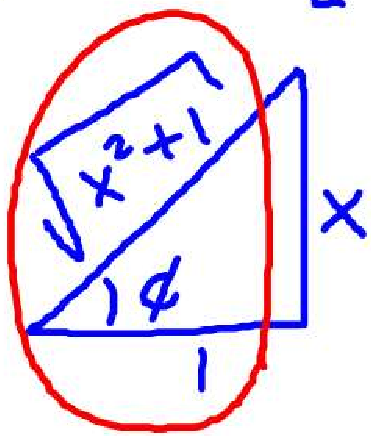
$$= \int \frac{1}{\left(\sqrt{x^2+1}\right)^4}\,dx =$$

$$\sqrt{a^2+u^2} \rightarrow u = a\tan\phi$$

$$\boxed{x = \tan\phi}$$

$$dx = \sec^2\phi\, d\phi$$

$$\tan\phi = \frac{x}{1}$$



$$\sec\phi = \sqrt{x^2+1}$$

$$=$$

$$= \int \frac{1}{\sec^{\cancel{4}\,2}\phi}\,\cancel{\sec^2\phi}\, d\phi = \int \cos^2\phi\, d\phi =$$

$$= \int \frac{1+\cos 2\phi}{2}\, d\phi = \frac{1}{2}\int \cancel{d\phi} + \frac{1}{2}\frac{1}{2}\int \cos 2\phi \, \overset{2}{d}\phi =$$

$$u = 2\phi$$

$$\cancel{2}\,\text{sen}\,\phi\cos\phi$$

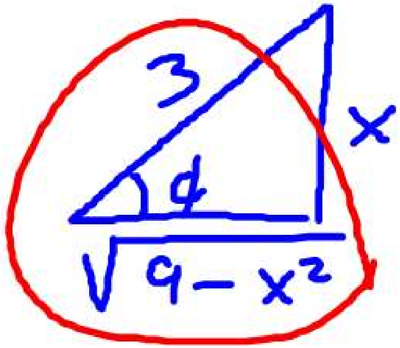
$$= \frac{1}{2}\phi + \frac{1}{\cancel{4}}\,\text{sen}\,2\phi + C =$$

$$= \frac{1}{2}\,\text{angtan}\, x + \frac{1}{2}\,\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}} + C =$$

$$= \frac{1}{2}\,\text{angtan}\, x + \frac{x}{2(x^2+1)} + C$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \left[\begin{array}{l} \sqrt{a^2-u^2} \rightarrow u = a\,\mathrm{sen}\,\phi \\ \boxed{x = 3\,\mathrm{sen}\,\phi} \\ dx = 3\cos\phi\, d\phi \end{array} \right.$$

$$\mathrm{sen}\,\phi = \frac{x}{3}$$



$$\cos\phi = \frac{\sqrt{9-x^2}}{3}$$

$$\left.\sqrt{9-x^2} = 3\cos\phi \right] =$$

$$= \int \frac{9\,\mathrm{sen}^2\phi}{\cancel{3\cos\phi}}\,\cancel{3\cos\phi}\, d\phi = 9\int \mathrm{sen}^2\phi\, d\phi =$$

$$= 9\int \frac{1-\cos 2\phi}{2}\,d\phi = \frac{9}{2}\int d\phi - \frac{1\,9}{2\,2}\int \cos \overset{=u}{2\phi}\, 2\, d\phi =$$

$$= \frac{9}{2}\phi - \frac{9}{\cancel{4}} \overset{\cancel{2}\,sen\phi\cos\phi}{sen 2\phi} + C =$$

$$= \frac{9}{2} \text{angsen}\left(\frac{x}{3}\right) - \frac{\cancel{9}}{2}\,\frac{x}{\cancel{3}} \cdot \frac{\sqrt{9-x^2}}{\cancel{3}} + C =$$

$$= \frac{9}{2} \text{angsen}\left(\frac{x}{3}\right) - \frac{1}{2} x \sqrt{9-x^2} + C$$