4.7.1. Función de función.

Sea $z = f(y)$ y $y = g(x)$

$$z = f(g(x))$$

z es función de función de la variable x.

$$z = f(u,v);\ u = u(x,y)\ ,\ v = v(x,y)$$

$$z = f(u(x,y), v(x,y))$$

z es función de función de las variables x, y.

Tipos de variables

Variables dependientes: z

Variables intermedias: u, v

Variables independientes: x, y

4.7.2. Regla de la cadena.

Sea $z = z(y_1, y_2, \ldots, y_n)$

a su vez

$$y_1 = y_1(x_1, x_2, \ldots, x_m)$$
$$y_2 = y_2(x_1, x_2, \ldots, x_m)$$
$$\vdots$$
$$y_n = y_n(x_1, x_2, \ldots, x_m)$$

V. dep. : z

V. int. : $y_1, y_2, \ldots, y_n$

V. indep. : $x_1, x_2, \ldots, x_m$

$$z = z(y_1(x_1, x_2, \ldots, x_m), y_2(x_1, x_2, \ldots, x_m), \ldots, y_n(x_1, x_2, \ldots, x_m))$$

z es función de función de las variables $x_1, x_2, \ldots, x_m$

Teorema. La regla de la cadena general.

Suponga que z es una función diferenciable de las n variables $y_1, y_2, \ldots, y_n$, y que cada una de estas variables es a su vez una función de las m variables $x_1, x_2, \ldots, x_m$. Suponga además que cada una de las derivadas parciales $\dfrac{\partial y_i}{\partial x_j}$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ existe.

Entonces z es una función de función de $x_1, x_2, \ldots, x_m$ y

$$\boxed{\frac{\partial z}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial z}{\partial y_i} \frac{\partial y_i}{\partial x_j}} \quad j = 1, 2, \ldots, m.$$

Ejemplos:

1) Sea $z = x^2 y$, donde $x = uv$, $y = u^2 - v^2$

obtenga $\dfrac{\partial z}{\partial u}$.

V. dep. : z

V. int. : x, y

V. indep. : u, v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = 2xy(v) + x^2(2u)$$

2) Sea $z = xe^{y}$, donde $x = senhu + \tan v$, $y = 3^{u} + \cos v$

obtenga $\dfrac{\partial z}{\partial v}$.

V. dep. : z

V. int. : x, y

V. indep. : u, v

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = e^{y}(\sec^2 v) + xe^{y}(-\operatorname{sen} v)$$

3) Sea $z = f\left(x^2 + y^2, \dfrac{x}{y}\right)$, obtenga $\dfrac{\partial z}{\partial y}$.

Interpretación

$U = x^2 + y^2$

$v = \dfrac{x}{y}$

v. dep. : z

v. int. : u, v

v. indep. : x, y

$z = f(u, v)$; $u = x^2 + y^2$

$v = \dfrac{x}{y}$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}\left(-\frac{x}{y^2}\right)$$

4) Sea $z = f(senh(xy))$, obtenga $\frac{\partial z}{\partial x}$.

Interpretación

V. dep.: z

V. int.: u

V. indep.: x, y

$U = senh(xy)$

$z = f(u)$, $u = senh(xy)$

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial x} = f'(u)\, y\cosh(xy)$$

$$\frac{\partial z}{\partial x} = y\cosh(xy)\, f'$$

Permanencia de la forma de la diferencial total.

Teorema

La forma de la diferencial de una función escalar de variable vectorial, compuesta, se conserva. Es decir, dada la función $f(y_1, y_2, \ldots, y_n)$ en donde

$$y_1 = y_1(x_1, x_2, \ldots, x_m)$$
$$y_2 = y_2(x_1, x_2, \ldots, x_m)$$
$$\vdots$$
$$y_n = y_n(x_1, x_2, \ldots, x_m)$$

Nota: Aquí $\Delta y_i \neq dy_i$

su diferencial está dada por

$$df = \frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2 + \cdots + \frac{\partial f}{\partial y_n} dy_n$$

A través de las variables intermedias

Ejemplo:

Sea $z = 2u + v$ a su vez $u = x^2 + y,\ v = x + y^2$ con los valores $x = 10,\ y = 10,\ dx = 0.1,\ dy = 0.2$ determine dz.

$a)$ A través de las variables intermedias.

$b)$ A través de las variables independientes.

V. dep. : z

V. int. : u, v

V. indep. : x, y

a) A través de las variables intermedias.

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$$

$$dz = 2du + dv$$

$$u = x^2 + y \qquad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$du = 2x dx + dy$$

$$du = (2)(10)(0.1) + 0.2 = 2.2$$

$$v = x + y^2 \qquad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = dx + 2y dy$$

$$dv = 0.1 + 2(10)(0.2) = 4.1$$

$$dz = 2(2.2) + 4.1 = 8.5$$

b) A través de las variables independientes.

$$z = 2(x^2 + y) + x + y^2$$

$$z = 2x^2 + 2y + x + y^2$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = (4x + 1) dx + (2 + 2y) dy$$

$$dz = (4 \cdot 10 + 1)(0.1) + (2 + 2(10))\, 0.2$$

$$dz = 4.1 + 4.4 = 8.5$$

Derivada total.

Sea $z = f\left(y_1, y_2, \ldots, y_n\right)$ a su vez

$$y_1 = y_1(t)$$

$$y_2 = y_2(t)$$

$$\vdots$$

$$y_n = y_n(t)$$

V. dep.: z

V. int.: $y_1, y_2, \ldots, y_n$

V. indep.: t

$$z = f(y_1(t), y_2(t), \ldots, y_n(t))$$

z es función de función de la variable t.

Derivada de esta función.

Se tiene una derivada ordinaria

$$\frac{dz}{dt}=\frac{\partial z}{\partial y_1}\frac{dy_1}{dt}+\frac{\partial z}{\partial y_2}\frac{dy_2}{dt}+\cdots+\frac{\partial z}{\partial y_n}\frac{dy_n}{dt}$$

$$\frac{dz}{dt}=\underbrace{\sum_{i=1}^{n}\frac{\partial z}{\partial y_i}\frac{dy_i}{dt}}_{\text{Derivada total}}$$

Su diferencial:

$$dz=\frac{\partial z}{\partial y_1}dy_1+\frac{\partial z}{\partial y_2}dy_2+\cdots+\frac{\partial z}{\partial y_n}dy_n$$

$$dz = \frac{\partial z}{\partial y_1}\frac{dy_1}{dt}dt + \frac{\partial z}{\partial y_2}\frac{dy_2}{dt}dt + \cdots + \frac{\partial z}{\partial y_n}\frac{dy_n}{dt}dt$$

$$dz = \underbrace{\left(\frac{\partial z}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial z}{\partial y_2}\frac{dy_2}{dt} + \cdots + \frac{\partial z}{\partial y_n}\frac{dy_n}{dt}\right)}_{\text{Derivada total}}dt$$

$$dz = \frac{dz}{dt}\,dt$$

Ejemplo:

Sea $z = 4x^2 - y^2;\ x = 3^t,\ y = \cosh\left(t^2\right),$

obtenga su derivada total y su diferencial.

v. dep. : z , v. int. : x, y , v. indep. : t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

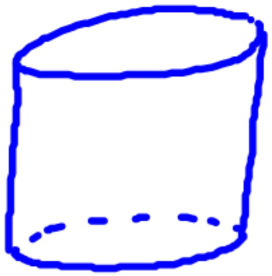
$$\frac{dz}{dt} = 8x\ 3^t \ln 3 + (-2y)\ 2t\ \operatorname{senh}(t^2)$$

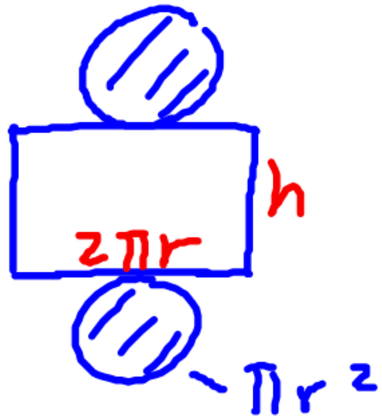
$$dz = \frac{dz}{dt}\,dt$$

$$dz = \left(8x\,3^t \ln 3 + (-2y)\,2t\,\operatorname{senh}(t^2)\right)dt$$

Ejemplo:

Suponga que se calienta un cilindro circular recto sólido y que su radio aumenta a razón de 0.2 $= \frac{dr}{dt}$ centímetros por hora y su altura a 0.5 $= \frac{dh}{dt}$ centímetros por hora. Encuentre la razón de aumento del área con respecto al tiempo, cuando el radio mide 10 centímetros y la altura 100.





$$A = 2\pi r^2 + 2\pi r h$$

$$r = r(t)$$
$$h = h(t)$$

V. dep. : A

V. int. : r, h

V. indep. : t

$$\frac{dA}{dt} = \frac{\partial A}{\partial r}\frac{dr}{dt} + \frac{\partial A}{\partial h}\frac{dh}{dt}$$

$$\frac{dA}{dt} = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt}$$

$$\frac{dA}{dt} = (4\cdot\pi\cdot 10 + 2\cdot\pi\cdot 100)(0.2) + 2\cdot\pi\cdot 10\cdot 0.5$$

$$\frac{dA}{dt} = (240\pi)(0.2) + (20\pi)\,0.5$$

$$\frac{dA}{dt} = 58\pi \quad \frac{cm^2}{hr}$$

4.8.1. Función implícita.

Sea la ecuación $F(x,y,z)=0,$ la cual define a una de las variables como función de las otras dos variables, por ejemplo $z=f(x,y)$ como función de x e y, $F(x,y,f(x,y))\equiv 0.$ Se dice que z es función implícita de x e y para distinguirla de la función explícita f.

Ejemplos:

$$1)\ x^2+y^2+z^2-1=0, \quad z=\sqrt{1-x^2-y^2}$$

$$x^2+y^2+\left(\cancel{\sqrt{1-x^2-y^2}}\right)^{\cancel{2}}-1\equiv 0$$

$$2)\ x^2+y^2+z^2-1=0, \quad x=\sqrt{1-y^2-z^2}$$

$$\left(\cancel{\sqrt{1-y^2-z^2}}\right)^{\cancel{2}}+y^2+z^2-1\equiv 0$$

$$3)\ xy - e^{z} = 0, \qquad z = \ln xy$$

$$xy - \cancel{e}^{\cancel{\ln} xy} \equiv 0$$

Derivación de funciones implícitas.

Sean $F\left(x_1, x_2, \ldots, x_n, z\right) = 0$, donde $z = f\left(x_1, x_2, \ldots, x_n\right)$

es la función implícita de las variables $x_1, x_2, \ldots, x_n$.

Encuentre $\dfrac{\partial z}{\partial x_1}$.

Hagamos $w = F\left(x_1, x_2, \ldots, x_n, z\right)$, $z = f\left(x_1, x_2, \ldots, x_n\right)$

y apliquemos la regla de la cadena.

$$\frac{\partial w}{\partial x_1}=\frac{\partial F}{\partial x_1}\frac{\partial x_1}{\partial x_1}+\frac{\partial F}{\partial x_2}\frac{\partial x_2}{\partial x_1}+\cdots+\frac{\partial F}{\partial x_n}\frac{\partial x_n}{\partial x_1}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x_1}$$

$$\underbrace{\frac{\partial w}{\partial x_1}}_{\substack{\parallel\\0}}=\frac{\partial F}{\partial x_1}\underbrace{\frac{\partial x_1}{\partial x_1}}_{\substack{\parallel\\1}}+\frac{\partial F}{\partial x_2}\underbrace{\frac{\partial x_2}{\partial x_1}}_{\substack{\parallel\\0}}+\cdots+\frac{\partial F}{\partial x_n}\underbrace{\frac{\partial x_n}{\partial x_1}}_{\substack{\parallel\\0}}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x_1}$$

$$0=\frac{\partial F}{\partial x_1}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x_1}$$

$$\frac{\partial z}{\partial x_1}=-\frac{\dfrac{\partial F}{\partial x_1}}{\dfrac{\partial F}{\partial z}},\quad \frac{\partial F}{\partial z}\neq 0$$

Las derivadas parciales de z (la función implícita) están dadas por

$$\frac{\partial z}{\partial x_i} = -\frac{\dfrac{\partial F}{\partial x_i}}{\dfrac{\partial F}{\partial z}}, \quad i = 1, 2, \ldots, n, \quad \frac{\partial F}{\partial z} \neq 0$$

Ejemplos:

1) Sea $\overbrace{x^2 + y^2 + z^2 - 3}^{F} = 0$ encuentre $\dfrac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{\cancel{2}x}{\cancel{2}z}$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

2) Sea $\overbrace{x^y + y^z + z^y}^{F} = 0$ encuentre $\dfrac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{y\,x^{y-1}}{y^z \ln y + y\,z^{y-1}}$$

4.8.2. Derivación implícita en sistemas de ecuaciones.

Sean $\begin{matrix} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{matrix}$ donde $\left.\begin{matrix} u=f(x,y) \\ v=g(x,y) \end{matrix}\right\}$ funciones implícitas

Sustituyendo tenemos $\begin{matrix} F\big(x,y,f(x,y),g(x,y)\big)\equiv 0 \\ G\big(x,y,f(x,y),g(x,y)\big)\equiv 0 \end{matrix}$

Son ciertas para todos los valores x, y en alguna región.

Con $\begin{matrix} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{matrix}$ donde $\begin{matrix} u=f(x,y) \\ v=g(x,y) \end{matrix}$

encuentre $\dfrac{\partial u}{\partial x}$.

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial G}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial G}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial G}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{\substack{\parallel \\ 1}}+\frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{\substack{\parallel \\ 0}}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial G}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{\substack{\parallel \\ 1}}+\frac{\partial G}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{\substack{\parallel \\ 0}}+\frac{\partial G}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial G}{\partial x}+\frac{\partial G}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=-\frac{\partial F}{\partial x}$$

$$\frac{\partial G}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial x}=-\frac{\partial G}{\partial x}$$

$$a_1x+b_1y=c_1$$

$$a_2x+b_2y=c_2$$

$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Por el método de Cramer

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix}-\dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial v}\\ -\dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial v}\end{vmatrix}}{\begin{vmatrix}\dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v}\\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial v}\end{vmatrix}}$$

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial v} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial v} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v} \\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial v} \end{vmatrix}}$$

Determinantes
Jacobianos

Notación:

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}$$

← Jacobiano de F y G con respecto de x y v.

← Jacobiano de F y G con respecto de u y v.

$$\frac{\partial u}{\partial x}=-\frac{J\left(\frac{F,G}{x,v}\right)}{J\left(\frac{F,G}{u,v}\right)}\begin{array}{l}\text{Jacobiano de } F \text{ y } G \text{ con respecto de } x \text{ y } v\\ \text{Jacobiano de } F \text{ y } G \text{ con respecto de } u \text{ y } v\end{array}$$

$$\frac{\partial u}{\partial y}=-\frac{J\left(\frac{F,G}{y,v}\right)}{J\left(\frac{F,G}{u,v}\right)}\begin{array}{l}\text{Jacobiano de } F \text{ y } G \text{ con respecto de } y \text{ y } v\\ \text{Jacobiano de } F \text{ y } G \text{ con respecto de } u \text{ y } v\end{array}$$

Ejemplo:

Sean $\begin{matrix} F = u^2 - v - 3x - y = 0 \\ G = u - 2v^2 - x + 2y = 0 \end{matrix}$ determine $\dfrac{\partial u}{\partial x}$.

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}} = -\frac{\begin{vmatrix} -3 & -1 \\ -1 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4v \end{vmatrix}}$$

$$\frac{\partial u}{\partial x} = -\frac{12v - 1}{-8uv + 1} \qquad \frac{\partial u}{\partial x} = \frac{1 - 12v}{1 - 8uv}$$

Generalizando

$$\text{Sean } \begin{matrix} F(x,y,z,u,v,w)=0 \\ G(x,y,z,u,v,w)=0 \\ H(x,y,z,u,v,w)=0 \end{matrix} \quad \text{donde} \quad \begin{matrix} u=f(x,y,z) \\ v=g(x,y,z) \\ w=h(x,y,z) \end{matrix}$$

La derivada $\dfrac{\partial u}{\partial x}$ está dada por $\dfrac{\partial u}{\partial x}=-\dfrac{\dfrac{\partial(F,G,H)}{\partial(x,v,w)}}{\dfrac{\partial(F,G,H)}{\partial(u,v,w)}}$

La derivada $\dfrac{\partial w}{\partial y}$ está dada por $\dfrac{\partial w}{\partial y}=-\dfrac{\dfrac{\partial(F,G,H)}{\partial(u,v,y)}}{\dfrac{\partial(F,G,H)}{\partial(u,v,w)}}$

Ejemplo:

$$F = u^3 - 2v + w - 2x + y + z = 0$$

Sean $G = w^2 - 2u + v - x + 2y - 3z = 0$, obtenga $\dfrac{\partial u}{\partial x}$.

$$H = 3v + 2u - 4w + 2x - 3y + 2z = 0$$

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G,H)}{\partial(x,v,w)}}{\dfrac{\partial(F,G,H)}{\partial(u,v,w)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{vmatrix}}$$

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} -2 & -2 & 1 \\ -1 & 1 & 2w \\ 2 & 3 & -4 \end{vmatrix}}{\begin{vmatrix} 3u^2 & -2 & 1 \\ -2 & 1 & 2w \\ 2 & 3 & -4 \end{vmatrix}}$$