3.2 Integrales de expresiones trigonométricas e integración por sustitución trigonométrica.

Integrales del tipo $\int sen^m x \cos^n x dx$

Caso 1. $m = 2k+1$, entero impar positivo

$n =$ cualquier entero

$$\int sen^{2k+1} x \cos^n x dx = \int sen^{2k} x \cos^n x \underline{sen x dx} =$$

$$= \int (sen^2 x)^k \cos^n x \, sen x dx =$$

$$= -\int (1-\cos^2 x)^k \cos^n x \overset{(-)}{sen x} dx = \begin{bmatrix} u = \cos x \\ du = -sen x dx \end{bmatrix} =$$

$$= -\int (1-u^2)^k u^n du = \cdots$$

Ejemplo: $\int sen^3 x dx$

$$\int sen^3 x\,dx = \int sen^2 x\; sen x\,dx =$$

$$= -\int (1-\cos^2 x)\,(-)\,sen x\,dx = \begin{bmatrix} u=\cos x \\ du=-sen x\,dx \end{bmatrix} =$$

$$= -\int (1-u^2)\,du = -u + \frac{u^3}{3} + c =$$

$$= \underline{-\cos x + \frac{1}{3}\cos^3 x + c}$$

$$\int sen^m x \cos^n x dx$$

Caso 2. $m =$ cualquier entero

$n = 2l + 1$, entero impar positivo

$$\int sen^m x \cos^{2l+1} x\, dx = \int sen^m x \cos^{2l} x \underline{\cos x\, dx} =$$

$$= \int sen^m x (\cos^2 x)^l \cos x dx =$$

$$= \int sen^m x (1 - sen^2 x)^l \cos x dx = \left[\begin{array}{c} u = sen x \\ du = \cos x\, dx \end{array} \right] =$$

$$= \int u^m (1 - u^2)^l du = \ldots$$

Ejemplo: $\int sen^2 x \cos^3 x dx$

$$\int sen^2 x \cos^3 x\,dx = \int sen^2 x \cos^2 x \cos x\,dx =$$

$$= \int sen^2 x\,(1 - sen^2 x)\cos x\,dx = \begin{bmatrix} u = sen x \\ du = \cos x\,dx \end{bmatrix} =$$

$$= \int u^2 (1 - u^2)\,du = \int (u^2 - u^4)\,du =$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c = \frac{1}{3}sen^3 x - \frac{1}{5}sen^5 x + c$$

$$\int sen^m x \cos^n x dx$$

Caso 3. Tanto m como n enteros positivos pares

$a)\ m = 2k,\quad n = 0$

$$\int sen^{2k} x\, dx = \int (sen^2 x)^k dx =$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^k dx = \dots$$

$b)\ m = 0,\quad n = 2k$

$$\int \cos^{2k} x dx = \int (\cos^2 x)^k dx =$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^k dx = \dots$$

$c)\ m = 2k,\quad n = 2l$

$$\int sen^{2k}x\cos^{2l}x\,dx = \int (sen^2x)^k(\cos^2x)^l\,dx =$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^k\left(\frac{1+\cos 2x}{2}\right)^l dx = \dots$$

Ejemplo: $\int sen^2 x dx$

$$\int sen^2x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\frac{1}{2}\int \cos \overset{u}{2x}\, 2\,dx =$$

$$= \frac{1}{2}x - \frac{1}{4}sen\,2x + c$$

Integrales que contienen productos seno-coseno de ángulos diferentes.

$$\int sen(mx)\,sen(nx)dx, \int sen(mx)\cos(nx)dx, \int \cos(mx)\cos(nx)dx$$

utilice las identidades trigonométricas

$$sen(mx)\,sen(nx) = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$sen(mx)\cos(nx) = \frac{1}{2}\left[sen(m-n)x + sen(m+n)x\right]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

Ejemplo: $\int \cos 2x \cos 5x\,dx$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

$$\int \cos \overset{m}{2}x \cos \overset{n}{5}x\,dx = \frac{1}{2}\int \left[\cos(2-5)x + \cos(2+5)x\right]dx$$

$$= \frac{1}{2}\frac{1}{3}\int \underset{\cos 3x}{\underset{\parallel}{\cos(-3)x}}\,3\,dx + \frac{1}{2}\frac{1}{7}\int \cos \overset{u}{7x}\,7\,dx =$$

$$= \frac{1}{6}\,\text{sen}\,3x + \frac{1}{14}\,\text{sen}\,7x + C$$

① $\operatorname{sen}(m+n)x = \operatorname{sen} mx \cos nx + \cos mx \operatorname{sen} nx$

② $\operatorname{sen}(m-n)x = \operatorname{sen} mx \cos nx - \cos mx \operatorname{sen} nx$

③ $\cos(m+n)x = \cos mx \cos nx - \operatorname{sen} mx \operatorname{sen} nx$

④ $\cos(m-n)x = \cos mx \cos nx + \operatorname{sen} mx \operatorname{sen} nx$

① + ②

$\operatorname{sen}(m+n)x + \operatorname{sen}(m-n)x = 2 \operatorname{sen} mx \cos nx$

Integrales del tipo:

$$\int \sec^m x \tan^n x dx, \int \csc^m x \cot^n x dx.$$

Caso 1. $m = 0,\ n =$ cualquier entero

$$\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx =$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) dx =$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

$u = \tan x$

$du = \sec^2 x dx$

$$\int \cot^{n}x\,dx = \int \cot^{n-2}x\,\cot^{2}x\,dx =$$

$$= \int \cot^{n-2}x\,(\csc^{2}x - 1)\,dx =$$

$$= -\int \cot^{n-2}x\,(-)\csc^{2}x\,dx - \int \cot^{n-2}x\,dx$$

$$u = \cot x$$

$$du = -\csc^{2}x\,dx$$

Ejemplo: $\int \tan^5 x dx$

$$\int \tan^5 x\, dx = \int \tan^3 x \tan^2 x\, dx =$$

$$= \int \tan^3 x (\sec^2 x - 1) dx = \int \tan^3 x \sec^2 x dx - \int \tan^3 x dx =$$

$u = \tan x$

$du = \sec^2 x dx$

$$= \frac{1}{4}\tan^4 x - \int \tan x \overset{\sec^2 x - 1}{\tan^2 x}\, dx =$$

$$= \frac{1}{4}\tan^4 x - \int \tan x \sec^2 x dx + \int \tan x dx =$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + c$$

$$\int \sec^m x \tan^n x\,dx, \int \csc^m x \cot^n x\,dx.$$

Caso 2. $m = 2k$ entero positivo par,

$n =$ cualquier entero

$$\int \sec^{2k} x \tan^n x\,dx = \int \sec^{2k-2} x \tan^n x \sec^2 x\,dx =$$

$$= \int (\sec^2 x)^{k-1} \tan^n x \sec^2 x\,dx =$$

$$= \int (1+\tan^2 x)^{k-1} \tan^n x \sec^2 x\,dx = \begin{bmatrix} u = \tan x \\ du = \sec^2 x\,dx \end{bmatrix}$$

$$= \int (1+u^2)^{k-1} u^n\,du = \ldots$$

$$\int \csc^{2k}x \cot^n x\,dx = \int \csc^{2k-2}x \cot^n x \underline{\csc^2 x\,dx} =$$

$$= \int (\csc^2 x)^{k-1} \cot^n x \csc^2 x\,dx$$

$$= (-)\int (1+\cot^2 x)^{k-1} \cot^n x\,(-)\csc^2 x\,dx = \begin{bmatrix} u = \cot x \\ du = -\csc^2 x\,dx \end{bmatrix}$$

$$= -\int (1+u^2)^{k-1} u^n\,du = \dots$$

Ejemplo: $\int \csc^4 x \cot^5 x dx$

$$\int \csc^4 x \cot^5 x\, dx = \int \csc^2 x \cot^5 x \underline{\csc^2 x\, dx} =$$

$$= (-)\int (1+\cot^2 x)\cot^5 x (-)\csc^2 x\, dx = \left[\begin{array}{c} u=\cot x \\ du=-\csc^2 x\, dx \end{array}\right]$$

$$= -\int (1+u^2)u^5\, du = -\int (u^5+u^7)\, du =$$

$$= -\frac{1}{6}u^6 - \frac{1}{8}u^8 + c = \underline{-\frac{1}{6}\cot^6 x - \frac{1}{8}\cot^8 x + c}$$

$$\int \sec^m x \tan^n x\,dx,\ \int \csc^m x \cot^n x\,dx.$$

Caso 3. $m =$ cualquier entero

$n = 2l + 1$ entero positivo impar

$$\int \sec^m x \tan^{2l+1} x\,dx = \int \sec^{m-1} x \tan^{2l} x \ \underline{\sec x \tan x}\,dx =$$

$$= \int \sec^{m-1} x\,(\tan^2 x)^l \sec x \tan x\,dx =$$

$$= \int \sec^{m-1} x\,(\sec^2 x - 1)^l \sec x \tan x\,dx = \begin{bmatrix} u = \sec x \\ du = \sec x \tan x\,dx \end{bmatrix}$$

$$= \int u^{m-1} (u^2-1)^l\,du = \ldots$$

$$\int \csc^{m} x \cot^{2\ell+1} x \, dx = \int \csc^{m-1} x \cot^{2\ell} x \, \underline{\csc x \cot x} \, dx$$

$$= \int \csc^{m-1} x \left(\cot^{2} x\right)^{\ell} \csc x \cot x \, dx =$$

$$= (-)\int \csc^{m-1} x \left(\csc^{2} x - 1\right)^{\ell} (-)\csc x \cot x \, dx = \left[\begin{array}{l} u = \csc x \\ du = -\csc x \cot x \, dx \end{array}\right]$$

$$= -\int u^{m-1} \left(u^{2} - 1\right)^{\ell} du = \dots$$

Ejemplo: $\int \sec^7 x \tan^5 x dx$

$$\int \sec^7 x \tan^5 x\, dx = \int \sec^6 x \tan^4 x \underline{\sec x \tan x}\, dx =$$

$$= \int \sec^6 x (\tan^2 x)^2 \sec x \tan x\, dx =$$

$$= \int \sec^6 x (\sec^2 x - 1)^2 \sec x \tan x\, dx = \left[\begin{array}{l} u = \sec x \\ du = \sec x \tan x\, dx \end{array}\right]$$

$$= \int u^6 \overset{u^4 - 2u^2 + 1}{(u^2 - 1)^2}\, du = \int (u^{10} - 2u^8 + u^6)\, du =$$

$$= \frac{1}{11} u^{11} - \frac{2}{9} u^9 + \frac{1}{7} u^7 + C = \underline{\frac{1}{11} \sec^{11} x - \frac{2}{9} \sec^9 x + \frac{1}{7} \sec^7 x + C}$$

$$\int \sec^m x \tan^n x\,dx,\ \int \csc^m x \cot^n x\,dx.$$

Caso 4. $m = 2k+1$ (entero positivo impar)

$n = 2l$ entero no negativo

$$\int \sec^{2k+1} x \tan^{2l} x\,dx = \int \sec^{2k+1} x \left(\tan^2 x\right)^{l} dx =$$

$$= \int \sec^{2k+1} x \left(\sec^2 x - 1\right)^{l} dx = \left[\begin{array}{l}\text{A continuación} \\ \text{por partes}\end{array}\right]$$

$$\int \csc^{2k+1} x \cot^{2l} x\,dx = \int \csc^{2k+1} x \left(\csc^2 x - 1\right)^{l} dx =$$

$$= \left[\begin{array}{l}\text{A continuación} \\ \text{por partes}\end{array}\right]$$

Ejemplo: $\int \sec x \tan^2 x dx$

$$\int \sec x \tan^2 x dx = \int \sec x (\sec^2 x - 1) dx =$$

$$= \int \sec^3 x dx - \int \sec x dx$$

$$\int \sec^3 x dx = \int \sec x \sec^2 x dx = \begin{bmatrix} u = \sec x & du = \sec x \tan x dx \\ dv = \sec^2 x dx & v = \tan x \end{bmatrix}$$

$$\int u dv = uv - \int v du$$

$$= \sec x \tan x - \int \sec x \tan^2 x dx$$

$$\int \sec x \tan^2 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$
$$- \int \sec x\,dx$$

$$2\int \sec x \tan^2 x\,dx = \sec x \tan x - \int \sec x\,dx$$

$$\int \sec x \tan^2 x\,dx = \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$