

TEMA 2

LAS INTEGRALES DEFINIDA E INDEFINIDA

2.1 Concepto de sumas de Riemann.

El problema del área.

Área (palabra)- concepto intuitivo

Parte del problema: precisar una idea intuitiva mediante una definición exacta

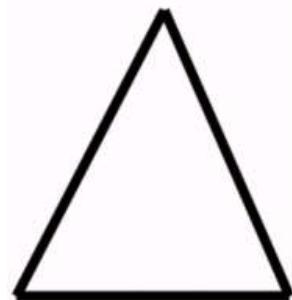
De geometría elemental:

Área = número de cuadrados de lado unidad que caben en una región.

Ejemplos:



$$A = b \cdot h$$

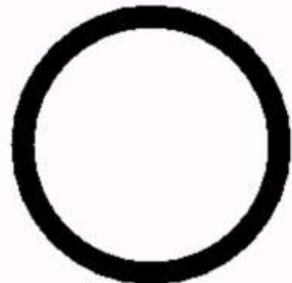


$$A = \frac{b \cdot h}{2}$$



$$A_T = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7$$

1. El círculo



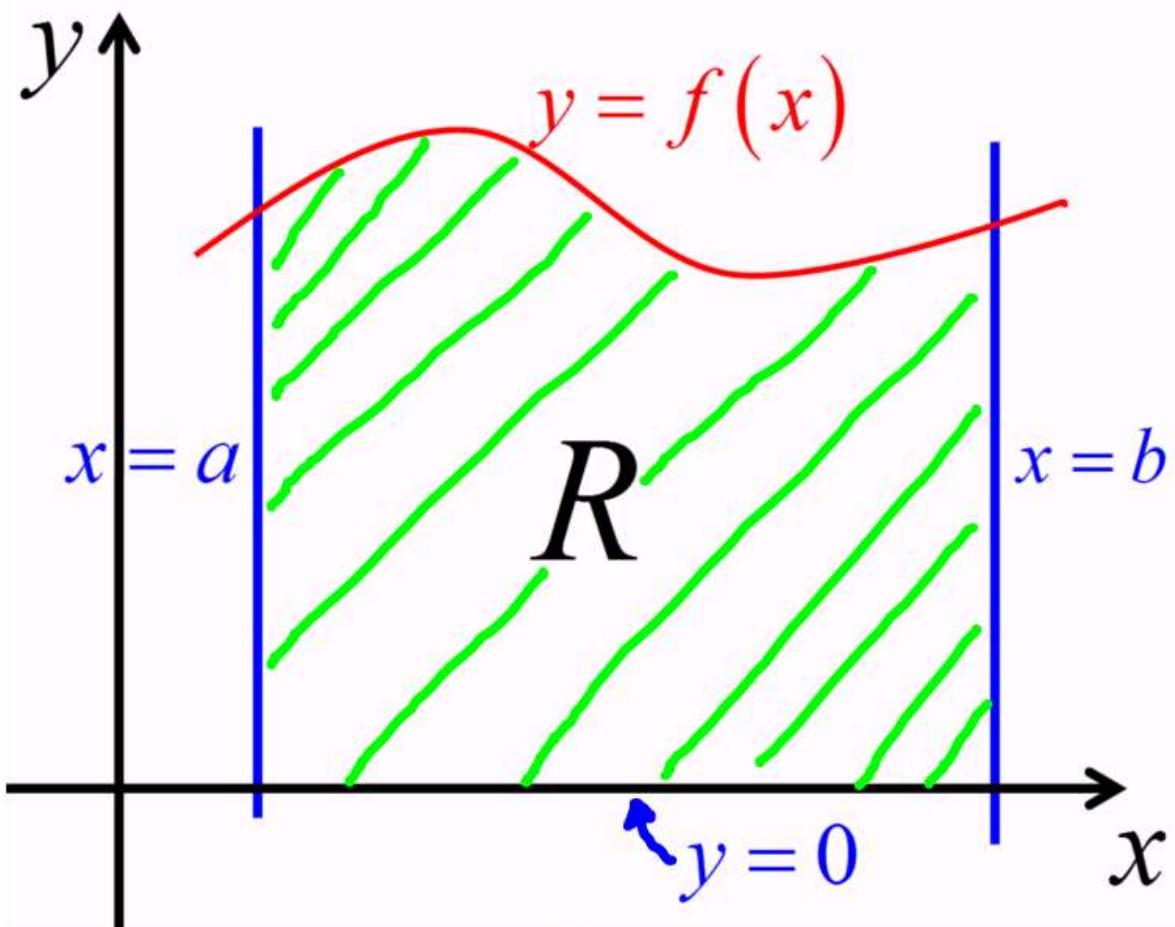
$$A = \pi r^2$$

$$r = 1, \quad A = \pi \text{ cuadrados}$$

$$r = \frac{1}{\sqrt{\pi}}, \quad A = 1 \text{ cuadrado}$$

La definición de área es inadecuada

2. Lados curvos



$$A_R = ?$$

Notación "suma abreviada"

Sea $f(x) = x^2$ ó $y = x^2$

$$D_f = \mathbb{R} \quad R_f = [0, \infty)$$

s, t, u, v x, y, z } variables
que toman
valores
reales

i, j, k, l, m, n } variables
que toman
valores
enteros

Al calcular áreas y evaluar integrales aparecen sumas como

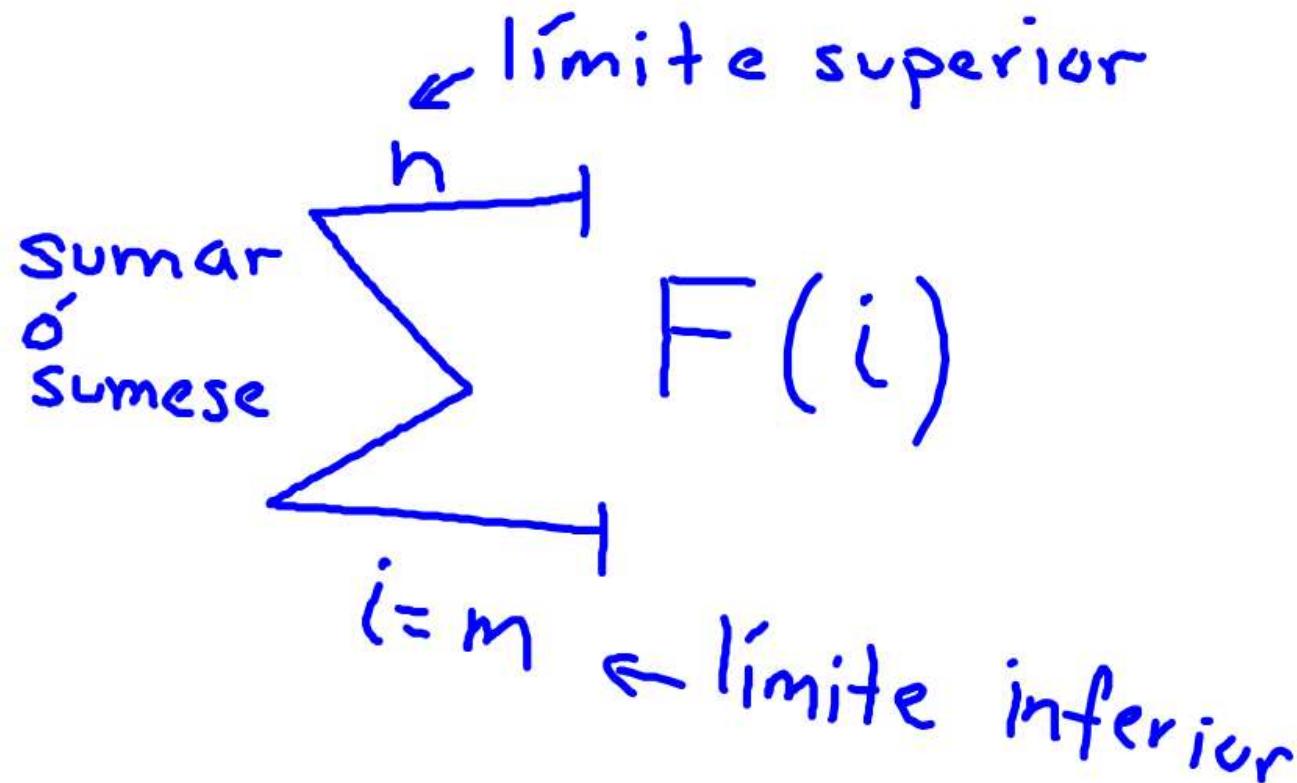
$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + 100^2$$

$$a_1 + a_2 + a_3 + \cdots + a_n$$

Definición de la notación sigma.

$$\sum_{i=m}^n F(i) = F(m) + F(m+1) + F(m+2) + \cdots + F(n-1) + F(n)$$

donde m y n son números enteros, y $m \leq n$



Ejemplos:

$$1) \sum_{i=1}^n i = 1 + 2 + 3 + \dots + (n-1) + n$$

$$F(i) = i$$

$$i=1, F(1) = 1$$

$$i=2, F(2) = 2$$

$$2) \sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2$$

$$F(i) = i^2$$

$$F(1) = 1^2 \quad F(2) = 2^2$$

$$3) \sum_{i=1}^n i^3 = 1^3 + 2^3 + 3^3 + \dots + n^3$$

$$4) \sum_{i=1}^n i^4 = 1^4 + 2^4 + \dots + n^4$$

$$5) \sum_{i=1}^5 2 = 2 + 2 + 2 + 2 + 2$$

$$F(i) = 2$$

$$F(1) = 2$$

$$F(2) = 2$$

$$6) \sum_{k=1}^n \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

$$7) \sum_{i=1}^n A_i = A_1 + A_2 + \cdots + A_n$$

$$8) \sum_{i=1}^n f(x_i^*) \Delta x_i = f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \\ f(x_3^*) \Delta x_3 + \cdots + f(x_n^*) \Delta x_n$$

Teorema

Si c es cualquier constante; esto es, si no depende de i , entonces

$$1) \sum_{i=1}^n cF(i) = c \sum_{i=1}^n F(i)$$

$$2) \sum_{i=1}^n [F(i) + G(i)] = \sum_{i=1}^n F(i) + \sum_{i=1}^n G(i)$$

$$3) \sum_{i=1}^n [F(i) - G(i)] = \sum_{i=1}^n F(i) - \sum_{i=1}^n G(i)$$

$$4) \sum_{i=a}^b F(i) = \sum_{i=a+c}^{b+c} F(i - c)$$

$$5) \sum_{i=a}^b F(i) = \sum_{i=a-c}^{b-c} F(i + c)$$

Teorema

Sean c una constante y n un número entero positivo. Entonces

$$1) \sum_{i=1}^n 1 = n$$

$$\sum_{i=1}^n 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ términos}} = n(1)$$

$$2) \sum_{i=1}^n c = nc$$

$$3) \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$4) \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$5) \sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2 \quad \checkmark$$

$$6) \sum_{i=1}^n i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

Ejemplos:

1) Calcule la suma $\sum_{i=2}^5 (i-1)^3$

$$\begin{aligned}\sum_{i=2}^5 (i-1)^3 &= \sum_{i=2-1}^{5-1} (i-1+1)^3 = \\ &= \sum_{i=1}^4 i^3 = \left[\frac{4(4+1)}{2} \right]^2 = 100 \\ n &= 4\end{aligned}$$

$$2) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{2i}{n} \right)^2 + 1 \right] \frac{2}{n} = \lim_{h \rightarrow \infty} \sum_{i=1}^n \left(\frac{4i^2}{n^2} + 1 \right) \frac{2}{h} =$$

$$= \lim_{h \rightarrow \infty} \sum_{i=1}^n \left(\frac{8}{h^3} i^2 + \frac{2}{h} \right) =$$

$$= \lim_{h \rightarrow \infty} \sum_{i=1}^n \frac{8}{h^3} i^2 + \sum_{i=1}^n \frac{2}{h} =$$

$$= \lim_{h \rightarrow \infty} \frac{8}{h^3} \sum_{i=1}^n i^2 + \frac{2}{h} \sum_{i=1}^n 1 =$$

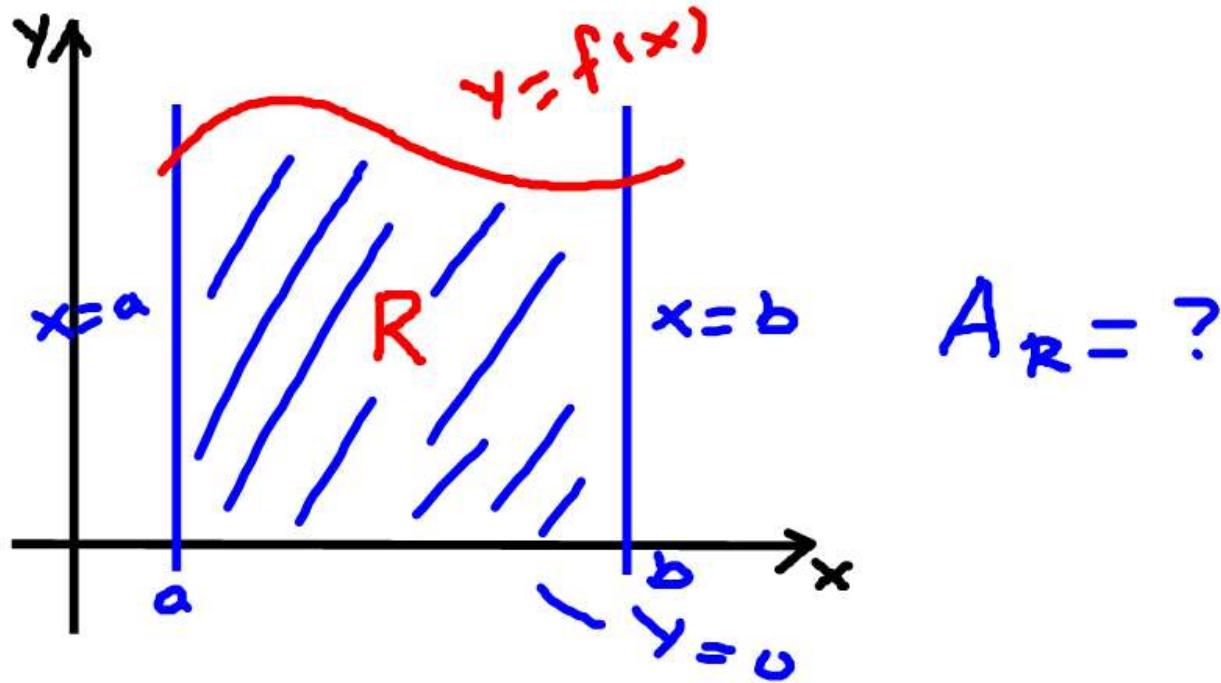
$$= \lim_{n \rightarrow \infty} \frac{\cancel{8} \cancel{n(n+1)(2n+1)}}{\cancel{n^3}^2 6} + \cancel{\frac{2}{n}}(n) =$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} \frac{(n+1)(2n+1)}{n^2} + 2 =$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} \frac{n+1}{n} \cdot \frac{2n+1}{n} + 2 =$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) + 2 = \frac{8}{3} + 2 = \frac{14}{3}$$

Sumas de Riemann



$$A_R = ?$$

$[a, b]$ - partición arbitraria

x_0, x_1, \dots, x_n - puntos de la partición

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ - subintervalos

$[x_{i-1}, x_i]$ – i-ésimo subintervalo

x_i – extremo derecho del subintervalo

x_{i-1} – extremo izquierdo del subintervalo

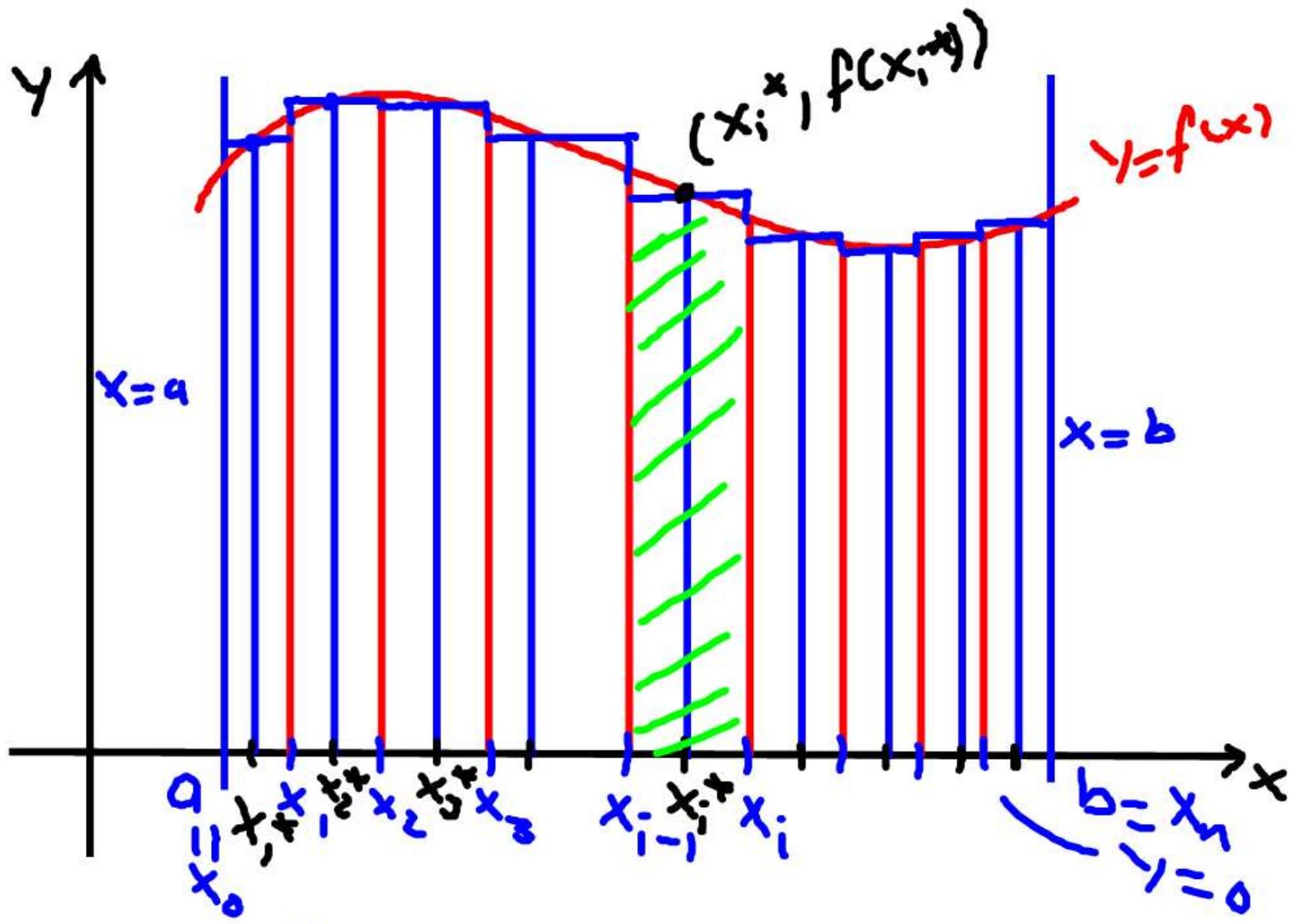
$\Delta x_i = x_i - x_{i-1}$ – la longitud del i-ésimo subintervalo

$\|P\|$ – norma de la partición

$\|P\| = \max\{\Delta x_i\}$ – la longitud del mayor de los subintervalos

x_i^* – punto muestra

$$[x_{i-1}, x_i] \quad \left\{ \begin{array}{l} x_i^* \in (x_{i-1}, x_i) \\ x_i^* = x_{i-1}, \quad x_i^* = x_i \end{array} \right.$$



$$A_i = (x_i - x_{i-1}) f(x_i^*)$$

$$A_i = f(x_i^*) \Delta x_i$$

$$A_{\text{aprox.}} = \sum_{i=1}^{n+1} A_i = \underbrace{\sum_{i=1}^{n+1} f(x_i^*) \Delta x_i}_{\text{Suma de Riemann}}$$

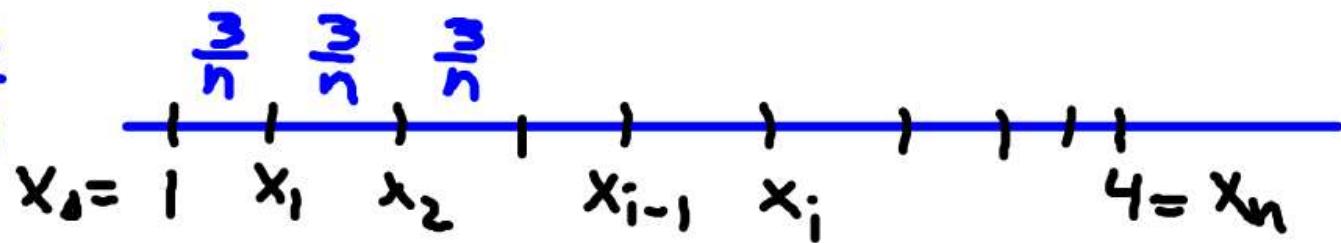
$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^{n+1} f(x_i^*) \Delta x_i$$

Ejemplo:

Determine el área bajo la curva $f(x) = x^2 + 2$ en el intervalo $[1, 4]$, mediante sumas de Riemann.

$[1, 4]$ partición regular (n partes iguales)

$$\frac{4-1}{n} = \frac{3}{n}$$



$$x_0=1, x_1=1+\frac{3}{n}, x_2=1+2\left(\frac{3}{n}\right), x_3=1+3\left(\frac{3}{n}\right), \dots, x_i=1+i\left(\frac{3}{n}\right), \dots, x_n=1+n\left(\frac{3}{n}\right)$$

$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^{n-1} f(x_i^*) \Delta x_i$$

$$\|P\| = \frac{3}{n} \quad \|P\| \rightarrow 0, \quad n \rightarrow \infty$$

$$\Delta x_i = \frac{3}{n}$$

$$x_i^* = x_i = 1 + i \left(\frac{3}{n} \right)$$

$$f(x_i^*) = f\left(1 + i \frac{3}{n}\right) = \left[1 + i \frac{3}{n}\right]^2 + 2$$

$$f(x) = x^2 + 2$$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\left[1 + i \frac{3}{n} \right]^2 + 2 \right) \frac{3}{n} =$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(3 + \frac{6}{n} i + \frac{9}{n^2} i^2 \right) \frac{3}{n} =$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{9}{n} + \frac{18}{n^2} i + \frac{27}{n^3} i^2 \right) =$$

$$= \lim_{n \rightarrow \infty} \frac{9}{n} \sum_{i=1}^n 1 + \frac{18}{n^2} \sum_{i=1}^n i + \frac{27}{n^3} \sum_{i=1}^n i^2 =$$

$$= \lim_{n \rightarrow \infty} \frac{\cancel{q}}{\cancel{n}} (n) + \frac{q}{\cancel{n^2}} \frac{n(n+1)}{2} + \frac{\cancel{27}}{\cancel{n^32}} \frac{n(n+1)(2n+1)}{6} =$$

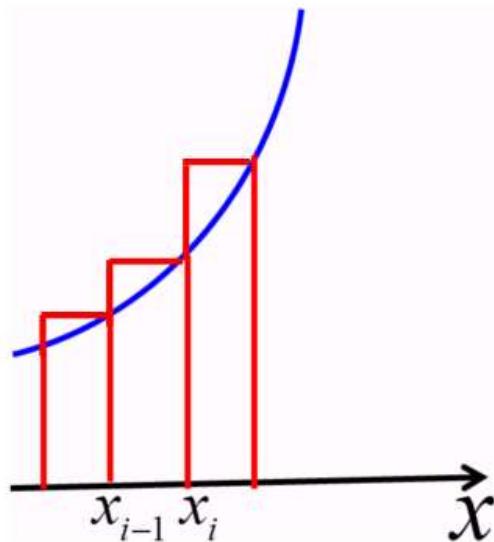
$$= \lim_{n \rightarrow \infty} q + q \frac{n+1}{n} + \frac{q}{2} \frac{(n+1)(2n+1)}{n^2} =$$

$$= \lim_{n \rightarrow \infty} q + q \left(1 + \frac{1}{n}\right)^{70} + \frac{q}{2} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)^{70} =$$

$$= q + q + q = 27$$

$A = 27$ unidades
de área

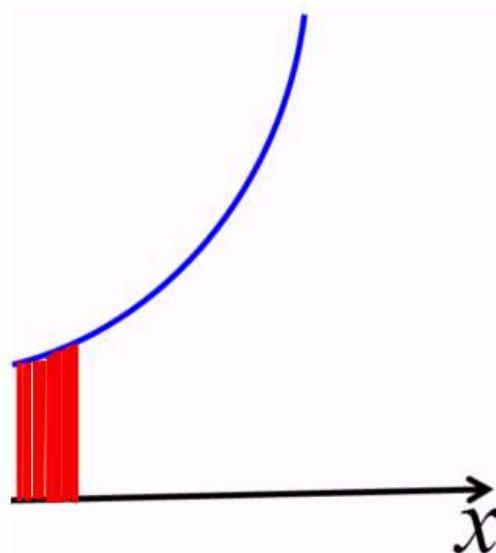
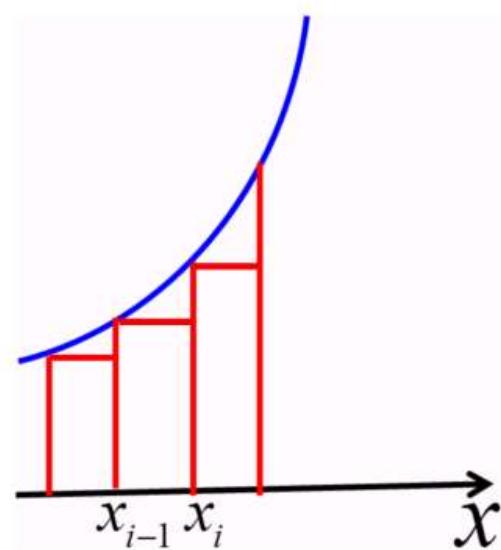
$$x_i^* = x_i$$



$$\|P\| \rightarrow 0$$

$$n \rightarrow \infty$$

$$x_i^* = x_{i-1}$$



$$x_{i-1}^* = 1 + (i-1) \frac{3}{n}$$