2.2 Enunciado e interpretación geométrica del Teorema del Valor Medio del Cálculo Integral

Teorema

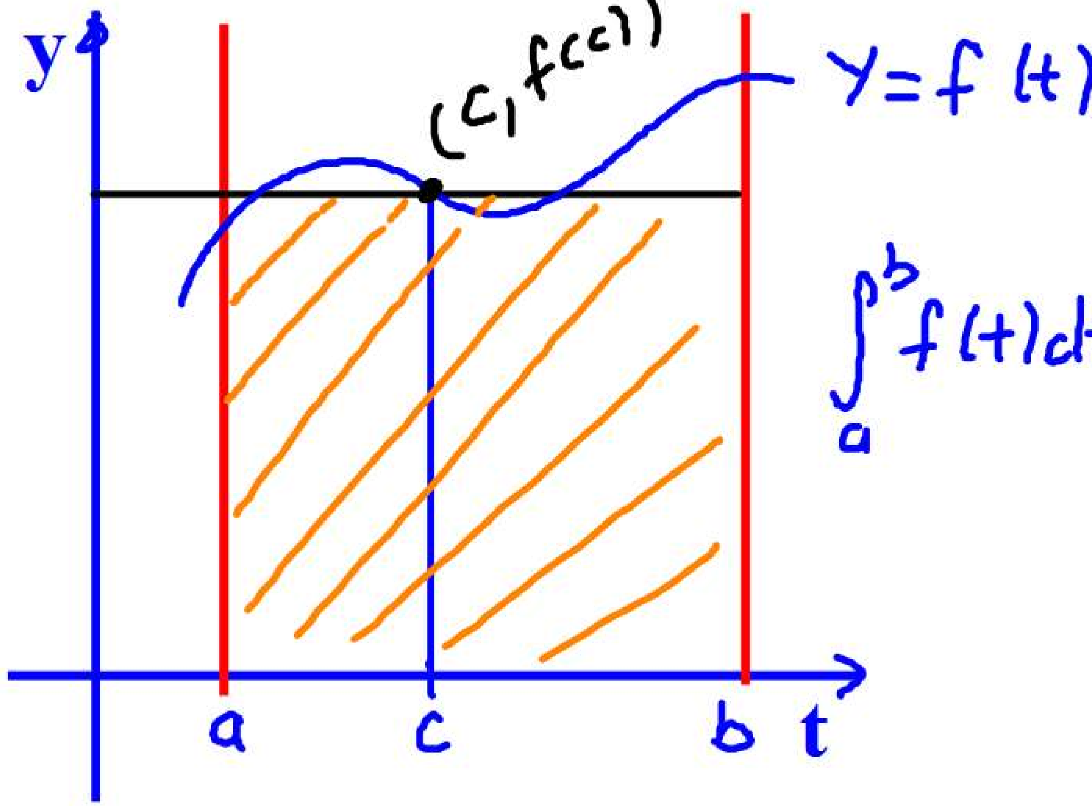
Si f es continua en el intervalo $[a,b]$, entonces existe un número c en $[a,b]$ tal que

$$\int_a^b f(t)\,dt = f(c)(b-a)$$

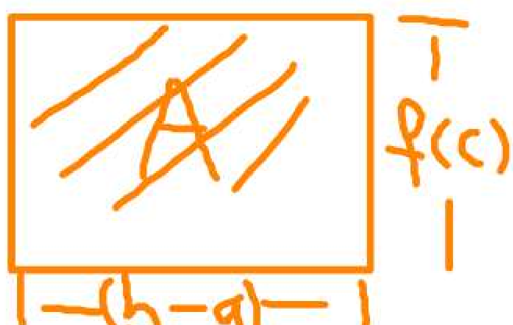
$$f(c) = \frac{\int_a^b f(t)\,dt}{b-a}$$

$f(c)$ - ordenada media, c - abscisa media

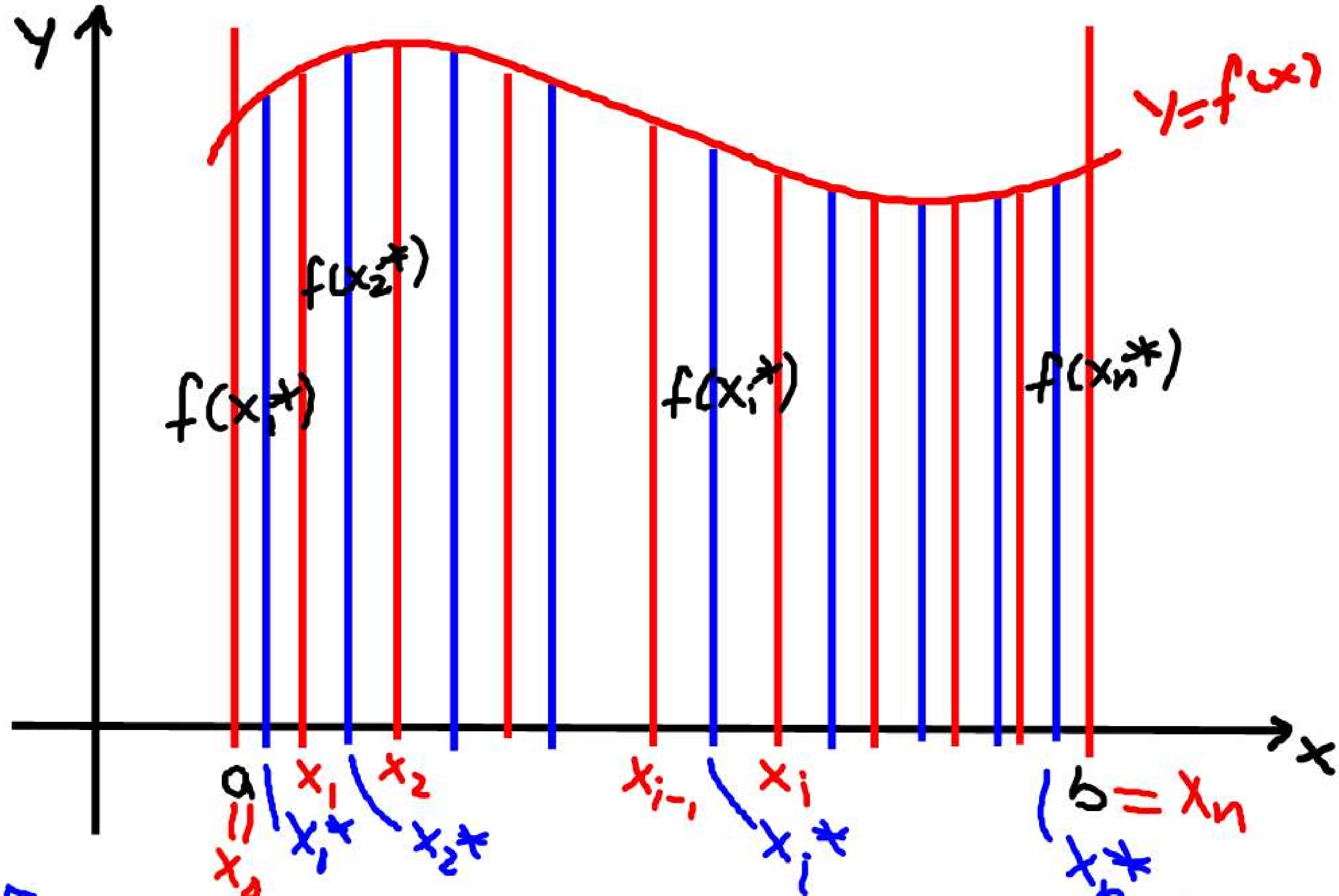
Interpretación geométrica



$$\int_a^b f(t)\,dt = \text{área bajo la curva}$$



$$= \int_a^b f(t)\,dt$$



[a,b] - partición regular (n partes iguales)

$x_i^* \in [x_{i-1}, x_i]$ | $\|P\| = \frac{b-a}{n}$, $\Delta x_i = \frac{b-a}{n}$

$$\bar{y} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$$

$$\bar{y} \approx \frac{1}{n} \sum_{i=1}^{n} f(x_i^*)$$

$$\bar{y} \approx \frac{1}{n} \frac{b-a}{b-a} \sum_{i=1}^{n} f(x_i^*)$$

$$\bar{y} \approx \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n}$$

$$\bar{y} \approx \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

$$\lim_{\|P\| \to 0} \bar{y} = \lim_{\|P\| \to 0} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\bar{y} = f(c)$$

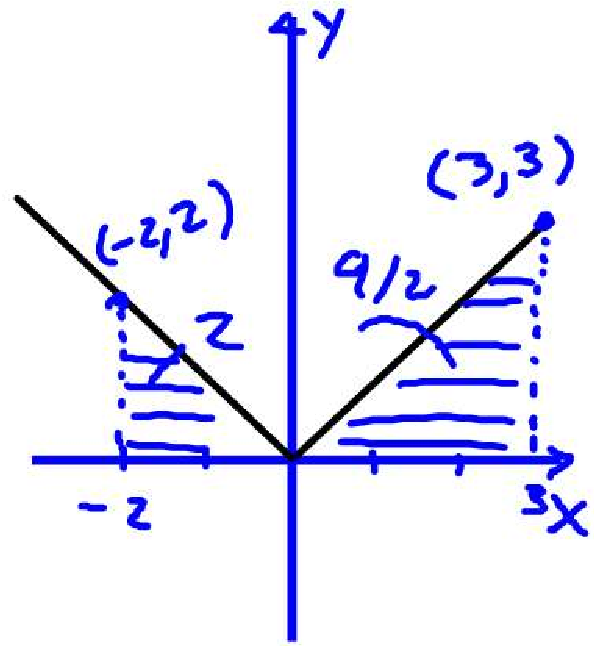
Ejemplo:

$f(c) = ?$

Determine el valor medio de la función $f(x) = |x|$ en el intervalo $[-2,3]$.

$$f(c) = \frac{\int_a^b f(t)\,dt}{b-a}$$

$$f(c) = \frac{\int_{-2}^{3} |x|\,dx}{3-(-2)} = \frac{1}{5}\int_{-2}^{3} |x|\,dx$$



$$\int_{-2}^{3} |x|\,dx = 2 + \frac{9}{2} = \frac{13}{2}$$

$$f(c) = \frac{1}{5}\left(\frac{13}{2}\right) = \frac{13}{10}$$

$$f(c) = \frac{13}{10}$$

2.3.1 Definición de integral indefinida a partir de la integral definida con el extremo superior variable.

Definición

Una función F, se denomina antiderivada de f en un intervalo I, si $F'(x) = f(x)$ para todo x en I.

Ejemplo:

$$F(x) = \frac{x^3}{3} + 2$$ es antiderivada de $f(x) = x^2$

$$F'(x) = x^2 = f(x)$$

$$f(x) = x^2$$

Otro ejemplo: $F(x) = \frac{x^3}{3} + \pi$

Teorema

Si F es una antiderivada de f en un intervalo I, la antiderivada más general de f en I es

$$F(x)+c$$

en donde c es una constante arbitraria.

Ejemplo:

$$f(x)=x^2$$

$$F(x)=\frac{x^3}{3}+c$$

su antiderivada general

Definición. Integral indefinida

Se llama integral indefinida de la función continua $f(x)$, a:

$$\int_a^x f(t)\,dt + C$$

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt + C\right] = f(x)$$

Notación:

$$\int \underbrace{f(x)}_{\text{integrando}} dx = \int_a^x f(t)\,dt + C = F(x) + C$$

donde C es la constante de integración

1) $\int f(x)dx \rightarrow \frac{d}{dx}\Big[\underbrace{\int f(x)dx}_{= \int_a^x f(t)dt + C}\Big] = f(x)$

2) $\frac{d}{dx} f(x) \rightarrow \int\left[\frac{d}{dx} f(x)\right] dx = \underbrace{f(x)+C}_{\frac{d}{dx}}$

$\int, \frac{d}{dx}$ son operaciones inversas

3) $\int f(x)\,dx$ Buscando su diferencial

$$d\left[\int f(x)\,dx\right]=\frac{d}{dx}\Big[\underbrace{\int f(x)\,dx}_{\int_a^x f(t)\,dt + c}\Big]dx = f(x)\,dx$$

4) $$df(x)=\left[\frac{d}{dx}f(x)\right]dx$$

$$\int df(x)=\int\left[\frac{d}{dx}f(x)\right]dx=\underbrace{f(x)+C}_{d/dx}$$

$\int$, d son operaciones inversas

Propiedades

$$1)\ \int k f(x)\,dx = k\int f(x)\,dx$$

$$2)\ \int \left[f(x)+g(x)\right]dx = \int f(x)\,dx + \int g(x)\,dx$$

$$3)\ \int \left[f(x)-g(x)\right]dx = \int f(x)\,dx - \int g(x)\,dx$$

2.3.2 Enunciado y demostración del teorema Fundamental del Cálculo.

Teorema. Teorema fundamental del Cálculo.

Sea f una función continua (y por lo tanto Integrable) en $[a,b]$ y sea F una antiderivada cualquiera de f en él. Entonces, $F' = f$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$

$[a,b]$ - partición arbitraria

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$$

$$F(b) - F(a) = F(x_n) - F(x_0) = [F(x_n) - F(x_{n-1})] +$$

$$[F(x_{n-1}) - F(x_{n-2})] + \cdots + [F(x_i) - F(x_{i-1})] +$$

$$\cdots + [F(x_2) - F(x_1)] + [F(x_1) - F(x_0)]$$

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$$

T. V. M. Cálculo Diferencial : $f(b)-f(a)=f'(c)(b-a)$

$F(x)$, $[x_{i-1}, x_i]$, $c = x_i^*$

$$F(x_i)-F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) \qquad F'=f$$

$$F(x_i)-F(x_{i-1}) = f(x_i^*)\Delta x_i$$

$$F(b)-F(a) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$$\lim_{\|P\|\to 0} F(b)-F(a) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

$$F' = f$$

$$\int_a^b \left(\frac{d}{dx} F(x)\right) dx = F(x) \Big]_a^b$$

$$\int_a^b f(x)\,dx = F(x) \Big]_a^b$$

2.4 Determinación de integrales indefinidas inmediatas. Integrales de funciones cuyo resultado involucra a la función logaritmo natural. Cambio de variable.

Fórmulas básicas de integración

1. $\int du = u + C$

2. $\int u^n du = \dfrac{u^{n+1}}{n+1} + C, n \neq -1$

3. $\int \dfrac{1}{u} du = \ln|u| + C$

4. $\int \dfrac{1}{\sqrt{u}} du = 2\sqrt{u} + C$

5. $\int \dfrac{1}{u^2} du = -\dfrac{1}{u} + C$

6. $\int e^u du = e^u + C$

7. $\int a^u du = \dfrac{a^u}{\ln a} + C$

8. $\int senu\, du = -\cos u + C$

9. $\int \cos u du = senu + C$

10. $\int \tan u du = \ln|\sec u| + C$

$$11.\ \int \cot u du = \ln|senu| + C$$

$$12.\ \int \sec u du = \ln|\sec u + \tan u| + C$$

$$13.\ \int \sec^2 u\ du = \tan u + C$$

$$14.\ \int \sec u \tan u du = \sec u + C$$

$$15.\ \int \csc u du = \ln|\csc u - \cot u| + C$$

$$16.\ \int \csc^2 u du = -\cot u + C$$

$$17.\ \int \csc u \cot u du = -\csc u + C$$

$$18.\ \int senhu\ du = \cosh u + C$$

$$19.\ \int \cosh u du = senhu + C$$

$$20.\ \int \tanh u du = \ln(\cosh u) + C$$

$$21.\ \int \coth u du = \ln|senhu| + C$$

$$22.\ \int \sec hu du = ang \tan(senhu) + C$$

$$23.\ \int \sec h^2 u\ du = \tanh u + C$$

$$24.\ \int \sec hu \tanh u du = -\sec hu + C$$

$$25.\ \int \operatorname{csch} u du = \ln\left|\tanh \frac{u}{2}\right| + C$$

$$26.\ \int \csc h^2 u du = -\coth u + C$$

$$27.\ \int \csc hu \coth u du = -\csc hu + C$$

$$28.\ \int \frac{1}{\sqrt{a^2 - u^2}} du = angsen\left(\frac{u}{a}\right) + C$$

$$29.\ \int \frac{1}{a^2 + u^2} du = \frac{1}{a} ang \tan\left(\frac{u}{a}\right) + C$$

$$30.\ \int \frac{1}{u\sqrt{u^2 - a^2}} du = \frac{1}{a} ang \sec\left(\frac{u}{a}\right) + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

$u = u(x)$

$du = \frac{du}{dx}\,dx$

$du = u'\,dx$

$$\int a^u u'\,dx = \frac{a^u}{\ln a} + c$$

$$\frac{d}{dx}\left[\frac{a^u}{\ln a} + c\right] = \frac{1}{\cancel{\ln a}}\,a^u \cancel{\ln a}\,u'$$

Teorema

Si u es una función derivable de x tal que $u \neq 0$, entonces $\frac{d}{dx}\left[\ln|u|\right] = \frac{u'}{u}$.

$u > 0$, $|u| = u$, $\frac{d}{dx}[\ln u] = \frac{u'}{u}$

$u < 0$, $|u| = -u$, $\frac{d}{dx}[\ln(-u)] = \frac{-u'}{-u} = \frac{u'}{u}$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

$$u = u(x)$$

$$du = \frac{du}{dx}\,dx$$

$$du = u'\,dx$$

$$\int \frac{1}{u}\,u'\,dx = \ln|u| + c$$

$$\int \sec u\,du = \ln\left|\sec u + \tan u\right| + C$$

$$\int \sec u\,du = \int \sec u \left(\frac{\sec u + \tan u}{\sec u + \tan u}\right) du = \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u}\, du =$$

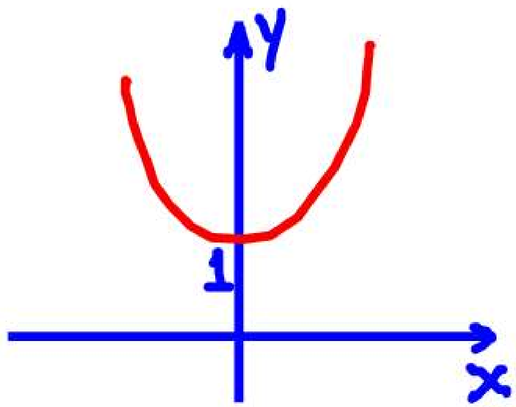
$$= \begin{bmatrix} z = \sec u + \tan u \\ dz = (\sec u \tan u + \sec^2 u)\,du \end{bmatrix} = \int \frac{dz}{z} = \ln|z| + C =$$

$$= \ln\left|\sec u + \tan u\right| + C$$

$$\int \tanh u\,du = \ln(\cosh u) + C$$

$$\int \tanh u\,du = \int \frac{\operatorname{senh} u}{\cosh u}\,du = \begin{bmatrix} z = \cosh u \\ dz = \operatorname{senh} u\,du \end{bmatrix} = \int \frac{dz}{z} = \ln|z| + c =$$

$$= \ln|\cosh u| + c = \ln(\cosh u) + c$$



Integración inmediata

Comparar directamente el integrando dado con el de una de las expresiones matemáticas anteriores, se le conoce a este procedimiento como integración inmediata

Procedimientos de ajuste de integrandos a las reglas básicas.

Técnica

1) Desarrollar (el numerador)

$$\int u^n du = \frac{u^{n+1}}{n+1} + c$$

$$\int\left(1+x^2\right)^2 dx = \int (1 + 2x^2 + x^4)\,dx =$$

$$= \cancel{\int dx} + \int 2x^2 dx + \int x^4 dx =$$

$$= x + 2\int x^2 dx + \frac{x^5}{5} + c =$$

$$= x + 2\frac{x^3}{3} + \frac{x^5}{5} + c$$

2) Separar el numerador

$$\int \frac{x+1}{\sqrt{x}}dx = \int \left[\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right] dx =$$

$$\int u^n du = \frac{u^{n+1}}{n+1} + c$$

$$= \int \sqrt{x}\, dx + \int x^{-\frac{1}{2}} dx =$$

$$= \int x^{\frac{1}{2}} dx + 2x^{\frac{1}{2}} + c =$$

$$= \frac{2}{3}x^{3/2} + 2x^{\frac{1}{2}} + c$$

3) Completar el cuadrado

$$\int \frac{1}{\sqrt{2x-x^2}}dx = \left[\begin{array}{l} 2x-x^2 = -(x^2-2x) = \\ = -(x^2-2x+1)+1 = 1-(x-1)^2 \end{array} \right]$$

$$= \int \frac{1}{\sqrt{1-(x-1)^2}}dx = \left[\begin{array}{ll} u = x-1 & y = f(x) \\ du = dx & dy = f'(x)\,dx \end{array} \right]$$

$$= \int \frac{1}{\sqrt{1-u^2}}du = \text{angsen}\left(\frac{u}{1}\right)+c =$$

$$= \text{angsen}(x-1)+c$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \text{angsen}\left(\frac{u}{a}\right)+c$$

$$a^2 = 1 \quad a = 1$$

4) Dividir la función racional impropia

$$\int \frac{x^{2}}{x^{2}+1}\,dx = \int \left[1 - \frac{1}{x^2+1} \right] dx =$$

$$\begin{array}{r|l} & \quad 1 \\ \cline{2-2} x^2+1 & \quad x^2 \\ & -x^2-1 \\ \cline{2-2} & \qquad -1 \end{array}$$

$$= \cancel{\int dx} - \int \frac{1}{x^2+1}\,dx =$$

$$= x - \text{angtan}\, x + C$$

5) Sumar y restar términos en el numerador

$$\int \frac{x^2}{x^2+1}dx = \int \frac{(x^2+1)-1}{x^2+1}dx =$$

$$= \int \left[\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} \right] dx =$$

$$= \int \left[1 - \frac{1}{x^2+1} \right] dx = x - \text{angtan}\, x + c$$

6) Usar identidades trigonométricas

$$\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx =$$

$$\sin^2 x + \cos^2 x = 1 \quad \div \sin^2 x$$

$$1 + \underline{\cot^2 x} = \csc^2 x$$

$$= \int \csc^2 x\,dx - \int dx =$$

$$= \underline{-\cot x - x + c}$$

7) Multiplicar el integrando por la unidad

$$\int \frac{1}{x\sqrt{x^4-1}}dx = \int \frac{1}{x\sqrt{(x^2)^2-1}}dx = \begin{bmatrix} u = x^2 \\ du = 2x\,dx \end{bmatrix} =$$

$$= \int \frac{1}{x\sqrt{(x^2)^2-1}}\left(\frac{2x}{2x}\right)dx = \frac{1}{2}\int \frac{2x}{x^2\sqrt{(x^2)^2-1}}dx =$$

$$= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2}\,\text{angsec}\,(x^2) + c$$

$$a^2 = 1$$

$$a = 1$$

8) Multiplicar y dividir por el conjugado pitagórico

$$\int \frac{1}{1+senx}dx = \int \frac{1}{1+senx}\cdot\left(\frac{1-senx}{1-senx}\right)dx =$$

$$= \int \frac{1-senx}{1-sen^2x}dx = \int \frac{1-senx}{cos^2x}dx =$$

$$= \int \left[\frac{1}{cos^2x} - \frac{senx}{cos^2x}\right]dx = \int sec^2x\,dx - \int \frac{senx}{cos^2x}dx =$$

$$= \tan x - \sec x + c$$

$$\int \frac{\operatorname{sen} x}{\cos^2 x}\,dx = \left[\begin{array}{l} u = \cos x \\ du = -\operatorname{sen} x\,dx \end{array}\right] =$$

$$= \int \frac{(-1)}{(-1)} \frac{\operatorname{sen} x}{\cos^2 x}\,dx = -\int \frac{-\operatorname{sen} x\,dx}{\cos^2 x} =$$

$$= -\int \frac{du}{u^2} = -\int u^{-2}\,du = -\frac{u^{-1}}{-1} + c = \frac{1}{u} + c$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + c$$

$$= \frac{1}{\cos x} + c = \underline{\sec x + c}$$

9) Reescribir la integral

$$\int \frac{senx}{\cos^2 x}dx = \int \frac{senx}{\cos x} \cdot \frac{1}{\cos x} dx =$$

$$= \int \tan x \sec x \, dx = \sec x + C$$

Teorema

Si F es una antiderivada de f, entonces

$$\int f\left(g(x)\right)g'(x)\,dx = F\left(g(x)\right)+C$$

$F' = f$

$y = F(u) \quad , u = g(x)$

$y = F(g(x))$

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))\,g'(x)$$

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C$$

f. interior

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C$$

f. exterior

$$u = g(x)$$

$$du = g'(x)\, dx$$

$$\int f(u)\, du = F(u) + C$$

Ejemplos:

1) $\int \left(x^2+1\right)^2 2xdx = \begin{bmatrix} u = x^2+1 \\ du = 2xdx \end{bmatrix} =$

$$= \int u^2 du = \frac{u^3}{3} + c = \frac{(x^2+1)^3}{3} + c$$

$$2)\int \frac{x}{\left(4x^2+3\right)^6}dx = \left[\begin{array}{l} u = 4x^2+3 \\ du = 8x\,dx \end{array}\right] =$$

$$= \int \frac{x}{(4x^2+3)^6}\left(\frac{8}{8}\right)dx = \frac{1}{8}\int \frac{8x}{(4x^2+3)^6}dx =$$

$$= \frac{1}{8}\int \frac{du}{u^6} = \frac{1}{8}\int u^{-6}du = \frac{1}{8}\frac{u^{-5}}{-5} + c =$$

$$= -\frac{1}{40u^5} + c = -\frac{1}{40(4x^2+3)^5} + c$$

$$3)\ \int sen\frac{x}{2}dx = \begin{bmatrix} u=\frac{x}{2} \\ du=\frac{1}{2}dx \end{bmatrix} =$$

$$= \int sen\left(\frac{x}{2}\right)\left(\frac{2}{2}\right)dx = 2\int sen\left(\frac{x}{2}\right)\frac{1}{2}dx =$$

$$= 2\int sen\,u\,du = -2\cos u + C =$$

$$= -2\cos\left(\frac{x}{2}\right) + C$$

$$4)\ \int \frac{x}{x^4+2x^2+2}dx = \int \frac{x}{(x^2+1)^2+1}dx =$$

$$= \begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} = \int \frac{x}{(x^2+1)^2+1}\left(\frac{2}{2}\right)dx =$$

$$= \frac{1}{2}\int \frac{2x}{(x^2+1)^2+1}dx = \frac{1}{2}\int \frac{du}{u^2+1} =$$

$$= \frac{1}{2}\text{angtan } u + c =$$

$$= \frac{1}{2}\text{angtan}(x^2+1) + c$$