2.6 La función logaritmo natural, sus propiedades y su representación gráfica.

$\log_{10} 10 = 1$, $10^1 = 10$

$\log_{10} 1000 = 3$, $10^3 = 1000$

$\log_{10} 0.001 = -3$, $10^{-3} = 0.001$

$\log_{10} 1 = 0$, $10^0 = 1$

$\log_2 16 = 4$, $2^4 = 16$

$\log_3 27 = 3, \quad 3^3 = 27$

$\log_4 64 = 3, \quad 4^3 = 64$

$\log_e e = 1, \quad e^1 = e$

$\log_e 1 = 0, \quad e^0 = 1$

$\log_e = \ln$

Definición. Función logaritmo natural

La función logaritmo natural, designada mediante ln, se define como

$$\ln x = \int_1^x \frac{1}{t}\,dt,\ \ x > 0.$$

Propiedades

Si a y b son números positivos y r es un número racional cualquiera, entonces

1) $\ln 1 = 0$

2) $\ln ab = \ln a + \ln b$

3) $\ln \dfrac{a}{b} = \ln a - \ln b$

4) $\ln a^{r} = r \ln a$

5) $\ln e = 1$

Ejemplos:

1) Aplique propiedades

$$\ln \frac{x^{1/2}(2x+7)^4}{(3x^2+1)^2} = \ln x^{\frac{1}{2}}(2x+7)^4 - \ln (3x^2+1)^2 =$$

$$= \ln x^{\frac{1}{2}} + \ln(2x+7)^4 - 2\ln(3x^2+1) =$$

$$= \frac{1}{2}\ln x + 4\ln(2x+7) - 2\ln(3x^2+1)$$

2) Evalúe la expresión

$$\log_{10} 1.25 + \log_{10} 80 = \log_{10} (1.25)(80) =$$

$$= \log_{10} 100 = 2$$

3) Despeje x de cada ecuación y calcule su valor

a) $\ln(x+6)+\ln(x-3)=\ln 5+\ln 2$

$$\ln(x+6)(x-3)=\ln(5)(2)$$

$$\ln(x^2+3x-18)=\ln 10$$

$$x^2+3x-18=10$$

$$x^2+3x-28=0$$

$$(x+7)(x-4)=0$$

$x_1=-7$ X

$x_2=4$ ✓

b) $\ln(\ln x) = 1$

$\ln = \log_e$

$\log_e(\ln x) = 1$, $e^1 = \ln x$

$\ln x = e$

$\log_e x = e$ $\qquad e^e = x$

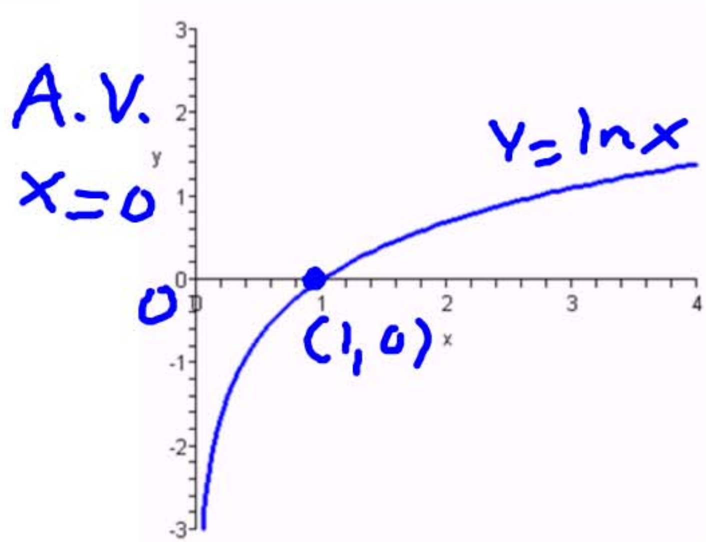
$$\underline{\underline{x = e^e}}$$

Gráfica del logaritmo natural

$$\ln x = \int_1^x \frac{1}{t}\,dt,\ \ x > 0$$

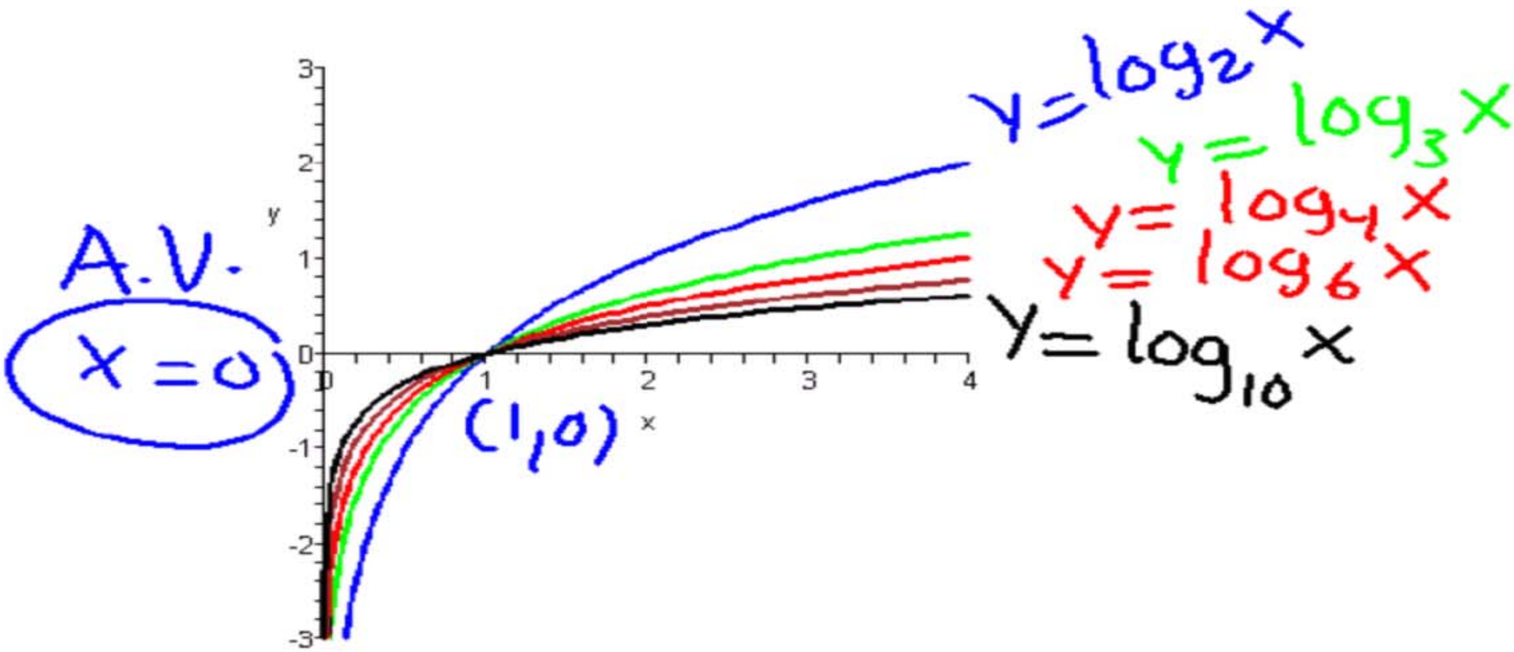
$D = \mathbb{R}^+$

$R = \mathbb{R}$

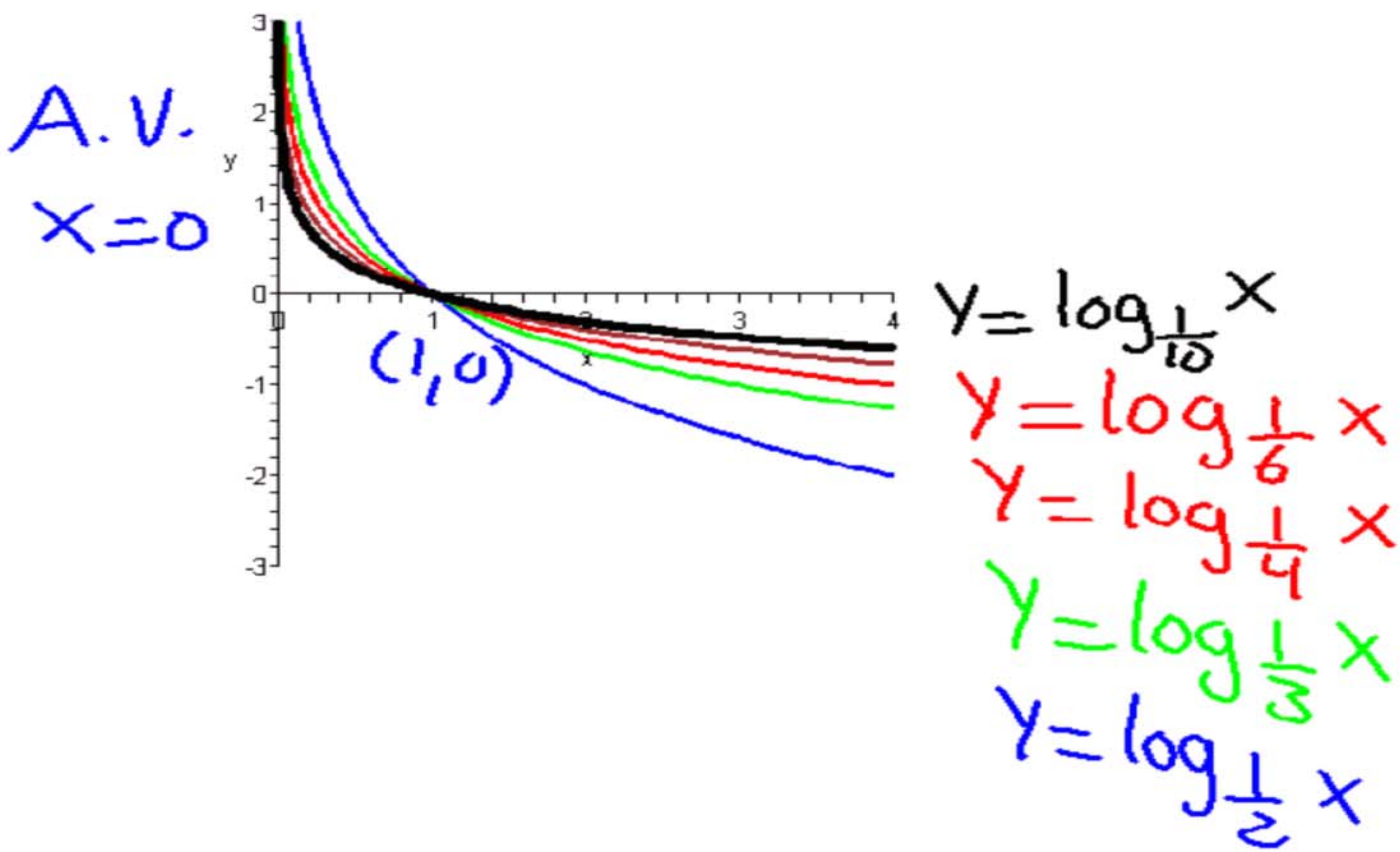


Gráficas de las funciones logarítmicas

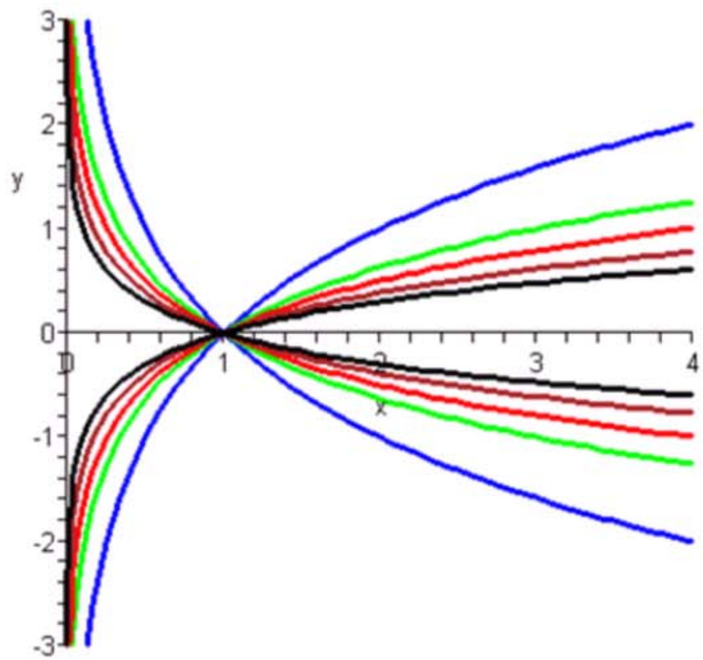
```
> plot([log[2](x),log[3](x),log[4](x),log[6](x),log[10](x)], x
  = 0..4, y =-3..3, discont =
  true,thickness=2,color=[blue,green,red,brown,black]);
```



```
> plot([log[1/2](x),log[1/3](x),log[1/4](x),log[1/6](x),log[1/1
  0](x)], x = 0..4, y =-3..3, discont =
  true,thickness=2,color=[blue,green,red,brown,black]);
```



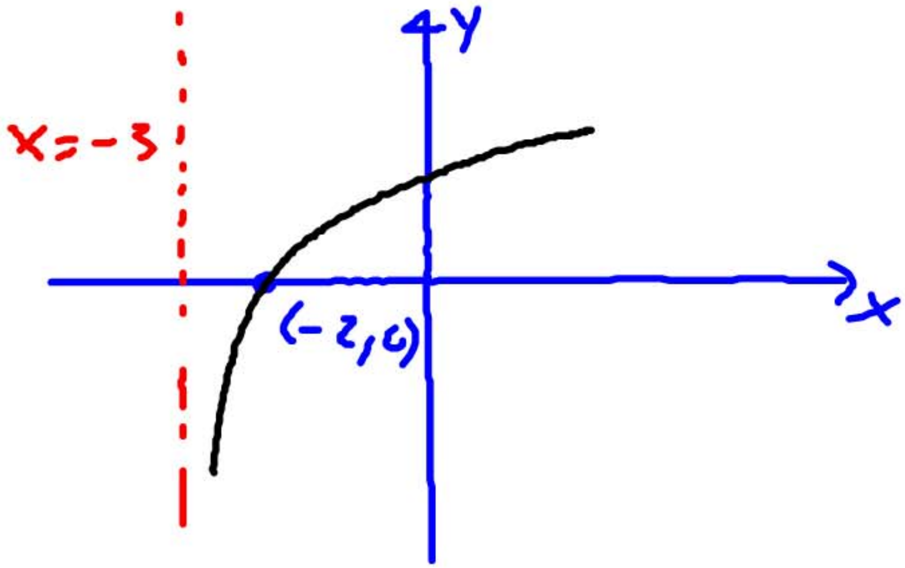
```
> plot([log[1/2](x),log[1/3](x),log[1/4](x),log[1/6](x),log[1/10](x),log[2](x),log[3](x),log[4](x),log[6](x),log[10](x)], x = 0..4, y =-3..3, discont = true,thickness=2,color=[blue,green,red,brown,black,blue,green,red,brown,black]);
```

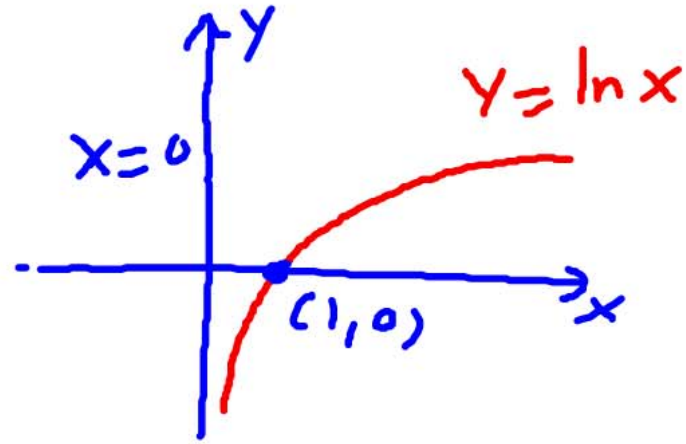


Ejemplos:

Bosqueje la gráfica

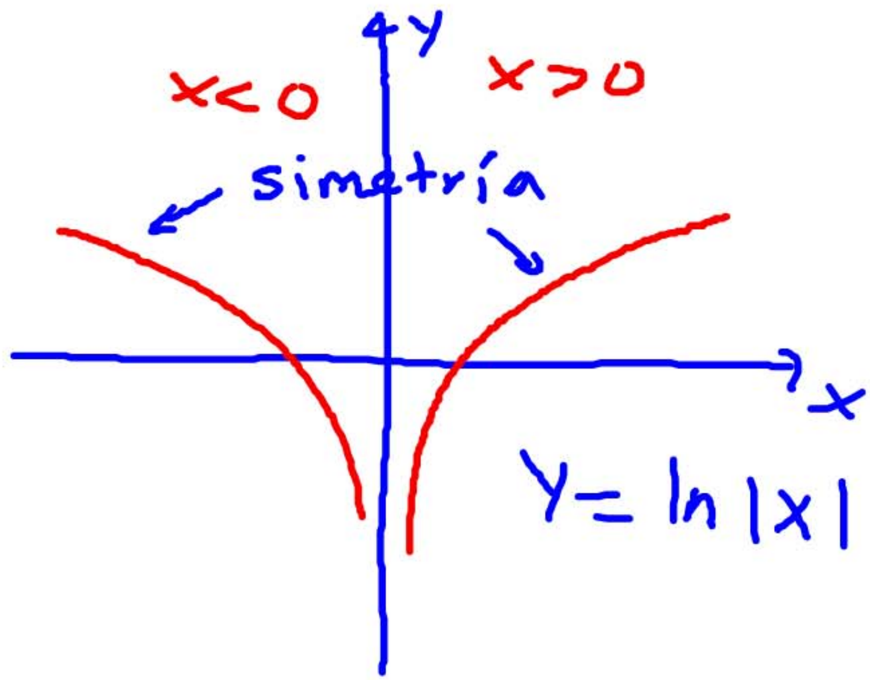
1) $y = \ln(x+3)$



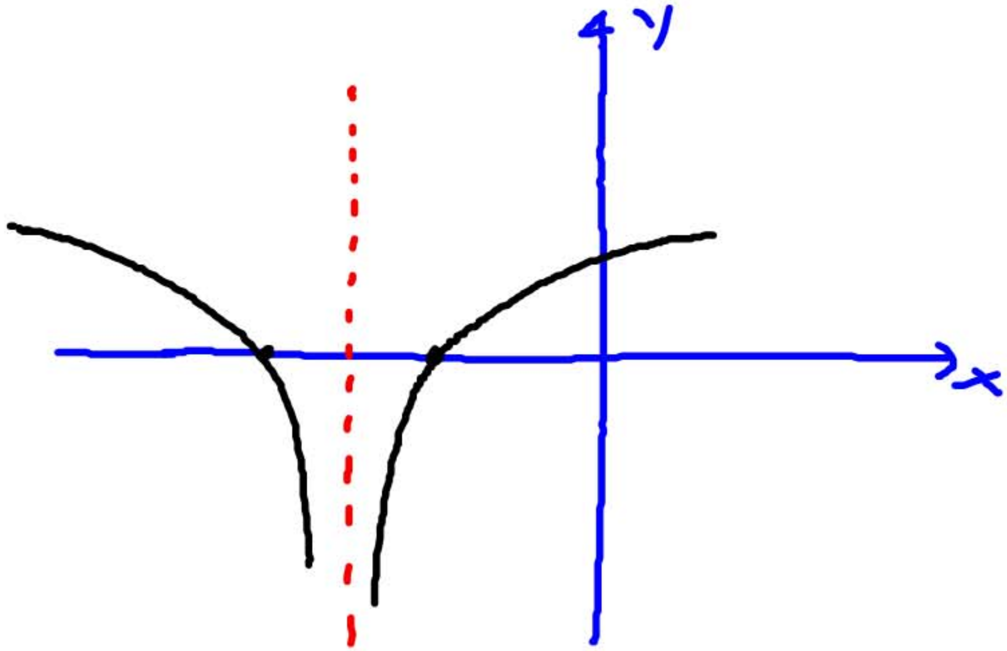


2) $y = \ln|x|$

$$\begin{cases} x>0, & y=\ln x \\ x<0, & y=\ln(-x) \end{cases}$$

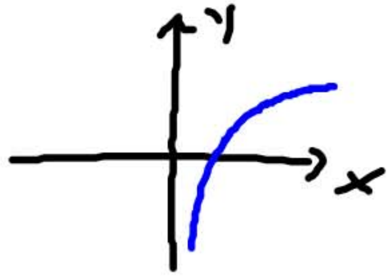


3) $y = \ln|x+3|$

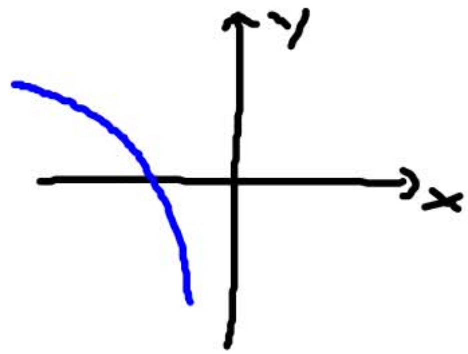


4) $y = -\ln(-x)$

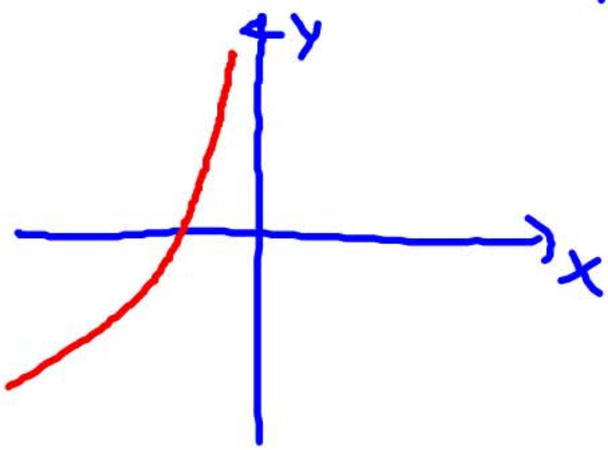
$y = \ln x$



$y = \ln(-x)$



$y = -\ln(-x)$



2.7.1 La función exponencial, sus propiedades y su representación gráfica.

Función exponencial

En general, una función exponencial es una función de la forma $f(x) = a^x$ donde a es una constante positiva, $x \in \mathbb{R}$.

Leyes de los exponentes

Si a y b son números positivos y x e y son cualesquiera números reales, entonces

1. $a^{x+y} = a^x a^y$

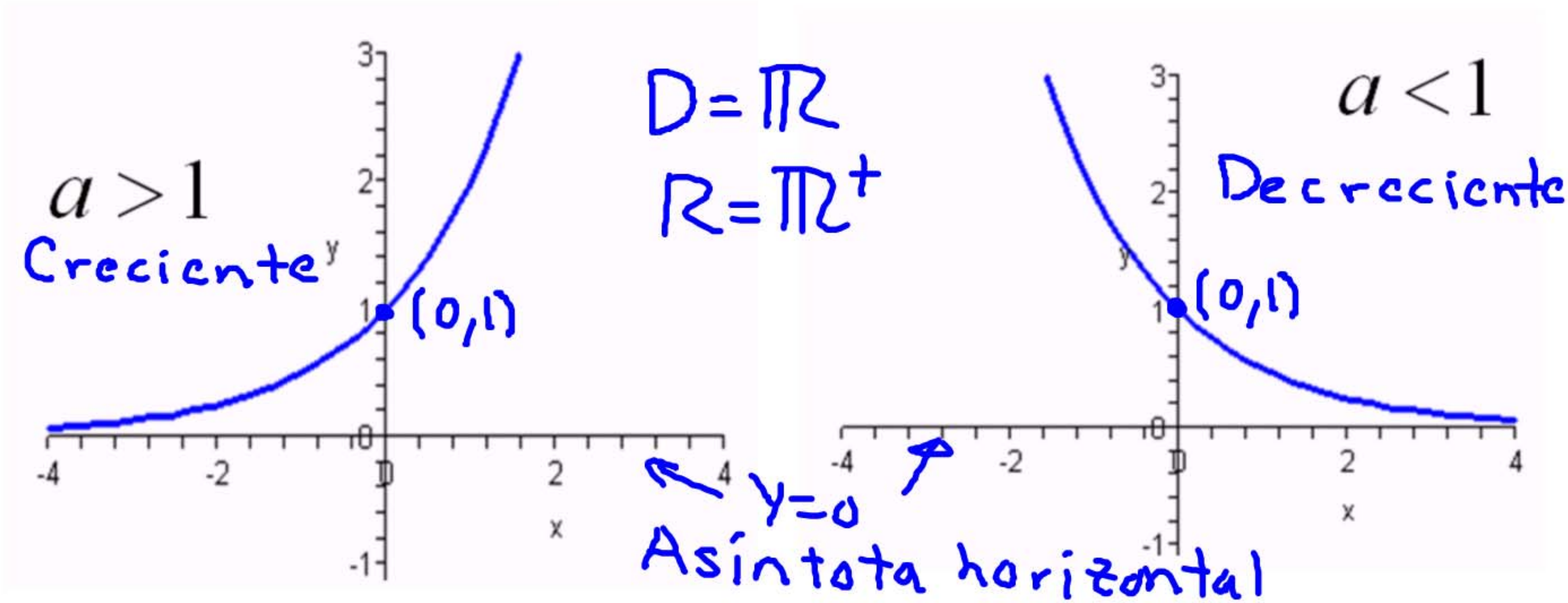
2. $a^{x-y} = \dfrac{a^x}{a^y}$

3. $\left(a^x\right)^y = a^{xy}$

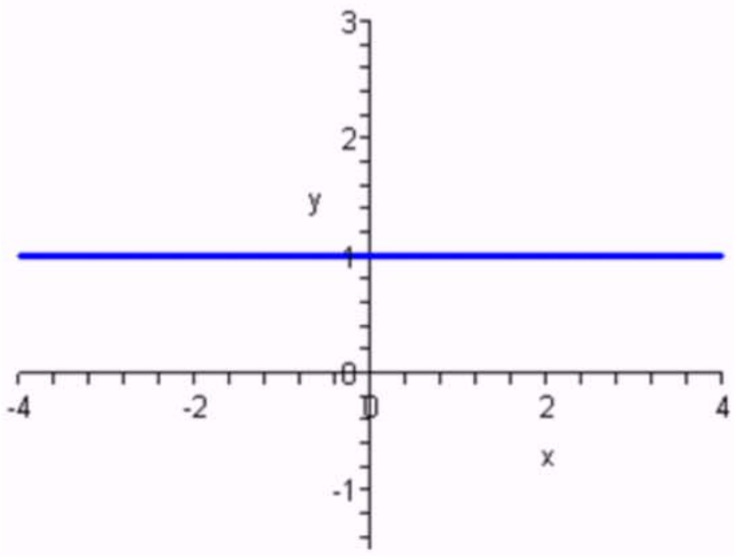
4. $\left(ab\right)^x = a^x b^x$

Gráfica de la función exponencial

$$y = a^x, a > 0$$



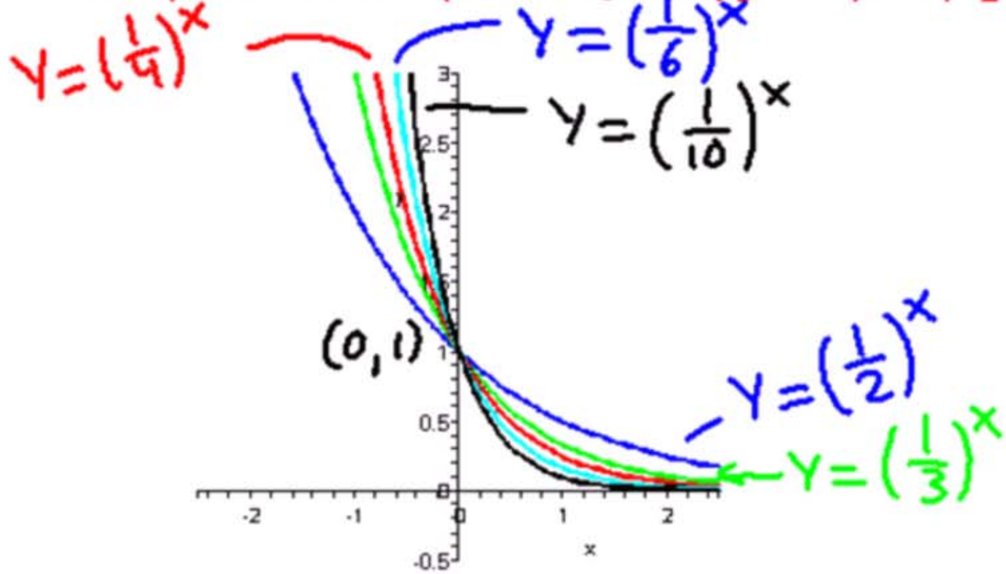
$$y = a^x, \ a = 1$$



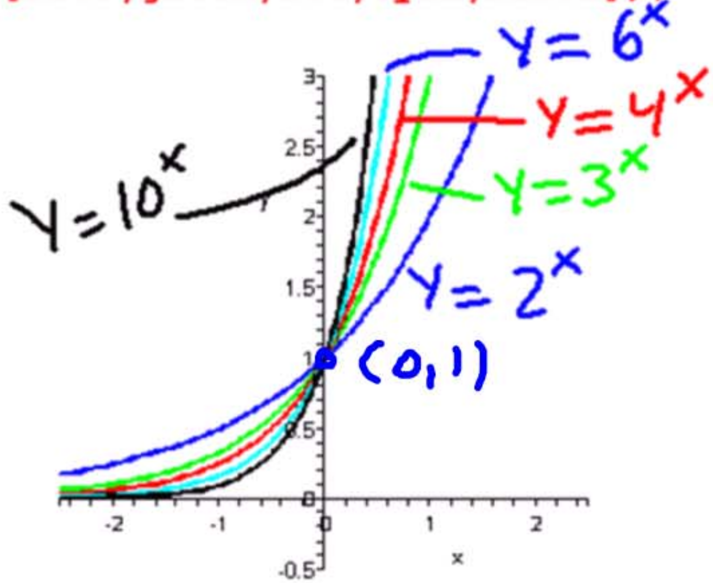
$Y = 1^x$
$D = \mathbb{R}$
$R = \{1\}$

Gráficas de funciones exponenciales

```
> plot([(1/2)**(x),(1/3)**(x),(1/4)**(x),(1/6)**(x),(1/10)**(x)], x = -2.5..2.5,
  y =-0.5..3, discont = true,thickness=2,color=[blue,green,red,cyan,black]);
```

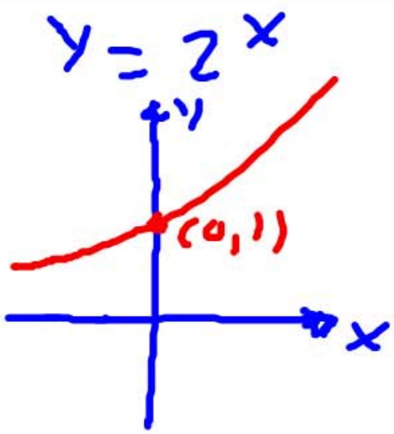


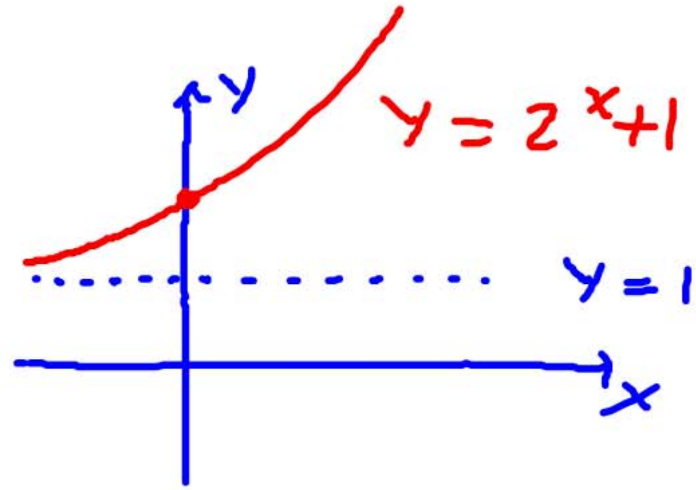
```
> plot([2**(x),3**(x),4**(x),6**(x),10**(x)], x = -2.5..2.5, y =-0.5..3, discont
  = true,thickness=2,color=[blue,green,red,cyan,black]);
```



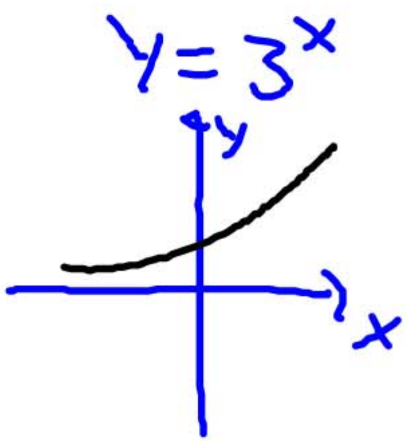
Bosqueje la gráfica

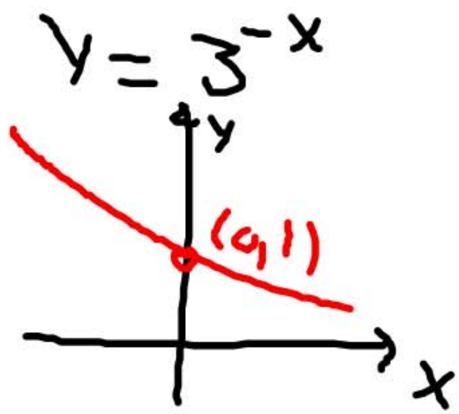
1) $y = 2^x + 1$



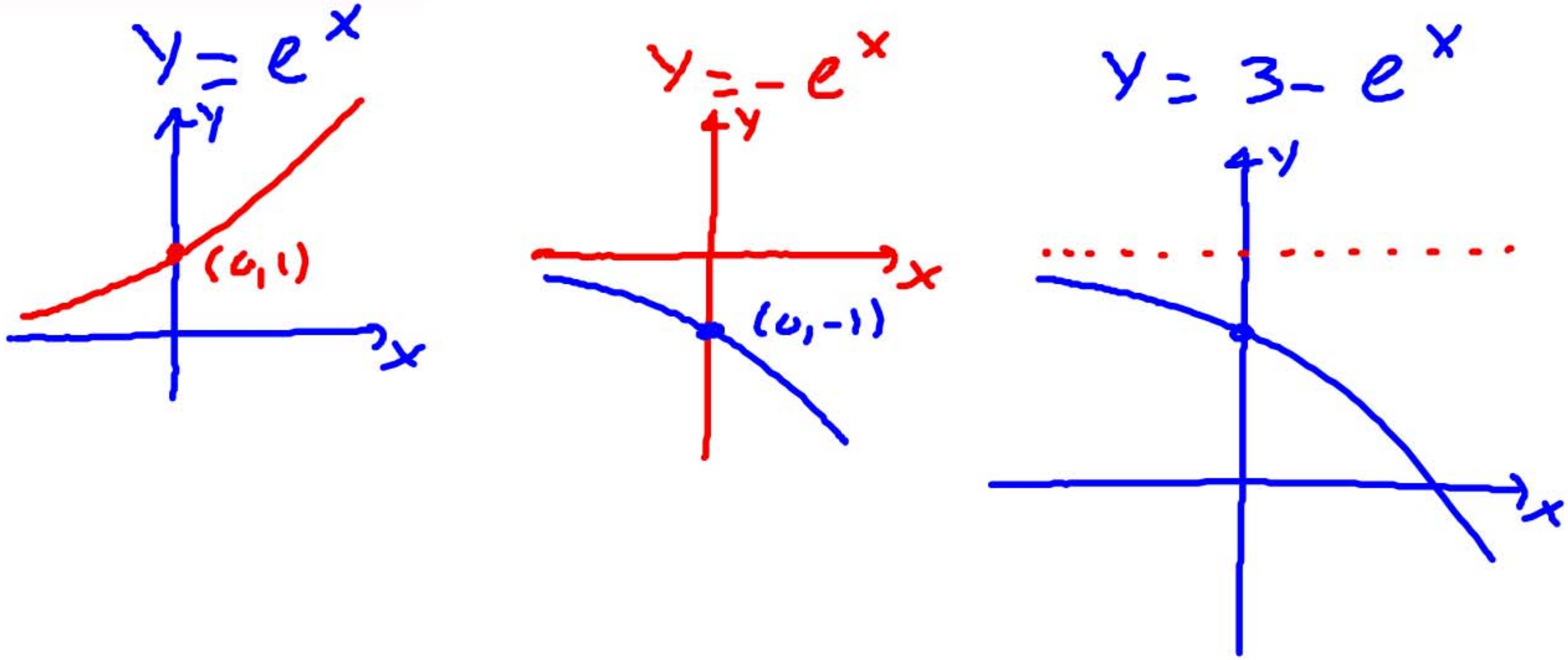


2) $y = 3^{-x}$



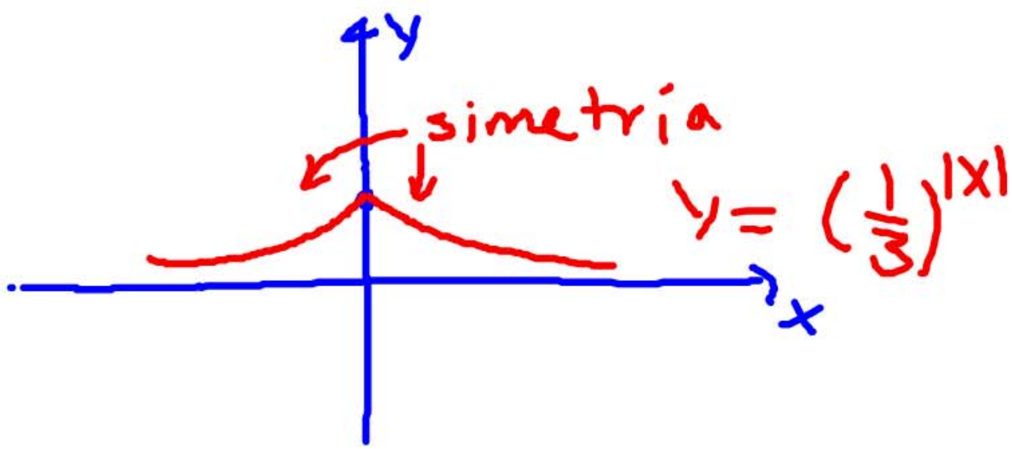


3) $y = 3 - e^x$

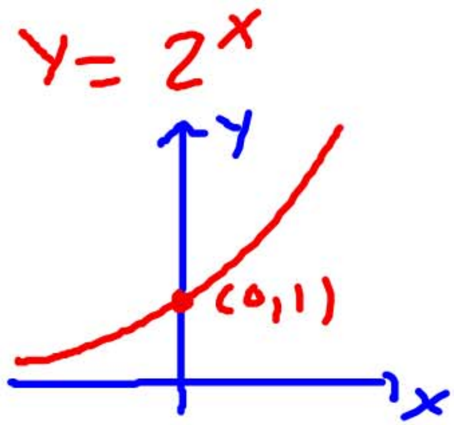


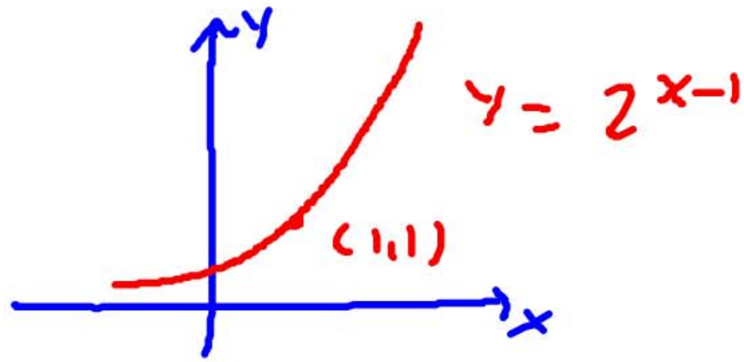
$$4)\ y=\left(\frac{1}{3}\right)^{|x|}$$

$$y=\left(\frac{1}{3}\right)^{|x|}\begin{cases} x\geq 0, & y=\left(\frac{1}{3}\right)^{x} \\ x<0, & y=\left(\frac{1}{3}\right)^{-x}\end{cases}$$



5) $y = 2^{x-1}$

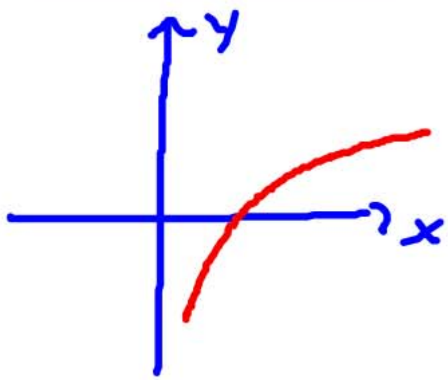


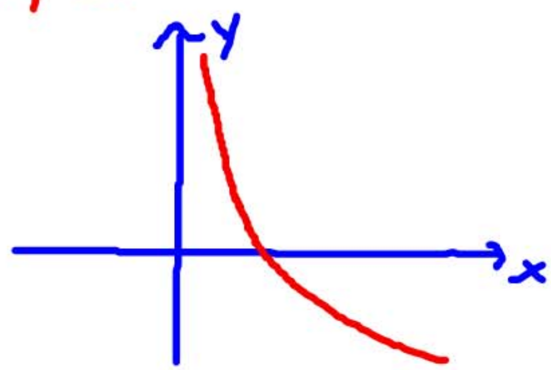


2.7.2 Las funciones logaritmo natural y exponencial, como inversas. Cambios de base.

Inversa del logaritmo

$y = \log_a x$ es inyectiva





$\log_a x = y \rightarrow a^y = x$

$a^x = y$

$\underline{y = a^x}$ - su inversa

Ecuaciones de anulación

$$\log_a a^x = x, \quad \forall x \in \mathbb{R}$$

$$a^{\log_a x} = x, \quad \forall x > 0$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

Cambio de base

Para cualquier número positivo a $(a \neq 1)$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$y = \log_a x$$

$$a^y = x$$

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$y = \frac{\ln x}{\ln a} \rightarrow \underline{\log_a x = \frac{\ln x}{\ln a}}$$

Ejemplos:

1) $\log_3 x = 2$

$$3^{\log_3 x} = 3^2$$

$$\underline{x = 9}$$

2) $3^x = 9$

$$\log_3 3^x = \log_3 9$$

$$\underline{x = 2}$$

2.8 Las funciones hiperbólicas, directas e inversas.

Definición. Funciones hiperbólicas.

El seno hiperbólico, coseno hiperbólico y las cuatro funciones relacionadas se definen como:

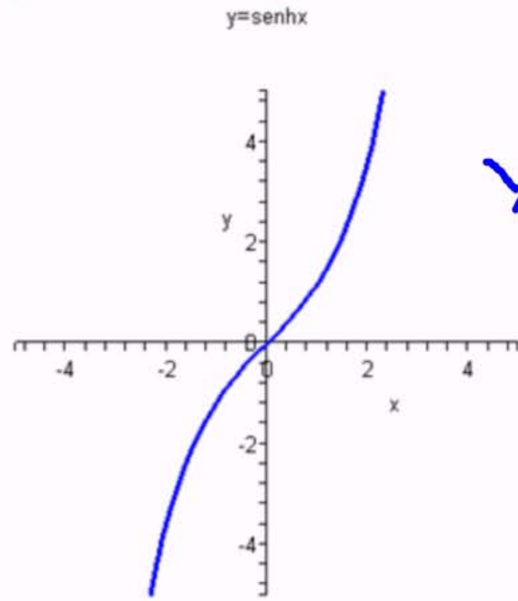
$$senhx = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{senhx}{\cosh x} \qquad \coth x = \frac{\cosh x}{senhx}$$

$$\sec hx = \frac{1}{\cosh x} \qquad \csc hx = \frac{1}{senhx}$$

Gráficas

```
> plot(sinh(x),x=-5..5,y=-5..5,color=blue,thickness=2,title="y=senhx");
```

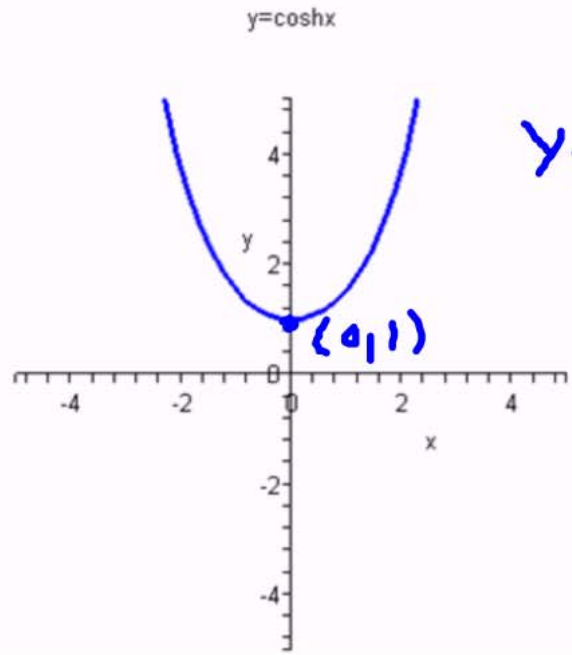


$Y = \operatorname{senh} x$

$D = \mathbb{R}$

$R = \mathbb{R}$

```
> plot(cosh(x),x=-5..5,y=-5..5,color=blue,thickness=2,title="y=coshx");
```

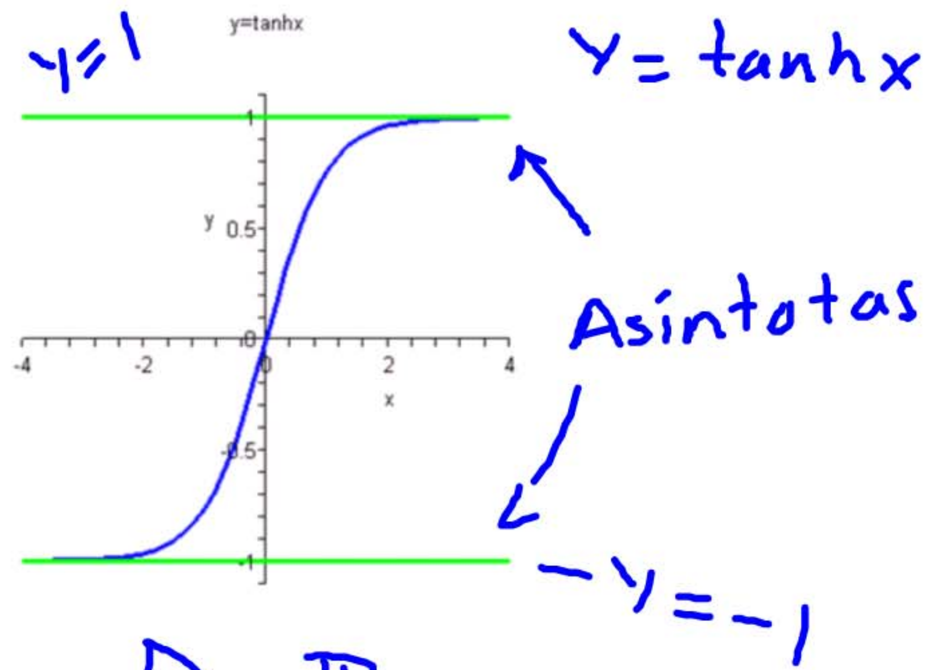


Y= coshx

$D = \mathbb{R}$

$R = [1, \infty)$

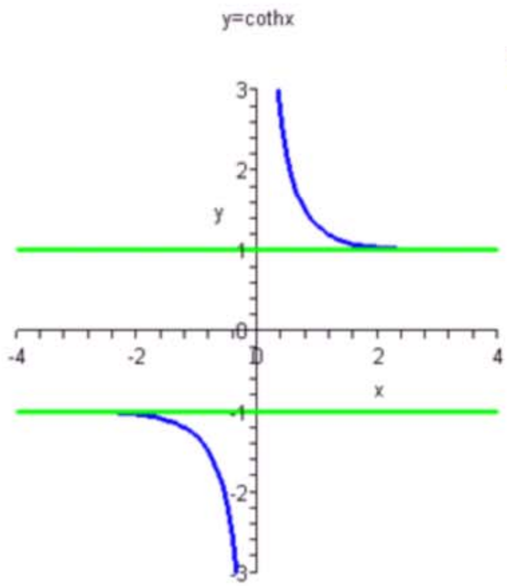
```
> plot([tanh(x),1,-1],x=-4..4,y=-1.1...1.1,color=[blue,green,green],thickness=2,title="y=tanhx");
```



$D = \mathbb{R}$

$R = (-1, 1)$

```
> plot([coth(x),1,-1],x=-4..4,y=-3...3,color=[blue,green,green],thickness=2,discon
  t = true,title="y=cothx");
```



$Y = \coth x$

$Y = 1$

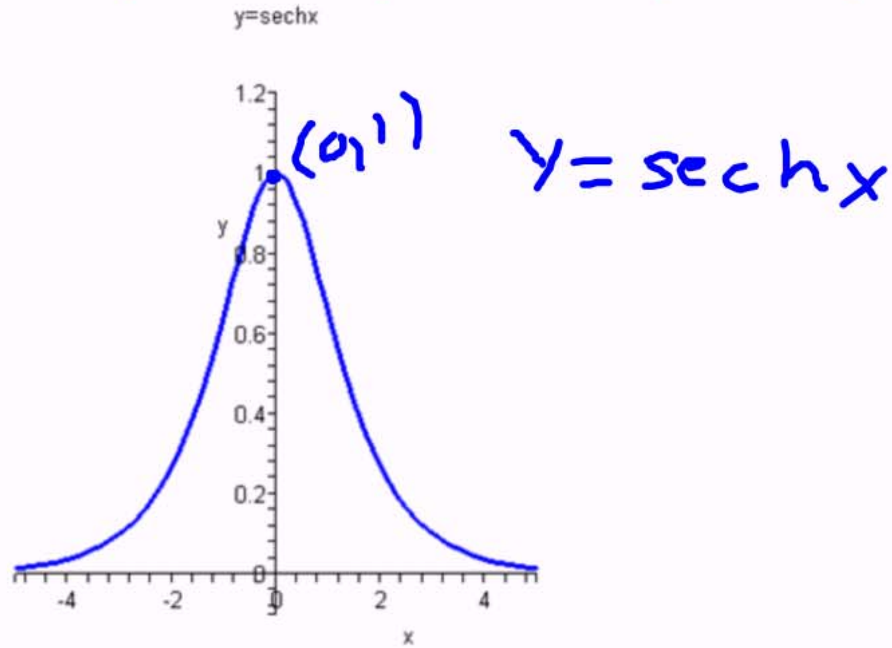
Asíntotas

$Y = -1$

$$D = \mathbb{R} - \{0\}$$

$$R = (-\infty, -1) \cup (1, \infty)$$

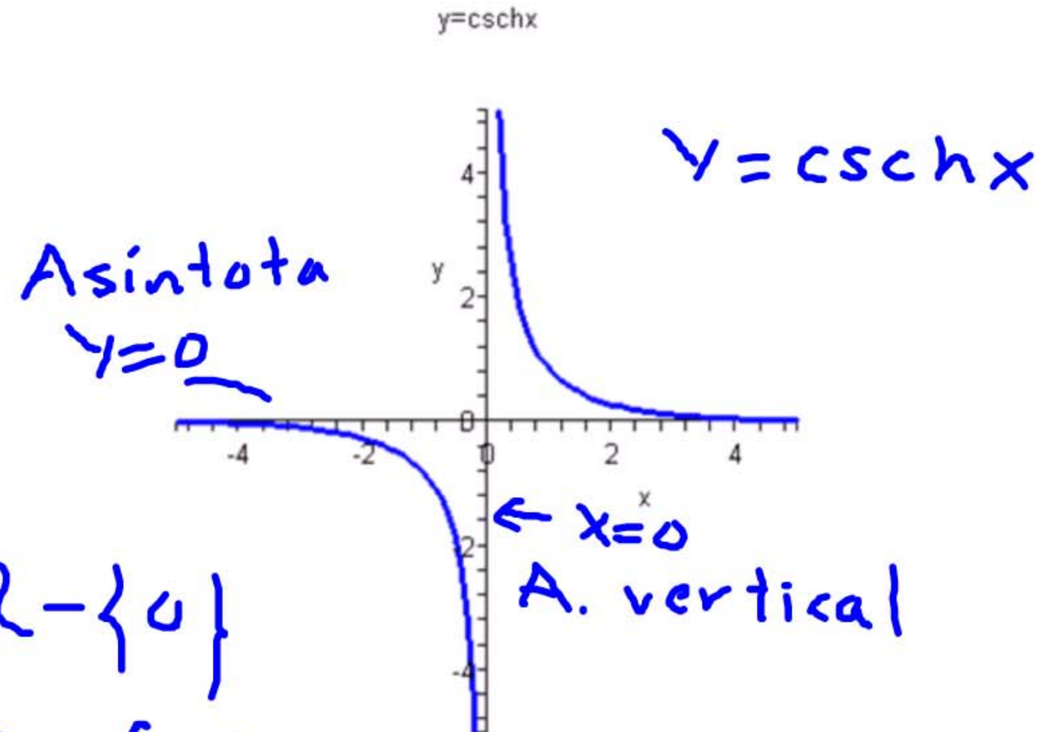
```
> plot(sech(x),x=-5..5,y=-0.1..1.2,color=blue,thickness=2,title="y=sechx");
```



$D = \mathbb{R}$

$R = (0, 1]$

```
> plot(csch(x),x=-5..5,y=-5..5,color=blue,thickness=2,discont =
  true,title="y=cschx");
```



$D = \mathbb{R} - \{0\}$

$R = \mathbb{R} - \{0\}$

Catenaria

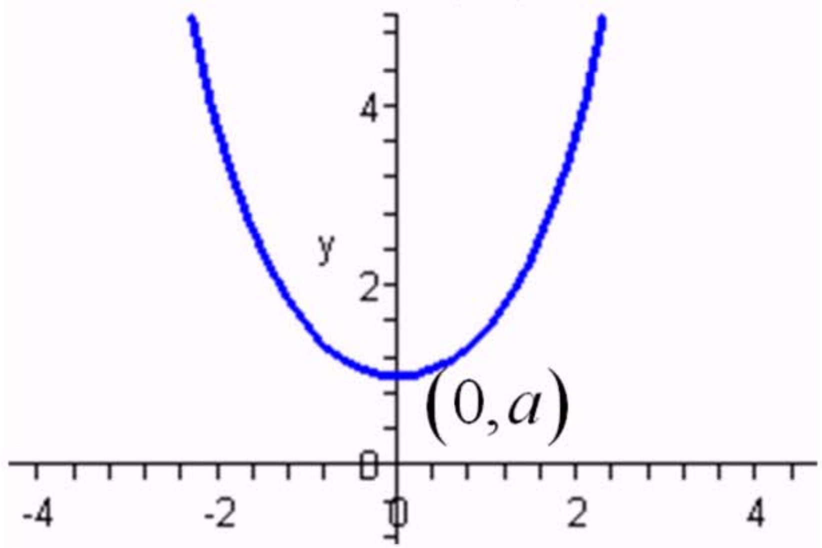
Es una curva formada por un cable flexible de densidad uniforme que cuelga libremente de dos puntos bajo su propio peso.

Ejemplos:

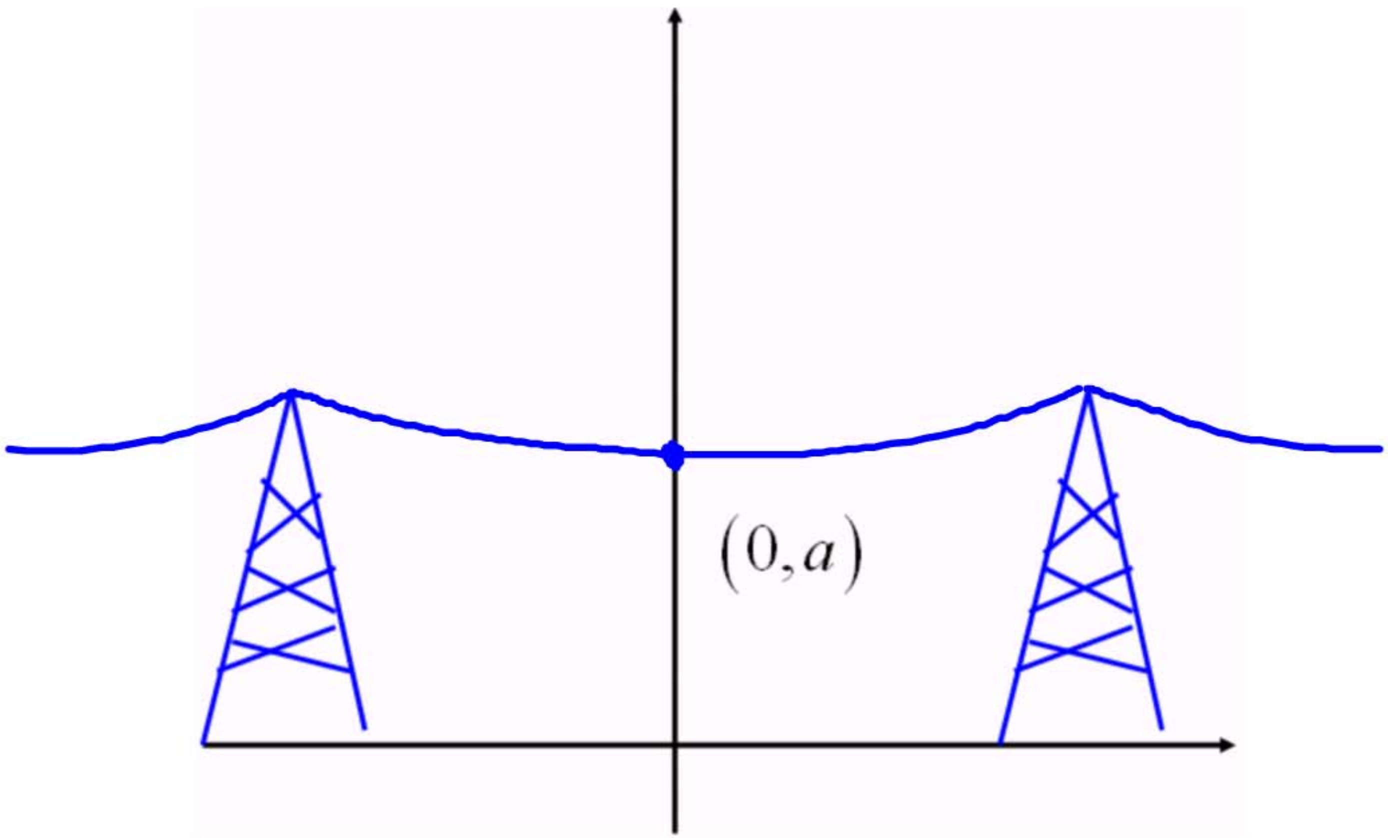
-algunos cables de puentes colgantes

-cables con corriente eléctrica para trolebuses

$$y = a\cosh\left(\frac{x}{a}\right)$$



$$y = a\cosh\left(\frac{x}{a}\right)$$



Identidades hiperbólicas

1) $senh(-x) = -senhx$ — impar

2) $\cosh(-x) = \cosh x$ — par

3) $\cosh x + senhx = e^{x}$

4) $\cosh x - senhx = e^{-x}$

5) $\cosh^2 x - senh^2 x = 1$

6) $1 - \tanh^2 x = \sec h^2 x$

7) $\coth^2 x - 1 = \csc h^2 x$

8) $senh(x+y) = senhx \cosh y + \cosh x senhy$

9) $senh(x-y) = senhx \cosh y - \cosh x senhy$

$$10)\ senh2x = 2senhx\cosh x \checkmark$$

$$11)\ \cosh(x+y) = \cosh x\cosh y + senhxsenhy$$

$$12)\ \cosh(x-y) = \cosh x\cosh y - senhxsenhy$$

$$13)\ \cosh 2x = \cosh^2 x + senh^2 x \checkmark$$

$$14)\ senh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$15)\ \cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$16)\ \tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x\tanh y}$$

$$17)\ \tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x\tanh y}$$

$$\operatorname{senh} x = \frac{e^{x} - e^{-x}}{2}$$

$$\operatorname{senh}(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} =$$

$$= \frac{-(-e^{-x} + e^{x})}{2} = -\frac{e^{x} - e^{-x}}{2} = -\operatorname{senh} x$$

$$\cosh x + \operatorname{senh} x = \frac{e^{x} + e^{-x}}{2} + \frac{e^{x} - e^{-x}}{2} =$$

$$= \frac{e^{x} + \cancel{e^{-x}} + e^{x} - \cancel{e^{-x}}}{2} = e^{x}$$

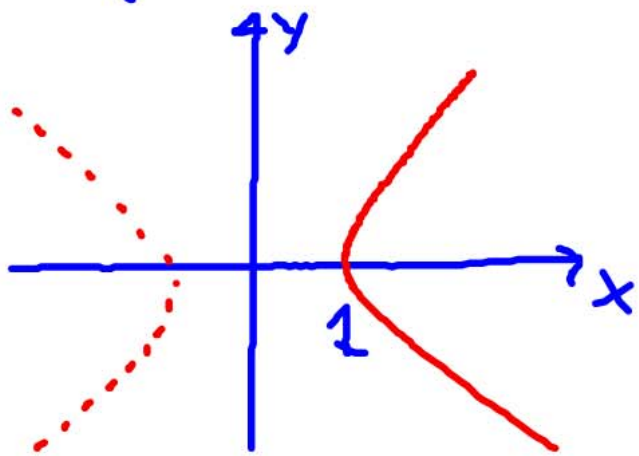
$$(\cosh x + \operatorname{senh} x)(\cosh x - \operatorname{senh} x) = e^{x} e^{-x}$$

$$\underline{\cosh^2 x - \operatorname{senh}^2 x = 1}$$

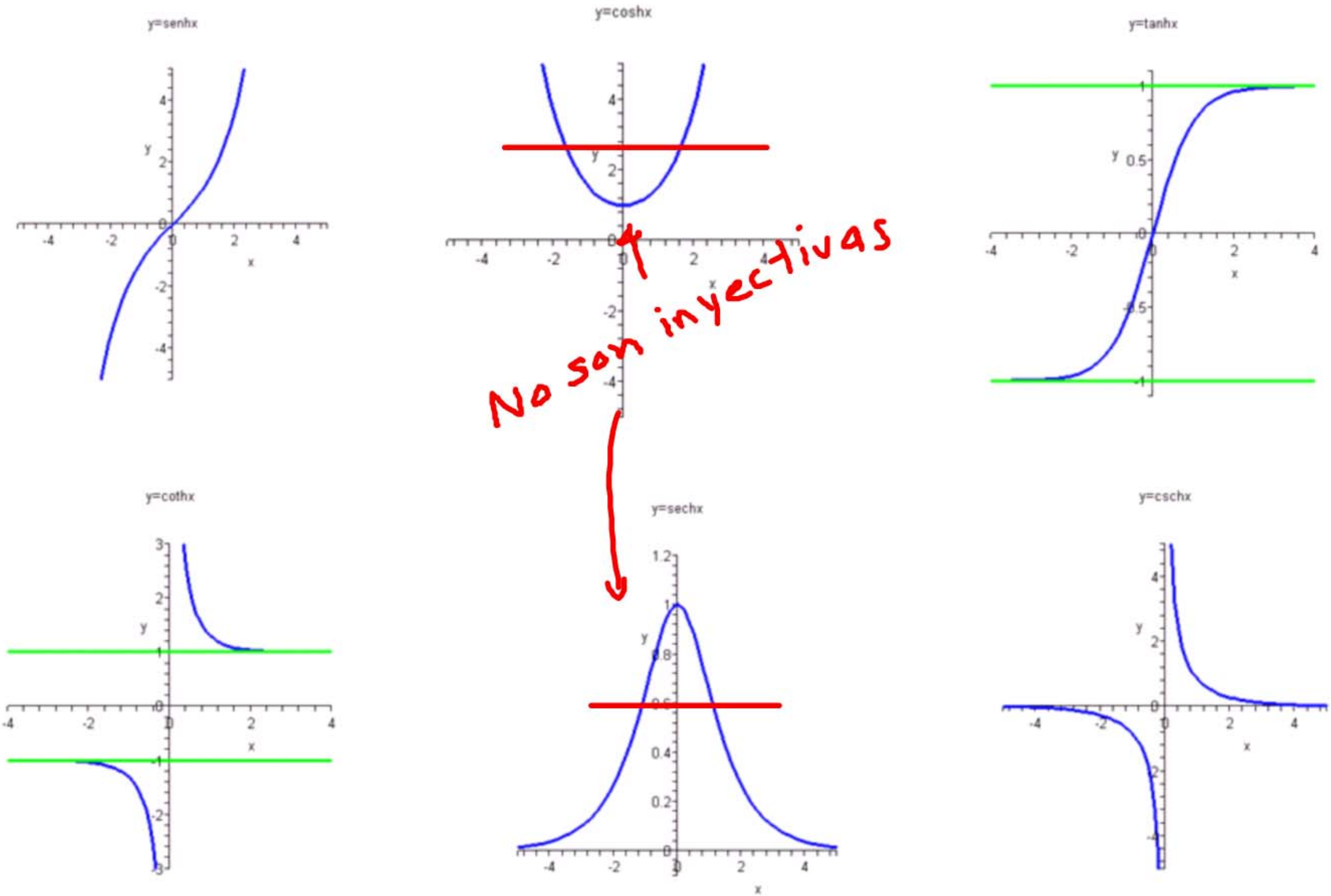
Identidad fundamental

$$C: \begin{cases} x = \cosh t \\ y = \operatorname{senh} t \end{cases}$$

$$\begin{array}{r} x^2 = \cosh^2 t \\ - \quad y^2 = \operatorname{senh}^2 t \\ \hline x^2 - y^2 = 1 \end{array}$$



Funciones hiperbólicas inversas



Teorema

$$senh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\sec h^{-1}x = \ln\frac{1+\sqrt{1-x^2}}{x}$$

$$\csc h^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

$$y = senh^{-1}x$$

$$senh\, y = x$$

$$\frac{e^y - e^{-y}}{2} = x$$

$$\left(e^y - e^{-y} = 2x\right)e^y$$

$$e^{2y} - 2xe^y - 1 = 0$$

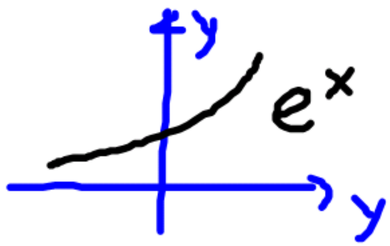
$$(e^y)^2 - 2x(e^y) - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4(1)(-1)}}{2}$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$



$$e^y = x + \sqrt{x^2+1}$$

$$y = \ln\left(x+\sqrt{x^2+1}\right) \rightarrow \operatorname{senh}^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$$

$$y = \tanh^{-1} x$$

$$\tanh y = x$$

$$\frac{\operatorname{senh} y}{\cosh y} = x$$

$$\frac{\frac{e^{y} - e^{-y}}{\cancel{2}}}{\frac{e^{y} + e^{-y}}{\cancel{2}}} = x$$

$$e^{y}\left(e^{y} - e^{-y} = xe^{y} + xe^{-y}\right)$$

$$e^{2y} - 1 = xe^{2y} + x$$

$$e^{2y} - xe^{2y} = x + 1$$

$$e^{2y}(1 - x) = x + 1$$

$$e^{2y} = \frac{1 + x}{1 - x}$$

$$2y = \ln\left(\frac{1 + x}{1 - x}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

Identidades para funciones hiperbólicas inversas

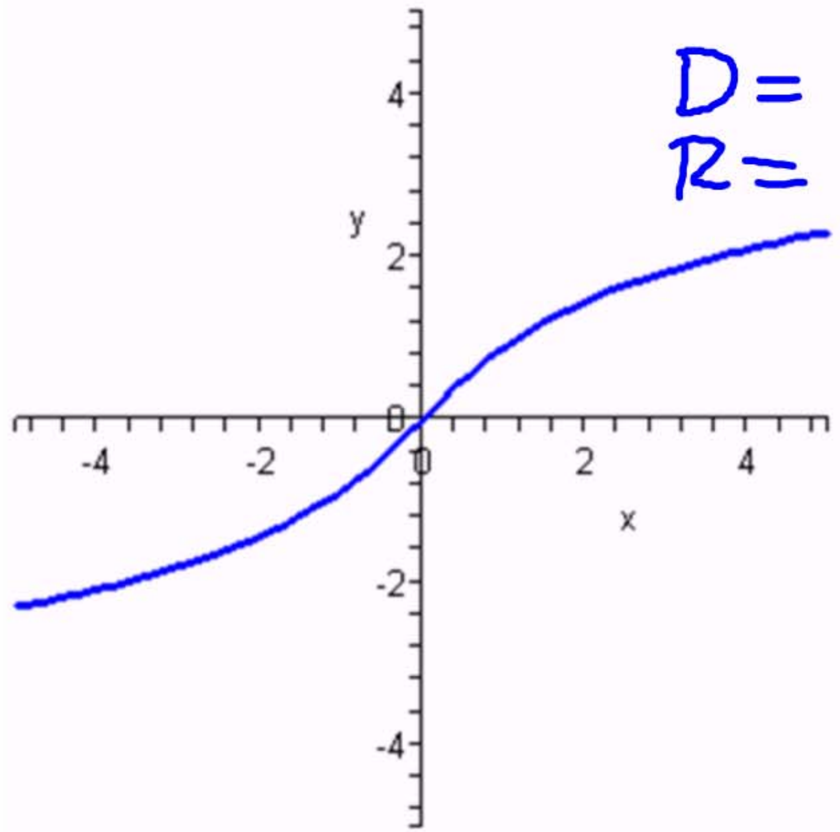
$$1)\ \csc h^{-1}x = senh^{-1}\left(\frac{1}{x}\right)$$

$$2)\ \sec h^{-1}x = \cos h^{-1}\left(\frac{1}{x}\right)$$

$$3)\ \cot h^{-1}x = \tan h^{-1}\left(\frac{1}{x}\right)$$

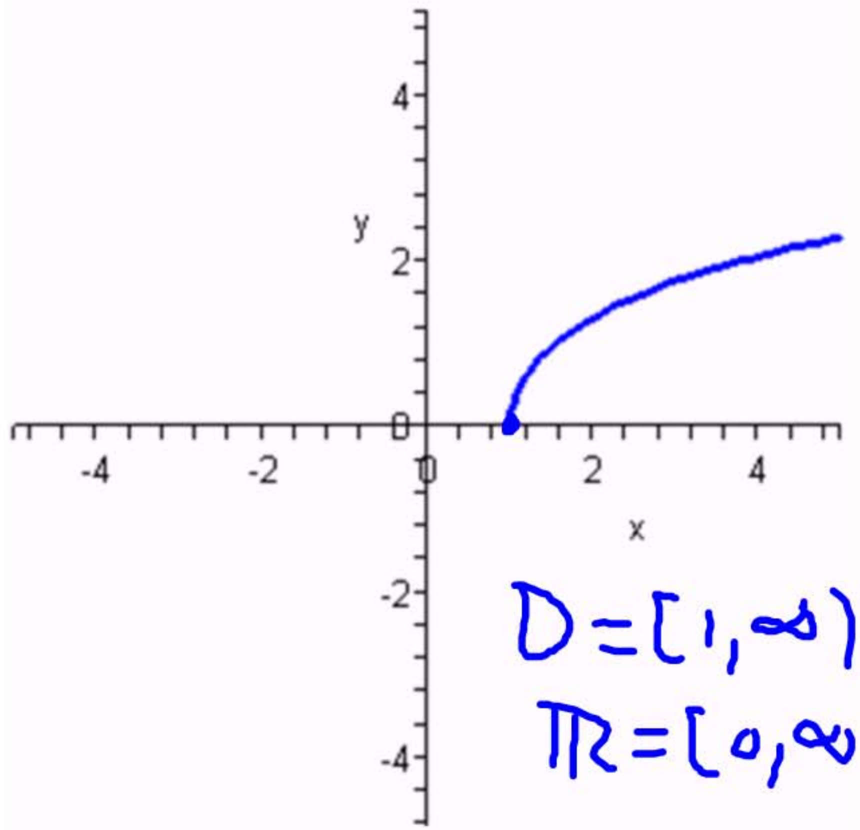
Gráficas

$$y = senh^{-1}x$$



$D = \mathbb{R}$
$R = \mathbb{R}$

$$y = \cosh^{-1} x$$



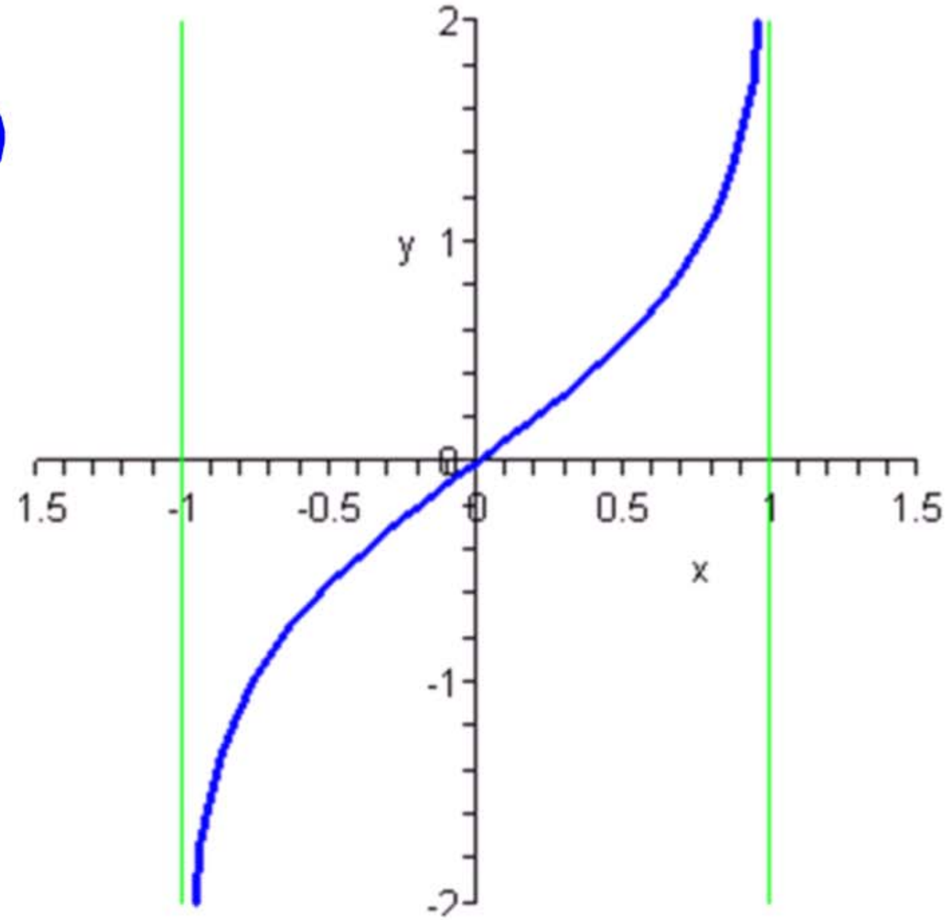
$D = [1, \infty)$

$R = [0, \infty)$

$$y = \tanh^{-1} x$$

$D = (-1, 1)$

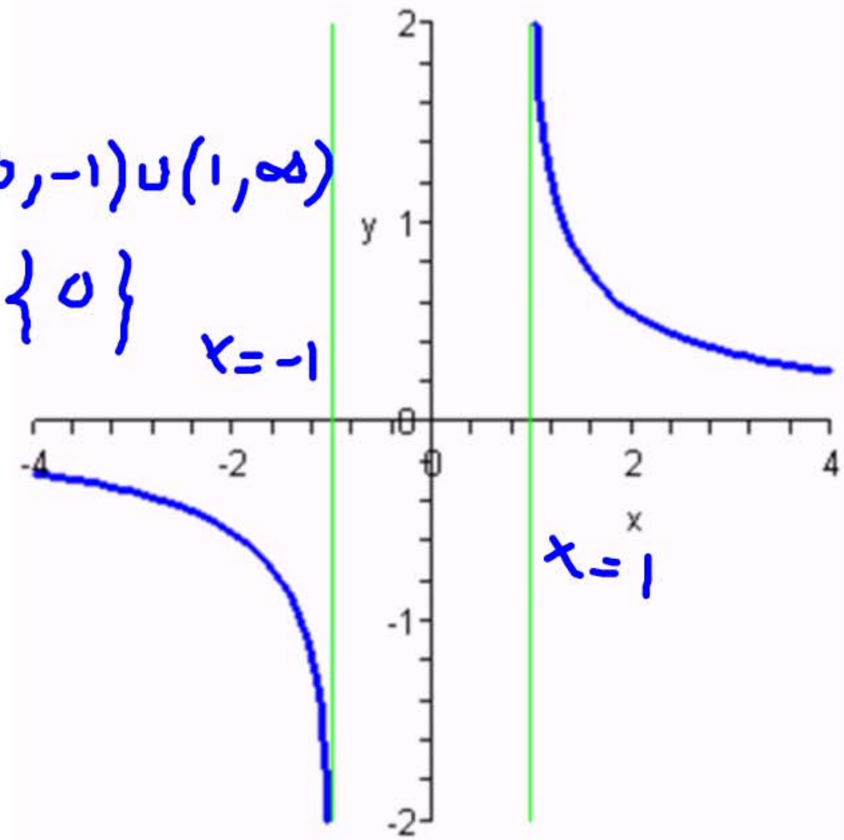
$R = \mathbb{R}$



$$y = \coth^{-1} x$$

$D = (-\infty, -1) \cup (1, \infty)$

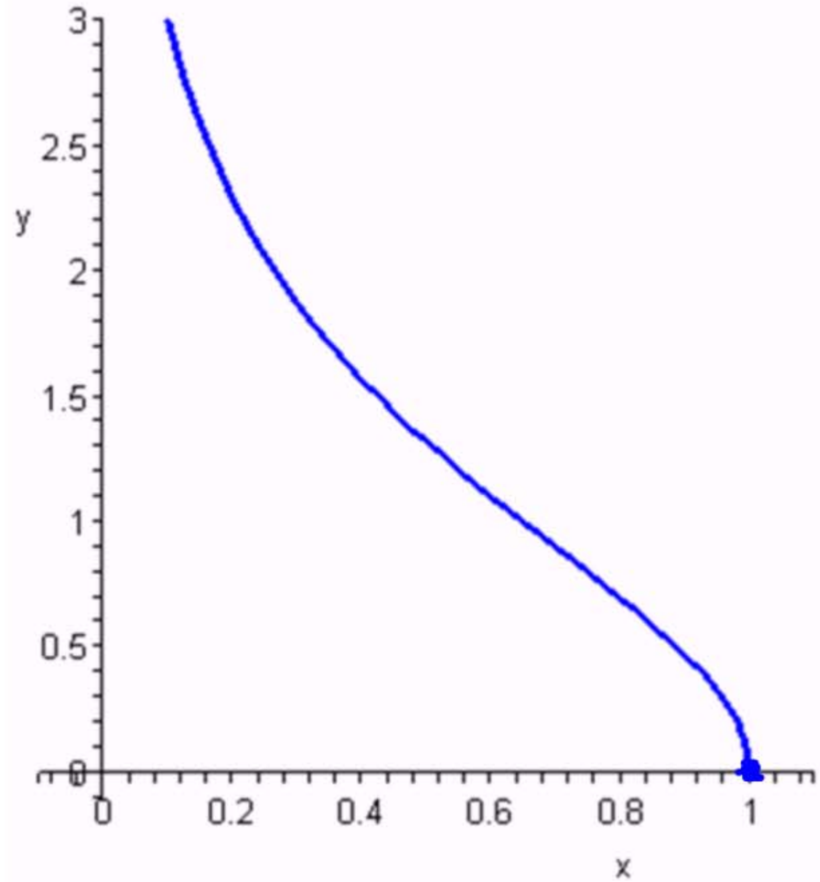
$R = \mathbb{R} - \{0\}$



$$y = \operatorname{sech}^{-1} x$$

$D = (0, 1]$

$R = [0, \infty)$



$$y = \csc h^{-1} x$$

$D = \mathbb{R} - \{0\}$

$R = \mathbb{R} - \{0\}$