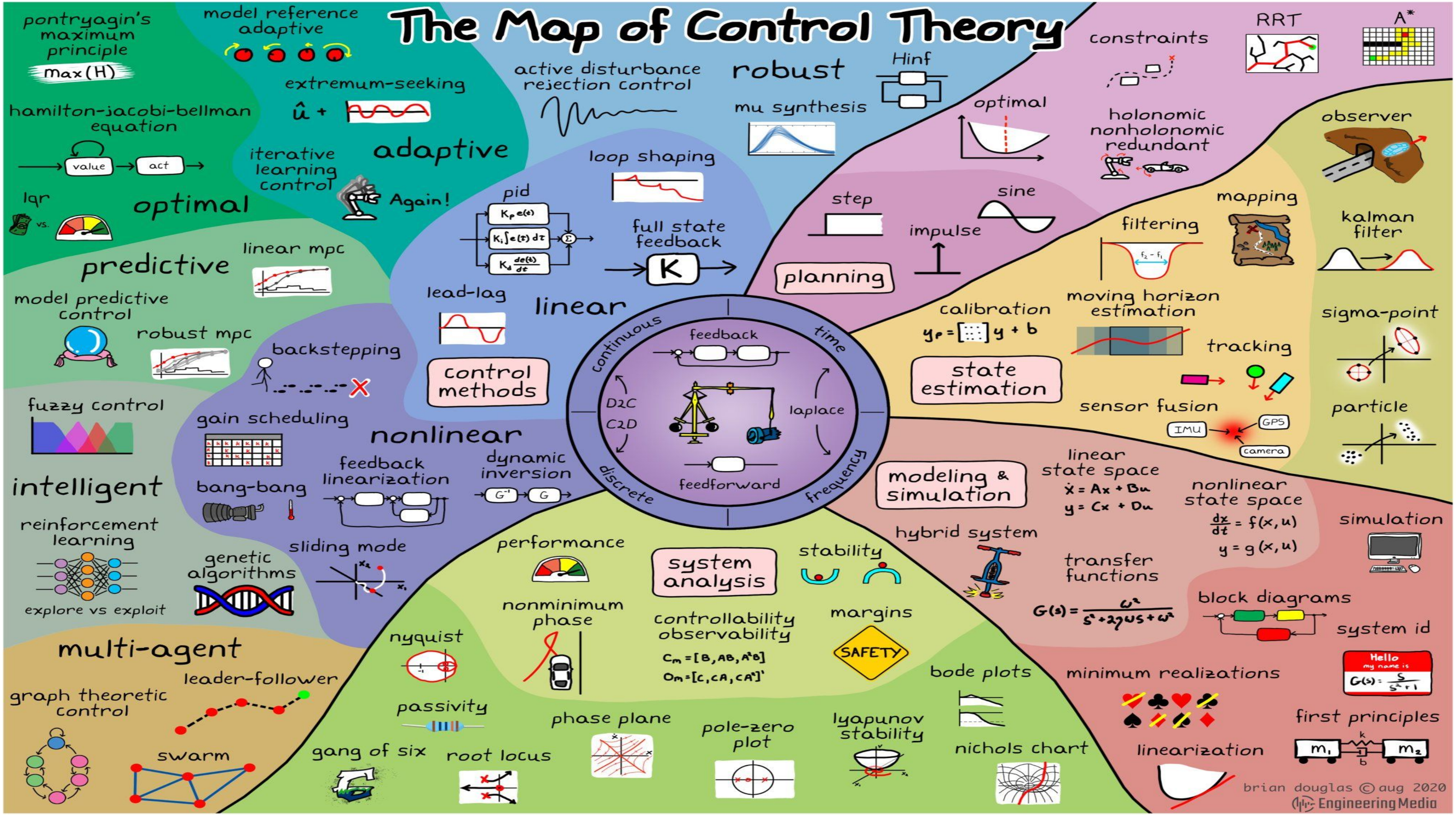

The Map of Control Theory
pontryagin's maximum principle
Max(H)
model reference adaptive
extremum-seeking
$\hat{u}$ +
hamilton-jacobi-bellman equation
value
act
iterative learning control
Again!
adaptive
lqr
vs.
optimal
active disturbance rejection control
robust
Hinf
mu synthesis
loop shaping
pid
$K_p e(t)$
$K_i \int e(t)\,dt$
$K_d \frac{de(t)}{dt}$
Σ
full state feedback
K
linear mpc
predictive
model predictive control
robust mpc
lead-lag
linear
control methods
backstepping
fuzzy control
gain scheduling
nonlinear
intelligent
bang-bang
feedback linearization
dynamic inversion
G^{-1}
G
reinforcement learning
explore vs exploit
genetic algorithms
sliding mode
x_2
x_1
multi-agent
leader-follower
graph theoretic control
swarm
continuous
time
discrete
frequency
feedback
feedforward
D2C
C2D
laplace
constraints
RRT
A*
optimal
holonomic
nonholonomic
redundant
observer
step
sine
impulse
planning
mapping
filtering
$f_2 - f_1$
kalman filter
calibration
$y_p = [\,\vdots\,]\,y + b$
moving horizon estimation
sigma-point
tracking
state estimation
sensor fusion
IMU
GPS
camera
particle
modeling & simulation
linear state space
$\dot{x} = Ax + Bu$
$y = Cx + Du$
nonlinear state space
$\frac{dx}{dt} = f(x,u)$
$y = g(x,u)$
simulation
hybrid system
transfer functions
$G(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$
block diagrams
system id
Hello my name is
$G(s) = \frac{5}{s^2+1}$
minimum realizations
first principles
m_1
k
b
m_2
linearization
performance
system analysis
stability
nonminimum phase
controllability observability
$C_m = [B, AB, A^2B]$
$O_m = [C, CA, CA^2]'$
margins
SAFETY
nyquist
-1
passivity
phase plane
pole-zero plot
lyapunov stability
bode plots
nichols chart
gang of six
root locus
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SRA Domain Session: Control Systems

By: Jash and Ayush

What we will cover in this session?

1. What is a controller ?
2. Why the need for a controller ?
3. Feedback loop & Feedforward loop
4. Stability
5. Modelling a System (State space equations) linear/non-linear model
6. Controllability and Observability, State Estimator (Kalman Filter)
7. Control Systems:
    a. PID
    b. Pole Placement
    c. LQR
    d. MPC
8. Control Systems using RL

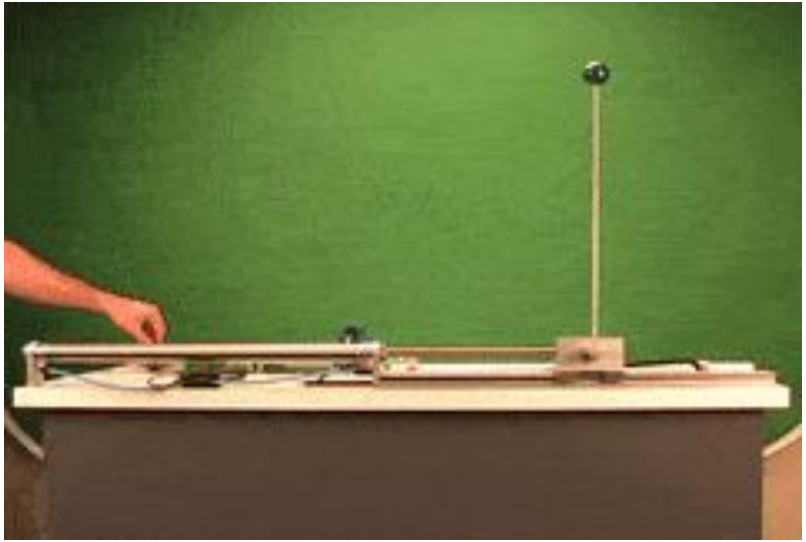
What is a Controller ?

A Controller is a mechanism that seeks to minimize the difference between the actual value of a system (i.e. the process variable) and the desired value of the system (i.e. the setpoint)

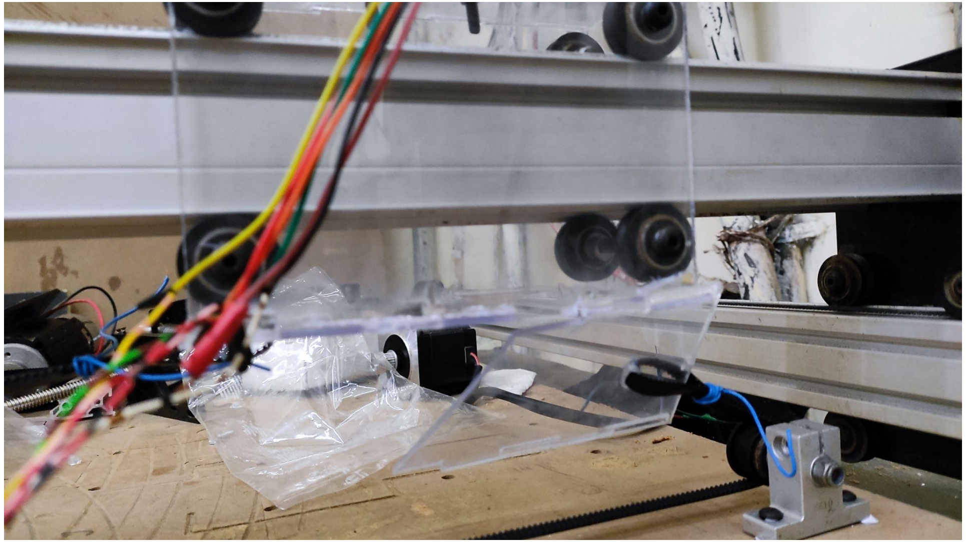
What is a Control System ?

A system of devices that manages, commands, directs, or regulates the behavior of other devices or systems to achieve a desired result.

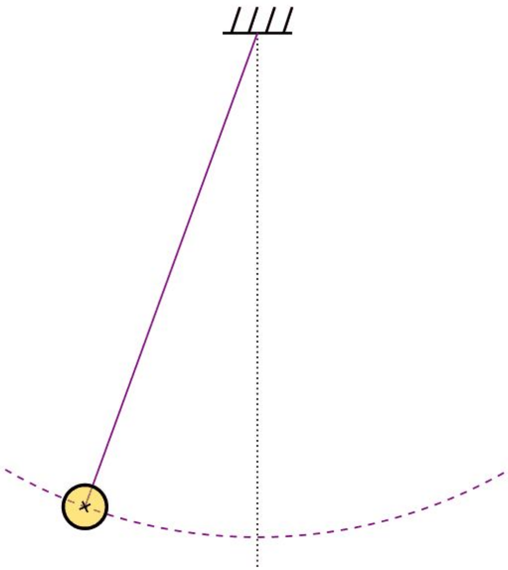
With Controller



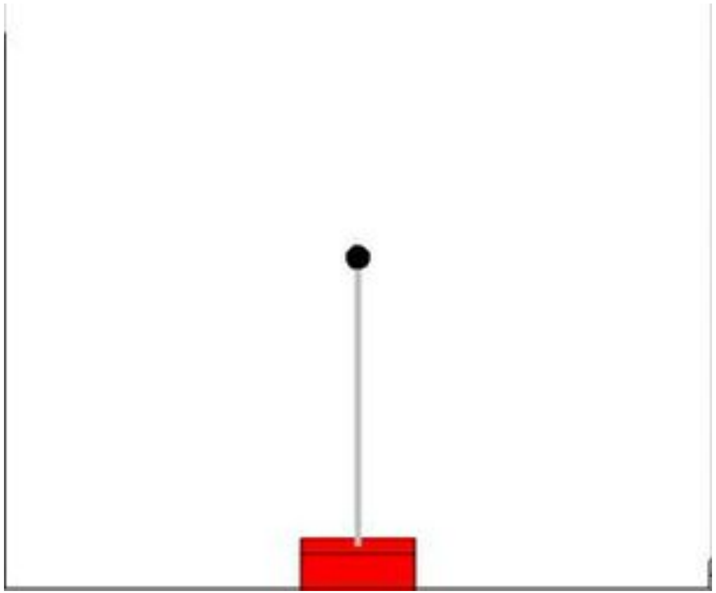
Without Controller



Pendulum



Inverted Pendulum



What is the major physical difference between the two?

Pendulum

Trying to balance around **stable equilibrium point**. So no need of any external actuation

Inverted Pendulum

Trying to balance around **unstable equilibrium point**. So external actuation is required

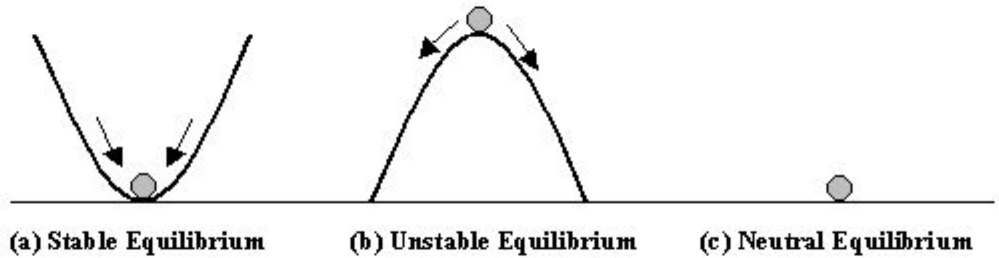


Figure 1 – Three Types of Equilibria

Why the need for a Control System ?

1. Controllers improve the steady-state accuracy by **decreasing the steady state error.**

2. As the steady-state accuracy improves, the stability also improves.

3. Controllers also help in **reducing the unwanted offsets** produced by the system.

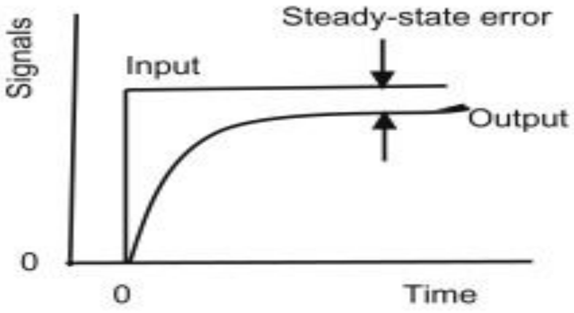
4. Controllers can **control the maximum overshoot of the system**.

5. Controllers can help in **reducing the noise signals produced by the system**.

6. Controllers can help to **speed up the slow response of an overdamped system**.

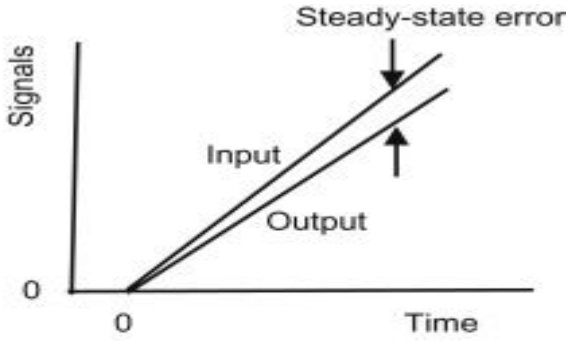
Steady State Error

The **difference between the desired value and the actual value** of a system output in the limit **as time goes to infinity** (i.e. when the response of the control system has reached steady-state).



(a)

For Step Input

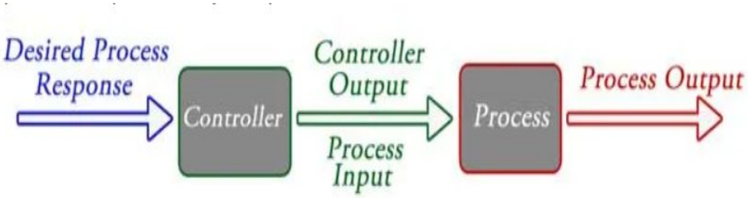


(b)

For Ramp Input

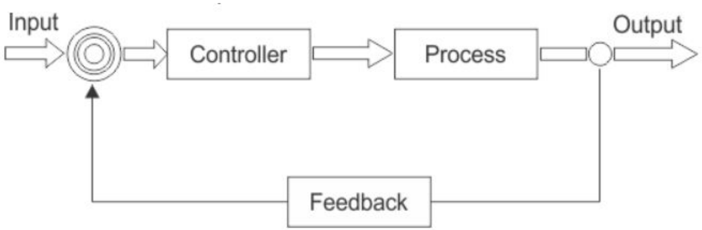
Open Loop

A control system in which the <u>**control action is totally independent of the output of the system**</u> then it is called an open-loop control system.



Closed Loop

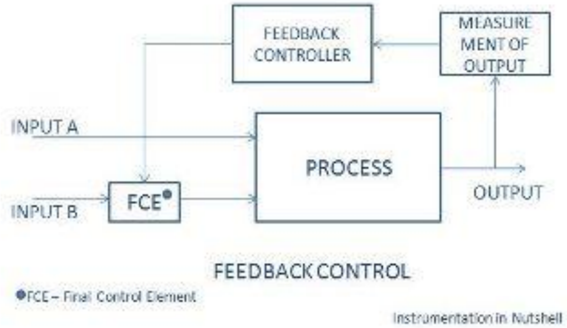
Control systems in which the <u>**output has an effect on the input quantity**</u> in such a manner that the input quantity will adjust itself based on the output generated is called a closed-loop control system.



Feedback

Measures the output of a process, calculates the error in the process and then adjusts one or more inputs to get the desired output value.

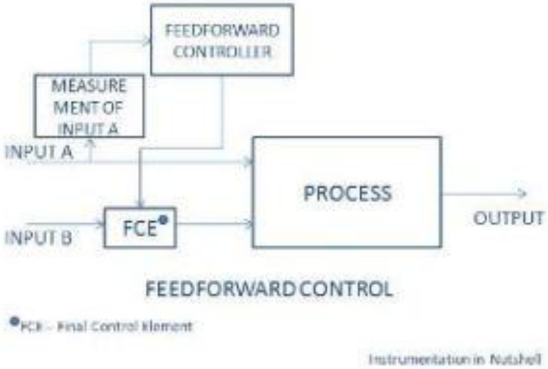
<u>**REACTIVE CONTROL**</u>



Feedforward

Measures one or more inputs of a process, calculates the required value of the other inputs and then adjusts it.

<u>**PREDICTIVE CONTROL**</u>



Modelling a System

The process of representing a physical system in terms of an idealized mathematical model.

- **Newton - Euler Equations**

$$F_{\text{ext}} = \frac{d\mathbf{p}}{dt}. \quad \mathbf{M} = \frac{d\mathbf{L}}{dt}. \qquad \begin{pmatrix} \mathbf{F} \\ \boldsymbol{\tau} \end{pmatrix} = \begin{pmatrix} m\mathbf{I}_3 & 0 \\ 0 & \mathbf{I}_{\text{cm}} \end{pmatrix} \begin{pmatrix} \mathbf{a}_{\text{cm}} \\ \boldsymbol{\alpha} \end{pmatrix} + \begin{pmatrix} 0 \\ \boldsymbol{\omega} \times \mathbf{I}_{\text{cm}}\, \boldsymbol{\omega} \end{pmatrix}$$

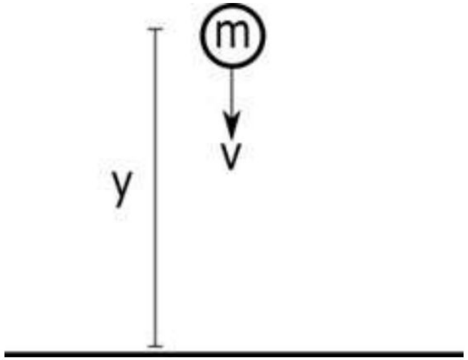
- **Euler - Lagrange Equations**

$$L = K.E - P.E \qquad S[\{q_i(t)\}] = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i; t) \qquad \frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = 0$$

- **Hamiltonian Equations**

$$H = \sum_i p_i \dot{q}_i - L. \qquad \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}, \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}.$$

Euler Lagrange Method - Example 1



$$PE = mgy \qquad (1)$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2 \qquad (2)$$

$$L = KE - PE = \frac{1}{2}m\dot{y}^2 - mgy \qquad (3)$$

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2}m\dot{y}^2 - mgy\right) = -mg \qquad (4)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2 - mgy\right)\right) = m\ddot{y} \qquad (5)$$

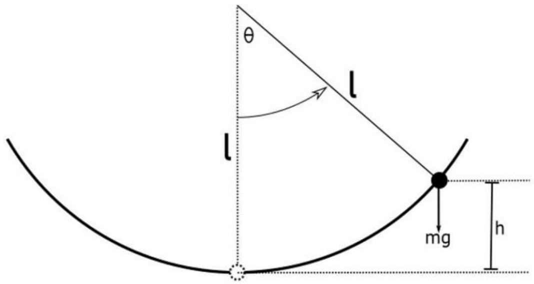
Hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$=> m\ddot{y} - (-mg) = 0$$

$$\boxed{\ddot{y} = -g}$$

Euler Lagrange Method - Example 2



$$PE = mgh = mg(l - l\cos\theta) = mgl(1-\cos\theta)$$

$$KE = \frac{1}{2}I\omega^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$L = KE - PE = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right)\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right) = 0$$

$$=> \quad ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\boxed{\ddot{\theta} = \frac{-g}{l}\sin\theta}$$

Representing a Control System

$$\dot{x}_1 = f_1(t, x_1, \ldots\ldots, x_n, u_1, \ldots\ldots, u_p)$$

$$\dot{x}_2 = f_2(t, x_1, \ldots\ldots, x_n, u_1, \ldots\ldots, u_p)$$

$$\vdots$$

$$\dot{x}_n = f_n(t, x_1, \ldots\ldots, x_n, u_1, \ldots\ldots, u_p)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_p \end{bmatrix}, f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

We can rewrite the n first-order differential equations as one n-dimensional first-order vector differential equation

$$\dot{x} = f(t, x, u)$$

Representing a Control System

Linearizing Systems using Jacobian

(b) Linearize around these equilibrium points:

i.e Non-Linear system acts linear when we look rally close to equilibrium points.

$$\text{Let } \bar{X} => \text{Equilibrium point}$$

For Non-Linear System,

$$\dot{X} = f(x)$$

where f(x) is a non-linear combination of state variables.
Around Equilibrium point,

$$\therefore \dot{X} = f(\bar{X}) + \frac{Df}{Dx}\Big|_X (X - \bar{X}) + \frac{D^2 f}{DX^2}\Big|_X (X - \bar{X})^2 + \ldots.$$

This is the Taylor-Series expansion around $\bar{X}$
where, $\frac{Df}{DX}\Big|_X$ is the Jacobian of $\dot{X} = f(X) at \bar{X}$

$$\frac{Df}{DX}\Big|_{\bar{X}} = \begin{bmatrix} \frac{\delta f1}{\delta x1} & \frac{\delta f1}{\delta x2} & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\delta fn}{\delta x1} & . & . & \frac{\delta fn}{\delta xn} \end{bmatrix} at \bar{X}$$

Taking only the linear component of this expansion,

$$\dot{X} = \frac{Df}{DX}\Big|_X (X - \bar{X})$$

$$\therefore \dot{X} = A \cdot X$$

,

$$A = \begin{bmatrix} \frac{\delta f1}{\delta x1} & \frac{\delta f1}{\delta x2} & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\delta fn}{\delta x1} & . & . & \frac{\delta fn}{\delta xn} \end{bmatrix} at \bar{X}$$

we come to realize that for a Non-Linear system, the matrix defines our state space model linearly is basically just the Ja- ı of the system taken at the equilibrium point.

Representing a Control System

1 State Space Analysis

1. **Select States of System**

$$X = \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

2. **State Space Model**

$$\dot{X} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \cdot \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

$$\therefore \dot{X} = A \cdot X$$

$$\therefore \frac{dX}{dt} = A \cdot X$$

$$\therefore X(t) = e^{At} \cdot X(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ... \qquad (1)$$

But this expansion will be tough to calculate and hence the equations become unnecessarily complex.

Representing a Control System

So instead we can make use of diagonalization to linearly transform our system to eigenvalue co-ordinates.

$$\text{i.e } X = P \cdot Z \text{ (P = Modal Matrix)}$$

$$\therefore \dot{X} = P \cdot \dot{Z} = A \cdot X$$

$$\therefore P \cdot \dot{Z} = A \cdot P \cdot Z$$

$$\therefore \dot{Z} = P^{-1} \cdot A \cdot P \cdot Z$$

Representing a Control System

Considering P is orthogonal Matrix

$$\dot{Z} = D \cdot Z$$

D = Diagonal Matrix

$$\therefore \dot{Z} = \begin{bmatrix} \lambda 1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \lambda n \end{bmatrix} \cdot Z$$

Equations become decoupled and thus have simple to calculate solutions.

$$i.e\ \dot{z1} = \lambda 1.z1$$

$$\dot{z2} = \lambda 2.z2$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

Also,

$$Z(t) = e^{Dt} \cdot Z(0)$$

$$\therefore Z(t) = \begin{bmatrix} e^{\lambda 1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & e^{\lambda n} \end{bmatrix} \cdot Z(0)$$

Representing a Control System

Now,
Converting back to original state since it is important to preserve our required system state variables.
Thus, Using Power of Matrix Theorem in equation (1),

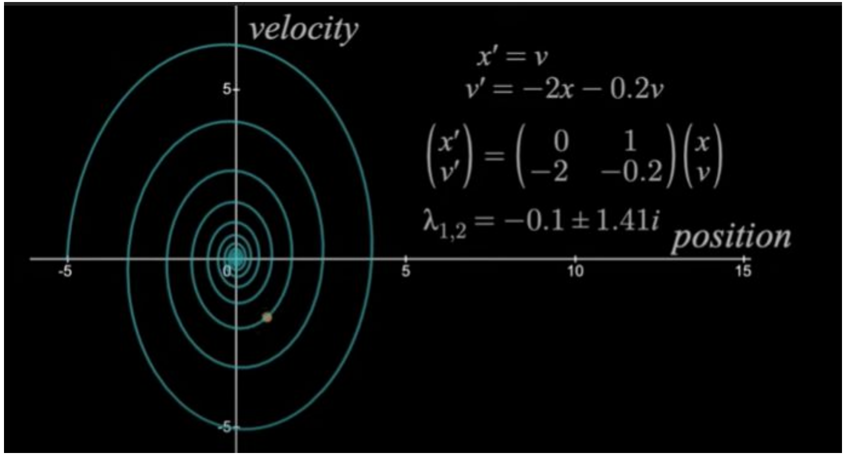
$$e^{At} = P^{-1} \cdot P + P^{-1} \cdot D \cdot Pt + \frac{P^{-1} \cdot D^2 \cdot Pt^2}{2!} + ...$$

$$\therefore e^{At} = P\left[I + D + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + ...\right]P^{-1}$$

$$\therefore e^{At} = P^{-1} \cdot e^{Dt} \cdot P$$

$$\therefore \boxed{X(t) = P^{-1} \cdot e^{Dt} \cdot P \cdot X(0)} \tag{2}$$

Effect of Eigenvalues on Stability



Negative Real Part corresponding to damping of system which results in Stability

Studying the Effect of Eigenvalues on Stability of System

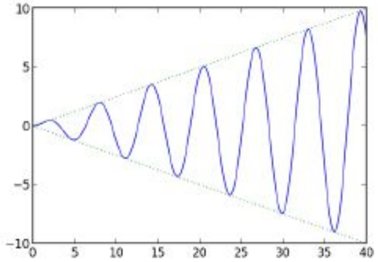
From equation (2), we can infer that X(t) will be some linear combinatior of $e^{\lambda t}$ terms.

Now,

$$\therefore \lambda = a + ib$$

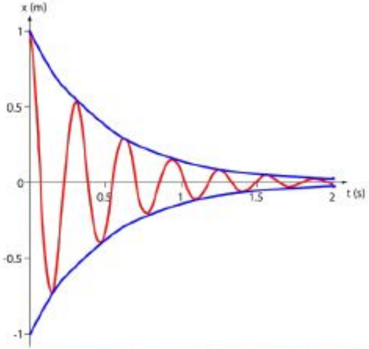
$$\therefore e^{\lambda t} = e^{at}\underbrace{(cos(bt) + i.sin(bt))}_{\text{always 1}}$$

if $a > 0$



System oscillations grow overtime so unstable.

if $a < 0$



System oscillations increase overtime so reaches stability.

However, this modelling only works if the system is linear. But in most practical applications, the systems are non-linear. So, in order to linearize them we make use of Jacobians.

What about Eigenvalues having positive real part?

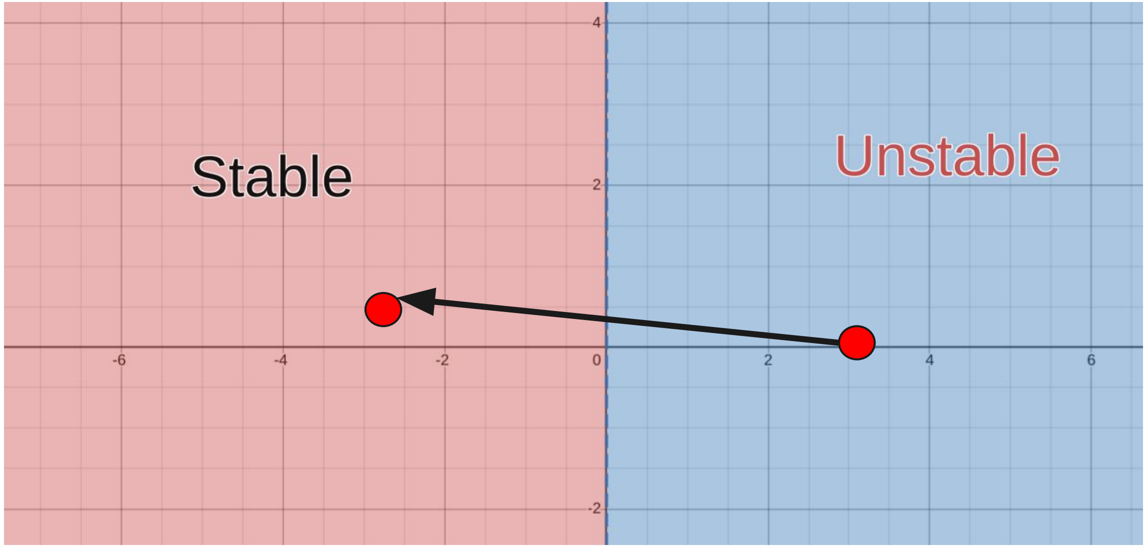
Ax => Ax + Bu

u = -Kx

So,

Ax => Ax - BKx

Ax => (A-B.K)x



Representing a Control System

$$\text{State Equation} \Rightarrow \dot{x}(t) = Ax(t) + Bu(t)$$
$$\text{Output Equation} \Rightarrow y(t) = Cx(t) + Du(t)$$

Here

$x(t)$- State Vector (n x 1 matrix)

$y(t)$- Output Vector (p x 1 matrix)

$u(t)$- Input Vector (m x 1 matrix)

A - State (or system) matrix (n x n matrix)

B - Input matrix (n x m matrix)

C - Output Matrix (p x n matrix)

D - Feed-forward matrix (p x m matrix)

Controllability

Now that we have out System Modelled, we can work towards making a controller for the same.

$$\boxed{\dot{X} = A \cdot X + B \cdot u} \tag{3}$$

where,

$$X \in \Re^n;$$

$$A \in \Re^{nxn};$$

$$B \in \Re^{nxq};$$

$$u \in \Re^q;$$

Controllability Matrix C

$$C = \begin{bmatrix} B & A.B & A^2.B & . & . & . & A^{n-1}.B \end{bmatrix}$$

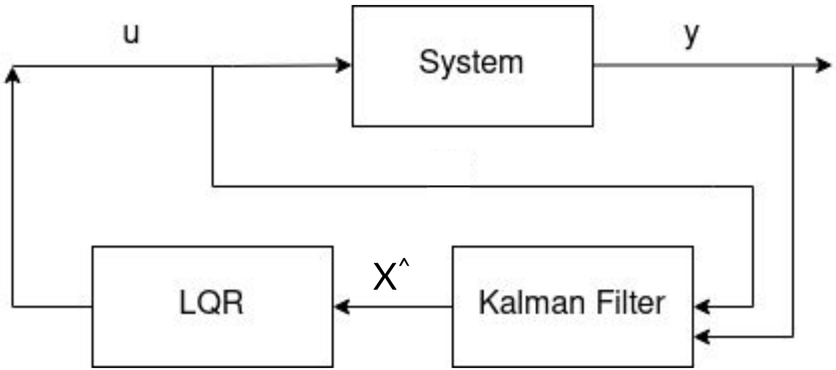
For system to be controllable,

$$\boxed{rank(C) = n}$$

Observability and Kalman Filter

$$\mathcal{O}_v = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{v-1} \end{bmatrix}.$$

- Observability of a control system is the ability of the system to determine the internal states of the system by observing the output in a finite time interval when input is provided to the system.
- If and only if the column rank of the *observability matrix* is equal to N, then the system is observable



Kalman Filters are a form of predictor-corrector used extensively in control system design for estimating the unmeasured states of a process.

PID vs LQR Controller

- LQR is an optimal control regulator expected to be more robust than PID algorithm.
- LQR focuses on non-linear models rather than the classical linear equation approach of PID but it requires linearization and model of system.
- LQR is deals with Multiple Input Multiple Output (MIMO) Systems. For the same system (MIMO) we will require separate PID Controllers for each state variable, whereas LQR provides weights or “knobs” to control multiple state variables which are part of the model.

Pole Placement Method

One method to ensure that the system is stable is to select the gain matrix K in such a way so that the eigenvalues of the (A-BK) matrix are purely real and negative.

We can select the desired eigenvalues for the system and calculate the K matrix such that (A-BK) has our desired eigenvalues.

For Pendulum with external torque:

$$(A-BK) = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$(A-BK) = \begin{bmatrix} 0 & 1 \\ \left(\frac{g}{l} - \frac{k_1}{ml^2}\right) & \frac{-k_2}{ml^2} \end{bmatrix}$$

$\Rightarrow$

Characteristic Equation:

$$|A - BK - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ \left(\frac{g}{l} - \frac{k_1}{ml^2}\right) & \frac{-k_2}{ml^2} - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + \frac{k_2}{ml^2}\lambda + \left(\frac{k_1}{ml^2} - \frac{g}{l}\right) = 0$$

Pole Placement Method

Select arbitrary values of ƛ as roots of the equation.

Substitute the values of ƛ in the equation obtained and find the values of K1, K2, K3, K4 to form the K matrix.

Note that while selecting values of ƛ, it's real part must be negative for it to be stable and positive for it to be unstable.

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

$$(\lambda + 2)(\lambda + 1) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

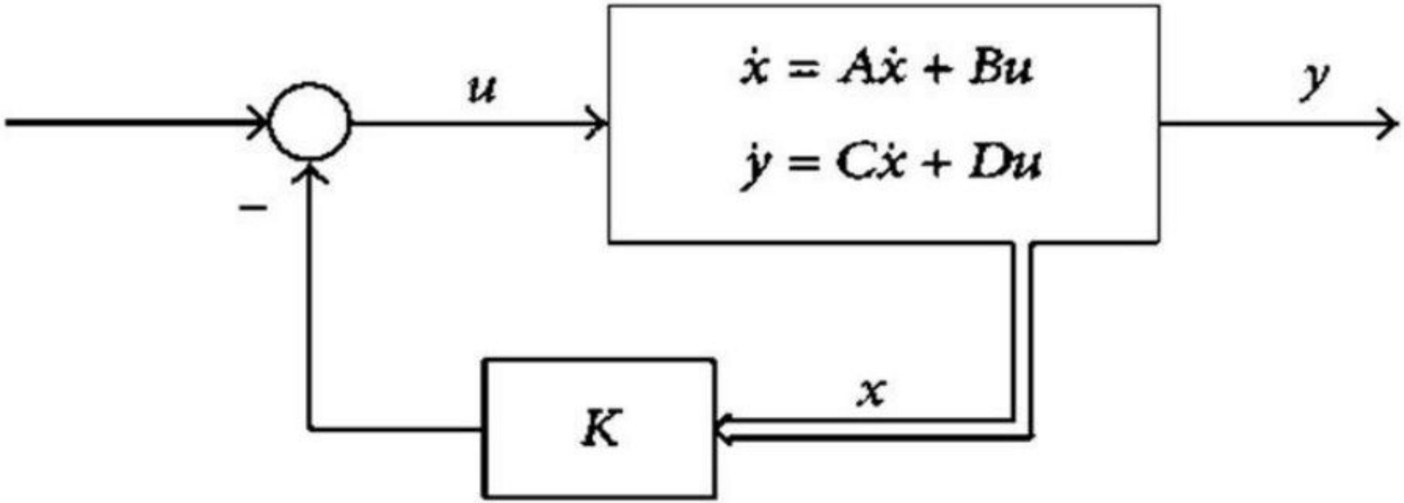
$$\frac{k_2}{ml^2} = 3 \;; \qquad \frac{k_1}{ml^2} - \frac{g}{l} = 2$$

$$k_2 = 3ml^2; \qquad k_1 = 2ml^2 + mgl$$

Hence,

$$K = \begin{bmatrix} 2ml^2 + mgl & 3ml^2 \end{bmatrix}$$

Intuition for LQR



i.e. U = -K . X

$$\dot{X} = (A - B.K).X$$

Intuition for LQR

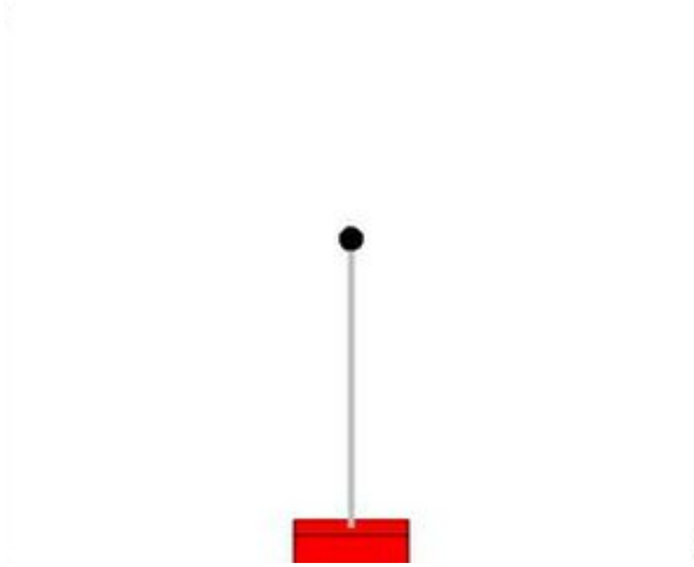
- **Linear Quadratic Regulator(LQR)** helps us optimize the K matrix according to our desired response.
- Here we use a cost function,

$$J = \int_0^\infty (x^T Q x + u^T R u)$$

- Where, Q and R are positive semi-definite diagonal matrices and x and u are the state vector and input vector respectively.
- The controller is of the form $u = -Kx$ which is a **Linear** controller and the underlying cost function is **Quadratic** in nature and hence the name **Linear Quadratic Regulator**.

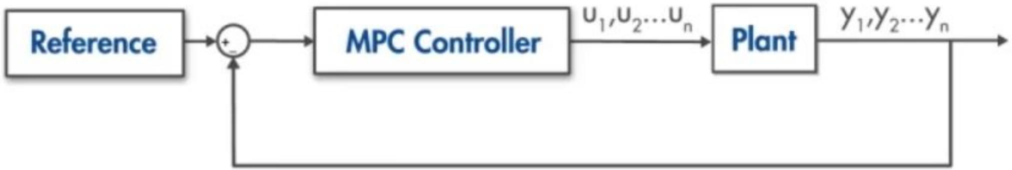
- Each Qi are the weights for the respective states xi.
- The trick is to choose weights Qi for each state xi so that the desired performance criteria is achieved. Greater the state objective is, greater will be the value of Q corresponding to the said state variable.
- LQR minimizes this cost function J based on the chosen matrices Q and R.
- In real life applications we use **Discrete LQR**

Demo : Applying a Controller to a Physical System

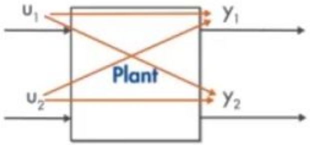


Model Predictive Control (MPC)

- MIMO systems

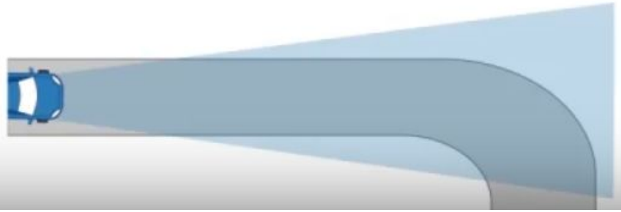


- Input-output interactions

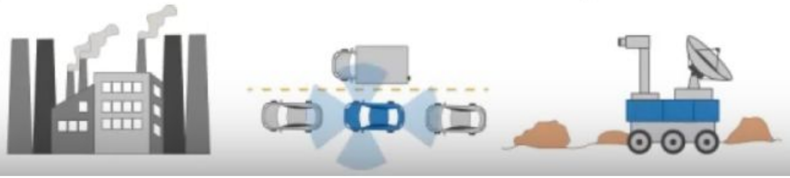


- Constraints

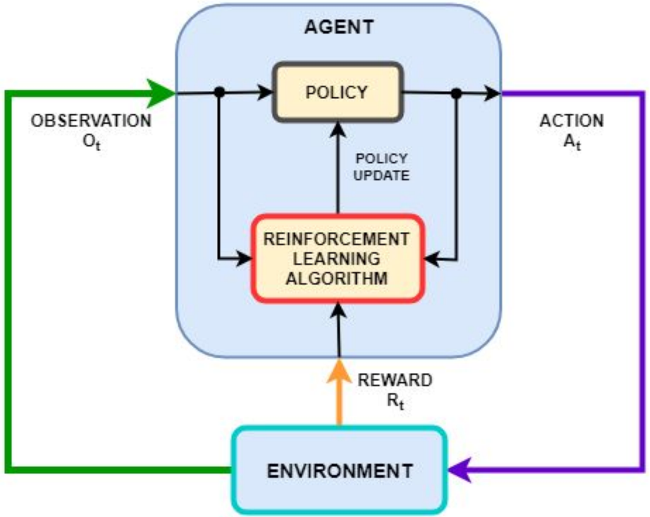
- Preview

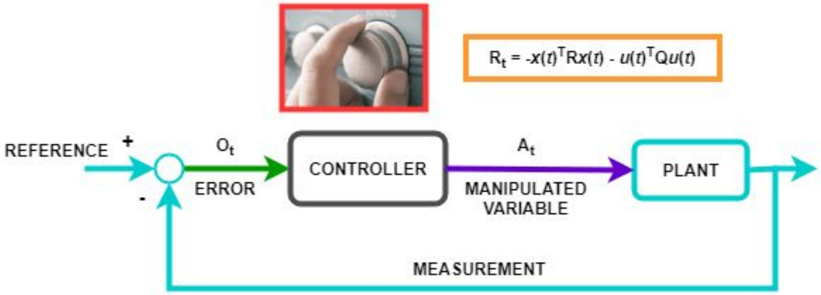


- Has been used in many industries such as process, automotive, and aerospace



Control Systems using RL





- The behavior of a reinforcement learning policy—that is, how the policy observes the environment and generates actions to complete a task in an optimal manner—is similar to the operation of a controller in a control system.
- Many control problems encountered in areas such as robotics and automated driving require complex, nonlinear control architectures. Techniques such as gain scheduling, robust control, and MPC can be used for these problems, but often require significant domain expertise from the control engineer. For example, gains and parameters are difficult to tune. The resulting controllers can pose implementation challenges, such as the computational intensity of nonlinear MPC.
- You can use deep neural networks, trained using reinforcement learning, to implement such complex controllers. These systems can be self-taught without intervention from an expert control engineer. Also, once the system is trained, you can deploy the reinforcement learning policy in a computationally efficient way.

References

- Steve Brunton https://www.youtube.com/c/Eigensteve
- Matlab Tech Talks https://in.mathworks.com/videos/tech-talks/controls.html