Module II: Multiple Integrals

Curve Tracing

* Tracing of Cartesian curves *

* $f(x,y)=0$

(i) Symmetry :-

(a) The curve is symmetric about x-axis if the powers of y occurring in the equation are all even i.e $f(x,-y) = f(x,y)$.

(b) The curve is symmetric about y-axis if the powers of x occurring in the equation are all even i.e $f(-x,y) = f(x,y)$.

(c) The curve is symmetric about the line $y=x$, if on interchanging x and y, the equation remains unchanged i.e $f(y,x) = f(x,y)$.

(d) The curve is symmetric about the line $y=-x$ if on replacing x by $-y$ and y by $-x$, the eq^n remains unchanged, i.e $f(-y,-x) = f(x,y)$.

(e) The curve is symmetric in opposite quadrants or about origin if on replacing x by $-x$ & y by $-y$, the eq^n remains unchanged i.e $f(-x,-y) = f(x,y)$.

(ii) Origin *

The curve passes through the origin if there is no constant term in the eqⁿ.

(a) if the curve passes through the origin, the tangents at the origin are obtained by equating the lowest degree term in x and y to zero.

(b) If there are two or more tangents at the origin, it is called a multiple point. The multiple point is called a node, a cusp or an isolate point if the tangents at this points are real and distinct, real and coincident or imaginary respectively.

(iii) Points of Intersection *

(a) The points of intersection of the curve with x & y axis are obtained by putting $x=0$ and $y=0$ respectively in the eqⁿ of the curve.

(b) Tangent at the point of intersection is obtained by shifting the origin to this point and then equating the lowest degree term to zero.

(iv) Region of Existence *

The region is obtained by expressing one variable in terms of other, i.e, $y = f(x)$ [or $x = f(y)$] and then finding the value of x (or y) at which y (or x) becomes imaginary. The curve doesnot exist in the region which lies between these values of x (or y).

(v) Asymptotes :-

ⓐ Asymptotes parallel to x- axis are obtained by equating the coefficient of highest degree term of x in the equation to zero.

ⓑ Asymptotes parallel to y-axis are obtained by equating the coefficient of highest degree term of y in the equation to zero.

ⓒ Oblique asymptotes are obtained by the following method:-

Let $y = mx + c$ be the asymptote to the curve and $\phi_2(x,y)$, $\phi_3(x,y)$ are the second and third degree terms in the eqn.

putting $x = 1$ and $y = m$ in $\phi_2(x,y)$ and $\phi_3(x,y)$

$$\therefore \phi_2(x,y) = \phi_2(1,m) \text{ or } \phi_2(m)$$
$$\phi_3(x,y) = \phi_3(1,m) \text{ or } \phi_3(m).$$

Oblique Asymptote (OA) occurs when the polynomial in the numerator is a higher degree than the polynomial in the denominator. To find OA we must divide the numerator by the denominator using long division.

Find $C = -\dfrac{\phi_2(m)}{\phi_3'(m)}$ Solve $\phi_3(m) = 0$

$m = m_1, m_2, \ldots$

calculate c at $m_1, m_2, \ldots$

Substituting the values of m and c in $y = mx + c$, we get oblique asymptotes to the curve.

(vi) Interval of Increasing decreasing function :-

ⓐ The curve increases strictly in the interval in which $\frac{dy}{dx} > 0$.

ⓑ the curve decreases strictly in the interval in which $\frac{dy}{dx} < 0$.

ⓒ The curve attains its maximum and minimum values at the points where $\frac{dy}{dx} = 0$.

eg Trace the cissoid $y^2(a-x) = x^3$, $a>0$.

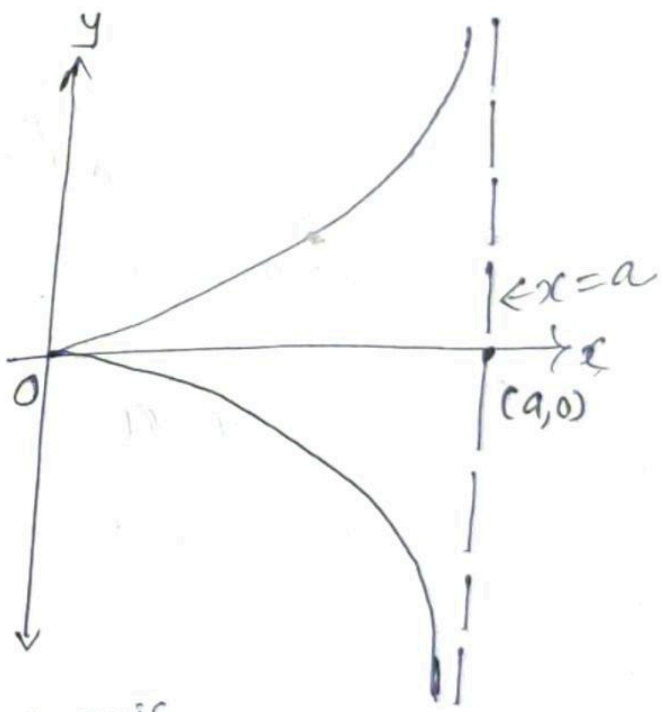
Soln: ① Symmetry: The curve is symmetric about x-axis.

② origin: The curve passes through origin.
Equating the lower degree term i.e ay^2 to zero we get $y=0$. Thus x-axis is the tangent at the origin.

③ Points of Intersection: Putting $y=0$, we get $x=0$. Thus the curves meets the coordinate axes only at the origin.

④ Region of Existence: from the eqn of the curve

$y = \pm x\sqrt{\dfrac{x}{a-x}}$ which becomes imaginary when $x<0$ or $x>a$. Hence, the curve does not exist in the region $-\infty < x < 0$ and $a < x < \infty$. Thus, the curve lie in the region $0<x<a$.



(v) Asymptotes:

ⓐ Since coefficient of highest degree term of x is constant, there is no asymptote parallel to x-axis.

ⓑ Equating the coefficient of highest degree term of y to zero we get $a-x=0$ Thus $x=a$ is the asymptote parallel to y axis.

(vi) Interval of Increasing - decreasing function:

$$\frac{dy}{dx} = \frac{x^2(3a-2x)}{2y(a-x)^2}$$

Since the curve is symmetric about x-axis considering the part of the curve above x-axis ($y>0$).

$\therefore \frac{dy}{dx} > 0$, when $x < \frac{3}{2}a$ i.e $0 < x < a$ [In the region of existence]

Thus, curve is strictly increasing in this interval.

OR

① The curve is symmetrical about the x axis.

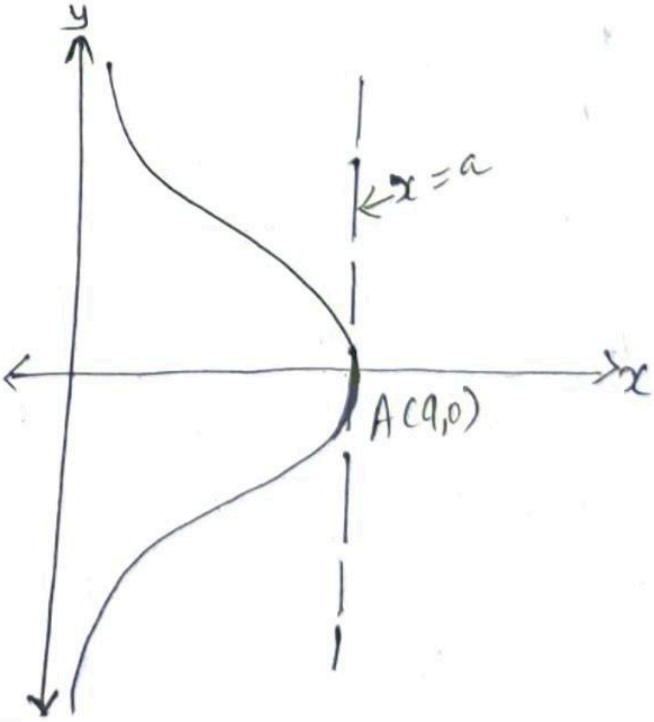
② the curve passes through the origin and the tangents at the origin are $y^2 = 0$. i.e the x-axis is a double tangent at the origin.

③ Since $y^2 = \frac{x^3}{2a-x}$ as $x \to 2a$, $y \to \infty$ the line $x = a$ is an asymptote.

④ when $x > a$ y^2 is negative, Hence the curve doesnot exist when $x > 2a$.

Eg. Trace the witch of agnesi
$$x^2y^2 = 4a^2(a-x).$$

Solⁿ: (i) Symmetry :- The curve is symmetry about x-axis
(ii) origin :- the curve does not pass through the origin.
(iii) Points of Intersection :- Putting $y=0$, we get $x=a$. Thus, the curve meets x-axis at $A(a,0)$.
⇒ Shifting the origin to $A(a,0)$ by putting $x = X+a$ and $y = Y+0$ in the Eqⁿ of the curve, $(X+a)Y^2 = 4a^2(-X)$,
$\therefore (X+a)Y^2 + 4a^2X = 0.$
⇒ Equating the lowest degree term i.e $4a^2X$ to zero, we get $X=0$, i.e $x-a=0 \Rightarrow \boxed{x=a}$ is the tangent at $A(a,0)$.
(iv) Region of Existence :- from the Eqⁿ of the curve
$y = \pm 2a\sqrt{\frac{a-x}{x}}$ which becomes imaginary when $x<0$ or $x>a$. Hence, the curve does not exist in the region $-\infty<x<0$ and $a<x<\infty$. Thus the curve lies in the region $0<x<a$.



(v) Asymptotes :-
(a) Equating the coefficient of highest degree term of x to zero, we get $y^2+4a^2=0$ which gives imaginary values. Thus there is no asymptote parallel to x-axis.

(B) Equating the coefficient of highest degree term of y to zero, we get $x=0$. Thus y-axis is the asymptote.

(vi) Interval of Increasing-decreasing function.

$$\frac{dy}{dx} = -\frac{2a^3}{x^2 y}$$

Since the curve is symmetric about x axis considering the part of the curve above x-axis. $(y>0)$, $\frac{dy}{dx} < 0$ for all values of x.

Thus, the curve is strictly decreasing in the $0<x<a$.

Eg: Trace the strophoid $y^2(a+x) = x^2(b-x)$.

Sol^n: (i) Symmetry: The curve is symmetry about x-axis.

(ii) origin: The curve passes through the origin. Equating the lowest degree term i.e $ay^2 - bx^2$ to zero, we get $y = \pm\sqrt{\frac{b}{a}}\,x$. Thus $y = \pm\sqrt{\frac{b}{a}}\,x$ are two tangents at origin.

(iii) Points of Intersection: Putting $y=0$, we get $x=0, b$. Thus, the curve meets x-axis at $O(0,0)$ and $A(b,0)$. Shifting the origin to $A(b,0)$ by putting $x = X+b$ and $y = Y+0$ in the eq^n of the curve, $Y^2(a+X+b) = (X+b)^2(-X)$

$$Y^2(a+b+X) + X(X^2+2bX+b^2) = 0.$$

Equating the lowest degree term,

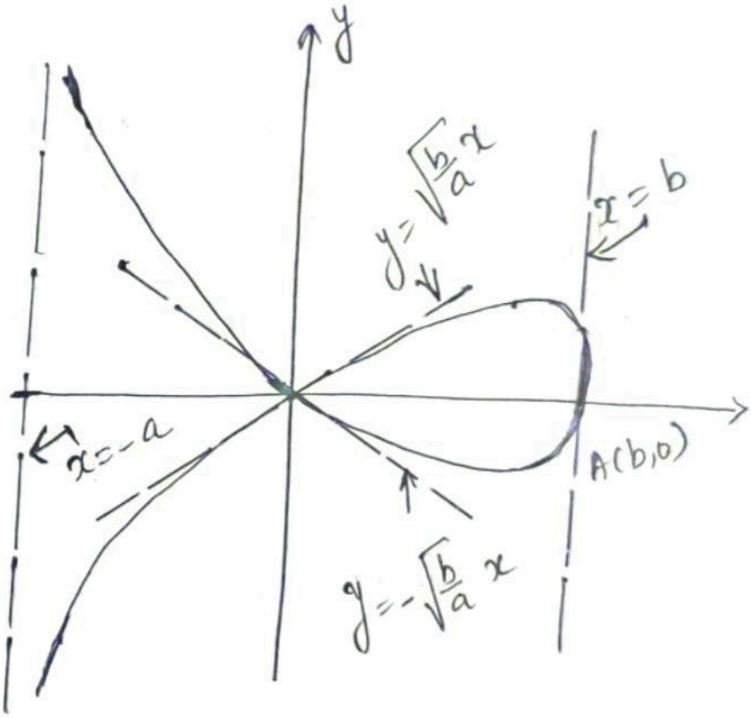
i.e. b^2x to zero, we get $X=0$, $x-b=0$. Thus, $x=b$ is the tangent at $A(b,0)$.

(iv) <u>Region of Existence</u> :- from the eqn of the curve $y=\pm x\sqrt{\frac{b-x}{a+x}}$ which becomes imaginary when $x<-a$ or $x>b$. Hence, the curve does not exist in the region $-\infty<x<-a$ and $b<x<\infty$. Thus, the curve lies in the region $-a<x<b$.

(v) <u>Asymptotes</u>:

ⓐ Since the coefficient of highest degree term of x is constant, there is no asymptote parallel to x-axis.

ⓑ Equating the coefficient of highest degree term of y to zero, we get $x+a=0$ thus $x=-a$ is the asymptotes parall to y axis.



$\Rightarrow$ Since the curve meets x-axis at two points $O(0,0)$ and $A(b,0)$, a loop exists in the region $0<x<b$. Also, $y=\pm\sqrt{\frac{b}{a}}x$ are the tangents at the origin, hence after passing through the origin, the curve extends towards the asymptote $x=-a$ in the region $-a<x<0$.

Eg: Trace the catenary $y = c\cosh\frac{x}{c}$.

Sol^n: $y = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right)$

(i) Symmetry: The curve is symmetric about y axis, since on replacing x by $-x$ eq^n remains unchanged.

(ii) origin: the curve does not pass through the origin.

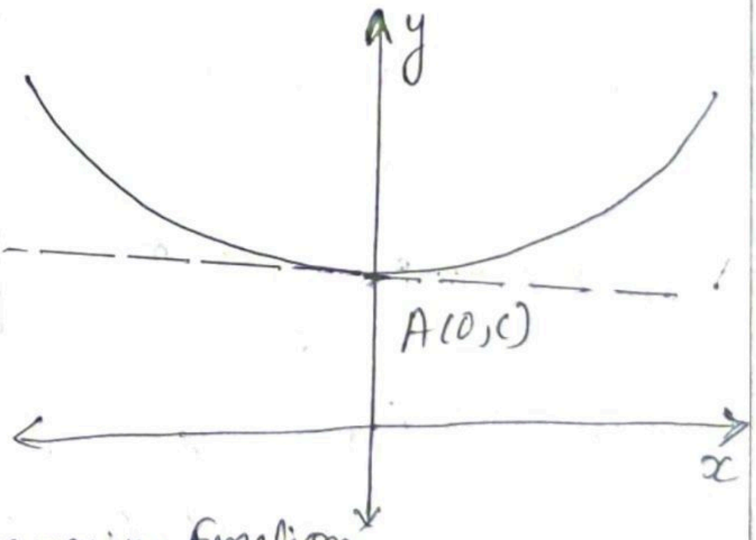
(iii) Points of intersection: Putting $x=0$, we get $y=c$.

from the Eq^n of the curve, $\frac{dy}{dx} = \frac{c}{2}\left(\frac{1}{c}e^{x/c} - \frac{1}{c}e^{-x/c}\right)$

$$\therefore \left.\frac{dy}{dx}\right|_{(0,c)} = 0$$

Thus the tangent is parallel to x-axis at $A(0,c)$.
The curve does not meet x-axis.

(iv) Region of Existence: Since $1 \le \cosh\frac{x}{c} < \infty$ for $-\infty < x < \infty$, the curve lies in the region $c \le y < \infty$, $-\infty < x < \infty$.



(v) There is no asymptote to the curve.

(vi) Interval of Increasing-decreasing function:

$\frac{dy}{dx} = \sinh\frac{x}{c}$

(a) $\frac{dy}{dx} > 0$ when $x > 0$. Thus, the curve is strictly increasing in $0 < x < \infty$

(b) $\frac{dy}{dx} < 0$, when $x < 0$. Thus, the curve is strictly decreasing in $-\infty < x < 0$.

Ex: Trace the Folium of Descartes

$$x^3+y^3=3axy.$$

Sol^n: Symmetry: The curve is not symmetric about the coordinate axes but is symmetric about the line $y=x$, since after interchanging y and x, equation of the curve remains unchanged.

(ii) Origin: The curve passes through the origin. Equating the lowest degree term i.e xy to zero, we get $x=0$ and $y=0$. Thus, $x=0$ and $y=0$ are the tangents at the origin.

(iii) Points of Intersection:

(a) putting $y=0$ we get $x=0$. Thus the curve meets the coordinate axes only at the origin.

(b) putting $y=x$ we get $2x^3=3ax^2$, $x=0, \frac{3a}{2}$ and $y=0, \frac{3a}{2}$. thus, the curve meets the line $y=x$ at $O(0,0)$ and $A\left(\frac{3a}{2}, \frac{3a}{2}\right)$. Tangent at $A\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is given by,

$$\left(y-\frac{3a}{2}\right)=\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)}\left(x-\frac{3a}{2}\right)$$

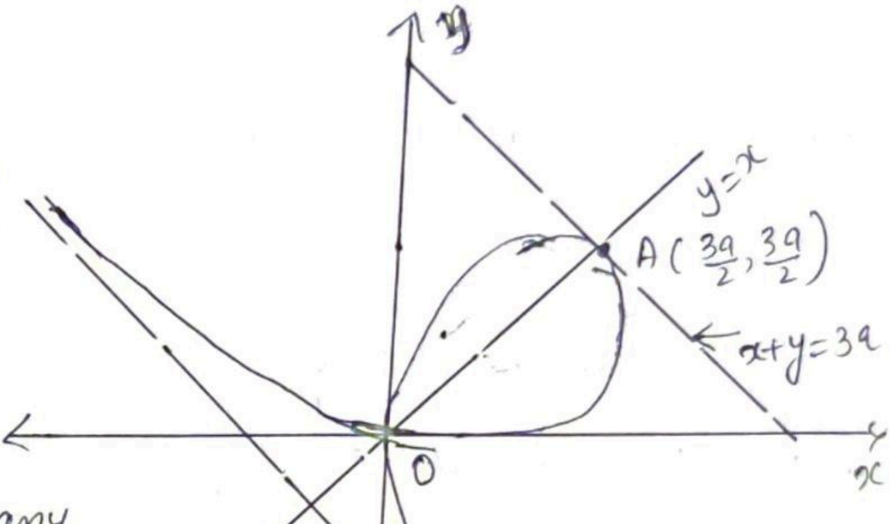
$$=\left[\frac{ay-x^2}{y^2-ax}\right]_{\left(\frac{3a}{2},\frac{3a}{2}\right)}\left(x-\frac{3a}{2}\right)=-1\left(x-\frac{3a}{2}\right)$$

 Thus, $x+y=3a$ is the tangent at $A\left(\frac{3a}{2},\frac{3a}{2}\right)$.

(iv) Region of Existence :- In the Eq^n of the curve x & y can not be negative simultaneously, otherwise Eq^n of the curve will not be satisfied. Thus the curve does not exist in the region where $x<0$ and $y<0$ i.e. third quadrant.

(v) Asymptotes:

(a) Since coefficients of highest degree term of x and y are constant, the curve does not have any asymptotes parallel to coordinate axis.



(b) oblique Asymptotes:

Let $y=mx+c$ be the asymptote of the curve.

Putting $x=1$ and $y=m$ in the third and second degree of the Eq^n separately.

$\phi_3(m)=1+m^3$, $\phi_2(m)=-3ma$

$\therefore c=-\frac{\phi_2(m)}{\phi_3'(m)}$

Solving $\phi_3(m)=0$, $1+m^3=0$, $m=-1$

$=-\frac{-3am}{3m^2}=\frac{a}{m}=-a$

Thus $y=-x-a$ i.e $x+y+a=0$ is the asymptote of the curve. Since, no part of the curve lies in the third quadrant and coordinate axes are the tangents at the origin, after passing through the origin, the curve extends towards the asymptote $x+y+a=0$ in the second & fourth quadrants.