Tracing of Polar Curves

Relation between Polar and Cartesian Coordinates

The polar coordinates r and θ can be converted to the cartesian coordinates x and y by using the trignometric sine and cosine functions,

$$x = r\cos\theta, \quad y = r\sin\theta$$

The Cartesian coordinates x and y can be converted to polar coordinates r and θ with $r \geq 0$ and θ in the interval $(-\pi, \pi)$ by relation.

$$\therefore r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The polar curves are given by polar coordinates (r, θ) and is written as $r = f(\theta)$ or $\theta = f(r)$.

* The points to be taken into consideration while tracing a polar curve $r = f(\theta)$ are as follows :-

* (i) Symmetry :- (a) A curve is symmetric about the initial line $\theta = 0$ (x-axis) if the eq^n remains unchanged after replacing θ by $-\theta$.

(b) A curve is symmetric about the line $\theta = \frac{\pi}{2}$ (line through pole perpendicular to the initial line), if the eq^n remains unchanged after replacing θ by $\pi - \theta$.

(c) A curve is symmetric about the pole (opposite quadrant) if the eq^n remains unchanged when θ is replaced by $\pi + \theta$ (or r is replaced by $-r$).

(a) A curve is symmetric about the line $\theta = \frac{\pi}{4}$, if the eq^n remains unchanged after replacing θ by $\frac{\pi}{2} - \theta$.

* (ii) Pole :- The pole lies on the curve, if for $r=0$, there exists at least one real value of θ. and that real value of θ, is also tangent at the pole.

* (iii) Points of Intersection :- Determine the points where the curve meets initial line $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$.

(iv) Direction of Tangent :- Determine ϕ, i.e angle between the radius vector and the tangent at the points of intersection using $\tan\phi = r\frac{d\theta}{dr}$.

The angle ϕ gives the direction of the tangent at the point of intersection.

* (v) Region : (a) Determine the max^m & min^m value of r if exists. If min^m value of r is a then no part of the curve lies inside the circle with radius a and centre at pole. If max^m value of r is b then the whole curve lies within the circle of radius b and centre at the pole.

(b) Determine the range of θ in which $r^2 < 0$, i.e r is imaginary, then curve does not exists in this range.

(vi) **Asymptotes :-** If $r \to \infty$ for some $\theta = \theta_1$, then the asymptote of the curve may exist and is given by $r\sin(\theta - \theta_1) = f'(\theta_1)$

where θ_1 is the Soln of $\frac{1}{f(\theta)} = 0$.

* (vii) **Variation of r :-**

Trace the variation of r for some suitable values of θ. If $\frac{dr}{d\theta} > 0$ then r increases as θ increases.

and if $\frac{dr}{d\theta} < 0$ then r decreases as θ increases.

If the curve meets the line of symmetry at two points, then a loop exists between these two points.

Note :- Curve of the type $r = a\sin n\theta$ or $r = a\cos n\theta$ consists of (i) n similar loops, if n is odd, and (ii) $2n$ similar loops, if n is even.

⇒ If $n = 1$ then the curve becomes a circle.

Trace the cardioid $r = a(1-\cos\theta)$

Sol^n: (i) Symmetry: The curve is symmetric about the initial line $\theta = 0$, since when θ is replaced by $-\theta$, Eq^n of the curve remains unchanged.

(ii) Pole: Pole lies on the curve since when $\theta=0$, $r=0$ $\therefore$ Tangent at the pole is the initial line $\theta = 0$.

(iii) Points of Intersection: Putting $\theta = \frac{\pi}{2}, \pi$ we get $r = a, 2a$ respectively. Thus the curve meets the line $\theta = \frac{\pi}{2}$ and $\theta = \pi$ at $A(a, \frac{\pi}{2})$ and $B(2a, \pi)$ respectively.

(iv) Direction of Tangent: $r = a(1-\cos\theta)$

$\therefore \frac{dr}{d\theta} = a\sin\theta$ $\qquad \therefore \tan\phi = r\frac{d\theta}{dr}$

$$= \frac{a(1-\cos\theta)}{a\sin\theta}$$

$$= \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = \tan\frac{\theta}{2}$$

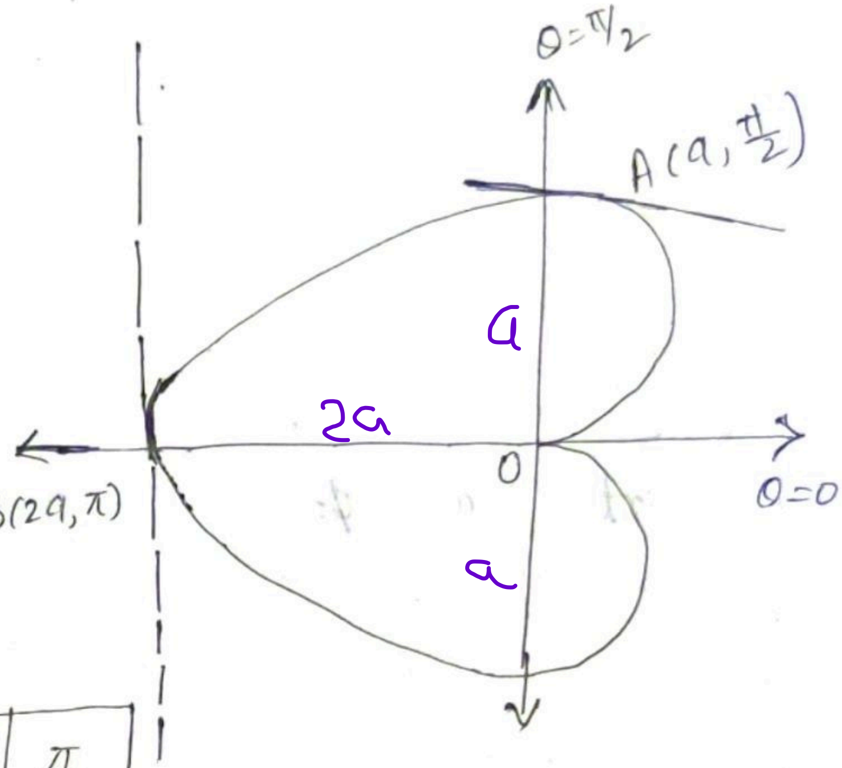
$\therefore \phi = \frac{\theta}{2}$

At point $A(a, \frac{\pi}{2})$; $\phi = \frac{\pi}{4}$, thus, the tangent makes an angle $\frac{\pi}{4}$ with the line $\theta = \frac{\pi}{2}$.

At point $B(2a, \pi)$: $\phi = \frac{\pi}{2}$, thus, the tangent is perpendicular to the line $\theta = \pi$.

(v) Region:- Since minimum value of $\cos\theta$ is -1, the max^m value of r is $2a$. Thus, the whole curve lies within a circle with centre at the pole and radius $2a$. $[0 \le \theta \le 2\pi]$

×(vi) Asymptote:- there is no asymptote of the curve since r is finite for all values of θ



(vii) Variation of r:-

θ	0	$\pi/3$	$\pi/2$	$\frac{2\pi}{3}$	π
r	0	$\frac{a}{2}$	a	$\frac{3}{2}$	$2a$

Eg:- Trace the lemniscate of Bernoulli $r^2 = a^2\cos 2\theta$.

Sol^n:- symmetry:- The curve is symmetric about the initial line $\theta = 0$ and the line $\theta = \frac{\pi}{2}$, since when θ is replaced by $-\theta$ and by $\pi - \theta$ respectively, eq^n of the curve remains unchanged.

* The curve is also symmetric about the pole since power of r is even.

(ii) Pole:- It lies on the curve since $r=0$ at $\theta = \pm\frac{\pi}{4}$.

Tangents at the pole are the lines $\theta = \pm\pi/4$.

(iii) Points of Intersection: The curve meets the initial line $\theta = 0$ at $A(a,0)$ and $B(-a,0)$.

(iv) Direction of Tangent:- $r^2 = a^2 \cos 2\theta$

$$\therefore 2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

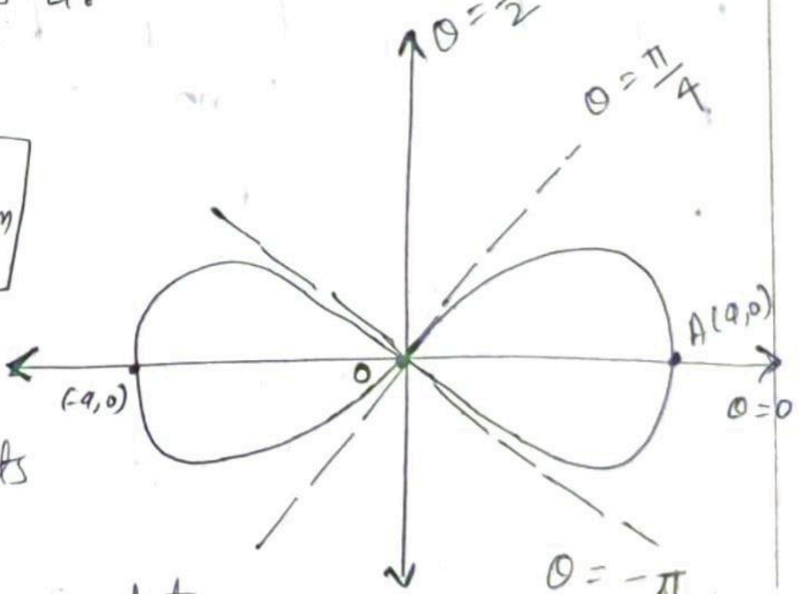
$$\therefore \tan\phi = r\frac{d\theta}{dr} = \frac{-r^2}{a^2 \sin 2\theta} = -\frac{a^2 \cos 2\theta}{a^2 \sin 2\theta} = -\cot 2\theta$$
$$= \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\boxed{\therefore \phi = \frac{\pi}{2} + 2\theta}$$

At Point $A(a,0)$, $\phi = \frac{\pi}{2}$. Thus, the tangent is perpendicular to the initial line $\theta = 0$.

Due to Symmetry, the curve is disussed only between $\theta = 0$ to $\theta = \frac{\pi}{2}$.

(v) Region:- (a) Since $\max^m$ value of $\cos 2\theta$ is 1, the $\max^m$ value of r is a. thus the whole curve lies within a circle with centre at pole and radius a.



(b) $\cos 2\theta < 0$, if $\frac{\pi}{2} < 2\theta < \pi$

i.e $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ [Due to Symmetry considering θ between 0 and $\pi/2$]

Thus $r^2 < 0$, when $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

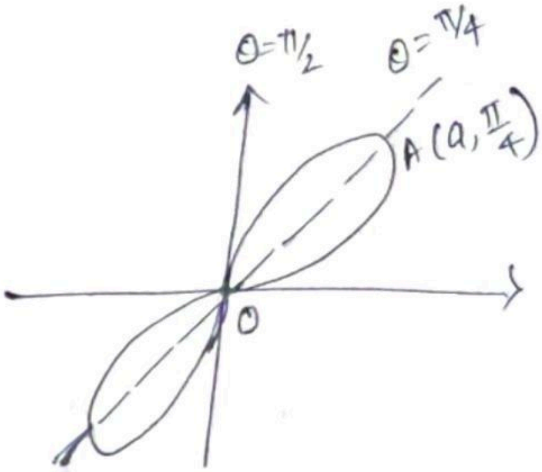
Hence, the curve does not exists in the region $\frac{\pi}{4} < \theta < \pi/2$.

X(vi) Asymptote:- There is no asymptote of the curve since r is finite for all values of θ.

(vi) Variation of r: Since the curve meets the initial line at two points $O\left(0, \frac{\pi}{4}\right)$ and $A(a,0)$ and is symmetric about the initial line, a loop exists between the points O and A.

θ	0	$\pi/8$	$\pi/6$	$\pi/4$
r	a	$\frac{a}{(2)^{1/4}}$	$\frac{a}{\sqrt{2}}$	0

H.W (eg) Trace the lemniscate $r^2 = a^2 \sin 2\theta$ $\Longrightarrow$



(eg): Trace the three-leaved rose $r = a\sin 3\theta$.

Solⁿ: Here $n=3$ (odd). The curve consists of three similar loops.

(i) Symmetry: The curve is symmetric about the line $\theta = \frac{\pi}{2}$, since on replacing θ by $\pi-\theta$, equation of the curve remains unchanged.

(ii) Pole: It lies on the curve since $r=0$ at $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$. Tangents at the pole are the lines $\theta = 0$, $\theta = \frac{\pi}{3}$, $\theta = \frac{2\pi}{3}$, $\theta = \pi$.

(iii) Points of Intersection: The curve meets the line $\theta = \frac{\pi}{2}$ at $A\left(-a, \frac{\pi}{2}\right)$.

(iv) Direction of Tangent: $r = a\sin 3\theta$

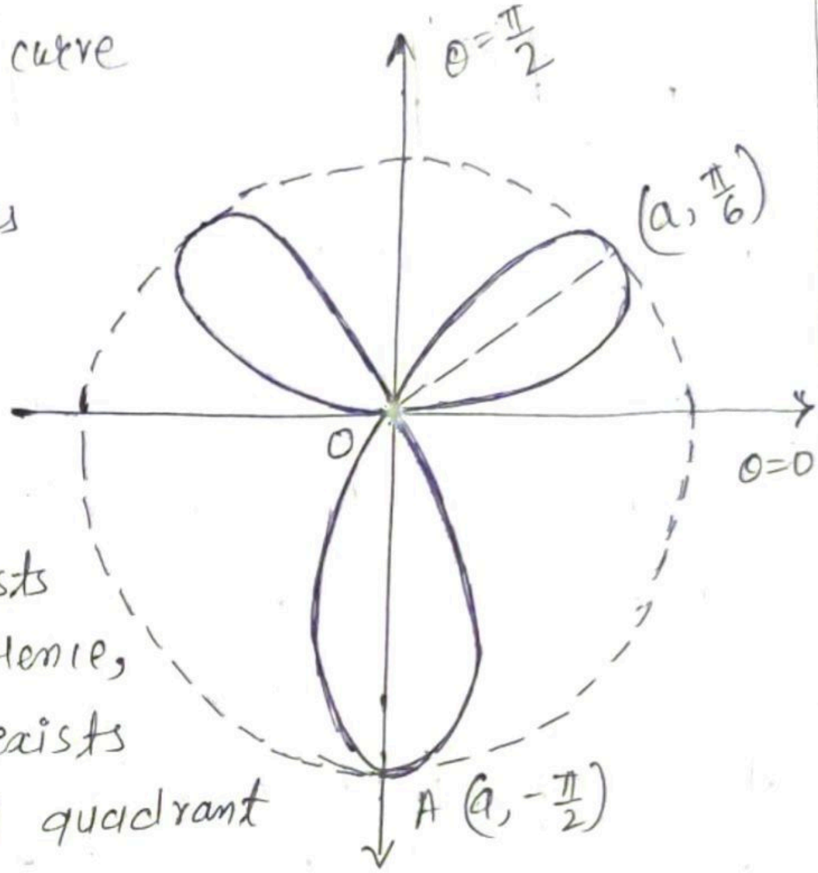
$\therefore \frac{dr}{d\theta} = 3a\cos 3\theta$ $\quad \therefore \tan\phi = r\frac{d\theta}{dr} = \frac{a\sin 3\theta}{3a\cos 3\theta} = \frac{1}{3}\tan 3\theta$

At point $A\left(-a, \frac{\pi}{2}\right)$, $\tan\phi = \frac{1}{3}\tan\frac{3\pi}{2} \to \infty$, $\phi = \frac{\pi}{2}$. Thus, the tangent is perpendicular to the line $\theta = \frac{\pi}{2}$.

(v) Region :- Since, the max^m value of $\sin 3\theta$ is 1, the max^m value of r is a. Thus, the whole curve lies within a circle of radius a and centre at the pole.

× (vi) Asymptote :- There is no asymptote of the curve since r is finite for all values of θ.

(vii) Variation of r :- the curve symmetric above the line $\theta = \pm \frac{\pi}{2}$. and also passes through the pole O. Hence a loop exists between the points O and A. This curve consists of three similar loops. Hence, two more similar loops, exists in the first and second quadrant due to symmetry.



θ	0	$\pi/6$	$\pi/3$	$\pi/2$
r	0	a	0	$-a$

HW: Trace the four leaved rose $r = a\cos 2\theta,\ a > 0$.