* Tracing of Parametric curves *

The points to be taken into consideration while tracing a parametric curve $y = f_1(t)$, $x = f_2(t)$, where t is a parameter.

(i) **Symmetry** :~ ⓐ The curve is symmetric about x-axis if x is an even function and y is an odd function of t.

ⓑ The curve is symmetric about y-axis if y is an even function and x is an odd function of t.

ⓒ The curve is symmetric about y axis if after replacing t by $\pi - t$, x becomes negative and y remains positive.

(ii) **Origin** :~ The curve passes through the origin if there exists at least one real value of t at which $x = 0$ and $y = 0$.

(iii) **Points of intersection** :~

ⓐ Points of intersection with x-axis: Find the value of t at which $y = 0$ and then find x for this value of t.

ⓑ Points of intersection with y-axis: Find the value of t at which $x = 0$ and then find y for this value of t.

(iv) Tangents :-

(a) Tangent is parallel to x-axis at the point where

$$\frac{dy}{dx} = 0.$$

(b) Tangent is parallel to y-axis at the point where

$$\frac{dy}{dx} \to \infty.$$

(v) maximum and minimum values :-

Determine the maximum and minimum values of x and y if exists.

(vi) Region :- Determine the region where x & y are real. The curve doesnot exist in the region, where x or y is imaginary.

(vii) Variation of x and y :-

Determine the values of x and y for some suitable values of t.

Note :- If x and y are periodic functions of t having the same period, then the curve is traced for one period only.

v) Asymptotes: $\lim_{t \to t_0} x = \infty$, $\lim_{t \to t_1} y = \infty$. Then $t = t_1$ is an asymptote.

(1)

Eg: Trace the hyplocycloid $x = a\cos^3 t$, $y = b\sin^3 t$

Sol^n: x & y are periodic functions of t with period 2π. Hence the curve is traced between 0 to 2π

(i) **Symmetry:** The curve is symmetry about x-axis since x is an even function of t and y is an odd function of t. Also the curve is symmetric about y-axis since after replacing t by $\pi - t$, x becomes negative but y remains positive.

(ii) **Origin:** The curve does not pass through the origin.

(iii) **Points of Intersection:**

(a) At $t = 0$, $y = 0$ and $x = a$

(b) At $t = \pi/2$, $x = 0$ and $y = b$

Thus the curve meets the x-axis at $A(a, 0)$ and y-axis at $B(0, b)$.

(iv) **Tangents:** $\frac{dx}{dt} = -3a\cos^2 t \sin t$, $\frac{dy}{dt} = 3b\sin^2 t \cos t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3b\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{b}{a}\tan t$$

* $\therefore \frac{dy}{dx} = 0$, when $t = 0$

Thus, the tangent is x-axis at $t = 0$ i.e. at $A(a, 0)$.

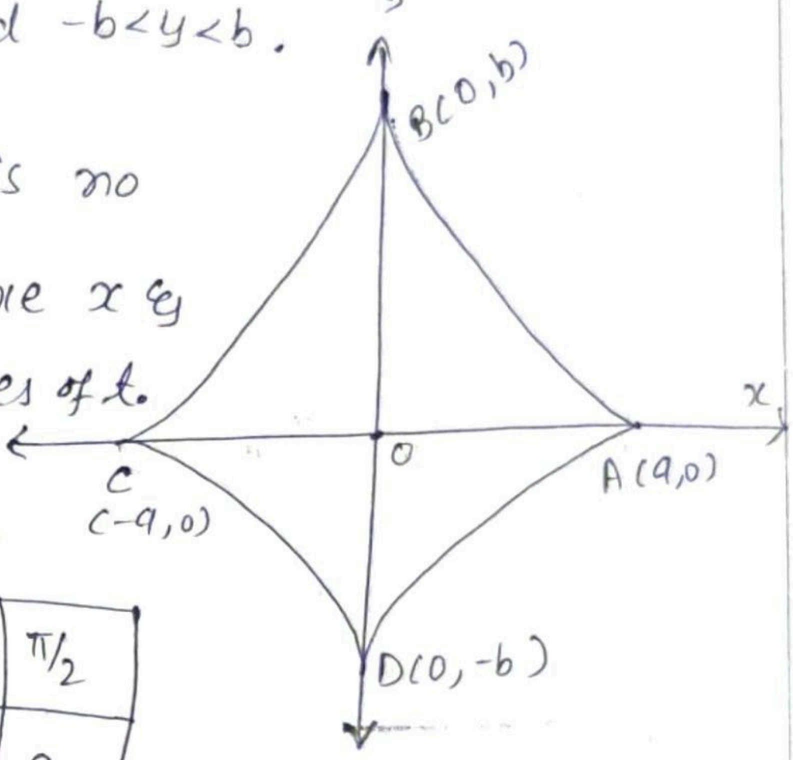
* $\therefore \frac{dy}{dx} \to \infty$ when $t = \pi/2$

Thus, the tangent is y-axis at $t = \pi/2$, i.e at $B(0, b)$.

(v) Maximum & Minimum Values: max^m values of x and y are a and b respectively since maximum value of $\cos t$ and $\sin t$ is 1. Minimum values of x and y are $-a$ and $-b$ respectively since minimum value of $\cos t$ and $\sin t$ is -1.

(vi) Region:- The curve lies in the region $-a < x < a$ and $-b < y < b$.

(vii) Asymptotes:- There is no asymptote of the curve since x & y are finite for all values of t.



(viii) Variation of x & y:

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
x	a	$\frac{3\sqrt{3}a}{8}$	$\frac{a}{2\sqrt{2}}$	$\frac{a}{8}$	0
y	0	$\frac{b}{8}$	$\frac{b}{2\sqrt{2}}$	$\frac{3\sqrt{3}b}{8}$	b

Eg: Trace the tractrix

$$x = a\left[\cos t + \log\left|\tan\left(\frac{t}{2}\right)\right|\right],\ y = a\sin t.$$

Sol^n: Symmetry: ⓐ The curve is symmetric about x-axis since x is an even function of t and y is an odd function of t.

ⓑ Replacing t by $\pi - t$.

$$x = a\left[\cos(\pi - t) + \log\left|\tan\left(\frac{\pi}{2} - \frac{t}{2}\right)\right|\right]$$

$$= a\left[-\cos t + \log\left|\cot\frac{t}{2}\right|\right]$$

$$= a\left[-\cos t - \log\left|\tan\frac{t}{2}\right|\right]$$

$$= -x$$

$$y = a\sin(\pi - t) = a\sin t = y$$

Thus, the curve is symmetric about y-axis.

(ii) Origin: The curve does not pass through the origin.

(iii) Points of Intersection:

ⓐ At $t = 0$, $y = 0$ and $x \to -\infty$ $[\because \log 0 \to -\infty]$

ⓑ At $t = \pi/2$, $x = 0$ and $y = a$.

Thus, the curve meets the y-axis at $A(0, a)$ and doesnot meet x-axis.

(iv) <u>Tangents:</u> $x = a\left[\cos t + \frac{1}{2}\log \tan^2 \frac{t}{2}\right]$

$$\therefore \frac{dx}{dt} = a\left[-\sin t + \frac{1}{2}\,\frac{1}{\tan^2\frac{t}{2}}\cdot 2\tan\frac{t}{2}\sec^2\frac{t}{2}\cdot\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = \frac{a\cos^2 t}{\sin t}$$

$$\therefore \frac{dy}{dt} = a\cos t \qquad \therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t\sin t}{a\cos^2 t} = \tan t$$

$\therefore$ At point $A(0,a)$: $\frac{dy}{dx} = \tan\frac{\pi}{2} \to \infty$

Thus, the tangent is y axis.

(v) <u>max</u> ^{m} & <u>min</u>^{m} <u>values</u>: max^m & min^m values of y are a and $-a$ respectively sine max^m & min^m values of $\sin t$ ~~and~~ are 1 and -1 respectively.

$\therefore x$ lies between $-\infty$ to ∞.

(vi) <u>Region:</u> The curve lies in the region $-a < y < a$ and $-\infty < x < \infty$.

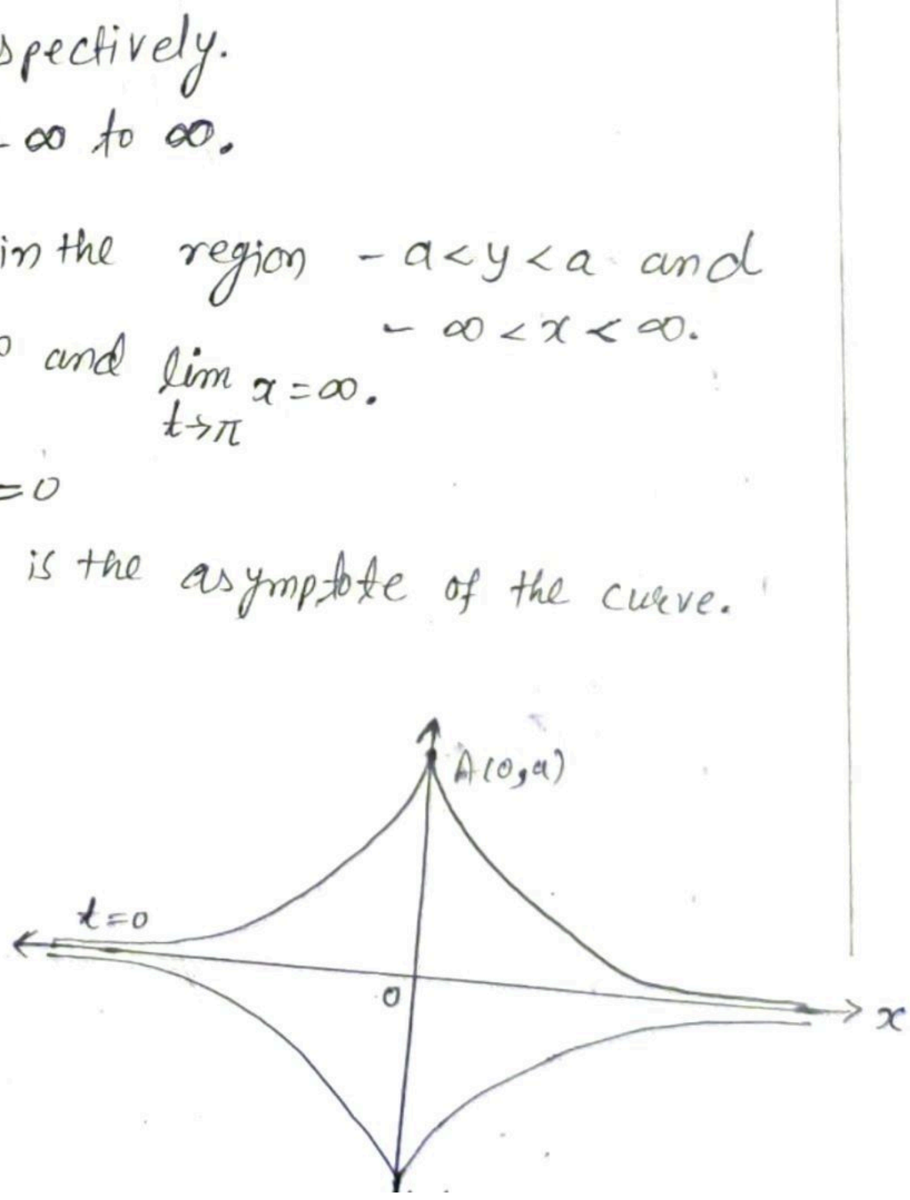
(vii) <u>Asymptotes:</u> $\lim_{t\to 0} x = -\infty$ and $\lim_{t\to\pi} x = \infty$.

At $t \to 0$ and $t \to \pi$, $y = 0$

Thus $y=0$, i.e x-axis is the asymptote of the curve.

(viii) Variation of x & y:

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
x	$-\infty$	$-0.45a$	$-0.17a$	$-0.04a$	0
y	0	$0.5a$	$0.71a$	$0.87a$	a



Eg: Trace the cycloid $x = a(t + \sin t)$,
$y = a(1 - \cos t)$.

Sol^n: (i) Symmetry: The curve is symmetric about y-axis since x is an odd function of t and y is an even function of t.

(ii) Origin: At $t=0$; $x=0$ and $y=0$. Thus, the curve passe through the origin.

(iii) Points of Intersection:

(a) $x=0$, only at $t=0$, Thus the curve meets the y-axis only at origin.

(b) If $y=0$, $\cos t = 1$, $t = 0, \pm 2\pi, \pm 4\pi, \ldots$

then $x = 0, \pm 2a\pi, \pm 4a\pi, \ldots$

Thus, the curve meets the x axis at $(0,0), (\pm 2a\pi, 0)$ $(\pm 4a\pi, 0) \ldots$

(iv) Tangents: $\frac{dx}{dt} = a(1+\cos t)$, $\frac{dy}{dt} = a\sin t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin t/2 \cos t/2}{2\cos^2 t/2} = \tan\frac{t}{2}$$

(a) $\frac{dy}{dx} = 0$ at $t = 0, \pm 2\pi, \pm 4\pi, \ldots$

Thus tangent is x-axis at $(0,0)$, $(\pm 2a\pi, 0)$, $(\pm 4a\pi, 0) \ldots$

(b) $\frac{dy}{dx} \to \infty$ at $t = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

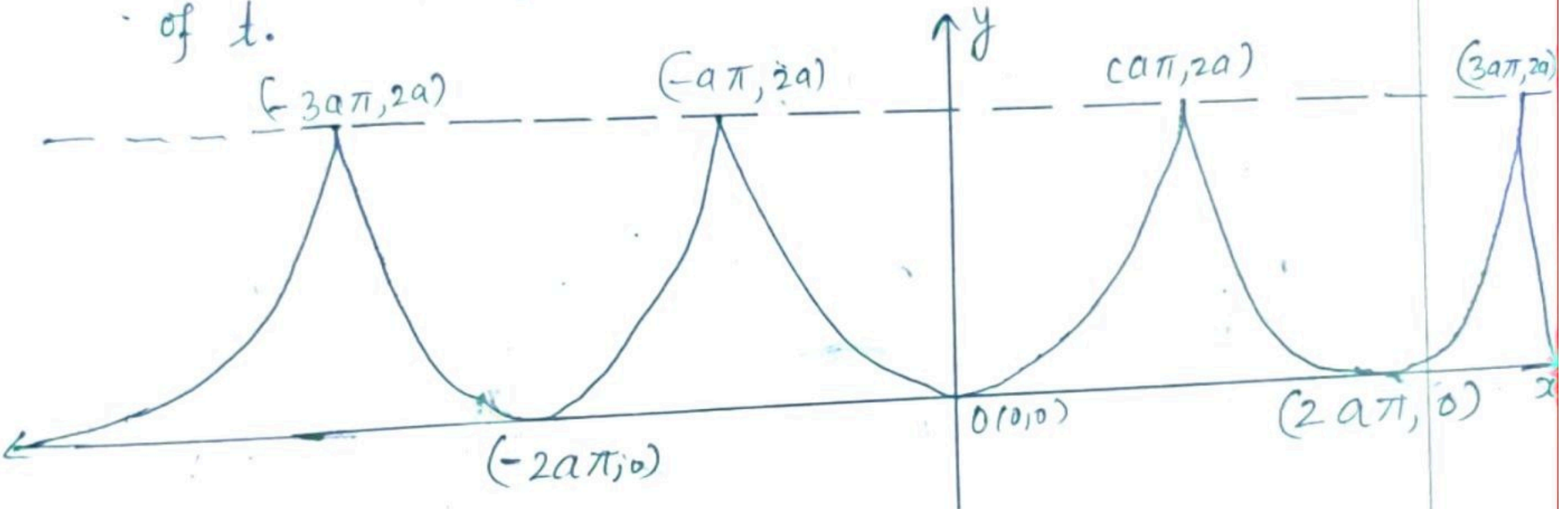
thus, tangent is parallel to y axis at $(\pm a\pi, 2a)$, $(\pm 3a\pi, 2a)$, $(\pm 5a\pi, 2a), \ldots$

(v) Maximum & Minimum values:-

max^m & minimum values of y are $2a$ and 0 since minimum and maximum values of $\cos t$ are -1 and 1.

(vi) Region:- curve lies in the region $0 < y < 2a$ and $-\infty < x < \infty$.

(vii) Asymptotes:- There is no asymptote of the curve since x and y are finite for all finite values of t.



(viii) Since $\sin t$ is a periodic function of period 2π, y is the periodic function of period 2π. Thus the curve repeat itself in the interval $[0, \pm 2a\pi]$, $[\pm 2a\pi, \pm 4a\pi]$ ....

(ix) Variation of x and y:

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	$a\left(\frac{\pi}{2}+1\right)$	$a\pi$	$a\left(\frac{3\pi}{2}+1\right)$	$2a\pi$
y	0	a	$2a$	a	0

Q. Trace the curve $x = a\sin 2t(1+\cos 2t)$
$y = a\cos 2t(1-\cos 2t)$.

Solⁿ:- x and y are periodic functions of t with period π, hence, we will discuss the curve only in the interval $0 \le t < \pi$.

① **Symmetry**:- The curve is symmetric about y axis since x is an odd function of t and y is an even function of t.

(ii) **Origin**: At $t=0$, $x=0$ & $y=0$. thus, origin lies on the curve.

(iii) **Points of Intersection**:- (a) $x=0$ at $t=0, \pm\frac{\pi}{2}$,
then $y=0, -2a$

Thus, the curve meets the y-axis at $(0,0)$, $(0,-2a)$.

(b) $y=0$ at $t=0, \pm\pi/4$ then $x=0, \pm a$

Thus, the curve meets the x-axis at $(0,0)$, $(a,0)$, $(-a,0)$.

(iv) **Tangents**: $\frac{dx}{dt} = 2a\cos 2t(1+\cos 2t) + a\sin 2t(-2\sin 2t)$

$= 2a\cos 2t + 2a\cos 4t$

$= 2a \cdot 2\cos 3t \cdot \cos t$

$$\frac{dy}{dt} = -2a\sin 2t(1-\cos 2t) + a\cos 2t(2\sin 2t)$$

$$= 4a\sin 2t\cos 2t - 2a\sin 2t$$

$$= 2a(\sin 4t - \sin 2t)$$

$$= 2a \cdot 2\cos 3t \sin t$$

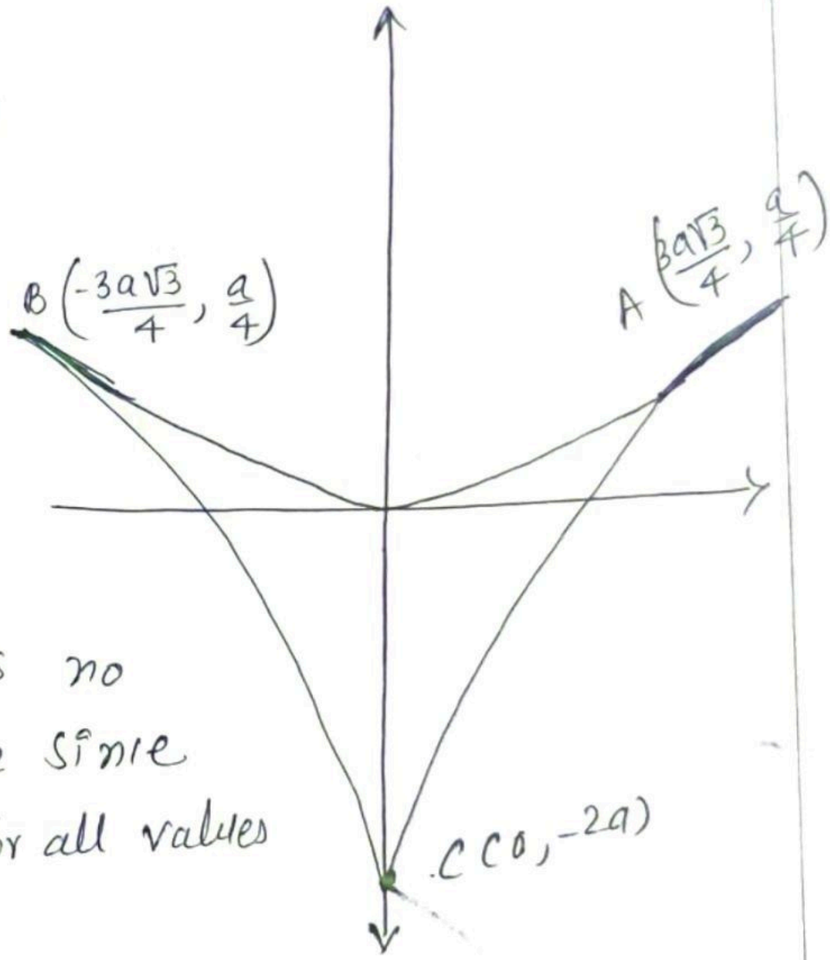
$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \tan t$$

$\therefore$ ⓐ $\frac{dy}{dx} = 0$ at $t=0$

Thus, the tangent is x-axis at $(0,0)$.

ⓑ $\frac{dy}{dx} \to \infty$ at $t = \pi/2$

Thus, the tangent is y-axis at $(0,-2a)$



(v) <u>Asymptotes</u>: there is no asymptote of the curve since x and y are finite for all values of t.

(vi) Variation of x and y:

t	0	$\pi/6$	$\pi/3$	$\pi/2$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
x	0	$\frac{3a\sqrt{3}}{4}$	$\frac{a\sqrt{3}}{4}$	0	$\frac{-a\sqrt{3}}{4}$	$\frac{-3\sqrt{3}}{4}$	0
y	0	$\frac{a}{4}$	$\frac{-3a}{4}$	$-2a$	$\frac{-3a}{4}$	$\frac{a}{4}$	0