Lecture CCS WS 23/24 Exercises

Exercise II: Random walk

Distributed: Friday, 2023-10-27

Hand back solutions until: Friday, 2023-11-03, 10:15 h via ILIAS

Background: In the lecture, you were introduced to the "random walk" as simple model for e.g. diffusion processes, and how it results in a Gaussian probability distribution concerning its endpoints. In this exercise, you will:

a) perform numerical calculations to show how a Gaussian distribution emerges from a random walk, and how the distribution's first three moments depend on the number of both steps and independent walks.
b) **(THEORY EXERCISE)** show analytically that a random walk is a solution to the diffusion equation. Please load up the written solution to this task along with your solution of a).

-Task I: implementation

- Generate a function that produces a random walk in one dimension x:
    - For step $N = 0$, Start at $x = 0$
    - Draw a number r from a uniform distribution between 0 and 1
    - If $r \geq 0.5$ add 1 to x, else add -1 to x
- Organize the output such that the full random walk is written out.

-Task II: simulation and evaluation

- Run $n = 10000$ random walks of $N = 20000$ steps each (that should only take ca. 2 minutes calculation time)
- Plot histograms of the distribution of x for all $n = 10000$ random walks for $N = 100, 1000, 10000$ and 20000.
- For $N = 100, 1000, 10000$ and 20000, check the convergence of all walks check in dependence of n for
    - the first moment, i.e. the mean $\langle x \rangle$,
    - the second moment, i.e. the variance $\langle (x - \langle x \rangle)^2 \rangle$ and
    - the third moment, giving the skewness $\langle (x - \langle x \rangle)^3 \rangle$.

-Task III: random walks and the diffusion equation (THEORY EXERCISE)

Usually, random walks are not rationalized in the form of number of steps N, but in form of the time increment ("step") Δt an individual step takes and the walk time $t = N \, \Delta t$.

In the limit of long length and time scales (i.e., a large number of steps), the evolution of random walks is described by the *diffusion equation*

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}$$

with the evolving probability density $P(x,t)$ to find the "walker" at a x of choice at time t and the diffusion constant D.

Using the probability density of a random walk $P(x) = (2\pi N)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2N}\right)$ (see the lecture / the lecture script), show that a random walk is a solution to the diffusion equation with $D = x_0^2/2\Delta t$ with step size x_0^2. *Hint*: bear in mind that P(x) then turns into a time-dependent distribution P(x,t).