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MULTIPLE INPUT ORTHOGONAL POLYNOMIAL PARAMETER ESTIMATION

H. VAN DER AUWERAER AND J. LEURIDAN

Leuven Measurement and Systems, Interleuvenlaan 65, B-3030 Heverlee (Leuven), Belgium

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The object of this paper is the development of a so-called global modal parameter estimation technique capable of analysing frequency response functions (FRFs) between several input and response stations simultaneously. The technique analyses the FRFs in their natural domain, the frequency domain. Highly consistent estimates of all modal parameters, including repeated modes, can be obtained. The effect of modes outside the analysis band can also be accounted for by explicitly locating these modes, by including residual terms, or by a combination of both.

The use of orthogonal polynomials improves the numerical properties of the calculation procedure. It also reduces the order of the identification problem. All pertinent equations have a size which is proportional to the number of modes in the data, and are independent of the number of response and input stations for which data are analysed simultaneously. As a consequence, the technique lends itself readily to implementation with microcomputers.

1. INTRODUCTION

Parameter estimation technology for experimental modal analysis remains a rapidly evolving subject area. Many new techniques [1]-[6] are capable of estimating the modal parameters of a structure by analysing data, mostly in the form of frequency response functions (FRFs), between several response and input stations simultaneously. Such techniques are commonly referred to as global modal parameter estimation techniques. They have proved to be viable solutions for increasing the accuracy and consistency of modal parameter estimates, while allowing for an overall reduced analysis time.

In reviewing such global parameter estimation techniques, one can distinguish time and frequency domain implementations. Time domain implementations analyse equally spaced sampled time data, mostly in the form of impulse responses (IRs) or free decay data. They take advantage of the fact that IRs or free decay responses of a linear viscously damped structure can be modelled as a linear superposition of the characteristic solutions of the systems governing equations of motion. These characteristic solutions are actually the modes of vibration of the system. For an isolated IR, the characteristic solutions take the form of a complex exponential. Considering a set of IRs between a single input station and many response stations, they take the form of a mode shape multiplied with a complex exponential. Finally, considering the set of IRs between a single response station and many input stations, they take the form of modal participation factors multiplied with a complex exponential (the modal participation factors are proportional to the systems left eigenvector coefficients at the input stations; for a system that satisfies reciprocity, they are proportional to the systems right eigenvector or mode shape coefficients at the input stations [4]). The underlying concept of such global modal parameter estimation techniques consists therefore of identifying the coefficients of a systems characteristic equation from which the characteristic solutions can then be calculated [6].

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Time domain implementations have two inherent constraints. First of all the data consists most frequently of FRFs, which therefore need to be transformed to the time domain to obtain corresponding IRs by inverse FFT. This may cause truncation bias errors, especially if sub-bands of a measured frequency band are to be analysed. It is then also impossible to analyse data with unequal frequency resolution, such as typically measured with step sine excitation [7, 8]. Secondly, since sampled time data are analysed, the characteristic equation has to be approximated by a finite difference equation. This equation can only describe modes with damped natural frequencies in the frequency band of analysis. This implies that the effect of modes outside the frequency band cannot be described, possibly causing errors on the estimates of the modes inside the frequency band [9]. The possibility of using compact, recursive and extremely stable solution algorithms [6] has, however, made time domain implementations very attractive.

Global frequency domain techniques analyse data in the frequency domain. These techniques are based on identifying the coefficients in the characteristic polynomial of the systems characteristic equation, following which the characteristic solutions can be calculated. This allows the description of the effects of modes outside the frequency band of analysis by locating modes outside the band of analysis, by use of residual terms, or a combination of both. Two approaches have been used successfully: direct parameter identification techniques [4, 5, 10, 11] and techniques based on the use of orthogonal polynomials [12-15]. The former techniques can be used to analyse data consistently, relative to several input stations simultaneously. Their application, especially with minicomputers, is however, limited by the fact that the size of the largest equation systems to be solved is proportional to the number of response stations for which data are analysed simultaneously. The latter techniques do not have this limitation, and are therefore easy to use with a minicomputer; current implementations do not however allow us to consistently analyse data relative to several input locations simultaneously.

The subject of this paper is a frequency domain parameter estimation technique, capable of analysing FRFs relative to several input stations simultaneously. All pertinent equations have a size which is proportional to the number of modes in the data, and are independent of the number of response and input stations for which data are analysed simultaneously, making the technique suitable for implementation with a minicomputer. Using a theoretical development, it is also demonstrated that the technique forms a dual formulation to the time domain multiple input least squares complex exponential, or polyreference, technique. The inherent capability of describing the influence of modes outside the analysis band explicitly represents a significant advantage over corresponding time domain implementations, especially for the analysis of zoom band data on structures with highly coupled, highly damped modes, frequently encountered in servo systems.

2. DEVELOPMENT OF THE TECHNIQUE

Let $[H]$ represent the matrix of FRFs between N_0 response stations and N_i input stations. Abstraction is made here whether or not the FRFs were measured with simultaneous multiple input excitation.

If the data originates from the same linear viscously damped mechanical system in which N modes are excited, then $[H]$ can be expressed by following partial fraction expansion $[H]$,

$$\begin{aligned} [H] &= \sum_{k=1}^{2N} \{V\}_k (s-\lambda_k)^{-1} \lfloor L \rfloor_k \\ [H] &= [V][sI-\Lambda]^{-1}[L] \end{aligned} \tag{1}$$

where: s is the independent variable of the Laplace domain; $\lambda_k = \alpha_k + j\omega_k$; Δ is the diagonal matrix with system poles λ_k; $[V]$ is the matrix with mode shapes and $[L]$ is the matrix with modal participation factors.

Note that the N modes appear in complex conjugate pairs which is further implied by the summation over $2N$ in the equation. Equation (1) can be reorganised in the following rational matrix polynomial form [4],

$$[H][I_s{}^p + A_{p-1}s^{p-1} + \cdots + A_0] = [B_q s^q + \cdots + B_0]. \tag{2}$$

The denominator coefficients, $[A_{p-1}] \cdots [A_0]$ are (N_i, N_i) matrices, the numerator coefficients $[\bar{B}_q] \cdots [B_0]$ are (N_0, N_i) matrices. The order p of the denominator polynomial is such that,

$$pN_i \geq 2N. \tag{3}$$

The order q of the numerator polynomial is at least equal to $p-1$. As will be explained further on, q can have a higher value, which will have the effect of including residual terms to describe the modes outside the frequency band of analysis.

The modal parameters in equation (1) can be derived from the matrix coefficients in equation (2), [4]. This is done by expressing the system poles λ_k and the modal participation factors $\lfloor L_k \rfloor$ as the eigensolutions of the characteristic system equation,

$$\lfloor L \rfloor_k \lambda_k^p + \lfloor L \rfloor_k \lambda_k^{p-1} A_{p-1} + \cdots + \lfloor L \rfloor_k A_0 = 0. \tag{4a}$$

In order to calculate the eigenvalues and vectors, equation (4) is transformed to follow equivalent linear eigenvalue problem,

$$\lfloor L' \rfloor_k \lambda_k I + \lfloor L' \rfloor_k [A] = 0 \tag{4b}$$

with

$$\lfloor L' \rfloor_k = \lfloor \lambda_k^{p-1} \lfloor L \rfloor_k \cdots \lfloor L \rfloor_k \rfloor$$

$$[A] = \begin{bmatrix} A_{p-1} & I & 0 & \cdots & 0 \\ A_{p-2} & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ A_0 & 0 & 0 & \cdot\cdot I & 0 \end{bmatrix}.$$

The matrix $[A]$ is usually referred to as the companion matrix of the matrix polynomial defined by equation (4a). The mode shape coefficients can then be calculated from,

$$\{V\}_k = \lim_{s \to \lambda_k} \{[B][L]^{-1}\}_k \tag{5}$$

with

$$[B] = [[B_q s^q + \cdots + B_0][0] \cdots [0]]$$

$$[L'] = \begin{bmatrix} \lfloor L' \rfloor_1 \\ \lfloor L' \rfloor_2 \\ \lfloor L' \rfloor_{Ni} \end{bmatrix}.$$

It is possible therefore to obtain global estimates of all modal parameters by first estimating the coefficients in the rational matrix polynomial expression for $[H]$, equation (2), using all available data, following which equations (4) and (5) are used to calculate a unique estimate of all modal parameters.

Estimating the coefficients in this rational matrix polynomial directly is, however, impractical. Firstly, the estimation of coefficients in a polynomial represents an ill-conditioned problem for higher orders of the polynomial. Secondly, both the coefficients of the denominator and the numerator polynomial have to be estimated simultaneously. The dimension of the numerator matrix coefficients is related to the number of response stations for which data are analysed simultaneously. The size of the equations from which these coefficients are to be estimated is, therefore, also related to the number of response stations, which may be prohibitive if data from a large number of response stations is to be analysed simultaneously.

Both problems can be resolved by using orthogonal polynomials to describe the denominator and numerator polynomials in equation (2). A proper choice of the orthogonality condition causes the estimation equation to separate into one equation from which the denominator coefficients are calculated and one equation from which the numerator coefficients are calculated. The size of the former equation will be directly related to the number of modes in the data, and independent of the number of response and input stations for which data are analysed simultaneously.

To derive these equations, consider the rational matrix polynomial expression for the FRFs between a particular response station and all input stations. From equation (2)

$$\lfloor H(s)\rfloor_i[Is^p + A_{p-1}s^{p-1} + \cdots + A_0] = \lfloor B_q s^q + \cdots + B_0\rfloor_i. \tag{6}$$

Using a basis of p orthogonal matrix polynomials $[\Theta_k(s)]$ with real coefficients to express the denominator polynomial and q orthogonal scalar polynomials $\phi_k(s)$ with real coefficients to express the numerator polynomial

$$\lfloor H(s)\rfloor_i[\Theta_p(s)I + \Theta_{p-1}(s)A_{p-1} + \cdots + \Theta_0(s)A_0] = \phi_q(s)\lfloor B_q\rfloor_i + \cdots + \phi_0(s)\lfloor B_0\rfloor_i. \tag{7}$$

Let FRFs be available at f frequencies, that is for values of $s = j\omega_1 \cdots j\omega_f$. Assuming Hermitian symmetry for the FRFs, one effectively also has data values for $s = -j\omega_1 \cdots -j\omega_f$. Equation (7) holds for each of these values of s, and for all response locations, $i = 1 \cdots N_0$. This therefore yields

$$[\alpha\beta]\begin{bmatrix} A' \\ B' \end{bmatrix} = [\gamma] \tag{8}$$

where

$$[\alpha\beta] = \begin{bmatrix} H(-j\omega_f)\Theta_{p-1}(-j\omega_f) & \cdots & H(-h\omega_f)\Theta(-j\omega_f) & -\Phi_q(-j\omega_f)\cdots-\Phi_0(j\omega_f) \\ \vdots & & \vdots & \\ H(-j\omega_1)\Theta_{p-1}(-j\omega_1) & \cdots & H(-j\omega_1)\Theta(-j\omega_1) & -\Phi_q(-j\omega_1)\cdots-\Phi_0(-j\omega_1) \\ H(j\omega_1)\Theta_{p-1}(j\omega_1) & \cdots & H(j\omega_1)\Theta(j\omega_1) & -\Phi_q(j\omega_1)\cdots-\Phi_0(j\omega_1) \\ \vdots & & \vdots & \vdots \\ H(j\omega_f)\Theta_{p-1}(j\omega_f) & \cdots & H(j\omega_f)\Theta(j\omega_f) & -\Phi_q(j\omega_f)\cdots-\Phi_0(j\omega_f) \end{bmatrix}$$

$$[\gamma] = \begin{bmatrix} -H(-j\omega_f)\Theta(-j\omega_f) \\ \vdots \\ -H(-j\omega_1)\Theta(-j\omega_1) \\ -H(j\omega_1)\Theta(j\omega_1) \\ \vdots \\ -H(j\omega_f)\Theta_p(j\omega_f) \end{bmatrix}$$

$$[A'] = \begin{bmatrix} A'_{p-1} \\ \vdots \\ A'_0 \end{bmatrix}, \qquad [B'] = \begin{bmatrix} B'_q \\ \vdots \\ B'_0 \end{bmatrix}$$

$$\Phi_k = \mathrm{diag}\,(\phi_k \cdots \phi_k), \text{ size } N_0.$$

By including data at sufficient response locations and at sufficient frequencies, the above equation can always be over-determined. It can therefore be solved in a least squares sense for $[A']$ and $[B']$ from the normal equation

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}[\alpha\beta]\begin{bmatrix} A' \\ B' \end{bmatrix}=\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}[\gamma] \tag{9}$$

or

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\begin{bmatrix} A' \\ B' \end{bmatrix}=\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

with

$$[X_{11}]=\begin{bmatrix} \sum \Theta_{p-1}^h H^h H \Theta_{p-1} & \cdots & \sum \Theta_{p-1}^h H^h H \Theta_0 \\ \vdots & & \vdots \\ \sum \Theta_0^h H^h H \Theta_{p-1} & \cdots & \sum \Theta_0^h H^h H \Theta_0 \end{bmatrix}$$

$$[X_{22}]=\begin{bmatrix} \sum \Phi_q^h \Phi_q & \cdots & \sum \Phi_q^h \Phi_0 \\ \vdots & & \vdots \\ \sum \Phi_0^h \Phi_q & \cdots & \sum \Phi_0^h \Phi_0 \end{bmatrix}$$

$$[X_{12}]=\begin{bmatrix} -\sum \Theta_{p-1}^h H^h \Phi_q & \cdots & -\sum \Theta_{p-1}^h H^h \Phi_0 \\ \vdots & & \vdots \\ -\sum \Theta_0^h H^h \Phi_q & \cdots & -\sum \Theta_0^h H^h \Phi_0 \end{bmatrix}$$

$$[X_{21}]=[X_{12}]^h$$

$$[Y_1]=\begin{bmatrix} \sum \Theta_{p-1}^h H^h H \Theta_p \\ \vdots \\ \sum \Theta_0^h H^h H \Theta_p \end{bmatrix}; \qquad [Y_2]=\begin{bmatrix} -\sum \Phi_q^h H \Theta_0 \\ \vdots \\ -\sum \Phi_0^h H \Theta_p \end{bmatrix}.$$

The summation $\sum$ in all of the above equations is for values of $s=-j\omega_f\cdots-j\omega_1$ and $s=j\omega_1\cdots j\omega_f$. Let the set of $[\Theta_k(s)]$ matrix polynomials be orthonormal with respect to the weighting function H^hH,

$$\sum \Theta_k H^h H \Theta_1 = \lfloor \delta_{kl} \rfloor. \tag{10}$$

Let the set of $\phi_k(s)$ polynomials be orthonormal with respect to unity weighting,

$$\sum \phi_k^h \phi_1 = \delta_{kl}. \tag{11}$$

The generation of the $\phi_k(s)$ polynomials poses no specific problem: Forsyth polynomials can be used [16]. The generation of a set of matrix polynomials $[\Theta_k(s)]$ that satisfy the above orthonormality condition is discussed in the Appendix. Due to the Hermitian symmetry of the FRFs and the orthogonal polynomials, the summation $\sum$ actually implies twice the real part of the summation for $s=j\omega_1\cdots j\omega_f$. This means that both $[X_{12}]$ and $[Y_2]$ are real matrices. With the above orthonormality conditions, equation (9) becomes

$$\begin{bmatrix} I & X_{12} \\ X_{12}^t & I \end{bmatrix}\begin{bmatrix} A' \\ B' \end{bmatrix}=\begin{bmatrix} 0 \\ Y_2 \end{bmatrix} \tag{12}$$

This then allows the uncoupling of the calculations for $[A']$ and $[B']$

$$[I-X_{12}X_{12}^t][A']=[-X_{12}Y_2] \tag{13}$$

$$[B']=[Y_2-X_{12}A']. \tag{14}$$

Having calculated coefficients $[A']$ and $[B']$, the coefficients $[A]$ and $[B]$ in the rational matrix polynomial expressed by equation (2) can easily be resynthesized. Applying equations (4) and (5) will then yield global estimates of all modal parameters.

3. SOME PRACTICAL CONSIDERATIONS

The coefficient matrix in equation (13), from which the matrix coefficients $[A']$ are to be solved, is a (pN_i, pN_i) matrix. Therefore, from equation (3), its size is proportional to the number of modes in the data. The equation is therefore very suitable for implementation on a minicomputer.

Having calculated $[A']$, and therefore the coefficients of the denominator polynomial, estimates for the poles, λ_k, and the modal participation factors $\lfloor L \rfloor_k$ can be found using equation (4). Rather than solving equation (14) for $[B']$, followed by equation (5) to find estimates of the mode shapes $\{V\}_k$, one can also use equation (1) directly. Considering again the expression for the FRFs between a particular response station i and all input stations, and including residual terms to describe the effect of modes outside the frequency band of analysis

$$\lfloor H(j\omega) \rfloor_i = \sum_{k=1}^{2N} \frac{v_{ik} \lfloor L \rfloor_k}{(j\omega - \lambda_k)} + \lfloor UR \rfloor_i - \frac{\lfloor LR \rfloor_i}{\omega^2} \tag{15}$$

$\lfloor UR \rfloor_i$ approximates the effect of modes at higher frequencies on the analysis band (upper residual) and $\lfloor LR \rfloor_i$; describes the effect of modes at lower frequencies (lower residuals). After estimating λ_k and $\lfloor L \rfloor_k$, the remaining unknowns in equation (15) are v_{ik}, $\lfloor UR \rfloor_i$ and $\lfloor LR \rfloor_i$. They can be estimated from this equation in a linear least squares sense, using the FRFs for each of the response stations sequentially.

The order of the denominator polynomial should be high enough to describe all modes that contribute to the response in the frequency band of analysis, as expressed by equation (3). Possibly modes outside the analysis band can have a significant contribution to the response in the analysis band, and should be included in the mode count. By increasing the order of the denominator polynomial such modes outside the analysis band can be explicitly described, increasing therefore also the accuracy for the modes inside the analysis band. This capability is a definite advantage compared to corresponding time domain implementations. To illustrate this, consider the discrete time domain equivalent of equation (1)

$$\lfloor H_n \rfloor_i = \lfloor H(n\Delta t) \rfloor_i = \sum_{k=1}^{2N} v_{ik} z_k^n \lfloor L \rfloor_k \tag{16}$$

with

$$z_k = e^{\lambda_k \Delta t}. \tag{17}$$

This equation expresses the sampled IRs between any response station an all reference stations as a linear superposition of the same characteristics $z_k \lfloor L \rfloor_k$. Therefore, they can be considered as characteristic solutions of the following linear finite difference equation,

$$\lfloor H_n \rfloor_i I + \lfloor H_{n-1} \rfloor_i A_{p-1} + \cdots + \lfloor H_{n-p} \rfloor_i A_0 = 0 \tag{18}$$

if the coefficients $A_{p-1} \cdots A_0$ are such that

$$z_k^p \lfloor L \rfloor_k I + z_k^{p-1} \lfloor L \rfloor_k A_{p-1} + \cdots + \lfloor L \rfloor_k A_0 = 0. \tag{19}$$

Equation (18) forms the kernel equation of the multiple input least squares complex exponential, or polyreference [3] method. The coefficients in this homogeneous finite difference equation are estimated using IRs between all available response and input stations. The IRs are first calculated by the inverse FFT on the FRFs over the frequency band of analysis.

Having estimates for the coefficients in equation (18), global estimates for z_k, the z-domain representation of the system poles λ_k, and $\lfloor L \rfloor_k$ are calculated from equation (19). The system poles are finally calculated from equation (17). Note, that since

$$z_k = e^{\lambda_k \Delta t} = e^{(\lambda_k + jm2\pi/\Delta t)\Delta t}. \qquad (20)$$

The sample increment Δt is effectively determined by the width of the frequency band of analysis over which the inverse FFT is applied to calculate the IRs. If the frequency band covers the frequencies between ω_1 and ω_f, then Δt equals,

$$\Delta t = 1/2(\omega_1 - \omega_f). \qquad (21)$$

The practical consequence of this 'aliasing effect' is that with a time domain implementation it is impossible to compensate the influence of modes outside this band, however high the order p of the finite difference equation.

Finally, with the presented frequency domain technique the effects of modes outside the frequency band of analysis can also be compensated by the use of residual terms that approximate such effects.

Let $[UR]$ be a (N_0, N_i) matrix representing the residual effect of modes at higher frequencies (upper residuals) and $[LR]$ be a (N_0, N_i) matrix representing the effect of modes at lower frequencies (lower residuals). Including such terms in equation (6)

$$\lfloor H(s) - UR - s^{-2}LR \rfloor_i [Is^p + A_{p-1}s^{p-1} + \cdots + A_0] = \lfloor B_q s^q + \cdots + B_0 \rfloor_i. \qquad (22)$$

Assuming q equals $p-1$, equation (22) becomes after reorganisation

$$\lfloor s^2 H(s) \rfloor_i [Is^p + A_{p-1}s^{p-1} + \cdots + A_0] = \lfloor B_{p12} s^{p+2} + \cdots + B_0 \rfloor_i. \qquad (23)$$

This equation indicates that with an order difference of 2 between the numerator and denominator polynomial, and using $[s^2 H(s)]$ to calculate the weighting function, it is possible to compensate implicitly for the effect of modes outside the frequency band of analysis by lower and upper residual terms. This technique can be used as an alternative to, or complementary to the mentioned technique of increasing the order of the polynominals to describe modes outside the frequency band of analysis.

4. APPLICATION

The following examples will illustrate some features of the multiple input orthogonal polynomial parameter estimation technique.

An important characteristic of the technique is its capability of estimating the modal parameters of, in theory, perfectly repeated modes. Repeated modes require the analysis of FRFs relative to several input stations [4]. Each input station will only excite a combination of the independent mode shapes corresponding to the repeated mode. The number of independent mode shapes equals the multiplicity of the repeated mode; the combination constants are the modal participation factors (one for each of the independent mode shapes) associated with the input station. The technique analyses FRFs relative to several input stations simultaneously and identifies global estimates of poles and modal participation factors as eigenvalues and eigenvectors of a generalised eigenvalue problem, equation (4). Therefore it is possible to detect repeated modes with a multiplicity at most equal to the number of inputs for which FRFs are analysed simultaneously.

When the modes are so highly coupled, as to be almost repeated, it should be possible to identify them, by analysing FRFs relative to just one input station. In practice, the data are distorted and corrupted with noise, making it impossible to identify highly coupled modes correctly using techniques that are capable of analysing FRFs relative to just one input station.

This is illustrated by following example. FRFs are synthesised, according to the system parameters listed in Table 1, with a frequency resolution of 0·15 Hz (100 data points in the range from 25 to 40 Hz). The following three cases were analysed.

1. No noise added, analysing FRFs relative to one input station
2. 5% rms noise added, analysing FRFs relative to one input station
3. 5% rms noise added, analysing FRFs relative to three input stations.

The resulting modal parameters are listed in Table 2. Figure 1 compares original and synthesised FRFs for the various cases. The next example illustrates the capability of the above method to account explicitly and implicitly for the effect of modes outside the frequency band of analysis in the estimation of the system characteristics (modes and modal participation factors). FRFs are synthesised using the system parameters listed in Table 3. The data were contaminated by additive noise (5% rms). Modal parameters were estimated using only data in the frequency range from 25 to 40 Hz. In this range, only two resonances are visible. The estimation was done using different orders for the denominator (p) and numerator (q) matrix polynomials. The results are listed in Table 4; Figure 2 compares original and synthesised FRFs at a driving point. In the first case ($p=2$, $q-1$), not allowing for the effect of modes outside the analysis band, both modes are identified; the resulting error on the damping ratios is however important (20% for mode 1, 10% for mode 2), as shown in Fig. 2(a). Increasing the order of the numerator polynomial to 4 to describe the effect of modes outside the analysis band by residual terms improves the accuracy of the modal parameters and of the fit-result considerably (Fig. 2(b)); the damping ratio error is reduced to 3% for mode 1 and 5% for mode 2.

TABLE 1

Synthesis of semi-repeated poles: modal parameters

Mode	Frequency (Hz)	Damping (%)
1	30·00	0·500
2	35·00	1·000
3	35·05	1·000

TABLE 2

Estimated modal parameters from the data in Table 1

			Mode 1		Mode 2		Mode 3	
Case	Noise %rms	Inputs	Frequency (Hz)	Damping (%)	Frequency (Hz)	Damping (%)	Frequency (Hz)	Damping (%)
1	0	1	30·00	0·500	35·00	1·000	35·05	1·00
2	5	1	29·96	0·510	34·93	1·109	35·54	0·21
3	5	3	30·06	0·520	34·94	0·953	34·99	1·03

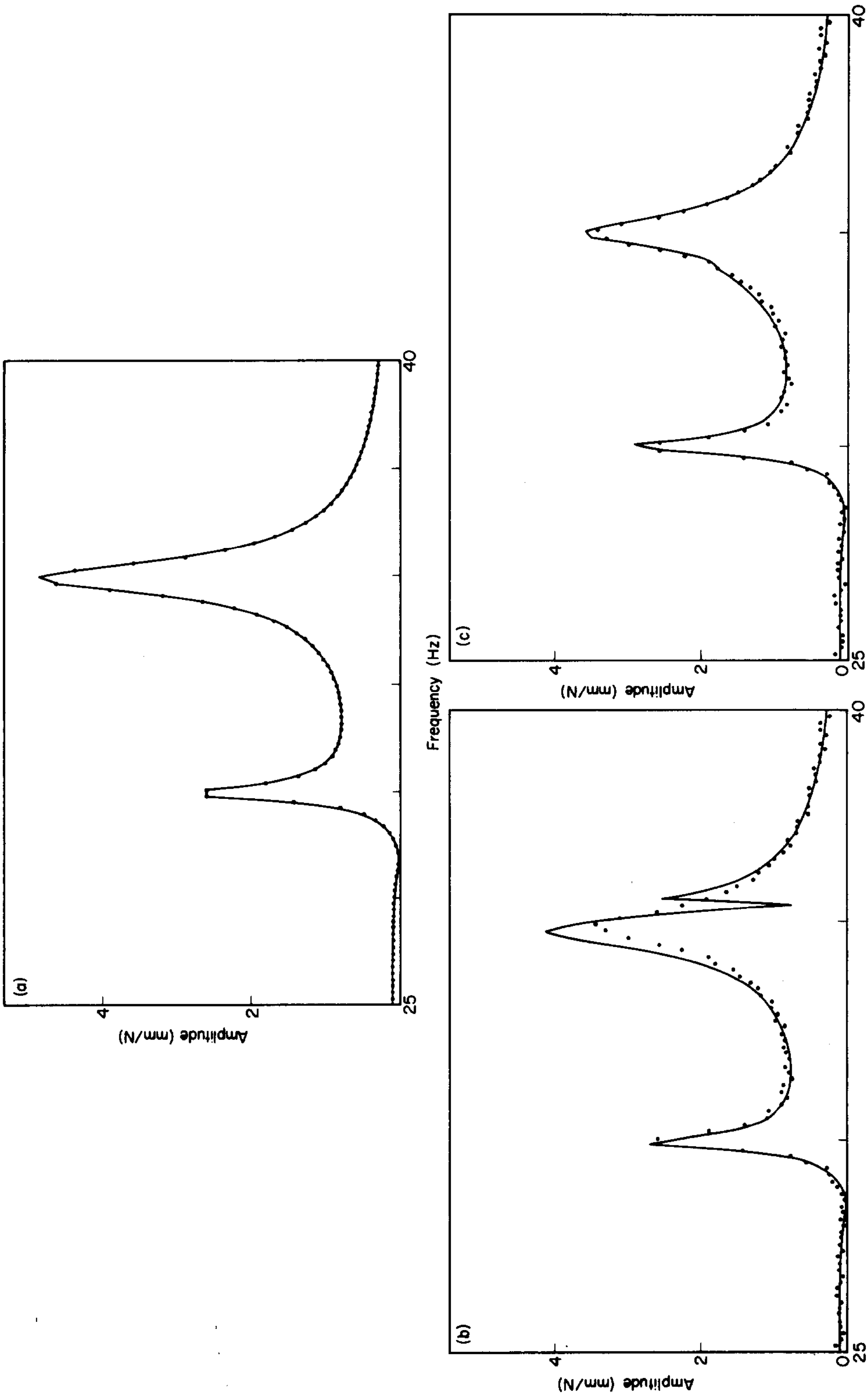

Figure 1. Semi-repeated poles. (a) One input, no noise; (b) one input, 5% noise; (c) three inputs, 5% noise.

TABLE 3

Synthesis of out-of-band effects: modal parameters

Mode	Frequency (Hz)	Damping ratio (%)
1	22	3
2	30	0·5
3	35	1
4	42	2

TABLE 4

Estimated modal parameters from the data in Table 3 with added noise (5% rms)

			Mode 1		Mode 2	
Case	p	q	Frequency (Hz)	Damping (%)	Frequency (Hz)	Damping (%)
1	2	1	30·01	0·600	34·89	1,101
2	2	4	30·02	0·485	34·96	0·953
3	4	3	30·01	0·513	34·98	0·972

In the third case, the effect of modes outside the analysis band is explicitly accounted for by increasing the number of poles in the system model. The two modes are now identified in the presence of two additional modes (24·24 Hz, 2·88% and 40·30 Hz, 1·70%). The numerical values of these poles are not accurate, as their contribution inside the analysis range is limited and the data are distorted with noise. The important consequence is that the accuracy of the parameters of both modes inside the analysis band is improved; the damping ratio error is reduced to 2% for mode 1 and 3% for mode 2. The improvement is also visible in Fig. 2(c).

Finally, one of the reasons for using orthogonal polynomials is to improve the numerical condition of the equation systems to be solved. A formulation of the problem in terms of standard matrix polynomials, as in equation (6), tends to yield poorly conditioned equation systems, especially when data, covering a broad frequency range are used [4]. When using a formulation in terms of orthogonal matrix polynomials, as in equation (7), this problem is greatly reduced.

In the following example, FRFs were synthesised in the frequency range from 0 to 40 Hz, with a frequency resolution of 0·40 Hz. Despite the broad frequency range and the contamination of the data by 5% rms noise (additive), the resulting modal parameters agree well with the theoretical values. The results are listed in Table 5; Fig. 3 compares an original FRF with a synthesised one.

5. CONCLUSIONS

The paper discusses a frequency domain global modal parameter estimation technique. The technique can be implemented as a two phase process using a minicomputer environment. In the first phase, the system's characteristic equation, expressed as a combination of orthogonal polynomials, is estimated using FRFs relative to several input stations

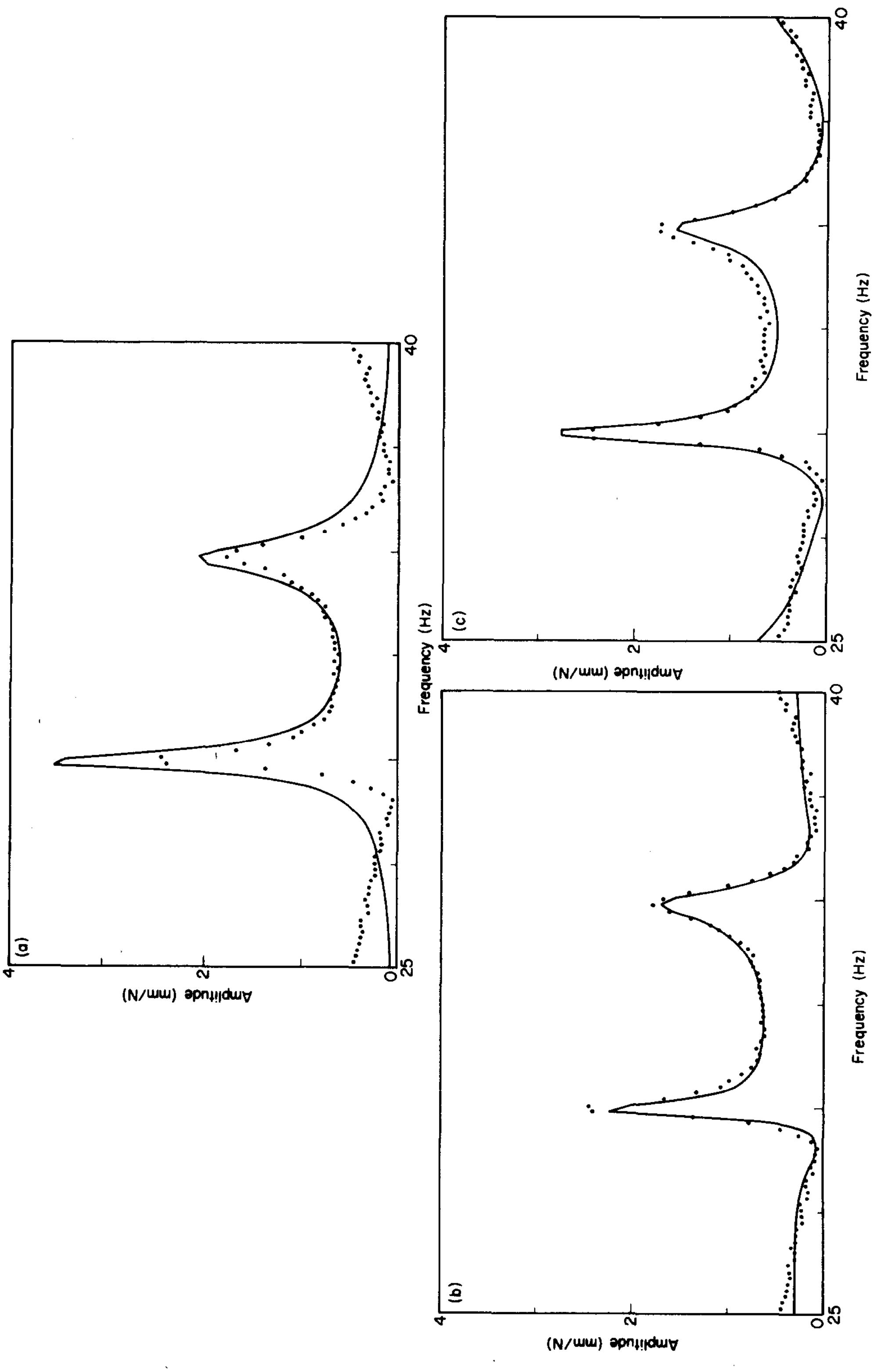

Figure 2. Out-of-band effects. (a) No precautions; (b) higher numerator order; (c) higher denominator order.

TABLE 5

Large frequency range estimation: estimated modal parameters

Mode	Estimated values Frequency (Hz)	Estimated values Damping (%)	Theoretical values Frequency (Hz)	Theoretical values Damping (%)
1	14·97	0·520	15·0	0·500
2	29·98	1·010	30·0	1·000
3	35·06	1·004	35·0	1·000

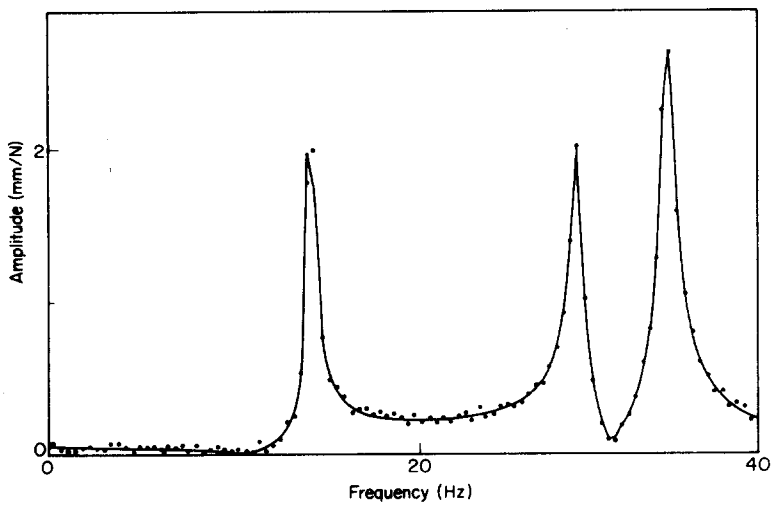

Figure 3. Large frequency range estimation.

simultaneously; the system characteristics, in terms of poles and modal participation factors, can be calculated from this equation. In a second phase, a unique set of mode shape coefficients, independent of input station, is estimated.

Examples of the applications demonstrate that the technique is capable of detecting highly coupled, repeated modes. They also indicate how estimation of poles can be improved considerably by taking the effect of modes outside the frequency band of analysis under consideration. The technique permits effects to be described explicitly by identifying such modes outside the analysis band, or implicitly by using residual terms.

Finally the technique can also be applied to data with non-equally spaced frequency resolution such as data measured with single or multiple input stepped sine excitation.

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REFERENCES

1. S. IBRAHIM and E. MUKULUK 1977 *The Shock and Vibration Bulletin*, **47**, 183–198. A method for direct identification of vibration parameters from the free response.

2. D. BROWN, R. ALLEMANG, R. ZIMMERMANN and M. MERGEAY 1979 *Society of Automotive Engineers Transactions*, **88**, 828-846. Parameter estimation techniques for modal analysis.
3. H. VOLD, J. KUNDRAT, T. ROCKLIN and R. RUSSELL 1982 *Society of Automative Engineers Transactions* **91**, 815-821. A multi-input modal parameter estimation algorithm for minicomputers.
4. J. LEURIDAN 1984 Ph.D. Dissertation, University of Cincinnati. Some direct parameter model identification methods applicable for multiple input modal analysis.
5. F. LEMBREGTS, J. LEURIDAN, L. ZHANG and H. KANDA 1986 *Proceedings of the 4th International Modal Analysis Conference, Los Angeles*, pp. 589-598. Multiple input modal analysis of frequency response functions based on direct parameter identification.
6. J. LEURIDAN, D. BROWN and R. ALLEMANG 1986 *American Society of Mechanical Engineers, Journal of Vibration Acoustics, Stress and Reliability in Design*, **108**, 1-8. Time domain parameter identification methods for linear modal analysis: a unifying approach.
7. H. VAN DER AUWERAER, P. SAS, P. VANHERCK and R. SNOEYS 1986 *Proceedings of the 4th International Modal Analysis Conference, Los Angeles*, pp. 572-580. Experimental modal analysis with stepped-sine excitation.
8. J. LEURIDAN, D. DE VIS, H. VAN DER AUWERAER and F. LEMBREGTS 1986 *Proceedings of the 4th International Modal Analysis Conference, Los Angeles*, pp. 908-918. A comparison of some frequency response function measurement techniques.
9. J. LEURIDAN, J. LIPKENS, H. VAN DER AUWERAER and F. LEMBREGTS 1986 *Proceedings of the 4th International Modal Analysis Conference, Los Angeles*, pp. 1586-1595. Global modal parameter estimation methods: an assessment of time versus frequency domain implementation.
10. B. COPPOLINO 1981 Aerospace Corporation, Los Angeles, Paper. A simultaneous frequency domain technique for estimation of modal parameters from measured data.
11. J. LEURIDAN and J. KUNDRAT 1982 *Proceedings of the 1st International Modal Analysis Conference, Orlando*, pp. 192-200. Advanced matrix methods for experimental modal analysis. A multi-matrix method for direct parameter extraction.
12. M. RICHARDSON and D. FORMENTI 1982 *Proceedings of the 1st International Modal Analysis Conference, Orlando*, pp. 167-186. Parameter estimation from frequency response measurements using rational fractional polynomials.
13. H. VAN DER AUWERAER, R. SNOEYS and J. LEURIDAN *American Society of Mechanical Engineers, Journal of Vibration, Stress and Reliability in Design*. To be published. A global frequency domain modal parameter estimation technique for minicomputers.
14. J. ADCOCK and R. POTTER 1985 *Proceedings of the 3rd International Modal Analysis Conference, Orlando*, pp. 541-547. A frequency domain curve fitting algorithm with improved accuracy.
15. R. JONES and Y. KOBAYASKI 1986 *Proceedings of the 4th International Modal Analysis Conference*, Global parameter estimation using rational fraction polynomials.
16. G. FORSYTHE 1957 *SIAM Journal* **5**, 74-88. Generation and use of orthogonal polynomials for data-fitting with a digital computer.

APPENDIX 1: MATRIX NOTATION USED

$[\cdot\cdot]$ matrix expressions
$\lfloor\cdot\cdot\rfloor$ row-vector expressions
$\{\cdot\cdot\}$ column-vector expressions
A upper case letters indicate matrices
$\lceil a \rfloor$ indicates a diagonal matrix with a on the diagonal.
A^t transpose of A
A^h hermitian transpose of A
$\|A\|$ Frobenius-norm of matrix A
$|a|$ Euclidian norm of element a

APPENDIX 2: GENERATION OF ORTHOGONAL MATRIX POLYNOMIALS

Let $[P_k(s)]$ be a polynomial of full order k, with real matrix coefficients, in an independent complex variable s

$$[P_k(s)] = \sum_{i=0}^{k} [A_i]s^i; \qquad [A_k]0. \tag{A.1}$$

Let $[Q(s)]$ be a matrix functional, with Hermitian symmetry

$$[Q(s^*)] = [Q(s)]^*. \tag{A2}$$

It will be proved that a set of polynomials $[P_k(s)]$ which are orthogonal with $[Q(s)]$ as the weighting function over a domain $(s_1 \cdots s_n) = (-j\omega_f \cdots -j\omega_1, j\omega_1 \cdots j\omega_f)$ can be calculated from following three term recurrence relation

$$[P_k(s)] = s[P_{k-1}(s)] - [P_{k-1}(s)][U_k] - [P_{k-2}(s)][V_k] \tag{A3}$$

obeying

$$\sum_{i=1}^{n} [P_1(s_i)]^h[Q(s_i)][P_k(s_i)] = [\delta_{kl}]. \tag{A4}$$

To prove the recurrence relationship, assume that orthogonal polynomials $[P_{k-1}] \cdots [P_0]$ have been found. Multiply equation (3) with $[P_l]^h[Q]$, and evaluate the summation over the domain defined above,

$$\begin{aligned} -\sum [P_1]^h[Q][P_k] &= \sum s[P_1]^h[Q][P_{k-1}] \\ -[\sum[P_1]^h[Q][P_{k-1}]][U_k] &- [\sum[P_1]^h[Q][P_{k-2}]][V_k]. \end{aligned} \tag{A5}$$

Evaluating this expression for 1 equal to $k-1$, and 1 equal to $k-2$ and imposing orthogonality of $[P_k]$ with respect to $[P_{k-1}]$ and $[P_{k-2}]$ yields

$$[U_k] = \sum s[P_{k-1}]^h[Q][P_{k-1}] \tag{A6}$$

$$[V_k] = \sum s[P_{k-2}]^h[Q][P_{k-1}]. \tag{A7}$$

Due to Hermitian symmetry of the polynomials (real matrix coefficients) and of the weighting function, the coefficients $[U_k]$ and $[V_k]$ are real. With these coefficients in the recurrence relation, $[P_k]$ will be orthogonal to $[P_{k-1}] \cdots [P_0]$. Furthermore, before normalisation

$$\sum [P_k]^h[Q][P_k] = [S] \tag{A8}$$

$[S]$ is real and can be proved to be positive definite if the weighting function $[Q]$ equals $[H]^h[H]$, and $N_0 \geqslant N_i$. Therefore

$$[S] = [R]^t[R]. \tag{A9}$$

Finally, the normalised polynomial $[P_{kn}]$ equals

$$[P_{kn}] = [P_k][R]^{-1}. \tag{A10}$$