Asymptotic Expansions of the Streamfunction for the Flow Past a Sphere at Small Reynolds Numbers

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1 Introduction

In this paper, we analyze matched asymptotic expansions of solutions to the incompressible, viscous Navier-Stokes equations for the steady flow past a sphere in the limit of small Reynolds number. The first investigations into incompressible, viscous, steady flows past finite bodies were performed in 1851 by George Stokes [1]. Low Reynolds numbers imply that inertial forces are small compared to viscous forces, so Stokes' approach was to find an approximate solution by omitting the inertial term from the Navier-Stokes momentum equation. The resulting linearized equation, along with the continuity equation, describes what is called Stokes flow, or creeping flow. In terms of dimensionless quantities, the complete Navier-Stokes equations for incompressible, viscous, steady flow are

$$\mathrm{Re}(\boldsymbol{v}^* \cdot \nabla^*)\boldsymbol{v}^* + \nabla^* p^* = \nabla^{*2}\boldsymbol{v}^*, \tag{1a}$$
$$\nabla^* \cdot \boldsymbol{v}^* = 0, \tag{1b}$$

where the relevant variable definitions are shown in Table 1. Dropping the inertial term, the Stokes equations are

$$\nabla^* p^* = \nabla^{*2}\boldsymbol{v}^*,$$
$$\nabla^* \cdot \boldsymbol{v}^* = 0.$$

The solution to these equations is commonly called the Stokes solution. In the case of flow over a sphere of radius a, the Stokes solution leads to the result

$$D = 6\pi\mu a U \quad \text{(Stokes' law)}$$

for the retarding drag force D parallel to the flow direction. The corresponding drag coefficient is

$$C_D \equiv \frac{D}{\rho U^2 a^2} = \frac{6\pi}{\mathrm{Re}}. \tag{2}$$

Since Eq. 2 is obtained from the Stokes equations, in which inertial effects are neglected entirely, it is actually the leading order term of an expansion

$$C_D = \frac{6\pi}{\mathrm{Re}}\left(1 + \sum_{i=1}^{\infty} \xi_i(\mathrm{Re})\right), \tag{3}$$

where the corrections are functions of the Reynolds number. In 1889, Alfred Whitehead attempted to improve upon the Stokes solution using a regular perturbation method, namely finding a leading order solution $\boldsymbol{v}_0^*$ from the Stokes equations and iteratively using the result to compute the inertia term in Eq. 1a [1]. The issue with this approach is that solutions beyond leading order cannot satisfy the uniform stream boundary conditions far from the sphere. The problem is thus said to exhibit singular behavior, in the sense that the straightforward perturbation scheme produces an approximation that ceases to be valid in some region of the flow. In fact, this issue arises not just for the flow over a sphere but for all problems with a uniform stream over a finite body and is referred to as Whitehead's paradox [1, 2].

The resolution of the paradox came from Carl Oseen in 1910. Oseen's idea was to approximate the neglected inertia term with a linear term, leading to the Oseen equations

$$\begin{aligned} \mathrm{Re}(\boldsymbol{U} \cdot \nabla\boldsymbol{v}^*) + \nabla^* p^* &= \nabla^{*2}\boldsymbol{v}^*, \\ \nabla^* \cdot \boldsymbol{v}^* &= 0. \end{aligned} \tag{4}$$

The linear term is small near the body compared to the viscous term, yet it provides the correct leading order balance far away, where the inertial and viscous effects become comparable. As a result, the solution to the Oseen equations turns out to be uniformly valid.

In the 1950s, Kaplun & Lagerstrom (1957) and Proudman & Pearson (1957) began to take up the challenge of finding higher-order corrections to the solutions of Stokes and Oseen. As mentioned in [1], this could have been done in the spirit of Whitehead's approach by iteratively substituting lower-order approximations into the Oseen equation (rather than the Stokes equation) for the neglected inertia terms. Instead, however, the dominant procedure used was that of matched asymptotic expansions. In particular, locally valid expansions called the Stokes and Oseen expansions were obtained and matched together in an intermediate region to create a uniformly valid approximation (The reason for naming the expansions this way is that their leading terms are related to the solutions of the Stokes and Oseen equations, as we will explain below).

Since the Stokes and Oseen expansions are only locally valid, not all of the physical boundary conditions can be satisfied by each expansion. The role of the matching process is to replace the inapplicable boundary conditions so that the expansions may be determined uniquely. The appeal of the method of matched asymptotic expansions is in part that the Stokes and Oseen expansions can be formally derived from the exact solution via limit processes, enabling one to clearly interpret the error at each term and determine the domains of validity. In addition, the fact that both expansions are formally derived in terms of the same underlying solution provides the intuitive grounds for the principle that governs the matching process.

In the sections that follow, we will first provide a brief overview of the ideas and notations necessary from the theory of asymptotic methods to understand the procedures used in the seminal works of [1] and [3]. Since Kaplun applies this formalism explicitly (in fact, he was one of the first to develop it), we will introduce the problem for the flow past a sphere as presented in [3]. When it comes time to complete the calculations, we will switch to follow the presentation of Proudman and Pearson in [1], which uses the streamfunction formulation that is common to most other literature on the topic, e.g. [4–6]. As we do so, we will describe the matching processes from the perspective of [2], which synthesizes a clear framework through which to understand the matching philosophies put forward in the original works. We will conclude by summarizing the complete approach.

2 Asymptotic Methods

2.1 Asymptotic Expansions and Singular Perturbation Problems

Asymptotic analysis methods seek to approximate the behavior of a solution to a physical or mathematical problem as a dimensionless parameter approaches a limiting value, which without loss of generality can be taken to be zero [6]. Specifically, if f represents the true solution of interest and ε represents a small parameter, then asymptotic methods seek to represent $f(\varepsilon)$ as

$$f(\varepsilon) = a_0 + a_1\delta_1(\varepsilon) + a_2\delta_2(\varepsilon) + a_3\delta_3(\varepsilon) + \cdots \tag{5}$$

in the limit $\varepsilon \to 0$. The functions $\delta_i(\varepsilon)$ are referred to as gauge functions and satisfy

$$\lim_{\varepsilon\to 0} \frac{\delta_1(\varepsilon)}{1} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\delta_{i+1}(\varepsilon)}{\delta_i(\varepsilon)} = 0$$

for all positive i. The coefficients a_i are defined formally by

$$a_0 = \lim_{\varepsilon\to 0} f(\varepsilon) \quad \text{and} \quad a_{i+1} = \lim_{\varepsilon\to 0} \frac{f(\varepsilon) - \sum_{j=1}^{i} a_j\delta_j(\varepsilon)}{\delta_{i+1}(\varepsilon)}, \tag{6}$$

where we define $\delta_0(\varepsilon) \equiv 1$. An asymptotic expansion is obtained by truncating the asymptotic series (Eq. 5) at a finite number of terms. The hallmark of this expansion is that it may not converge as more terms are added, yet it may provide a useful approximation with only a handful of terms.

We have so far written the solution $f(\varepsilon)$ as a function of one variable, but this need not be the case. In the context of fluid flow problems, for example, the solution may have the form $f(x_i, \varepsilon)$, where x_i denotes the position vector in index notation. The solution itself might also be a vector $\boldsymbol{f}(x_i, \varepsilon)$, e.g. if $\boldsymbol{f}$ is a velocity field. In this case, Eqs. 5 and 6 become

$$\boldsymbol{f}(x_i, \varepsilon) = \sum_{i=0}^{\infty} \boldsymbol{a}_i(x_i)\delta_i(\varepsilon) \tag{7}$$

and

$$\boldsymbol{a}_0(x_i) = \lim_{\varepsilon \to 0} \boldsymbol{f}(x_i, \varepsilon) \quad \text{and} \quad \boldsymbol{a}_{i+1}(x_i) = \lim_{\varepsilon \to 0} \frac{\boldsymbol{f}(x_i, \varepsilon) - \sum_{j=1}^{i} \boldsymbol{a}_j(x_i)\delta_j(\varepsilon)}{\delta_{i+1}(\varepsilon)}. \tag{8}$$

While the asymptotic series in Eq. 7 is still concerned with the limit $\varepsilon \to 0$, the expansion may not be valid for all x_i, i.e. spatially dependent dominant balances can arise, leading to the existence of boundary layers. If this happens, the asymptotic expansion is considered not to be uniformly valid, and the problem is referred to as a singular perturbation problem. Expansions in which the dependence on x_i and ε is separated in this manner are termed Poincaré expansions. The mathematical challenge for these problems is to find *inner* and *outer* asymptotic expansions that approximate the solution in different regions of the domain and match them together in a way that produces a uniformly valid result. In this spirit, the solution we seek for a singular perturbation problem is referred to as a matched asymptotic expansion.

A warning that singular behavior occurs in a problem is that the small parameter ε multiplies the highest order derivative, but this is not a sufficient condition. The source of singular behavior can be more subtle, e.g. when it comes from the boundary conditions or an infinite domain [2]. The more general physical criterion put forth by [2] is that singular behavior can be expected when the perturbation quantity ε is the ratio of two lengths or two times. In the case of flow past a sphere, two characteristic lengths are present: the sphere radius a and the viscous length ν/U. The ratio of these scales is the Reynolds number, so when we consider perturbations in the limit $\mathrm{Re} \to 0$, we expect to have a singular perturbation problem.

2.2 Inner Problems, Outer Problems, and Matching

The existence of spatially separated domains of validity in the theory of matched asymptotic expansions leads to the idea of distinguished limits, or variable scale factors that emphasize the dominant physics in different regions of a problem. The eminent dimension of a problem's geometry defines a primary reference length that can be used to nondimensionalize the governing equations and constitutes one scale over which the solution changes. The dimensionless quantities corresponding to this primary length are called the outer variables. The outer expansion is the asymptotic expansion that results from applying the outer limit, which is Eq. 8 with the outer variables held fixed. The inner variables are dimensionless quantities scaled by functions of ε so as to have order unity where the outer expansion is not uniform (Interpretations for the inner expansion and inner limit are analogous to those for the outer variables). We characterize these notions explicitly here because they directly govern the matching order, as we will now see.

The Van Dyke asymptotic matching principle states that

$$\begin{aligned} &m\text{-term inner expansion of (the } n\text{-term outer expansion)} \\ &= n\text{-term outer expansion of (the } m\text{-term inner expansion).} \end{aligned} \tag{9}$$

A latent element of this rule that must also be applied is the principle of minimum singularity, which states that a solution can only be matched to another expansion if it has the weakest possible singularity in its regions of nonuniformity [2]. Intuitively, Eq. 9 requires that the two expansions share the same functional form in some overlap domain in which both expansions are valid. The existence of such a region is grounded conceptually in the fact that both the inner and outer expansions are formally derived from the same underlying solution via Eq. 8, but Kaplun's extension theorem, given in [3], shows rigorously that an expansion's domain of validity may be extended to ensure that such an intermediate matching region exists.

In practice, the Van Dyke rule is applied by alternately writing one expansion in terms of the variables of the other and choosing the undetermined parameters of the $\boldsymbol{a}_i(x_i)$ such that the two expansions agree. The standard order in which this is done has to do with the definitions of the inner and outer variables given earlier. Van Dyke's convention is that the outer solution, being associated with variables changing over the dominant reference length, should be independent of the inner solution at leading order, i.e. a first-order change in the inner solution should not affect it. The first-order outer solution will thus be obtained first, followed by the first-order inner solution, as this one will be affected by changes in the latter, followed by the second-order outer solution, and so on. The standard order will sequentially determine not only the $\boldsymbol{a}_i(x_i)$ but also the gauge functions $\delta_i(\varepsilon)$.

Once the asymptotic expansions have been determined, a single uniformly valid expansion may be obtained by an additive or multiplicative composition. In this paper, we focus on additive compositions. Letting $\boldsymbol{f}_i^{(m)}$ denote the m-term inner expansion, $\boldsymbol{f}_o^{(n)}$ denote the n-term outer expansion, and so on, Van Dyke's rule for forming an additive composite expansion is

$$\hat{\boldsymbol{f}}^{(m,n)} = \begin{cases} \boldsymbol{f}_i^{(m)} + \boldsymbol{f}_o^{(n)} - [\boldsymbol{f}_o^{(n)}]_i^{(m)} \\ \boldsymbol{f}_o^{(n)} + \boldsymbol{f}_i^{(m)} - [\boldsymbol{f}_i^{(m)}]_o^{(n)} \end{cases}, \tag{10}$$

where $\hat{\boldsymbol{f}}^{(m,n)}$ is used to denote the uniformly valid composite expansion.

The general procedure of matched asymptotic expansions may be summarized as follows.

1. Identify the dimensionless perturbation parameter and limiting process.

2. Define appropriate inner and outer quantities. In each domain of validity, posit an asymptotic expansion to approximate the solution.

3. Alternately substitute the expansions into the governing equations, starting with the outer expansion.

4. At each step, solve the resulting equations for the unknown term of the expansion. Enforce the applicable physical boundary conditions, and use a matching principle to supply the remaining conditions necessary to uniquely specify the term.

5. Construct a uniformly valid solution by forming an additive (or multiplicative) composite expansion.

In the following, we apply this general framework to analyze solutions of the incompressible, viscous Navier-Stokes equations for the steady flow past a sphere at small Reynolds numbers.

3 Asymptotic Expansions of Navier-Stokes Solutions

3.1 Formalism of Kaplun 1957

The following formalism follows that presented in [3] and concerns the incompressible, viscous flow past a finite body in two or three dimensions. The relevant definitions are shown in Table 1.

$\boldsymbol{v}$	velocity field	$\mathrm{Re} = \frac{Ua}{\nu}$	Reynolds number
U	uniform stream velocity	θ	polar angle
ψ	streamfunction	$\boldsymbol{v}^* = \frac{\boldsymbol{v}}{U}$	scaled velocity
ρ	density (constant)	$p^* = \frac{pa}{\rho\nu U}$	viscous pressure scale
p	pressure	$x_i^* = \frac{x_i}{a} = \frac{\tilde{x}_i}{\mathrm{Re}}$	
ν	kinematic viscosity	$r^* = \frac{r}{a} = \frac{\tilde{r}}{\mathrm{Re}}$	Stokes variables ($*$)
μ	dynamic viscosity	$\psi^* = \frac{\psi}{a^2 U}$	
a	radius of sphere	$\tilde{x}_i = \frac{x_i U}{\nu}$	
x_i	Cartesian position	$\tilde{r} = \frac{rU}{\nu}$	Oseen variables ($=$)
r	$r^2 = x_i x_i$	$\tilde{\psi} = \mathrm{Re}^2 \psi^*$	

Table 1: Variable definitions. The variables shown on the left are dimensional, while those on the right are not.

The governing equations are the incompressible, viscous Navier-Stokes equations

$$\begin{gathered} \frac{D\boldsymbol{v}}{Dt} + \frac{1}{\rho}\nabla p = \nu \nabla^2 \boldsymbol{v}, \\ \nabla \cdot \boldsymbol{v} = 0, \end{gathered} \tag{11}$$

with boundary conditions

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{at body,} \tag{12a}$$

$$\boldsymbol{v} = U\boldsymbol{i}, \quad p = p_\infty \quad \text{at infinity,} \tag{12b}$$

where $\boldsymbol{i}$ is the unit vector in the x_1 direction. The general problem statement is as follows. We are interested in solving the mathematical problem comprised of Eqs. 11 and 12 in the limit $\mathrm{Re} \to 0$. The equations cannot be solved directly in their complete form, so we must seek an approximation. As the problem is characterized by two length scales in the limit as their ratio goes to zero, we expect that the approximation we introduce will lead to a singular perturbation problem. Accordingly, we will apply the method of matched asymptotic expansions.

The goal is to obtain an asymptotic expansion that is uniformly valid over the entire flow field in the limit $\mathrm{Re} \to 0$. We will follow the procedure described in Sec. 1. The dimensionless parameter is the Reynolds number, which is the ratio of the sphere radius a to the viscous length ν/U. The limiting process $\mathrm{Re} \to 0$ implies that the dominant scale is the viscous scale, so following the convention of Van Dyke, we choose the outer variables to be those that are nondimensionalized with ν/U (See Sec. 6.4 for a justification of this choice based on dominant balance). These are the Oseen variables. We then define the inner variables to be nondimensionalized by a so as to emphasize the physics on the other length scale. These are the Stokes variables.

The Oseen limit of a flow quantity $\boldsymbol{f}(x_i, \mathrm{Re})$ is defined by

$$\lim_O \boldsymbol{f} = \lim_{\mathrm{Re}\to 0} \boldsymbol{f}, \quad \tilde{x}_i = \text{constant} \neq 0. \tag{13}$$

Imagining that $\tilde{x}_i$ is held constant by fixing U and ν, this limit corresponds physically to looking at a fixed point in space while the sphere radius shrinks to zero. In other words, it describes the flow as it appears near the uniform stream at infinity. The Stokes limit is defined by

$$\lim_S \boldsymbol{f} = \lim_{\mathrm{Re}\to 0} \boldsymbol{f}, \quad x_i^* = \text{constant} = x_i^* \text{ on or outside the body}$$

and describes the flow at the sphere. We now posit two asymptotic expansions

$$\boldsymbol{v}^*(\tilde{x}_i, \mathrm{Re}) = \sum_{i=0}^{n} \boldsymbol{g}_i(\tilde{x}_i)\tilde{\delta}_i(\mathrm{Re}) \quad \text{(Oseen expansion)} \tag{14}$$

and

$$\tilde{\boldsymbol{v}}(\tilde{x}_i, \mathrm{Re}) = \sum_{i=0}^{n} \boldsymbol{h}_i(x_i^*)\delta_i^*(\mathrm{Re}) \quad \text{(Stokes expansion)}, \tag{15}$$

referred to as the Oseen and Stokes expansions, which are to be valid in the Oseen and Stokes limits, respectively. Eqs. 14 and 15 are written in the form of Eq. 7, so the coefficients $\boldsymbol{g}_i$ and $\boldsymbol{h}_i$ satisfy Eq. 8. The gauge functions $\tilde{\delta}_i$ and δ_i^* will be determined during the matching process. The Oseen limit describes the external flow away from the body, so the Oseen expansion should satisfy the boundary conditions in Eq. 12b. The Stokes limit describes the behavior of the boundary layer near the body, so the Stokes expansion should satisfy Eq. 12a.

At this point in the formalism, Kaplun proceeds to establish the existence of an intermediate limit, defined in terms of a partially ordered set of equivalence classes of functions, and postulates a matching principle that holds inside the overlap domain. We elect to follow the matching philosophy of Van Dyke in [2], so we omit these details and refer to [3] for their development.

The next step is to derive expressions for the $\boldsymbol{g}_i$ and $\boldsymbol{h}_i$ by substituting Eqs. 14 and 15 into Eq. 11. While Kaplun does this with the equations as written, the more common procedure is to use the streamfunction formulation of the Navier-Stokes equations and modify Eqs. 14 and 15 accordingly. We adopt this approach ourselves, following the general course of [1].

3.2 Formalism of Proudman 1957

In terms of Stokes variables, the incompressible, viscous, steady Navier-Stokes equations for the dimensionless streamfunction ψ^* are

$$\frac{\mathrm{Re}}{r^{*2}\sin\theta}\left(\frac{\partial\psi^*}{\partial\theta}\frac{\partial}{\partial r^*} - \frac{\partial\psi^*}{\partial r^*}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\psi^*}{\partial r^*} - \frac{2}{r^*}\frac{\partial\psi^*}{\partial\theta}\right)E_{r^*}^2\psi^* = E_{r^*}^2E_{r^*}^2\psi^*, \tag{16}$$

where

$$E_{r^*}^2 = \frac{\partial^2}{\partial r^{*2}} + \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

The no-slip boundary condition becomes

$$\psi^*(r^*=1,\theta) = \left.\frac{\partial\psi^*}{\partial r^*}\right|_{r^*=1} = 0, \tag{17}$$

and since the velocity components v_r^* and v_θ^* are related to ψ^* via

$$v_r^* = \frac{1}{r^{*2}\sin\theta}\frac{\partial\psi^*}{\partial\theta} \quad \text{and} \quad v_\theta^* = \frac{1}{r^*\sin\theta}\frac{\partial\psi^*}{\partial r^*},$$

the uniform stream conditions $v_{r^*} = \cos\theta$ and $v_\theta^* = -\sin\theta$ as $r^* \to \infty$ imply that ψ^* satisfies the boundary condition

$$\psi^* = \frac{r^{*2}}{2}\sin^2\theta \tag{18}$$

as $r^* \to \infty$. The governing equation in terms of Oseen variables is

$$\frac{1}{\tilde{r}^2\sin\theta}\left(\frac{\partial\tilde{\psi}}{\partial\theta}\frac{\partial}{\partial\tilde{r}} - \frac{\partial\tilde{\psi}}{\partial\tilde{r}}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\tilde{\psi}}{\partial\tilde{r}} - \frac{2}{\tilde{r}}\frac{\partial\tilde{\psi}}{\partial\theta}\right)E_{\tilde{r}}^2\tilde{\psi} = E_{\tilde{r}}^2E_{\tilde{r}}^2\tilde{\psi}, \tag{19}$$

where $E_{\tilde{r}}^2$ is the same operator as $E_{r^*}^2$ with r^* replaced by $\tilde{r}$. The boundary conditions in Oseen variables have the same form as Eqs. 17 and 18 with r^* replaced by $\tilde{r}$ and ψ^* replaced by $\tilde{\psi}$. Since the Stokes and Oseen expansions are only valid locally, however, we only require that the Stokes expansion satisfy the no-slip condition and that the Oseen expansion satisfy the uniform stream condition. The inapplicable boundary condition in each case is replaced by one generated from the matching principle. In accordance with the streamfunction formulation, instead of writing Stokes and Oseen expansions for the velocity $\boldsymbol{v}$, we write the streamfunction expansions

$$\psi^*(r^*, \theta, \mathrm{Re}) = \sum_{i=0}^{\infty} \psi_i^*(r^*, \theta)\delta_i^*(\mathrm{Re}) \quad \text{(Stokes expansion)}$$

and

$$\tilde{\psi}(\tilde{r}, \theta, \mathrm{Re}) = \sum_{i=0}^{\infty} \tilde{\psi}_i(\tilde{r}, \theta)\tilde{\delta}_i(\mathrm{Re}) \quad \text{(Oseen expansion).}$$

Let $\psi^{*(n)}$ and $\tilde{\psi}^{(n)}$ denote the n-term Stokes and Oseen expansions, respectively. In general, let the superscript (n) denote the n-term expansion of the relevant quantity. If either expansion is expressed in the variables of the other, we denote this by a subscript. For example, we denote the one-term Stokes expansion in terms of Oseen variables as $\psi_O^{*(1)}$. In general, let $[\cdot]_O$ denote the quantity in brackets in terms of Oseen variables and $[\cdot]_S$ denote the quantity in brackets in terms of Stokes variables. For example, $[\cdot]_O^{(1)}$ denotes the one-term expansion of the quantity in brackets written in Oseen variables. We now proceed to substitute the Stokes and Oseen expansions into the governing equations to determine ψ_i^*, $\tilde{\psi}_i$, and the gauge functions.

4 Stokes and Oseen Expansions

We consider the calculation of the Stokes and Oseen expansions to comprise two main tasks. The first is to solve the equations governing ψ_i^* and $\tilde{\psi}_i$. The second is to determine the unknown coefficients of ψ_i^* and $\tilde{\psi}_i$ and the gauge functions δ_i^* and $\tilde{\delta}_i$ via matching. As the former requires lengthy calculations to solve the relevant partial differential equations, we will provide a shortened version of the solution process here and place the full details in Sec.6. The latter represents the mathematical essence of the singular perturbation problem and the method of matched asymptotic expansions, so this is the task to which we direct our primary focus.

In a similar spirit, we will not provide the details of how to compute the drag coefficient from the streamfunction expansions. The drag coefficient is the main physical observable of interest for this problem, but if one already has access to a uniformly valid streamfunction, actually computing the drag becomes the subject of fluid dynamics rather than of asymptotic methods. However, unlike the calculation, the interpretation of the results is dictated by asymptotic approximation theory, so we will state the results for the drag coefficient and comment on how the number of terms kept in the expansions affects the quality of approximations.

4.1 First-Order Oseen Term $\tilde{\psi}_0$

We begin by calculating the first term of the Oseen expansion, $\tilde{\psi}_0$. The derivation of $\tilde{\psi}_0$ commonly proceeds via a physical argument based on the interpretation of the Oseen limit. The radial Oseen coordinate is defined as $\tilde{r} = rU/\nu$, and recall from Eq. 13 that the Oseen limit is defined by keeping $\tilde{x}_i$, and thereby $\tilde{r}$, fixed as $\mathrm{Re} \to 0$. If we imagine fixing U and ν, then the limiting process $\mathrm{Re} \to 0$ implies that the sphere radius $a \to 0$. Additionally, note that $\tilde{r}$ corresponds to a fixed point in space because r is the dimensional radial distance and does not participate in the limiting process. It follows that the limit in Eq. 13, taking $\boldsymbol{f}$ to be the velocity field, corresponds to the velocity at a fixed position in space as the spherical body vanishes to a point. As a point cannot create a finite disturbance, this velocity must be equal to the free stream velocity, implying that

$$\tilde{\psi}^{(1)} = \tilde{\psi}_0 = \frac{\tilde{r}^2}{2} \sin^2 \theta, \tag{20}$$

the uniform stream boundary condition [1, 5]. Notice that the derivation of Eq. 20 did not require reference to the leading order Stokes solution ψ_0^*, in agreement with Van Dyke's convention that the outer solution be independent of the inner one at leading order. We need this to be the case so that we have a starting point for the alternating matching procedure.

4.2 First-Order Stokes Term ψ_0^*

We now obtain the first term of the Stokes expansion, ψ_0^*. If we substitute ψ_0^* into Eq. 16, the limit $\mathrm{Re} \to 0$ leaves only the right side of the equation at leading order, such that ψ_0^* satisfies the Stokes equation

$$E_{r^*}^2 E_{r^*}^2 \psi_0^* = 0. \tag{21}$$

Since we are dealing with a sphere problem, we follow [6] in assuming a solution of the form

$$\psi_0^* = \sin^2 \theta F(r^*),$$

where F is a function to be determined. Substituting this ansatz into Eq. 21 leads to Euler's differential equation

$$F^{(\mathrm{iv})} - \frac{4}{r^{*2}} F'' + \frac{8}{r^{*3}} F' - \frac{8}{r^{*4}} F = 0. \tag{22}$$

The solutions to the Euler equation have the form $F_n = C_n r^{*n}$, where C_n is a constant. Substituting F_n into Eq. 22 leads to

$$C_n r^{*(n-4)} \left[n(n-1)(n-2)(n-3) - 4n(n-1) + 8n - 8\right] = 0,$$

which implies that solutions exist when n is -1, 1, 2, or 4 (The full details of this calculation are provided in Sec. 6.1). Hence, we have that the inner solution at leading order is

$$\psi^{*(1)} = \psi_0^* = \sin^2 \theta \left(C_{-1} \frac{1}{r^*} + C_1 r^* + C_2 r^{*2} + C_4 r^{*4} \right). \tag{23}$$

Since ψ_0^* solves the Stokes equation, it is commonly called the Stokes solution, and this motivates the nomenclature of the Stokes expansion. As it turns out, it is possible to apply both the no-slip and uniform stream boundary conditions to Eq. 23, such that ψ_0^* becomes a uniformly valid solution without considering the Oseen expansion. However, this is merely a coincidence, as the same cannot be done in the two-dimensional case of flow past a cylinder nor the present case beyond leading order. We will therefore proceed as we would otherwise by using the Van Dyke matching principle to complete the solution for ψ_0^*.

Applying Eq. 9 to the leading order terms we derived requires $[\tilde{\psi}_S^{(1)}]_O^{(1)} = \mathrm{Re}^2[\psi_O^{*(1)}]^{(1)}$. Let us explicitly describe what this means for clarity. On the left, $[\tilde{\psi}_S^{(1)}]_O^{(1)}$ indicates that we take the one-term Oseen expansion expressed in Stokes variables ($\tilde{\psi}_S^{(1)}$) and write the one-term expansion of the result in Oseen variables ($[\cdot]_O^{(1)}$). If we do this, we obtain the right side of Eq. 20. The notation

$\mathrm{Re}^2[\psi_O^{*(1)}]^{(1)}$ instructs us to take the one-term Stokes expansion expressed in Oseen variables and write the one-term expansion of the result. The reason that the Re^2 factor appears is that ψ^* and $\tilde{\psi}$ are defined differently and must be equated on equal footing.

Expressing Eq. 23 in terms of Oseen variables gives

$$\mathrm{Re}^2\psi_O^{*(1)} = \mathrm{Re}^2[\psi_0^*]_O = \sin^2\theta\left(C_{-1}\frac{\mathrm{Re}^3}{\tilde{r}} + C_1\mathrm{Re}\tilde{r} + C_2\tilde{r}^2 + C_4\frac{\tilde{r}^4}{\mathrm{Re}^2}\right).$$

In a one-term expansion, we have that

$$\mathrm{Re}^2[\psi_0^*]_O^{(1)} = \sin^2\theta\left(C_2\tilde{r}^2 + C_4\frac{\tilde{r}^4}{\mathrm{Re}^2}\right), \tag{24}$$

which must agree with Eq. 20. Hence, $C_2 = 1/2$ and $C_4 = 0$. We thus see that the common part of the Stokes and Oseen expansions at leading order is the uniform stream condition

$$\tilde{\psi}^{(1)} = \mathrm{Re}^2[\psi_O^{*(1)}]^{(1)} = \frac{\tilde{r}^2}{2}\sin^2\theta.$$

The remaining two coefficients C_{-1} and C_1 are found by applying the no-slip condition on the body to Eq. 23. From Eq. 17, we have

$$\psi_0^*(r^* = 1, \theta) = \sin^2\theta\left(C_{-1} + C_1 + \frac{1}{2}\right) = 0$$

and

$$\left.\frac{\partial\psi_0^*}{\partial r^*}\right|_{r^*=1} = \sin^2\theta\left(\frac{-C_{-1}}{r^{*2}} + C_1 + r^*\right) = 0,$$

leading to the system of equations

$$\begin{aligned} C_{-1} + C_1 &= -\frac{1}{2}, \\ -C_{-1} + C_1 &= -1. \end{aligned}$$

The solution to this is $C_{-1} = 1/4$ and $C_1 = -3/4$, so the inner solution becomes

$$\psi^{*(1)} = \psi_0^* = \frac{1}{4}\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta. \tag{25}$$

We mentioned above that this equation happens to be uniformly valid, but we can illustrate that this is indeed the case by explicitly forming the additive composite expansion. Adding Eqs. 25 and 20 and subtracting the common part, which would otherwise be counted twice, we find

$$\hat{\psi}^{(1,1)} = \psi^{*(1)} + \frac{1}{\mathrm{Re}^2}\tilde{\psi}_S^{(1)} - \frac{1}{\mathrm{Re}^2}[\tilde{\psi}_S^{(1)}]^{(1)} = \frac{1}{4}\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta = \psi_0^*,$$

where $\hat{\psi}^{(1,1)}$ denotes the uniformly valid leading order expansion, i.e. the additive expansion formed by the one-term Stokes expansion and the one-term Oseen expansion (All of the composite expansions $\hat{\psi}^{(m,n)}$ we construct in this paper are written in Stokes variables). We now proceed to obtain the second-order terms.

4.3 Second-Order Oseen Term $\tilde{\psi}_1$

The standard matching order of Van Dyke prescribes that we next seek the second term $\tilde{\psi}_1$ of the Oseen expansion. However, before we substitute the two-term expansion $\tilde{\psi}^{(2)} = \tilde{\psi}_0 + \tilde{\delta}_1(\mathrm{Re})\tilde{\psi}_1$ into Eq. 19, it is helpful to first determine the gauge function $\tilde{\delta}_1(\mathrm{Re})$. In theory, we could just proceed with the substitution and argue what $\tilde{\delta}_1(\mathrm{Re})$ must be to retain the correct balance in the governing equation, but this may not be straightforward. A better approach is to use the fact that the Van Dyke matching order will automatically dictate the form of successive gauge functions if Eq. 9 is applied at each stage.

So, let us take the leading Stokes solution ψ_0^* just obtained and express it in terms of Oseen variables to see what $\tilde{\delta}_1(\mathrm{Re})$ must be. We have

$$\mathrm{Re}^2[\psi_0^*]_O = \frac{\mathrm{Re}^2}{4}\left(2\frac{\tilde{r}^2}{\mathrm{Re}^2} - 3\frac{\tilde{r}}{\mathrm{Re}} - \frac{\mathrm{Re}}{\tilde{r}}\right)\sin^2\theta = \left(\frac{1}{2}\tilde{r}^2 - \mathrm{Re}\frac{3}{4}\tilde{r} - \mathrm{Re}^3\frac{1}{4\tilde{r}}\right)\sin^2\theta.$$

In a two-term expansion, we have

$$\mathrm{Re}^2[\psi_0^*]_O^{(2)} = \frac{1}{2}\tilde{r}^2\sin^2\theta - \mathrm{Re}\frac{3}{4}\tilde{r}\sin^2\theta. \tag{26}$$

We identify the first term as $\tilde{\psi}_0$, so a two-term expansion of the form $\tilde{\psi}^{(2)} = \tilde{\psi}_0 + \tilde{\delta}_1(\mathrm{Re})\tilde{\psi}_1$ must have $\tilde{\delta}_1(\mathrm{Re}) = \mathrm{Re}$.

We can now proceed to find $\tilde{\psi}_1$. Substituting $\tilde{\psi}^{(2)} = \tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1$ into Eq. 19 shows that $\tilde{\psi}_1$ satisfies the Oseen streamfunction equation

$$\left(\cos\theta\frac{\partial}{\partial\tilde{r}} - \frac{\sin\theta}{\tilde{r}}\frac{\partial}{\partial\theta}\right)E_{\tilde{r}}^2\tilde{\psi}_1 = E_{\tilde{r}}^2E_{\tilde{r}}^2\tilde{\psi}_1.$$

The details are found in Sec. 6.2 and Sec.6.3. Solving for $\tilde{\psi}_1$ gives

$$\tilde{\psi}_1 = -2c_2(1+\cos\theta)[1 - e^{-\frac{1}{2}\tilde{r}(1-\cos\theta)}].$$

We see that $\tilde{\psi}^{(2)} = \tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1$ automatically satisfies the uniform stream condition, for $\tilde{\psi}_1 \to 0$ as $\tilde{r} \to \infty$, leaving just the uniform stream $\tilde{\psi}_0$. So, to solve for c_2, we need to apply the matching principle. We will express the two-term Oseen expansion in terms of Stokes variables and expand the exponential for small Re. We have

$$\begin{aligned}
\frac{1}{\mathrm{Re}^2}[\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1]_S &= \frac{1}{\mathrm{Re}^2}\left[\frac{1}{2}\mathrm{Re}^2r^{*2}\sin^2\theta - \mathrm{Re}2c_2(1+\cos\theta)[1 - e^{-\frac{1}{2}\mathrm{Re}r^*(1-\cos\theta)}]\right] \\
&= \frac{1}{2}r^{*2}\sin^2\theta - \frac{1}{\mathrm{Re}}\left(2c_2(1+\cos\theta)\left[1 - \left(1 - \frac{1}{2}\mathrm{Re}r^*(1-\cos\theta) + \mathcal{O}(\mathrm{Re}^2)\right)\right]\right) \\
&= \frac{1}{2}r^{*2}\sin^2\theta - \frac{1}{\mathrm{Re}}[c_2\mathrm{Re}(1+\cos\theta)r^*(1-\cos\theta) + \mathcal{O}(\mathrm{Re}^2)] \\
&= \frac{1}{2}r^{*2}\sin^2\theta - c_2(1-\cos^2\theta)r^* + \mathcal{O}(\mathrm{Re}) \\
&= \frac{1}{2}r^{*2}\sin^2\theta - c_2r^*\sin^2\theta + \mathcal{O}(\mathrm{Re}).
\end{aligned} \tag{27}$$

Comparing this with Eq. 26 divided by Re^2 and written in Stokes variables, it must be that $c_2 = 3/4$. The two-term Oseen expansion is therefore

$$\tilde{\psi}^{(2)} = \tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1 = \frac{1}{2}\tilde{r}^2\sin^2\theta - \mathrm{Re}\frac{3}{2}(1+\cos\theta)[1 - e^{-\frac{1}{2}\tilde{r}(1-\cos\theta)}]. \tag{28}$$

We have now found the one-term Stokes expansion and the two-term Oseen expansion. The uniformly valid composition of these two expansions can now be found via Eq. 10. We have

$$
\begin{aligned}
\hat{\psi}^{(1,2)} &= \psi^{*(1)} + \frac{1}{\mathrm{Re}^2}\tilde{\psi}_S^{(2)} - \frac{1}{\mathrm{Re}^2}[\tilde{\psi}_S^{(2)}]^{(1)} = \psi_0^* + \frac{1}{\mathrm{Re}^2}[\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1]_S - \frac{1}{\mathrm{Re}^2}[\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1]_S^{(1)} \\
&= \frac{1}{4}\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta \\
&+ \frac{1}{\mathrm{Re}^2}\left[\frac{1}{2}r^{*2}\mathrm{Re}^2\sin^2\theta - \mathrm{Re}\frac{3}{2}(1+\cos\theta)[1-e^{-\frac{1}{2}\mathrm{Re}r^*(1-\cos\theta)}]\right] \\
&- \frac{1}{2}r^{*2}\sin^2\theta + \frac{3}{4}r^*\sin^2\theta \\
&= \left(\frac{1}{2}r^{*2} - \frac{3}{4}r^* + \frac{1}{4r^*}\right)\sin^2\theta + \frac{1}{2}r^{*2}\sin^2\theta - \frac{3}{2\mathrm{Re}}(1+\cos\theta)[1-e^{-\frac{1}{2}\mathrm{Re}r^*(1-\cos\theta)}] \\
&- \frac{1}{2}r^{*2}\sin^2\theta + \frac{3}{4}r^*\sin^2\theta \\
&= \frac{1}{4}\left(2r^{*2} + \frac{1}{r^*}\right)\sin^2\theta - \frac{3}{2\mathrm{Re}}(1+\cos\theta)[1-e^{-\frac{1}{2}\mathrm{Re}r^*(1-\cos\theta)}].
\end{aligned}
$$

which is the solution Oseen obtained in 1910 when he solved Eq. 4. We mentioned in Sec. 1 that, unlike the Stokes solution, Oseen's solution is uniformly valid, and we have confirmed this fact by explicitly constructing the composite expansion and recovering his solution.

The approximation of the drag coefficient that results from this solution (cf. Eq. 3) is

$$
C_D = \frac{6\pi}{\mathrm{Re}}\left(1 + \frac{3}{8}\mathrm{Re}\right). \tag{29}
$$

What is interesting about this formula is that it is valid beyond the extent that we would expect. As Proudman and Pearson discuss in [1], the Oseen equations are only a first approximation to Eq. 19. The Oseen term is introduced to account for inertial effects in the region approaching the uniform stream, but since it still neglects inertial effects in the Stokes region, the Stokes and Oseen descriptions are similar for the flow near the sphere. As a result, we should not expect a second approximation to the drag coefficient, and Eq. 29 should only be valid to the first term, in agreement with Stokes' law. However, the 3Re/8 term happens to be correct, for reasons we will discuss after we obtain ψ_1^*.

4.4 Second-Order Stokes Term ψ_1^*

As with the two-term Oseen expansion, we approach the two-term Stokes expansion $\psi^{*2} = \psi_0^* + \delta_1^*(\mathrm{Re})\psi_1^*$ by first determining the gauge function δ_1^*. Writing Eq. 28 in terms of Stokes variables and expanding the exponential gives

$$\begin{aligned}
\frac{1}{\mathrm{Re}^2}[\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1]_S &= \frac{1}{\mathrm{Re}^2}\left[\mathrm{Re}^2\frac{r^{*2}}{2}\sin^2\theta - \mathrm{Re}\frac{3}{2}(1+\cos\theta)\right. \\
&\quad \left.\times\left(1 - \left(1 - \frac{1}{2}\mathrm{Re}r^*(1-\cos\theta) + \mathrm{Re}^2\frac{1}{8}r^{*2}(1-\cos\theta)^2 + \mathcal{O}(\mathrm{Re}^3)\right)\right)\right] \\
&= \frac{1}{2}r^{*2}\sin^2\theta - \frac{3}{2\mathrm{Re}}(1+\cos\theta)\left[\mathrm{Re}\frac{r^*}{2}(1-\cos\theta) - \mathrm{Re}^2\frac{r^{*2}}{8}(1-\cos\theta)^2 + \mathcal{O}(\mathrm{Re}^3)\right] \\
&= \frac{1}{2}r^{*2}\sin^2\theta - \frac{3}{4}r^*\sin^2\theta + \frac{3}{16\mathrm{Re}}\mathrm{Re}^2 r^{*2}(1+\cos\theta)(1-\cos\theta)^2 + \mathcal{O}(\mathrm{Re}^2) \\
&= \frac{1}{2}r^{*2}\sin^2\theta - \frac{3}{4}r^*\sin^2\theta + \frac{3}{16}\mathrm{Re}r^{*2}\sin^2\theta(1-\cos\theta) + \mathcal{O}(\mathrm{Re}^2). \qquad (30)
\end{aligned}$$

Comparing this with Eq. 26 divided by Re^2 and written in Stokes variables, we see that ψ_0^* recovers the first two terms. Hence, in the two-term expansion $\psi^{*(2)} = \psi_0^* + \delta_1^*(\mathrm{Re})\psi_1^*$, it must be that $\delta_1^*(\mathrm{Re}) = \mathrm{Re}$. We can now proceed to determine ψ_1^* by substituting $\psi^{*(2)}$ into Eq. 16. Since ψ_0^* solves the Stokes equations, we have $E_{r^*}^2 E_{r^*}^2(\psi_0^* + \mathrm{Re}\psi_1^*) = \mathrm{Re}E_{r^*}^2 E_{r^*}^2\psi_1^*$, and Eq 16 becomes

$$\begin{aligned}
\mathrm{Re}E_{r^*}^2 E_{r^*}^2\psi_1^* &= -\frac{9}{4}\mathrm{Re}\left(\frac{2}{r^{*2}} - \frac{3}{r^{*3}} + \frac{1}{r^{*5}}\right)\sin^2\theta\cos\theta \\
&\quad + \frac{\mathrm{Re}^2}{r^{*2}\sin\theta}\left(\frac{\partial\psi^*}{\partial\theta}\frac{\partial}{\partial r^*} - \frac{\partial\psi^*}{\partial r^*}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\psi^*}{\partial r^*} - \frac{2}{r^*}\frac{\partial\psi^*}{\partial\theta}\right)E_{r^*}^2\psi_1^*
\end{aligned}$$

At $\mathcal{O}(\mathrm{Re})$, we can drop the second term on the right. Hence, ψ_1^* satisfies

$$E_{r^*}^2 E_{r^*}^2\psi_1^* = -\frac{9}{4}\left(\frac{2}{r^{*2}} - \frac{3}{r^{*3}} + \frac{1}{r^{*5}}\right)\sin^2\theta\cos\theta, \qquad (31)$$

which is seen to be Whitehead's equation with the factor of Re. It follows that Whitehead's solution in Eq. 52, without the factor of Re, is a particular integral of Eq. 31 (See Sec. 49 for the calculation). We must add to this the homogeneous solution of Eq. 31. Since $E_{r^*}^2 E_{r^*}^2\psi_1^* = 0$ is just the Stokes equation, we know that ψ_0^* in Eq. 25 is one term of the homogeneous solution, but as it turns, out it is the only one with the proper symmetry that satisfies the principle of minimum singularity [2]. We therefore have

$$\psi_1^* = C\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta - \frac{3}{32}\left(\frac{1}{r^{*2}} - \frac{1}{r^*} + 1 - 3r^* + 2r^{*2}\right)\sin^2\theta\cos\theta, \qquad (32)$$

where C is an unknown constant. The no-slip condition on the sphere is already satisfied by this equation, so C must be determined by matching. Writing Eq. 32 in terms of Oseen variables, we have

$$\begin{aligned}
\mathrm{Re}^2[\psi_0^* + \mathrm{Re}\psi_1^*]_O = \mathrm{Re}^2&\left[\frac{1}{4}\left(2\frac{\tilde{r}^2}{\mathrm{Re}^2} - 3\frac{\tilde{r}}{\mathrm{Re}} + \frac{\mathrm{Re}}{\tilde{r}}\right)\sin^2\theta + C\mathrm{Re}\left(2\frac{\tilde{r}^2}{\mathrm{Re}} - 3\tilde{r} + \frac{\mathrm{Re}^2}{\tilde{r}}\right)\sin^2\theta\right. \\
&\left.- \frac{3}{32}\mathrm{Re}\left(\frac{\mathrm{Re}^3}{r^{*2}} - \frac{\mathrm{Re}^2}{r^*} + \mathrm{Re} - 3r^* + 2\frac{2\tilde{r}^2}{\mathrm{Re}}\right)\sin^2\theta\cos\theta\right].
\end{aligned}$$

Dropping all terms of $\mathcal{O}(\mathrm{Re}^2)$ and higher yields

$$\begin{aligned}
\mathrm{Re}^2[\psi_0^* + \mathrm{Re}\psi_1^*]_O^{(2)} &= \frac{1}{4}(2\tilde{r}^2 - 3\tilde{r}\mathrm{Re})\sin^2\theta + 2C\tilde{r}^2\mathrm{Re}\sin^2\theta - \frac{3}{16}\tilde{r}^2\sin^2\theta\cos\theta \\
&= \frac{1}{2}\tilde{r}^2\sin^2\theta + \mathrm{Re}\left(-\frac{3}{4}\tilde{r} + 2C\tilde{r}^2 - \frac{3}{16}\tilde{r}^2\cos\theta\right)\sin^2\theta. \qquad (33)
\end{aligned}$$

Writing Eq. 30 in terms of Oseen variables and multiplying by Re^2 gives

$$
\begin{aligned}
[\tilde{\psi}_S^{(2)}]_O^{(2)} &= \frac{1}{2}\tilde{r}^2\sin^2\theta - \frac{3}{4}\tilde{r}\mathrm{Re}\sin^2\theta + \frac{3}{16}\mathrm{Re}\tilde{r}^2\sin^2\theta(1-\cos\theta) \\
&= \frac{1}{2}\tilde{r}^2\sin^2\theta + \mathrm{Re}\left(-\frac{3}{4}\tilde{r} + \frac{3}{16}\tilde{r}^2\theta(1-\cos\theta)\right)\sin^2 .
\end{aligned}
\tag{34}
$$

Comparing Eqs. 33 and 34, we see that

$$
-\frac{3}{4}\tilde{r} + 2C\tilde{r}^2 - \frac{3}{16}\tilde{r}^2\cos\theta = -\frac{3}{4}\tilde{r} + \frac{3}{16}\tilde{r}^2(1-\cos\theta),
$$

which implies that $C = 3/32$. The two-term Stokes expansion is thus

$$
\begin{aligned}
\psi^{*(2)} = \psi_0^* + \mathrm{Re}\psi_1^* &= \frac{1}{4}\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta + \frac{3}{32}\mathrm{Re}\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta \\
&- \frac{3}{32}\mathrm{Re}\left(\frac{1}{r^{*2}} - \frac{1}{r^*} + 1 - 3r^* + 2r^{*2}\right)\sin^2\theta\cos\theta \\
&= \frac{1}{4}(r^*-1)^2\sin^2\theta\left[\left(1+\frac{3}{8}\mathrm{Re}\right)\left(2+\frac{1}{r^*}\right) - \frac{3}{8}\mathrm{Re}\left(2+\frac{1}{r^*}+\frac{1}{r^{*2}}\right)\cos\theta\right].
\end{aligned}
\tag{35}
$$

We can now construct a uniformly valid composition from the two-term Stokes expansion and the two-term Oseen expansion. The result is

$$
\begin{aligned}
\hat{\psi}^{(2,2)} &= \psi^{*(2)} + \frac{1}{\mathrm{Re}^2}\tilde{\psi}_S^{(2)} - \frac{1}{\mathrm{Re}^2}[\tilde{\psi}_S^{(2)}]^{(2)} = \psi_0^* + \mathrm{Re}\psi_1^* + \frac{1}{\mathrm{Re}^2}[\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1]_S - \frac{1}{\mathrm{Re}^2}[\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1]_S^{(2)} \\
&= \frac{1}{4}\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta + \frac{3}{32}\mathrm{Re}\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta \\
&- \frac{3}{32}\mathrm{Re}\left(\frac{1}{r^{*2}} - \frac{1}{r^*} + 1 - 3r^* + 2r^{*2}\right)\sin^2\theta\cos\theta \\
&+ \frac{1}{2}r^{*2}\sin^2\theta - \frac{3}{2\mathrm{Re}}(1+\cos\theta)[1 - e^{-\frac{1}{2}\mathrm{Re}r^*(1-\cos\theta)}] \\
&- \frac{1}{2}r^{*2}\sin^2\theta + \frac{3}{4}r^*\sin^2\theta - \frac{3}{16}\mathrm{Re}r^{*2}\sin^2\theta(1-\cos\theta) \\
&= \frac{1}{4}\left(2r^{*2} + \frac{1}{r^*}\right)\sin^2\theta + \frac{3}{32}\mathrm{Re}\left(-3r^* - \frac{1}{r^*}\right)\sin^2\theta \\
&- \frac{3}{32}\mathrm{Re}\left(\frac{1}{r^{*2}} - \frac{1}{r^*} + 1 - 3r^*\right)\sin^2\theta\cos\theta - \frac{3}{2\mathrm{Re}}(1+\cos\theta)[1 - e^{-\frac{1}{2}\mathrm{Re}r^*(1-\cos\theta)}].
\end{aligned}
$$

The drag coefficient corresponding to this result is

$$
C_D = \frac{6\pi}{\mathrm{Re}}\left(1 + \frac{3}{8}\mathrm{Re}\right),
\tag{36}
$$

which is seen to be the same as Oseen's formula in Eq. 29. We can explain why this happens by reference to Eq. 35 and the discussion in [1, 7]. Examine the three terms in the second equality of Eq. 35. The first is the Stokes solution ψ_0^*, which we know yields the Stokes' law contribution to the drag coefficient, i.e. Eq. 2. The second term is the homogeneous part of the Whitehead solution in Eq. 32 and is a factor of $3\mathrm{Re}/8$ times the Stokes solution. It thus contributes an additional $3\mathrm{Re}/8$ times Stokes law to the drag coefficient. The third term, which is the particular integral of the Whitehead solution, does not contribute anything to the drag because of symmetry, yet the particular integral is what encapsulates the influence of the nonlinear inertial terms in Whitehead's equation. The particular integral having no contribution to the drag is thus equivalent to neglecting the inertial terms in Whitehead's equation, explaining why Eq. 36 agrees with Oseen's approximation in Eq. 29.

5 Summary

We presented derivations of the two-term Stokes expansion, the two-term Oseen expansion, and three uniformly valid composite expansions of the streamfunction describing the incompressible, viscous, steady flow past a sphere. The general procedure we followed was that outlined in Sec. 2 for the method of matched asymptotic expansions. In particular, we introduced the formalism of Kaplun (1957), carried out the calculations of Proudman and Pearson (1957), and applied the matching principle of Van Dyke. The essence of the calculation was to solve a singular perturbation problem that arose because the perturbation quantity, the Reynolds number, is the limit of two length scales, the viscous length and the sphere radius. We defined Stokes and Oseen expansions that are valid near and far from the sphere, respectively, and the principal task was to match them together in a manner that produces a uniformly valid result. We recount the applications of Eq. 9 below in summary of the calculation.

1. First order Stokes term, $\mathrm{Re}^2[\psi_O^{*(1)}]^{(1)} = [\tilde{\psi}_S^{(1)}]_O^{(1)}$: We compared Eq. 24 with Eq. 20.
2. Second order Oseen term, $[\psi_O^{*(1)}]_S^{(2)} = \frac{1}{\mathrm{Re}^2}[\tilde{\psi}_S^{(2)}]^{(1)}$: We compared Eq. 26, divided by Re^2 and expressed in Stokes variables, with Eq. 27.
3. Second order Stokes term, $\mathrm{Re}^2[\psi_O^{*(2)}]^{(2)} = [\tilde{\psi}_S^{(2)}]_O^{(2)}$: We compared Eq. 33 and Eq. 34.

6 Appendix

6.1 Solving the Stokes Equation for ψ_0^*

We substitute the ansatz $\psi_0^* = \sin^2\theta F(r^*)$ into the Stokes equation $E_{r^*}^2 E_{r^*}^2 \psi_0^* = 0$, obtaining

$$\begin{aligned}
0 &= E_{r^*}^2\left[\frac{\partial^2\psi_0^*}{\partial r^{*2}} + \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi_0^*}{\partial\theta}\right)\right] \\
&= E_{r^*}^2\left[\sin^2\theta F'' + \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}(2F\cos\theta)\right] \\
&= E_{r^*}^2\left[\sin^2\theta\left(F'' - \frac{2}{r^{*2}}F\right)\right] \\
&= \sin^2\theta\left[F^{(\mathrm{iv})} - \frac{\partial}{\partial r^*}\left(\frac{2}{r^{*2}}F' - F\frac{4}{r^{*3}}\right)\right] + \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left[2\cos\theta\left(F'' - \frac{2}{r^{*2}}F\right)\right] \\
&= \sin^2\theta\left[F^{(\mathrm{iv})} - \left(\frac{2}{r^{*2}}F'' - F'\frac{4}{r^{*3}} + F\frac{12}{r^{*4}} - F'\frac{4}{r^{*3}}\right)\right] - \sin^2\theta\frac{2}{r^{*2}}\left(F'' - \frac{2}{r^{*2}}F\right) \\
&= \sin^2\theta\left(F^{(\mathrm{iv})} - \frac{2}{r^{*2}}F'' + F'\frac{4}{r^{*3}} - F\frac{12}{r^{*4}} + F'\frac{4}{r^{*3}} - \frac{2}{r^{*2}}F'' + \frac{4}{r^{*4}}F\right) \\
&= \sin^2\theta\left(F^{(\mathrm{iv})} - \frac{4}{r^{*2}}F'' + \frac{8}{r^{*3}}F' - \frac{8}{r^{*4}}F\right).
\end{aligned}$$

The last equality implies

$$F^{(\mathrm{iv})} - \frac{4}{r^{*2}}F'' + \frac{8}{r^{*3}}F' - \frac{8}{r^{*4}}F = 0, \tag{37}$$

which is a Cauchy-Euler equation. Substituting the ansatz $F(r^*) = C_n r^{*n}$, where C_n is a constant into Eq. 37 gives

$$\begin{aligned}
0 &= n(n-1)(n-2)(n-3)C_n r^{*(n-4)} - \frac{4}{r^{*2}}n(n-1)C_n r^{*(n-2)} + \frac{8}{r^{*3}}nC_n r^{*(n-1)} - \frac{8}{r^{*4}}C_n r^{*n} \\
&= C_n r^{*(n-4)}\left(n(n-1)(n-2)(n-3) - \frac{4}{r^{*2}}n(n-1)r^{*2} + \frac{8}{r^{*3}}nr^{*3} - \frac{8}{r^{*4}}r^{*4}\right) \\
&= C_n r^{*(n-4)}(n^4 - 6n^3 + 7n^2 + 6n - 8),
\end{aligned}$$

which implies $n^4 - 6n^3 + 7n^2 + 6n - 8 = 0$. The solutions to this polynomial are $n = -1$, $n = 1$, $n = 2$, and $n = 4$, so the general solution for ψ_0^* is

$$\psi_0^* = \sin^2\theta\left(C_{-1}\frac{1}{r^*} + C_1 r^* + C_2 r^{*2} + C_4 r^{*4}\right),$$

as in Eq. 23.

6.2 $\tilde{\psi}_1$ Obeys the Oseen Equation

The two-term Oseen expansion is $\tilde{\psi}^{(2)} = \tilde{\psi}_0 + \tilde{\delta}_1(\mathrm{Re})\tilde{\psi}_1$. Substituting this into Eq. 19 gives

$$\begin{gathered}
\frac{1}{\tilde{r}^2\sin\theta}\left(\frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\theta}\frac{\partial}{\partial\tilde{r}} - \frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\tilde{r}}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\tilde{r}} - \frac{2}{\tilde{r}}\frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\theta}\right) \\
\times E_{\tilde{r}}^2(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1) = E_{\tilde{r}}^2E_{\tilde{r}}^2(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1).
\end{gathered} \tag{38}$$

Now, note that

$$E_{\tilde{r}}^2\tilde{\psi}_0 = \left[\frac{\partial^2}{\partial\tilde{r}^2} + \frac{\sin\theta}{\tilde{r}^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\frac{\tilde{r}^2}{2}\sin^2\theta = \sin^2\theta + \frac{\sin\theta}{\tilde{r}^2}\frac{\partial}{\partial\theta}(\tilde{r}^2\cos\theta) = 0.$$

Eq. 38 thus becomes

$$\begin{gathered}
\frac{1}{\tilde{r}^2\sin\theta}\left(\frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\theta}\frac{\partial}{\partial\tilde{r}} - \frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\tilde{r}}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\tilde{r}} - \frac{2}{\tilde{r}}\frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\theta}\right) \\
\times \mathrm{Re}E_{\tilde{r}}^2\tilde{\psi}_1 = \mathrm{Re}E_{\tilde{r}}^2E_{\tilde{r}}^2\tilde{\psi}_1.
\end{gathered} \tag{39}$$

We have

$$\frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\theta} = \tilde{r}^2\sin\theta\cos\theta + \mathrm{Re}\frac{\partial\tilde{\psi}_1}{\partial\theta} \quad \text{and} \quad \frac{\partial(\tilde{\psi}_0 + \mathrm{Re}\tilde{\psi}_1)}{\partial\tilde{r}} = \tilde{r}\sin^2\theta + \mathrm{Re}\frac{\partial\tilde{\psi}_1}{\partial\tilde{r}},$$

so Eq. 39 becomes

$$\begin{gathered}
\frac{1}{\tilde{r}^2\sin\theta}\left(\tilde{r}^2\sin\theta\cos\theta\frac{\partial}{\partial\tilde{r}} + \mathrm{Re}\frac{\partial\tilde{\psi}_1}{\partial\theta}\frac{\partial}{\partial\tilde{r}} - \tilde{r}\sin^2\theta\frac{\partial}{\partial\theta} - \mathrm{Re}\frac{\partial\tilde{\psi}_1}{\partial\tilde{r}}\frac{\partial}{\partial\theta}\right. \\
\left. + 2\cot\theta\tilde{r}\sin^2\theta + 2\cot\theta\mathrm{Re}\frac{\partial\tilde{\psi}_1}{\partial\tilde{r}} - 2\tilde{r}\sin\theta\cos\theta - \frac{2}{\tilde{r}}\mathrm{Re}\frac{\partial\tilde{\psi}_1}{\partial\theta}\right) \\
\times \mathrm{Re}E_{\tilde{r}}^2\tilde{\psi}_1 = \mathrm{Re}E_{\tilde{r}}^2E_{\tilde{r}}^2\tilde{\psi}_1.
\end{gathered}$$

Since the gauge function multiplying $\tilde{\psi}_1$ is $\tilde{\delta}_1(\mathrm{Re}) = \mathrm{Re}$, we can drop all the terms in parentheses that have a factor of Re, as they will be quadratic in the Reynolds number when multiplied by $\mathrm{Re} E_{\tilde{r}}^2 \tilde{\psi}_1$. Dropping these terms and distributing the leftmost factor, we obtain

$$\left(\cos\theta \frac{\partial}{\partial \tilde{r}} - \frac{\sin\theta}{\tilde{r}} \frac{\partial}{\partial \theta} \right) E_{\tilde{r}}^2 \tilde{\psi}_1 = E_{\tilde{r}}^2 E_{\tilde{r}}^2 \tilde{\psi}_1, \tag{40}$$

which is the streamfunction formulation of the Oseen equation.

6.3 Solving the Oseen Equation for $\tilde{\psi}_1$

We can solve Eq. 40 in two steps. The first is to set

$$E_{\tilde{r}}^2 \tilde{\psi}_1 = e^{\frac{1}{2}\tilde{r}\cos\theta} \tilde{\phi}_1, \tag{41}$$

which casts Eq. 40 into a simpler equation for $\tilde{\phi}_1$. The second is to solve Eq. 41 for $\tilde{\phi}_1$. We start by deriving the reduced equation for $\tilde{\phi}_1$. First, note that Eq. 40, with Eq. 41 substituted in for $E_{\tilde{r}}^2 \tilde{\psi}_1$, is equivalent to

$$\left(E_{\tilde{r}}^2 - \cos\theta \frac{\partial}{\partial \tilde{r}} + \frac{\sin\theta}{\tilde{r}} \frac{\partial}{\partial \theta} \right) e^{\frac{1}{2}\tilde{r}\cos\theta} \tilde{\phi}_1 = 0,$$

or

$$E_{\tilde{r}}^2 \left(e^{\frac{1}{2}\tilde{r}\cos\theta} \tilde{\phi}_1 \right) = \left(\cos\theta \frac{\partial}{\partial \tilde{r}} + \frac{\sin\theta}{\tilde{r}} \frac{\partial}{\partial \theta} \right) e^{\frac{1}{2}\tilde{r}\cos\theta} \tilde{\phi}_1. \tag{42}$$

Label Eq. 42 as LHS = RHS. Let us first address RHS. We have

$$\begin{aligned}
\mathrm{RHS} &= \cos\theta \left(e^{\frac{1}{2}\tilde{r}\cos\theta} \frac{\partial \tilde{\phi}_1}{\partial \tilde{r}} + e^{\frac{1}{2}\tilde{r}\cos\theta} \frac{\tilde{\phi}_1}{2} \cos\theta \right) - \frac{\sin\theta}{\tilde{r}} \left(e^{\frac{1}{2}\tilde{r}\cos\theta} \frac{\partial \tilde{\phi}_1}{\partial \theta} - e^{\frac{1}{2}\tilde{r}\cos\theta} \frac{\tilde{r}\tilde{\phi}_1}{2} \sin\theta \right) \\
&= e^{\frac{1}{2}\tilde{r}\cos\theta} \left(\frac{\partial \tilde{\phi}_1}{\partial \tilde{r}} \cos\theta + \frac{\tilde{\phi}_1}{2} \cos^2\theta - \frac{\partial \tilde{\phi}_1}{\partial \theta} \frac{\sin\theta}{\tilde{r}} + \frac{\tilde{\phi}_1}{2} \sin^2\theta \right) \\
&= e^{\frac{1}{2}\tilde{r}\cos\theta} \left(\frac{\partial \tilde{\phi}_1}{\partial \tilde{r}} \cos\theta - \frac{\partial \tilde{\phi}_1}{\partial \theta} \frac{\sin\theta}{\tilde{r}} + \frac{\tilde{\phi}_1}{2} \right).
\end{aligned}$$

For LHS, we have

$$
\begin{aligned}
\text{LHS} &= \left[\frac{\partial^2}{\partial\tilde{r}^2} + \frac{\sin\theta}{\tilde{r}^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right] e^{\frac{1}{2}\tilde{r}\cos\theta}\tilde{\phi}_1 \\
&= \frac{\partial}{\partial\tilde{r}}\left(e^{\frac{1}{2}\tilde{r}\cos\theta}\frac{\tilde{\phi}_1}{2}\cos\theta + e^{\frac{1}{2}\tilde{r}\cos\theta}\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}}\right) + \frac{\sin\theta}{\tilde{r}^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\left(e^{\frac{1}{2}\tilde{r}\cos\theta}\frac{\partial\tilde{\phi}_1}{\partial\theta} - e^{\frac{1}{2}\tilde{r}\cos\theta}\frac{\tilde{r}\tilde{\phi}_1}{2}\sin\theta\right)\right] \\
&= \frac{\partial}{\partial\tilde{r}}\left[e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\frac{\tilde{\phi}_1}{2}\cos\theta + \frac{\partial\tilde{\phi}_1}{\partial\tilde{r}}\right)\right] + \frac{\sin\theta}{\tilde{r}^2}\frac{\partial}{\partial\theta}\left[e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\frac{1}{\sin\theta}\frac{\partial\tilde{\phi}_1}{\partial\theta} - \frac{\tilde{r}\tilde{\phi}_1}{2}\right)\right] \\
&= e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\frac{1}{2}\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}}\cos\theta + \frac{\partial^2\tilde{\phi}_1}{\partial\tilde{r}^2}\right) + \frac{1}{2}e^{\frac{1}{2}\tilde{r}\cos\theta}\cos\theta\left(\frac{1}{2}\tilde{\phi}_1\cos\theta + \frac{\partial\tilde{\phi}_1}{\partial\tilde{r}}\right) \\
&+ \frac{\sin\theta}{\tilde{r}^2}\left[-\frac{\tilde{r}\sin\theta}{2}e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\frac{1}{\sin\theta}\frac{\partial\phi_1}{\partial\theta} - \frac{\tilde{r}}{2}\tilde{\phi}_1\right) + e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\frac{1}{\sin\theta}\frac{\partial^2\tilde{\phi}_1}{\partial\theta^2} - \frac{\partial\tilde{\phi}_1}{\partial\theta}\frac{\cos\theta}{\sin^2\theta} - \frac{\tilde{r}}{2}\frac{\partial\tilde{\phi}_1}{\partial\theta}\right)\right] \\
&= e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\frac{1}{2}\cos\theta\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}} + \frac{\partial^2\tilde{\phi}_1}{\partial\tilde{r}^2} + \frac{\tilde{\phi}_1}{4}\cos^2\theta + \frac{1}{2}\cos\theta\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}}\right) \\
&+ \frac{\sin\theta}{\tilde{r}^2}e^{\frac{1}{2}\tilde{r}\cos\theta}\left[-\frac{\tilde{r}}{2}\frac{\partial\tilde{\phi}_1}{\partial\theta} + \frac{\tilde{r}^2\sin\theta}{4}\tilde{\phi}_1 + \frac{1}{\sin\theta}\frac{\partial^2\tilde{\phi}_1}{\partial\theta^2} - \frac{\partial\tilde{\phi}_1}{\partial\theta}\frac{\cos\theta}{\sin^2\theta} - \frac{\tilde{r}}{2}\frac{\partial\tilde{\phi}_1}{\partial\theta}\right] \\
&= e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\cos\theta\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}} + \frac{\partial^2\tilde{\phi}_1}{\partial\tilde{r}^2} + \frac{\tilde{\phi}_1}{4}\cos^2\theta\right) \\
&+ \frac{\sin\theta}{\tilde{r}^2}e^{\frac{1}{2}\tilde{r}\cos\theta}\left[-\tilde{r}\frac{\partial\tilde{\phi}_1}{\partial\theta} + \frac{\tilde{r}^2\sin\theta}{4}\tilde{\phi}_1 + \frac{1}{\sin\theta}\frac{\partial^2\tilde{\phi}_1}{\partial\theta^2} - \frac{\partial\tilde{\phi}_1}{\partial\theta}\frac{\cos\theta}{\sin^2\theta}\right] \\
&= e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\cos\theta\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}} + \frac{\tilde{\phi}_1}{4}\cos^2\theta - \frac{\sin\theta}{\tilde{r}}\frac{\partial\tilde{\phi}_1}{\partial\theta} + \frac{\sin^2\theta}{4}\tilde{\phi}_1 + \frac{\partial^2\tilde{\phi}_1}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}^2}\frac{\partial^2\tilde{\phi}_1}{\partial\theta^2} - \frac{\partial\tilde{\phi}_1}{\partial\theta}\frac{\cot\theta}{\tilde{r}^2}\right) \\
&= e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\cos\theta\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}} + \frac{\tilde{\phi}_1}{4} - \frac{\sin\theta}{\tilde{r}}\frac{\partial\tilde{\phi}_1}{\partial\theta} + E^2_{\tilde{r}}\tilde{\phi}_1\right).
\end{aligned}
$$

In the last line we used the fact that

$$
\begin{aligned}
E^2_{\tilde{r}}\tilde{\phi}_1 &= \frac{\partial^2\tilde{\phi}_1}{\partial\tilde{r}^2} + \frac{\sin\theta}{\tilde{r}^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\tilde{\phi}_1}{\partial\theta}\right) \\
&= \frac{\partial^2\tilde{\phi}_1}{\partial\tilde{r}^2} + \frac{\sin\theta}{\tilde{r}^2}\left(\frac{1}{\sin\theta}\frac{\partial^2\tilde{\phi}_1}{\partial\theta^2} - \frac{\partial\tilde{\phi}_1}{\partial\theta}\frac{\cos\theta}{\sin^2\theta}\right) = \frac{\partial^2\tilde{\phi}_1}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}^2}\frac{\partial^2\tilde{\phi}_1}{\partial\theta^2} - \frac{\partial\tilde{\phi}_1}{\partial\theta}\frac{\cot\theta}{\tilde{r}^2}
\end{aligned}
$$

to identify an inverse product rule. Eq. 42 is thus

$$
e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}}\cos\theta - \frac{\partial\tilde{\phi}_1}{\partial\theta}\frac{\sin\theta}{\tilde{r}} + \frac{\tilde{\phi}_1}{2}\right) = e^{\frac{1}{2}\tilde{r}\cos\theta}\left(\cos\theta\frac{\partial\tilde{\phi}_1}{\partial\tilde{r}} + \frac{\tilde{\phi}_1}{4} - \frac{\sin\theta}{\tilde{r}}\frac{\partial\tilde{\phi}_1}{\partial\theta} + E^2_{\tilde{r}}\tilde{\phi}_1\right).
$$

or

$$
\left(E^2_{\tilde{r}} - \frac{1}{4}\right)\tilde{\phi}_1 = 0. \tag{43}
$$

The task is now to solve Eq. 43. Let us assume the ansatz $\tilde{\phi}_1 = \sin^2\theta F(\tilde{r})$, for some unknown function F. Substituting this into Eq. 43 yields

$$\sin^2\theta F'' + \frac{\sin\theta}{\tilde{r}^2}\frac{\partial}{\partial\theta}(2F\cos\theta) - \frac{1}{4}F\sin^2\theta = \sin^2\theta F'' - \frac{2\sin^2\theta}{\tilde{r}^2}F - \frac{1}{4}F\sin^2\theta = 0,$$

which implies

$$F'' - \left(\frac{2}{\tilde{r}^2} + \frac{1}{4}\right)F = 0. \tag{44}$$

We are thus faced with solving a second-order, linear, homogeneous ordinary differential equation with variable coefficients. We do not have access to initial conditions for F, so we cannot apply Laplace transforms to solve it. We could obtain a power series solution, but we want to find a Liouvillian solution so that we can rule out which term of the general solution is nonphysical. While no elementary methods exist to find such a solution, it can nonetheless be done. We use the Kovacic algorithm presented in [8]. The full details are provided for completeness, but as the calculation requires substantial familiarity with the notation and background of the Kovacic paper, one can skip to the result without loss of continuity.

We temporarily break from our notation and adopt that of [8]. In this respect, our goal is to solve $y'' = ry$, where

$$r = \frac{2}{x^2} + \frac{1}{4}. \tag{45}$$

Eq. 45 has one pole of order two at $r = 0$, and the order of r at infinity is zero, i.e. $\Gamma = \{0\}$. Hence, the necessary conditions of Case 1 in [8] hold (The order of every pole of r must be either even or one, and the order of r at infinity must be even or greater than two). We now carry out the procedure in Sec. 3.1 of [8], labeling the corresponding steps for ease of comparison.

1. Since $r = 0$ is a pole of order two, $[\sqrt{r}]_0 = 0$, and $b = 2$. So,

   $$\alpha_0^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b} = \frac{1}{2} \pm \frac{3}{2},$$

   or $\alpha_0^+ = 2$ and $\alpha_0^- = -1$. Since the order of r at infinity is $0 = -2\nu \leq 0$, we have $\nu = 0$, so $[\sqrt{r}]_\infty = a$. Comparing r and $([\sqrt{r}]_\infty)^2$, we see that $a = 1/2$. The coefficient of $x^{\nu-1} = x^{-1}$ in both r and $([\sqrt{r}]_\infty)^2$ is $b = 0$, so

   $$\alpha_\infty^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} - \nu\right) = 0.$$

2. Let $s(c)$ denote either $+$ or $-$ for $c \in \Gamma \cup \{\infty\}$. The equation

   $$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

   yields four possibilities:

   (a) When $s(c)$ is $-$ and $s(\infty)$ is $+$: $d = 0 - (-1) = 1$,
   (b) When $s(c)$ is $-$ and $s(\infty)$ is $-$: $d = 0 - (-1) = 1$,
   (c) When $s(c)$ is $+$ and $s(\infty)$ is $+$: $d = 0 - 2 = -2$,
   (d) When $s(c)$ is $+$ and $s(\infty)$ is $-$: $d = 0 - 2 = -2$.

As the latter two are negative, the only candidates for

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

are

$$\omega_{(a)} = \frac{1}{2} - \frac{1}{x} \quad \text{and} \quad \omega_{(b)} = -\frac{1}{2} - \frac{1}{x}.$$

3. We find a monic polynomial P of degree $d = 1$ that satisfies

$$P'' + 2\omega P' + (\omega' + w^2 - r)P = 0. \tag{46}$$

If a solution to P exists, a solution of the original differential equation $y'' = ry$ is given by $\eta = Pe^{\int \omega}$. Since $d = 1$, we let $P = x + s$ for some constant s. Substituting this into Eq. 46 for $\omega_{(a)}$ gives

$$2\left(\frac{1}{2} - \frac{1}{x}\right) + \left(\frac{1}{x^2} + \frac{1}{x^2} - \frac{1}{x} + \frac{1}{4} - \frac{2}{x^2} - \frac{1}{4}\right)(x+s) = -\frac{2}{x} - \frac{s}{x} = 0,$$

implying $s = -2$. A solution to the original differential equation is therefore

$$\eta = (x-2)\exp\left(\int \left(\frac{1}{2} - \frac{1}{x}\right) dx\right) = (x-2)\exp\left(\frac{x}{2} - \ln x\right) = \left(1 - \frac{2}{x}\right) e^{x/2}.$$

We have thus obtained one Liouvillian solution to $y'' = ry$. Since this is a second-order homogeneous equation, however, we need another linearly independent solution to form the general solution, but this can be done with reduction of order. Calling our first solution y_1, we write $y_2 = uy_1$ for the other solution. Skipping ahead, the second solution is

$$y_2 = y_1 \int \frac{dx}{y_1^2} = y_1 \int \left(\frac{x^2}{(x-2)^2} e^{-x}\right) dx = -\left(1 - \frac{2}{x}\right) e^{x/2} e^{-x} \frac{x+2}{x-2} = -e^{-x/2}\left(1 + \frac{2}{x}\right).$$

We now return to our standard notation. The general solution to Eq. 44 is

$$F(\tilde{r}) = c_1 e^{\tilde{r}/2}\left(1 - \frac{2}{\tilde{r}}\right) + c_2 e^{-\tilde{r}/2}\left(1 + \frac{2}{\tilde{r}}\right).$$

According to the principle of minimum singularity outlined in Sec. 2, we must take the solution that vanishes at large $\tilde{r}$. We therefore have

$$\tilde{\phi}_1 = c_2 e^{-\tilde{r}/2}\left(1 + \frac{2}{\tilde{r}}\right)\sin^2\theta,$$

so Eq. 41 becomes

$$E_{\tilde{r}}^2 \tilde{\psi}_1 = c_2 \left(1 + \frac{2}{\tilde{r}}\right) e^{-\frac{1}{2}\tilde{r}(1-\cos\theta)} \sin^2\theta.$$

A particular solution to this equation, which is quoted in [1, 2, 4], that posseses the proper symmetry and satisfies the principle of minimum singularity is

$$\tilde{\psi}_1 = -2c_2(1+\cos\theta)[1 - e^{-\frac{1}{2}\tilde{r}(1-\cos\theta)}].$$

6.4 Deriving and Solving Whitehead's Equation

We previously derived the Stokes solution, which satisfies $E^2_{r^*}E^2_{r^*}\psi^*_0 = 0$, as

$$\psi^*_0 = \frac{1}{4}\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\sin^2\theta. \tag{47}$$

As explained in Sec. 1, Whitehead's approach to improving upon the Stokes solution was to substitute ψ^*_0 back into the neglected inertial terms of the complete Navier-Stokes equation

$$\frac{\mathrm{Re}}{r^{*2}\sin\theta}\left(\frac{\partial\psi^*}{\partial\theta}\frac{\partial}{\partial r^*} - \frac{\partial\psi^*}{\partial r^*}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\psi^*}{\partial r^*} - \frac{2}{r^*}\frac{\partial\psi^*}{\partial\theta}\right)E^2_{r^*}\psi^* = E^2_{r^*}E^2_{r^*}\psi^*. \tag{48}$$

These are the same as Eqs. 25 and 16 but are reproduced here for convenience. Let us denote the left side as LHS. We now substitute Eq. 47 into the left side of Eq. 48, starting with the calculation of $E^2_{r^*}\psi^*_0$. We have

$$\begin{aligned}
E^2_{r^*}\psi^*_0 &= \left[\frac{\partial^2}{\partial r^{*2}} + \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left(\frac{1}{2}r^{*2} - \frac{3}{4}r^* + \frac{1}{4r^*}\right)\sin^2\theta \\
&= \sin^2\theta\frac{\partial}{\partial r^*}\left(r^* - \frac{3}{4} - \frac{1}{4r^{*2}}\right) + \frac{\sin\theta}{r^{*2}}\left(\frac{1}{2}r^{*2} - \frac{3}{4}r^* + \frac{1}{4r^*}\right)\frac{\partial}{\partial\theta}(2\cos\theta) \\
&= \sin^2\theta\left(1 + \frac{1}{2r^{*3}}\right) - \frac{2\sin^2\theta}{r^{*2}}\left(\frac{1}{2}r^{*2} - \frac{3}{4}r^* + \frac{1}{4r^*}\right) \\
&= \sin^2\theta\left(1 + \frac{1}{2r^{*3}} - 1 + \frac{3}{2r^*} - \frac{1}{2r^{*3}}\right) \\
&= \frac{3\sin^2\theta}{2r^*}.
\end{aligned}$$

We now apply the operator on the left of Eq. 48 to obtain

$$\begin{aligned}
\mathrm{LHS} = &\frac{\mathrm{Re}}{r^{*2}\sin\theta}\left[-\frac{1}{2}\sin\theta\cos\theta\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\frac{3\sin^2\theta}{2r^{*2}} - \frac{1}{4}\left(4r^* - 3 - \frac{1}{r^{*2}}\right)\sin^2\theta\frac{3\sin\theta\cos\theta}{r^*}\right. \\
&\left. + 2\cot\theta\frac{1}{4}\left(4r^* - 3 - \frac{1}{r^{*2}}\right)\sin^2\theta\frac{3\sin^2\theta}{2r^*} - \frac{2}{r^*}\frac{1}{2}\sin\theta\cos\theta\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\frac{3\sin^2\theta}{2r^*}\right] \\
= &\frac{\mathrm{Re}}{r^{*2}\sin\theta}\left[-\frac{3}{4r^{*2}}\sin^3\theta\cos\theta\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right) - \frac{3}{4r^*}\sin^3\theta\cos\theta\left(4r^* - 3 - \frac{1}{r^{*2}}\right)\right. \\
&\left. + \frac{3}{4}\left(4r^* - 3 - \frac{1}{r^{*2}}\right)\frac{\sin^3\theta\cos\theta}{r^*} - \frac{3}{2r^{*2}}\sin^3\theta\cos\theta\left(2r^{*2} - 3r^* + \frac{1}{r^*}\right)\right] \\
= &\frac{3\mathrm{Re}\sin^2\theta\cos\theta}{4r^{*2}}\left(\frac{3}{r^*} - 2 - \frac{1}{r^{*3}} - 4 + \frac{3}{r^*} + \frac{1}{r^{*3}} + 4 - \frac{3}{r^*} - \frac{1}{r^{*3}} - 4 + \frac{6}{r^*} - \frac{2}{r^{*3}}\right) \\
= &-\frac{9}{4}\mathrm{Re}\left(\frac{2}{r^{*2}} - \frac{3}{r^{*3}} + \frac{1}{r^{*5}}\right)\sin^2\theta\cos\theta.
\end{aligned}$$

Equating this with the right side of Eq. 48 shows that Whitehead's equation is

$$\left[\frac{\partial^2}{\partial r^{*2}} + \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi^*_W = -\frac{9}{4}\mathrm{Re}\left(\frac{2}{r^{*2}} - \frac{3}{r^{*3}} + \frac{1}{r^{*5}}\right)\sin^2\theta\cos\theta. \tag{49}$$

As an aside, we note that a simple dominant balance argument from [2] based on Eq. 49 can be used to understand why difficulties arise in the Stokes and Whitehead theories. As $r^* \to \infty$, we clearly see that the inertial terms on the right behave as $\mathcal{O}(\mathrm{Re}/r^{*2})$, while a typical term on the left computed with the Stokes solution ψ_0^* behaves as $\mathcal{O}(1/r^{*3})$. The ratio of these terms is $\mathcal{O}(\mathrm{Re}/r^*)$ and grows to $\mathcal{O}(1)$ when $r \approx \nu/U$, explaining both when the Stokes approximation fails and why ν/U is chosen as the primary reference length with which to define the outer variables.

We can proceed to solve Eq. 49 by assuming the ansatz $\psi_W^* = F(r^*)\sin^2\theta\cos\theta$, which is motivated by the form of the term on the right. Substituting the ansatz into Whitehead's equation gives

$$\begin{aligned}
E_{r^*}^2 E_{r^*}^2 \psi_W^* &= E_{r^*}^2\left[\frac{\partial^2}{\partial r^{*2}} + \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]F(r^*)\sin^2\theta\cos\theta \\
&= E_{r^*}^2\left[F''\sin^2\theta\cos\theta + F\frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\sin\theta}(2\cos^2\theta - \sin^2\theta)\right)\right] \\
&= E_{r^*}^2\left[F''\sin^2\theta\cos\theta + F\frac{\sin\theta}{r^{*2}}(-4\sin\theta\cos\theta - 2\sin\theta\cos\theta)\right] \\
&= E_{r^*}^2\left[F''\sin^2\theta\cos\theta - 6\frac{F}{r^{*2}}\sin^2\theta\cos\theta\right] \\
&= E_{r^*}^2\left[\sin^2\theta\cos\theta\left(F'' - 6\frac{F}{r^{*2}}\right)\right] \\
&= \left[\frac{\partial^2}{\partial r^{*2}} + \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left[\sin^2\theta\cos\theta\left(F'' - 6\frac{F}{r^{*2}}\right)\right] \\
&= \left[F^{(\mathrm{iv})} - 6\frac{\partial}{\partial r^*}\left(\frac{F'r^* - 2F}{r^{*3}}\right)\right]\sin^2\theta\cos\theta \\
&+ \frac{\sin\theta}{r^{*2}}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\sin\theta}(2\cos^2\theta - \sin^2\theta)\right)\left(F'' - 6\frac{F}{r^{*2}}\right) \\
&= \left[F^{(\mathrm{iv})} - 6\left(\frac{r^*(F' + r^*F'' - 2F') - 3(F'r^* - 2F)}{r^{*4}}\right)\right]\sin^2\theta\cos\theta \\
&- \frac{\sin\theta}{r^{*2}}\left(F'' - 6\frac{F}{r^{*2}}\right)6\sin\theta\cos\theta \\
&= \left[F^{(\mathrm{iv})} - 6\left(\frac{r^*F' + r^{*2}F'' - 2r^*F' - 3r^*F' + 6F}{r^{*4}}\right)\right]\sin^2\theta\cos\theta \\
&- \frac{6\sin^2\theta\cos\theta}{r^{*2}}\left(F'' - 6\frac{F}{r^{*2}}\right) \\
&= \left[F^{(\mathrm{iv})} - 6\left(-\frac{4F'}{r^{*3}} + \frac{F''}{r^{*2}} + \frac{6F}{r^{*4}}\right) - \frac{6}{r^{*2}}F'' + 36\frac{F}{r^{*4}}\right]\sin^2\theta\cos\theta \\
&= \left[F^{(\mathrm{iv})} + \frac{24}{r^{*3}}F' - \frac{6}{r^{*2}}F'' - \frac{36}{r^{*4}}F - \frac{6}{r^{*2}}F'' + \frac{36}{r^{*4}}F\right]\sin^2\theta\cos\theta \\
&= \left[F^{(\mathrm{iv})} - \frac{12}{r^{*2}}F'' + \frac{24}{r^{*3}}F'\right]\sin^2\theta\cos\theta = -\frac{9}{4}\mathrm{Re}\left(\frac{2}{r^{*2}} - \frac{3}{r^{*3}} + \frac{1}{r^{*5}}\right)\sin^2\theta\cos\theta.
\end{aligned}$$

Canceling the factor of $\sin^2\theta\cos\theta$, we are left with the ordinary differential equation

$$F^{(\mathrm{iv})} - \frac{12}{r^{*2}}F'' + \frac{24}{r^{*3}}F' = -\frac{9}{4}\mathrm{Re}\left(\frac{2}{r^{*2}} - \frac{3}{r^{*3}} + \frac{1}{r^{*5}}\right). \tag{50}$$

Our approach to Eq. 50 will be as follows. The left side has a Euler-Cauchy form, so we will assume the ansatz $F(r^*) = C_n r^{*n}$ to solve the homogeneous version of the equation. If we obtain four linearly independent solutions, we will have the fundamental system of solutions to the differential equation. We can then apply the method of variation of parameters to obtain a particular solution. The general solution will then be the linear combination of the fundamental system and the particular solution. Substituting $F(r^*) = C_n r^{*n}$ into the homogeneous counterpart of Eq. 50 gives

$$\begin{aligned}
0 &= n(n-1)(n-2)(n-3)C_n r^{*(n-4)} - \frac{12}{r^{*2}} n(n-1)C_n r^{*(n-2)} + \frac{24}{r^{*3}} n C_n r^{*(n-1)} \\
&= C_n r^{*(n-4)} \left(n(n-1)(n-2)(n-3) - \frac{12}{r^{*2}} n(n-1) r^{*2} + \frac{24}{r^{*3}} n r^{*3} \right) \\
&= C_n r^{*(n-4)} (n^4 - 6n^3 - n^2 + 30n),
\end{aligned}$$

which implies that solutions exist when n is -2, 0, 3, and 5. Hence, the homogeneous solution is

$$F_h(r^*) = C_{-2} \frac{1}{r^{*2}} + C_0 + C_3 r^{*3} + C_5 r^{*5}.$$

The particular solution is given by the method of variation of parameters by

$$F_p(r^*) = \sum_{i \in \{-2,0,3,5\}} r^{*i} \int \frac{\det[W_i(r^*)]}{\det[W(r^*)]} dr^*, \tag{51}$$

where $W(r^*)$ is the Wronskian matrix

$$W(r^*) = \begin{pmatrix} 1/r^{*2} & 1 & r^{*3} & r^{*5} \\ -2/r^{*3} & 0 & 3r^{*2} & 5r^{*4} \\ 6/r^{*4} & 0 & 6r^* & 20r^{*3} \\ -24/r^{*5} & 0 & 6 & 60r^{*2} \end{pmatrix}.$$

The $W_i(r^*)$ are the matrices formed by replacing the column corresponding to r^{*i} in $W(r^*)$ by $(0, 0, 0, f(x))^T$, where $f(x)$ is the right side of Eq. 50. The Wronskian determinant is $\det[W(r^*)] = 2100$, and computing the integrals in Eq. 51 gives

$$\begin{aligned}
F_p(r^*) &= \frac{1}{r^*} \left(\frac{9}{560} \mathrm{Re} r^* (2 - 2r^{*2} + r^{*3}) \right) - \frac{3\mathrm{Re}}{40\mathrm{Re}} (r^{*3} - 3r^{*2} - 1) \\
&\quad - r^{*3} \left(\frac{3\mathrm{Re}}{160 r^{*4}} (8r^{*3} - 6r^{*2}) + 1 \right) + r^{*4} \left(\frac{3\mathrm{Re}}{1120 r^{*6}} (8r^{*3} - 9r^{*2} + 2) \right) \\
&= \mathrm{Re} \left(\frac{3}{32 r^*} + \frac{9r^*}{32} - \frac{3r^{*2}}{16} \right).
\end{aligned}$$

The solution to Eq. 49 is therefore

$$\psi_W^* = \left[C_{-2} \frac{1}{r^{*2}} + C_0 + C_3 r^{*3} + C_5 r^{*5} + \mathrm{Re} \left(\frac{3}{32 r^*} + \frac{9r^*}{32} - \frac{3r^{*2}}{16} \right) \right] \sin^2\theta \cos\theta.$$

Since this solution cannot blow up as $r^* \to \infty$, $C_3 = C_5 = 0$. We can determine C_{-2} and C_0 by applying the no-slip conditions

$$\psi_W^*(r^* = 1, \theta) = \left[C_{-2} + C_0 + \frac{6}{32} \mathrm{Re} \right] \sin^2\theta \cos\theta = 0$$

and
$$\left.\frac{\partial \psi_W^*}{\partial r^*}\right|_{r^*=1} = -2C_{-2} - \frac{6}{32}\mathrm{Re} = 0.$$

The solution to the resulting linear system is

$$C_{-2} = C_0 = -\frac{3}{32}\mathrm{Re}.$$

With this, we have

$$\psi_W(r^*, \theta) = \left(-\frac{3\mathrm{Re}}{32r^{*2}} - \frac{3}{32}\mathrm{Re} + \frac{3\mathrm{Re}}{32r^*} + \frac{9}{32}\mathrm{Re}r^* - \frac{6}{32}\mathrm{Re}r^{*2}\right)\sin^2\theta\cos\theta.$$

or
$$\psi_W(r^*, \theta) = -\frac{3}{32}\mathrm{Re}\left(\frac{1}{r^{*2}} - \frac{1}{r^*} + 1 - 3r^* + 2r^{*2}\right)\sin^2\theta\cos\theta. \tag{52}$$

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