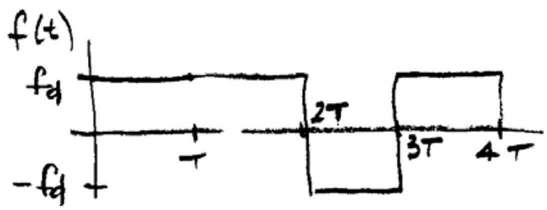
8.2 Start With CPFSK

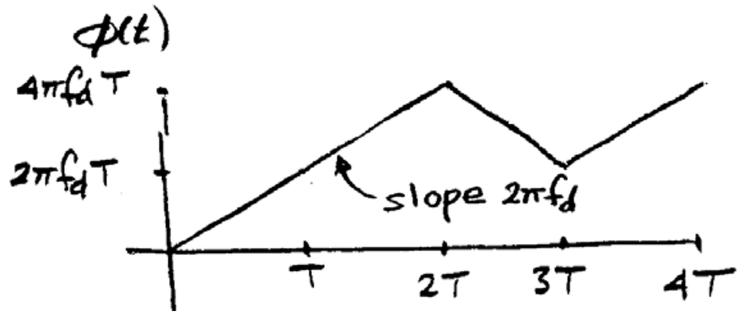
8.2.1 CPFSK Signals

- Three pictures of the same CPFSK signal:

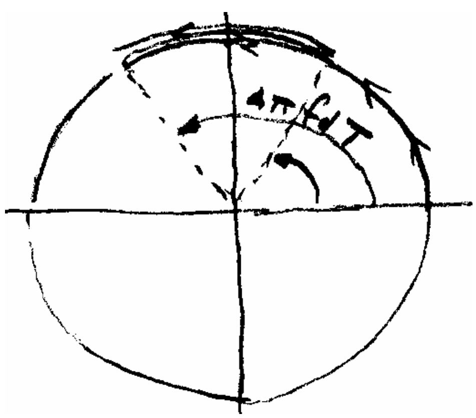
instantaneous frequency



instantaneous phase



trajectory



Trajectory is a circle with radius $A = \sqrt{P}$.

CPFSK moves at a steady speed around the circle, with reversals.

The starting phase of each symbol is determined by past bits. This memory can improve detection by giving additional hints about past symbol values.

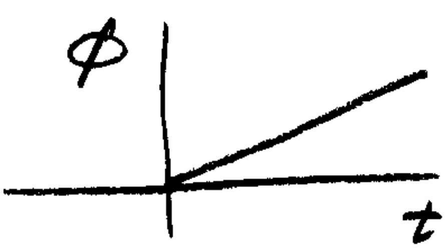
- Relation between bandpass signal and complex baseband:

$$\tilde{s}(t) = \sqrt{2}\,\mathrm{Re}\left[\underbrace{Ae^{j\phi(t)}}_{s(t)} e^{j2\pi f_c t}\right] = \sqrt{2}\,A\cos\left(2\pi f_c t + \phi(t)\right)$$

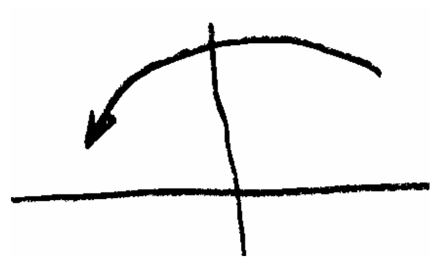
  - *Instantaneous* values:
    - total phase: $2\pi f_c t + \phi(t)$
    - frequency (rad/s): $2\pi f_c + \dot{\phi}(t) = 2\pi f_c \pm 2\pi f_d$
    - frequency (Hz): $f_c \pm f_d$

  - To shift frequency up or shift frequency down:

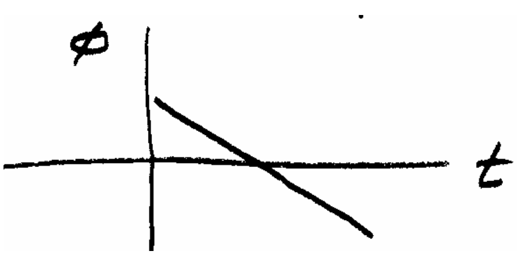
phase increases linearly



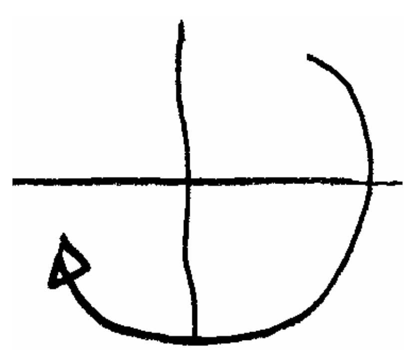
$s(t)$ rotates in positive direction, constant speed



or phase decreases linearly

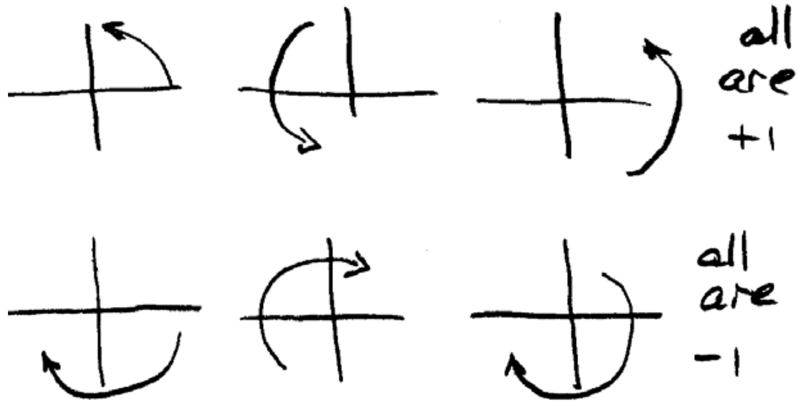


or $s(t)$ rotates in negative direction, constant speed



- PSK carries information in phase, FSK in the direction of travel:

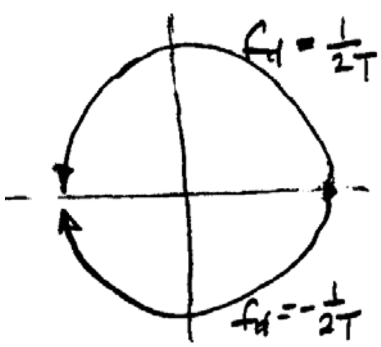
PSK FSK



- The product $f_d T$ has a strong effect on power spectrum:

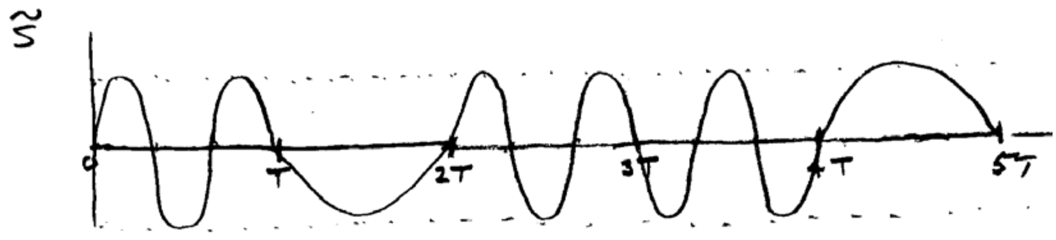
    - Generally, a greater max frequency deviation f_d produces a wider spectrum.

    - Multiples of 1/2 create discrete spectral components (tones). Consider the value $f_d T = \frac{1}{2}\text{cycle}$:



One half-cycle (half-circle) forward or backward in each bit. Start on the same phase, end on the same phase. No phase memory.

- Continuing with the $f_d T = \frac{1}{2}\text{cycle}$ example, suppose the carrier is $f_c = 1/T$ (for illustration only, f_c is not relevant to presence of tones). Then $f_c - f_d = \frac{1}{2T}$ and $f_c + f_d = \frac{3}{2T}$, and the bandpass signal is



A tuned circuit will ring at $f_c - f_d$ and another one will ring at $f_c + f_d$, hence there are tones in the signal.

- Ringing occurs if the total phase $2\pi f_c t + \phi(t)$ (the cosine argument) is the same at the end of a symbol (i.e., the start of the next symbol), no matter which data value was sent:

$$2\pi\left(f_c + f_d \right)T = 2\pi\left(f_c - f_d \right)T + 2\pi n \ , \quad n \text{ an integer}$$

$$f_d T = \frac{n}{2}$$

So we get tones if $f_d T = \frac{f_d}{R_s} = \frac{n}{2}$ is an integer multiple of $\frac{1}{2}$.

This extends to multilevel (i.e., non-binary) FSK, too.

- Why we don't like tones:

  - They mix in nonlinearities to produce unexpected and unwelcome images.

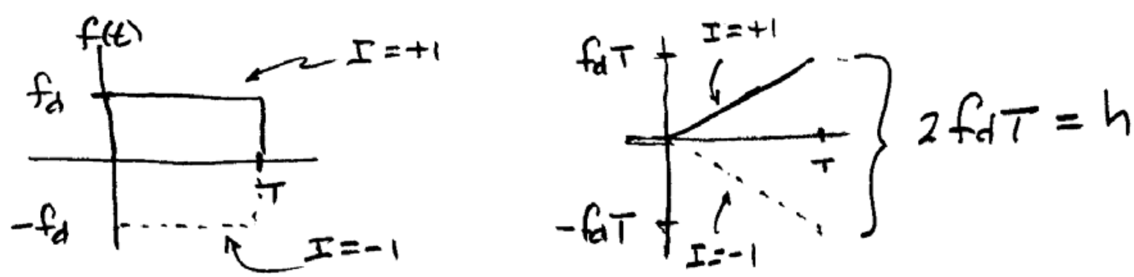
  - They consume power without carrying information.

- Formalize all these observations as expressions defining CPFSK.

  - Use $I_n = 2m - 1 - M$, for $m = 1, \ldots, M$ as the data at symbol time *n*. Odd integers from $-M + 1$ to $M - 1$ (e.g., -5, -3, -1, +1, +3, +5)

  - Alternative possibilities for the symbol (if $M = 2$):

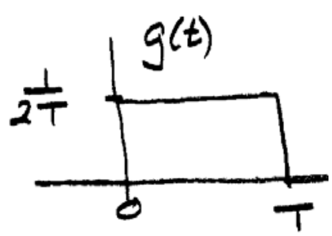
frequency phase



The modulation index *h* is the accumulated phase difference (in *cycles*) between the alternatives over one symbol. It is dimensionless. Integer *h* causes tones.

The modulation index h is related to the spacing Δf of CPFSK signal frequencies and the symbol rate R:

- Waveforms: Instantaneous frequency is a rectangular pulse of height $f_d I$, but we'd like to use h ($f_d = h/2T$). So normalize, by defining a frequency pulse $g(t)$ with area 1/2.



Then instantaneous frequency is $f(t) = h\,I\,g(t) = 2\,f_d T\,I\,g(t)$.

Larger $h \Leftrightarrow$ greater frequency swing.

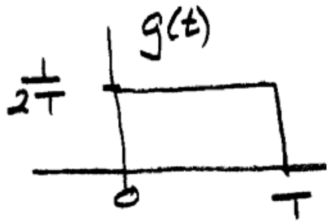
Extending this to a sequence of pulses,

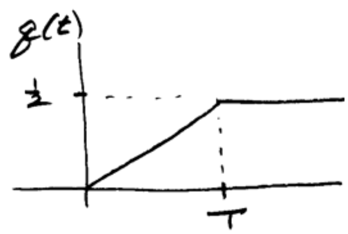
$$f(t) = h \sum_n I_n\, g(t - nT).$$

- Next, the instantaneous phase $\phi(t) = 2\pi \int_{-\infty}^{t} f(\alpha)\,d\alpha$ (radians):

$$\phi(t) = 2\pi h \sum_k I_k\, q(t-kT) = 4\pi f_d T \sum_k I_k\, q(t-kT)$$

where the phase pulse $q(t) = \int_{-\infty}^{t} g(\alpha)\,d\alpha$:





The saturation value of 1/2 means that the phase shift caused by I_n has a continuing effect in the future; i.e., the modulation has memory.

- The usual CPFSK defining equation writes phase in interval n as

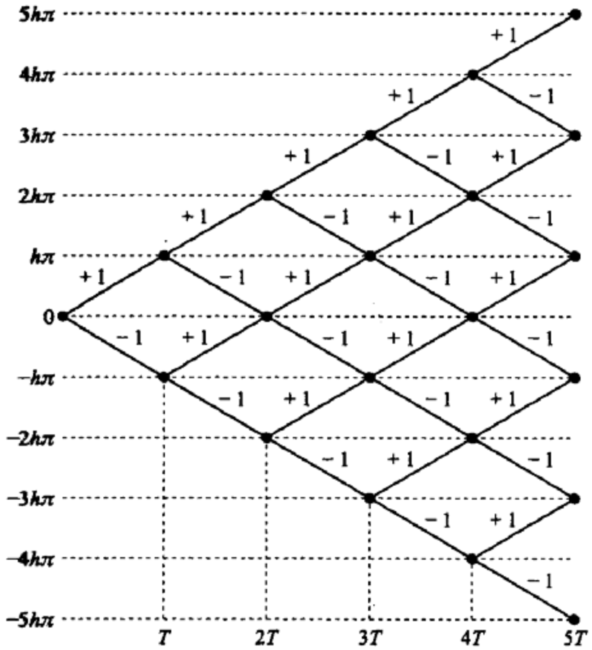
$$\phi(t, \boldsymbol{I}) = 2\pi h \sum_{k=-\infty}^{n} I_k\, q(t-kT) + \phi_o, \quad nT \le t \le (n+1)T$$

where ϕ_o is the phase at the beginning of time (but is zero if the receiver's phase tracking works) and $\boldsymbol{I}$ is the symbol sequence .
But all $q(t-kT)$ up to $k = n-1$ have already saturated at 1/2, so

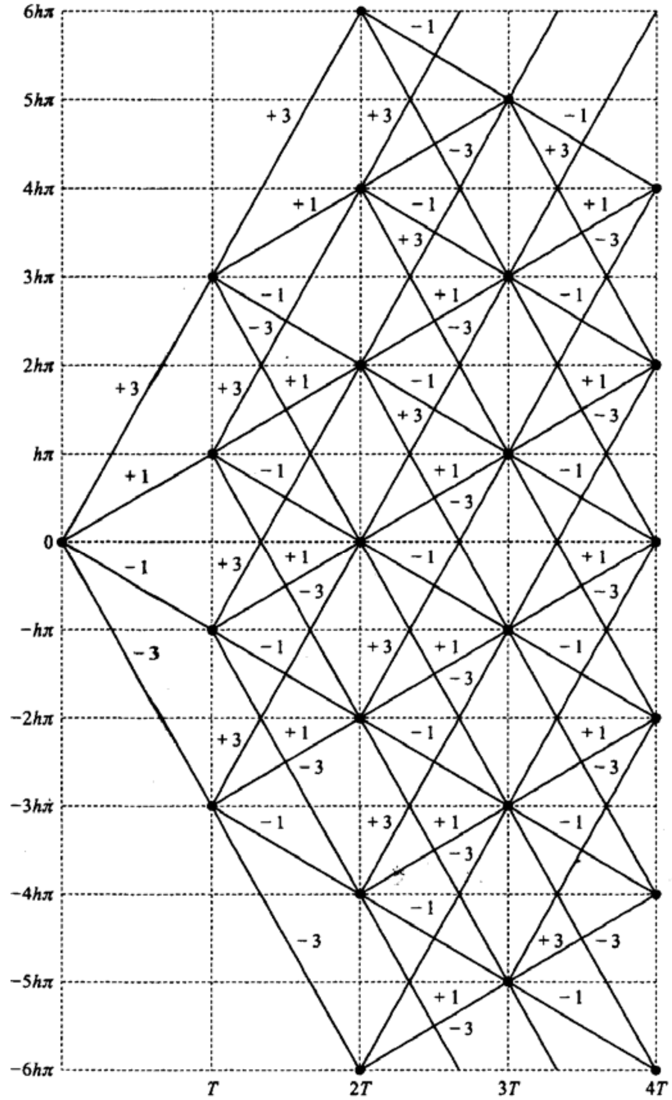
$$\phi(t, \boldsymbol{I}) = \theta_n + 2\pi h \left(\frac{t - nT}{2T} \right) I_n \quad \text{in } nT \le t \le (n+1)T$$

$$\theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k$$

- Examples of phase trajectories:



↑Above: binary ($M = 2$) CPFSK

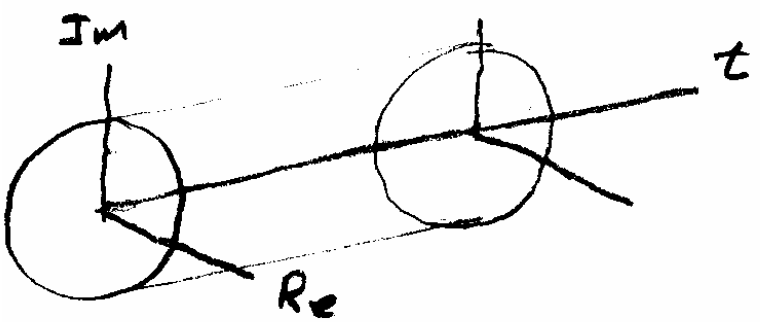


Right: quaternary ($M = 4$) $\rightarrow$ CPFSK

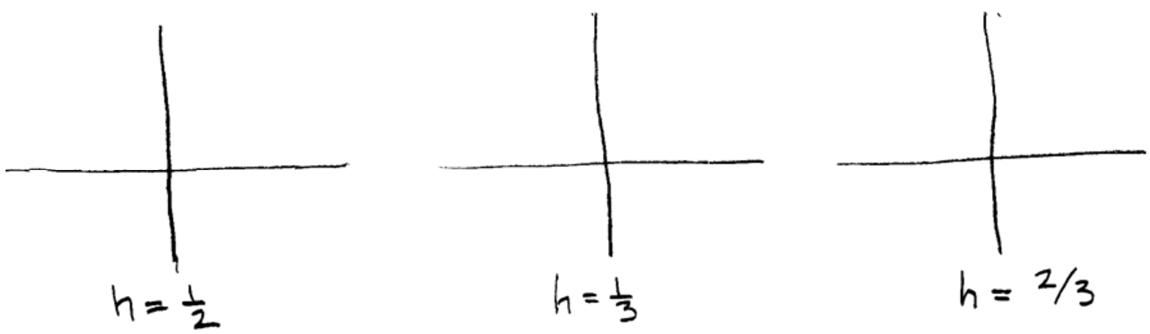
Note phase states.

Looks like a tree, but…

- …the phases are modulo 2π, so the signals wrap around a cylinder in time.



- If $h = 2 f_d T$ is rational, then phase state values recur, and we have a finite set of phase state values.



If $h = m/p$ (and *m*, *p* are coprime), then

- for even *m*, we get p different phase states

$$\theta_n \in \left\{ 0, \frac{\pi m}{p}, 2\frac{\pi m}{p}, \ldots, (p-1)\frac{\pi m}{p} \right\}$$

- for odd *m*, we get $2p$ different phase states, alternating even and odd

$$\theta_n \in \left\{ 0, \frac{\pi m}{p}, 2\frac{\pi m}{p}, \ldots, (2p-1)\frac{\pi m}{p} \right\}$$

- Finally, note that CPFSK is not a linear modulation.

  The phase is linear in the transmitted symbols

$$\phi(t) = 2\pi h \sum_n I_n q(t - nT)$$

but the actual signal is

$$s(t, \boldsymbol{I}) = \sqrt{P}\, e^{j\phi(t, \boldsymbol{I})}$$

8.2.2 CPFSK Signal Space, States and Trellis

- A set of basis functions for CPFSK:

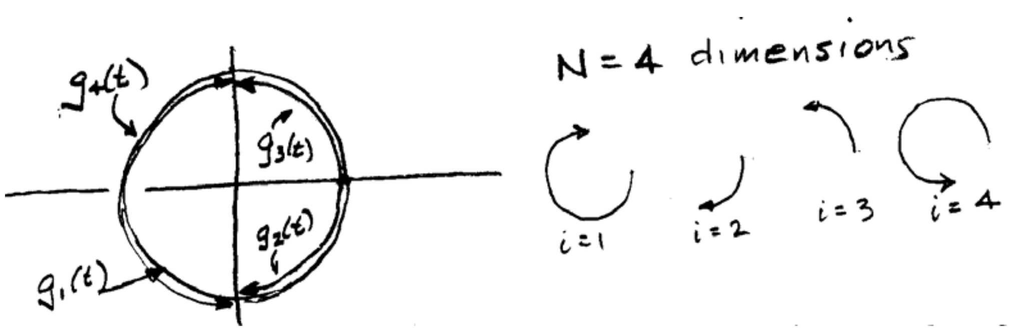
  - In interval *n*, the complex lowpass signal is

    $$s(t) = \sqrt{P}\, e^{j\theta_n} \exp\left[j2\pi h I_n \left(\frac{t - nT}{2T} \right) \right], \quad nT \le t < (n+1)T$$

  - Therefore, a basis set is the time translates of

    $$g_i(t) = \exp\left(j\pi h \left(2i - M - 1 \right) t/T \right),\ 0 \le t < T,\ i = 1, \ldots, M$$

    These basis functions are not orthogonal, except for some choices of *h*, but they are linearly independent. For example, for $M = 4$ and $h = 1/2$:



  - The coefficients $\sqrt{P}\, e^{j\theta_n}$ with respect to this basis carry the memory of past data values, so the current waveform sheds light on the past. Ignore this memory during detection (e.g., by use of a differential detector) and you pay a price of several dB in lost SNR margin.

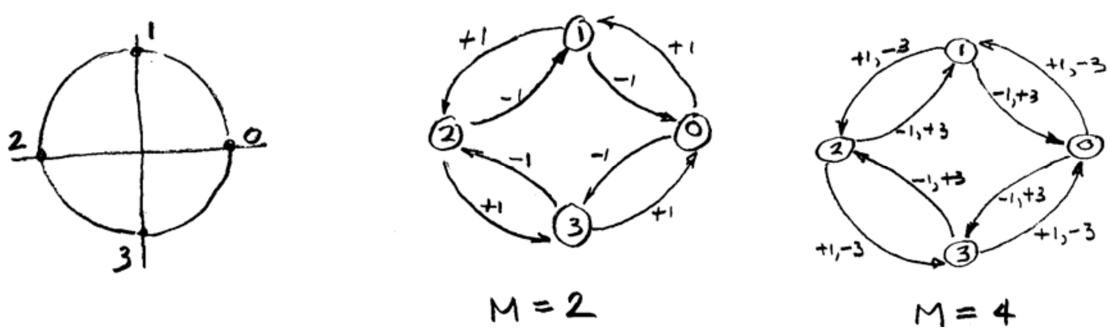
- Implementations either use a set of filters matched to the $g_i(t)$ or they compute those inner products implicitly, as part the metric calculations in tree or trellis search.

- The influence of the past on the response of a system is summarized in its state. Here θ_n acts as the state:

$$\theta_{n+1} = \theta_n + \pi h I_n \qquad \text{state transition equation}$$

$$s(t) = \sqrt{P}\, e^{j\theta_n} \exp\left[j2\pi h I_n \left(\frac{t - nT}{2T} \right) \right] \qquad \text{output equation in interval } n$$

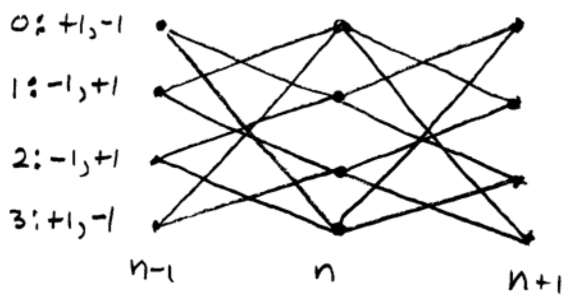
  - If h is rational, then there is a finite set of values for the phase state, and we can summarize the recurrence with a state diagram. For example, for $h = 1/2$ and $M = 2$ or 4,



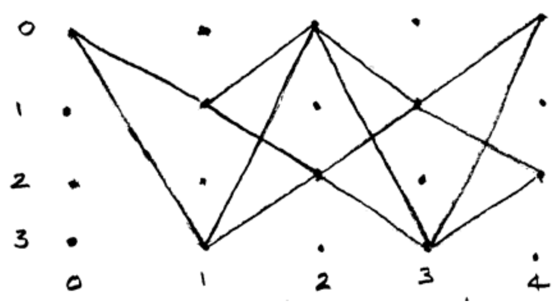
- The state transition diagram can be "unrolled" to form a trellis. Use the same example ($h = 1/2$ and $M = 2$ or 4).

  For $M = 2$ (binary signals):

  compacted

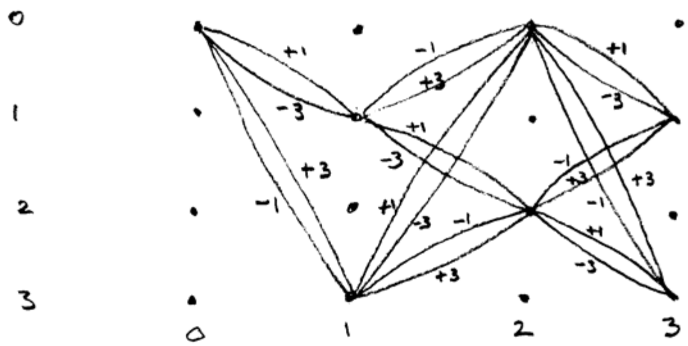


or starting from a known state



alternating subsets, even and odd

and for $M = 4$ (quaternary signals):



- Shown starting from state 0 at time 0.

  Note the parallel transitions.

  - For M=2, the sequence of states σ_n uniquely determines the data sequence I_n.

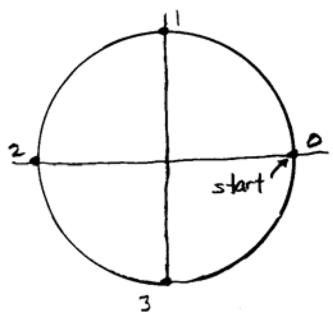
  - For $M = 4$, the parallel transitions mean the state sequence only partially determines the data. Also, which of the two I_n values in a (e.g.) +1, -3 transition leaves no trace in the memory (the state), so no help from context (the future and past) in deciding which it was.

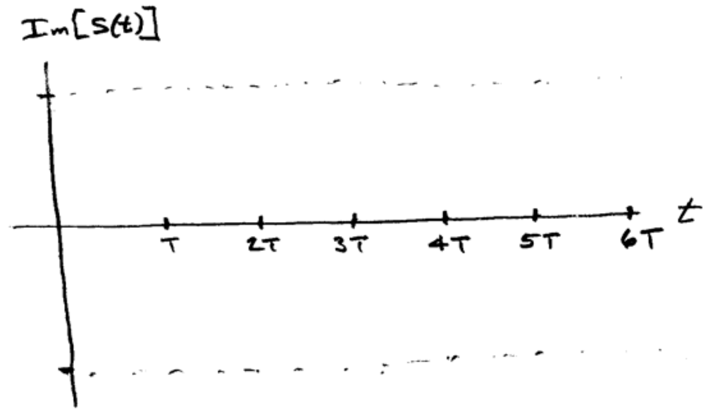
8.2.3 MSK – A Special Case

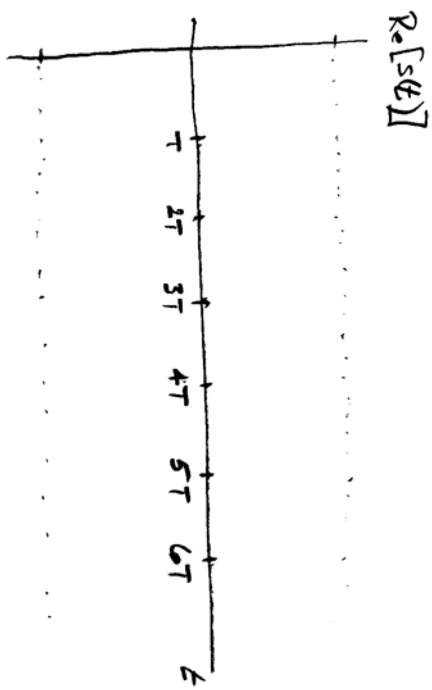
- FSK and QPSK are quite different – or are they?

  For one selection of parameters – MSK – CPFSK is equivalent to pre-coding applied to offset QPSK (OQPSK).

- MSK (minimum shift keying): $M = 2$ and $h = 1/2$.

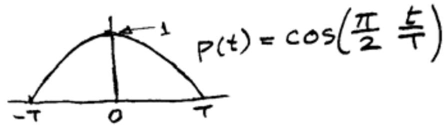






$$\mathrm{Re}\left[s(t)\right] = \sqrt{P} \sum_{n \text{ even}} J_n p(t - nT)$$

$$\mathrm{Im}\left[s(t)\right] = \sqrt{P} \sum_{n \text{ odd}} J_n p(t - nT)$$



where the $J_n \in \{-1, +1\}$, too.

Looks like 2PAM on each axis independently.

- It can also be considered as offset QPSK, in which the imaginary component is delayed by half a symbol. If

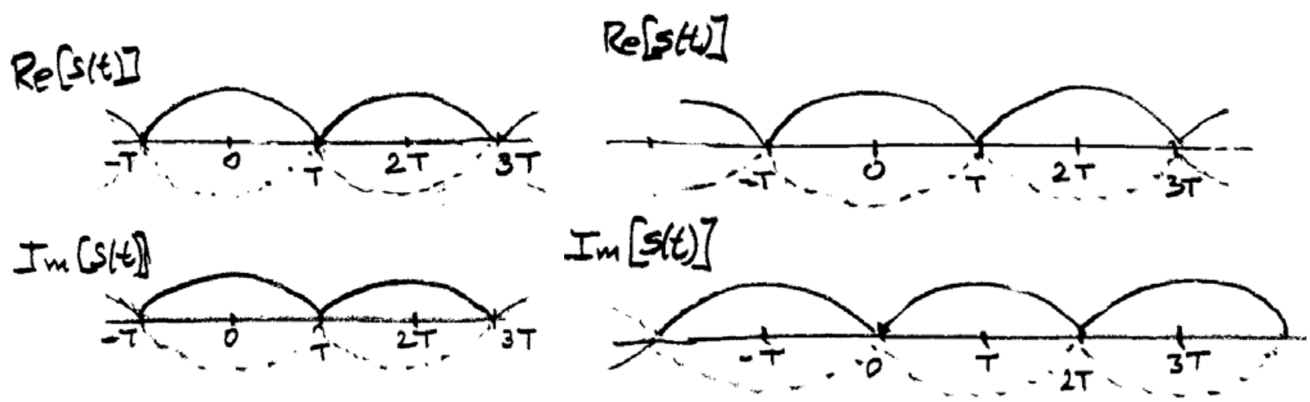
$$s(t)=\sqrt{P}\left(\sum_{n\ \text{even}} J_n p(t-nT)+j\sum_{k\ \text{odd}} J_k p(t-kT)\right)$$

then $s(t)$ is like 4QAM (or QPSK) of symbol duration $2T$

$$\sqrt{P}\sum_{n\ \text{even}} b_n p(t-nT), \quad b_n = J_n + j\,J_{n+1}$$

but with the imaginary part delayed by half a symbol (i.e., by T), so it's *offset* QPSK (OQPSK).

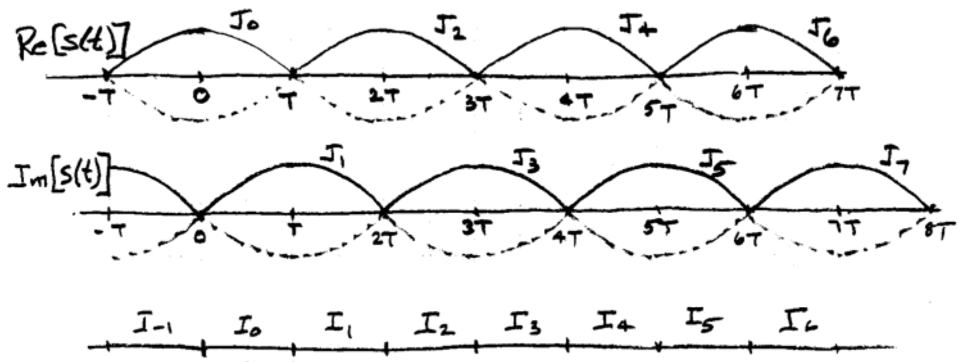
QPSK Offset QPSK

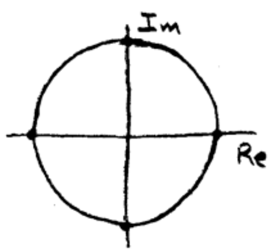


- If the receiver is coherent, it can tell $\mathrm{Re}\left[s(t)\right]$ from $\mathrm{Im}\left[s(t)\right]$, and it's easy to detect the J_n values, as the signs of the matched filter output samples.

But what is the relation between these J_n values and the I_n values of the MSK sequence, the ones we want to detect?

- How to get the I_n values from the J_n values?





Recall that $I_n = +1$ is anticlockwise rotation, $I_n = -1$ is clockwise. So:

- o for even *n*, $I_n = J_n J_{n+1}$;
- o for odd *n*, $I_n = -J_n J_{n+1}$;
- o hence, $I_n = (-1)^n J_n J_{n+1}$.
- o It's differential decoding with an alternating sign flip. Easy.

- From this, the BER follows as $P_b = 2Q\left(\sqrt{2E_b/N_0}\right) = 2Q\left(\sqrt{2PT/N_0}\right)$.

If we had detected I_n using the signal in interval *n* only, it would have been 2FSK with non-orthogonal signals, with the two alternative signals starting from the same unknown phase. See Assignment 2 from 2005 for its BER:

$P_b = Q\left(\sqrt{E_b/N_0}\right)$ (MSK detection without making use of memory)

- The similarity with QPSK also gives us an easy calculation of the MSK power spectrum: they are the same. See Section 8.6.