Last Time:

- Finished Quaternions

Today:

- Rigid Body Dynamics — Hughes Ch. 3, M+C Ch. 3.3

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Rigid Body Dynamics:

- What does "rigid" mean
  - Distances between points in the body are constant
  - Structural natural frequencies >> rigid body frequencies
  - We can get pretty far modeling spacecraft as rigid bodies.

* Conserved Quantities

- Angular Momentum

$$\|h\| = \text{constant}$$

- Kinetic Energy

$$T = \sum_i \frac{1}{2} m_i \, {}^N v_i^T \, {}^N v_i = \sum \frac{1}{2} m_i \left[ {}^N Q^B ({}^B\omega \times {}^B r_i) \right]^T \left[ {}^N Q^B ({}^B\omega \times {}^B r_i) \right]$$

$$= \sum \frac{1}{2} m_i (\omega \times r_i)^T \underbrace{Q^T Q}_{I} (\omega \times r_i)$$

$$= \sum \frac{1}{2} m_i \, {-\omega^T} \hat{r}^T (-\hat{r} \omega) = \sum_i -\frac{1}{2} m_i \, \omega^T \hat{r}_i \hat{r}_i \, \omega$$

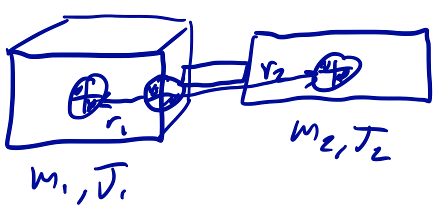
$$= \frac{1}{2} \omega^T \left[ \sum -m_i \hat{r}_i \hat{r}_i \right] \omega = \frac{1}{2} \omega^T J \omega$$

* Moment of Inertia

$${}^{B}J = -\sum_i m_i \hat{r}_i \hat{r}_i = \sum_i m_i (r_i^T r_i I - r_i r_i^T)$$

- Symmetric: $J^T = J$
- Equivalent to a diagonal matrix in some coordinate frame
- Coordinates where it is diagonal called "principle axes"
- Positive-definite (can't have negative energy)
- Sum $\rightarrow$ integral for continuous bodies

* How do we add inertias




$$J = J_1 + J_2 + m_1 (r_1^T r_1 I - r_1 r_1^T) + m_2 (r_2^T r_2 I - r_2 r_2^T)$$

1) Find new COM
2) Add original inertias
3) Add point mass terms from new COM

"Parallel Axis Theorem"

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Angular Momentum

$${}^{N}h = {}^{N}J\,{}^{N}\omega = \text{constant when there are no disturbances}$$

- When there are torques:

$${}^{N}\dot{h} = {}^{N}\tau = {}^{N}J\,{}^{N}\dot{\omega} + \underbrace{{}^{N}\dot{J}}_{\text{this is messy}}\,{}^{N}\omega$$

- From last time:

$$^N\dot{x} = {}^NQ^B\left({}^B\dot{x} + {}^B\omega \times {}^Bx\right)$$

$$\Rightarrow \quad {}^N\dot{h} = {}^NQ^B\left({}^B\dot{h} + {}^B\omega \times {}^Bh\right) = {}^N\tau$$

$$\Rightarrow \quad {}^B\dot{h} + {}^B\omega \times {}^Bh = {}^B\tau$$

Euler's Equation:

$$\boxed{{}^BJ\,{}^B\dot{\omega} + {}^B\omega \times {}^BJ\,{}^B\omega = {}^B\tau}$$

- In Principal Axes:

"Euler's Equations"

$$\begin{cases} {}^PJ_{11}\,{}^P\dot{\omega}_1 + \left({}^PJ_{33} - {}^PJ_{22}\right){}^P\omega_2\,{}^P\omega_3 = {}^P\tau_1 \\ J_{22}\dot{\omega}_2 + \left(J_{11} - J_{33}\right)\omega_1\omega_3 = \tau_2 \\ J_{33}\dot{\omega}_3 + \left(J_{22} - J_{11}\right)\omega_1\omega_2 = \tau_3 \end{cases}$$

* How many Equilibria Does a spinning body Have?

- Let's re-write Euler's equation in terms of $h$:

$$J\dot{\omega} + \omega \times J\omega = 0$$

$$\Rightarrow \quad \dot{h} + (J^{-1}h) \times h = 0$$

$$\Rightarrow \quad \dot{h} = h \times J^{-1}h$$

- Equilibrium $\Rightarrow$ $h \times J^{-1}h = 0$

$\Rightarrow$ $h$ is an eigenvector of $J$

$\Rightarrow$ $h$ is parallel to a principle axis

- 6 equilibria: $\pm p_1$, $\pm p_2$, $\pm p_3$

* Which Equilibria are stable?

- Linearize about a nominal spin in each axis, assume $J_{11} < J_{22} < J_{33}$

$\omega_1 = \omega_0 \gg \omega_2, \omega_3$

$$\dot{\omega}_2 = \underbrace{\omega_0\left(\frac{J_{33}-J_{11}}{J_{22}}\right)}_{\alpha_1 > 0}\omega_3, \quad \dot{\omega}_3 = \underbrace{\omega_0\left(\frac{J_{11}-J_{22}}{J_{33}}\right)}_{\alpha_2 < 0}\omega_2$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix}\omega_2\\ \omega_3\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}0 & \alpha_1\\ \alpha_2 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\omega_2\\ \omega_3\end{bmatrix}}_{x}$$

$$\lambda^2 - \lambda tr(A) + det(A) = 0 \Rightarrow \lambda^2 = -det(A)$$

$$\Rightarrow \lambda = \pm\sqrt{\alpha_1\alpha_2} \Rightarrow \text{pure imaginary}$$

$\boxed{\text{marginally stable}}$

- oscillatory motion is called nutation

$\omega_2 = \omega_0 \gg \omega_1, \omega_3$

$$\dot{\omega}_1 = \underbrace{\omega_0\left(\frac{J_{22}-J_{33}}{J_{11}}\right)}_{\alpha_1 < 0}\omega_3, \quad \dot{\omega}_3 = \underbrace{\omega_0\left(\frac{J_{11}-J_{22}}{J_{33}}\right)}_{\alpha_2 < 0}\omega_1$$

$\lambda = \pm\sqrt{\alpha_1\alpha_2} \Rightarrow$ <u>one positive</u>, one negative

unstable!

$\omega_3 = \omega_0 \gg \omega_1, \omega_2$:

$$\dot{\omega}_1 = \underbrace{\omega_0\left(\frac{J_{22} - J_{33}}{J_{11}}\right)}_{\alpha_1 < 0}\omega_2 \ , \quad \dot{\omega}_2 = \underbrace{\omega_0\left(\frac{J_{33} - J_{11}}{J_{22}}\right)}_{\alpha_2 > 0}\omega_1$$

$\lambda = \pm\sqrt{\alpha_1\alpha_2}$ pure imaginary $\Rightarrow$ marginally stable

$$\boxed{\text{In the absence of energy dissipation, the major and minor axes are stable, while the intermediate axis is unstable.}}$$