Last Time:

- MEKF with Star Tracker
- Environmental Disturbances
- Gravity Gradient Torque

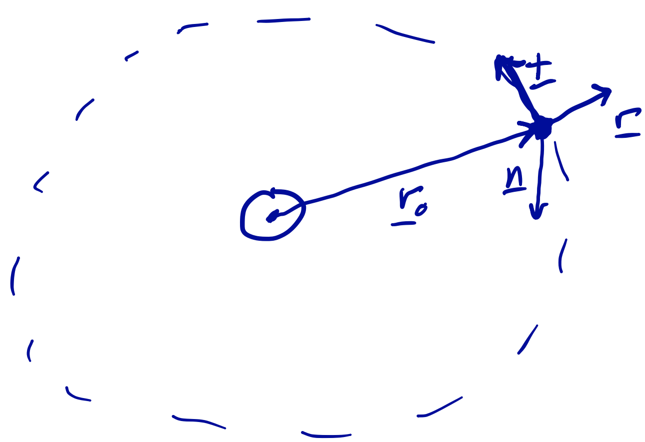
Today:

- Gravity Gradient Stabilization

The Orbital Frame:

- It is often convenient to work in a frame that moves along with the satellite in orbit.

- One common convention is "RTN"



$\underline{r}$ = radial direction

$\underline{t}$ = tangent (parallel to velocity)

$\underline{n}$ = normal (to orbit plane)

$\underline{r} \times \underline{t} = \underline{n}$

- This is NOT an inertial frame. It rotates once per orbit about the $\underline{n}$ vector.

- For a circular orbit the rotation rate is constant (mean motion) $\omega_n = {}^o\omega_3$

- Gravity gradient expression is most convenient in this frame

Gravity Gradient Torque:

- Last time:

$$\underline{\tau}_g = \frac{3\mu}{(\underline{r}_o \cdot \underline{r}_o)^{5/2}} \underline{r}_o \times J \cdot \underline{r}_o$$

- From basic orbit mechanics, mean motion can be written as:

$$\omega_n^2 = \frac{\mu}{\underbrace{a^3}_{\text{semimajor axis}}} = \frac{\mu}{(\underline{r}_o \cdot \underline{r}_o)^{3/2}}$$

$$\Rightarrow \underline{\tau}_g = 3\omega_n^2 \underline{r} \times J \cdot \underline{r}$$

- Note that $\tau_g = 0$ when $\underline{r}$ is parallel to a principal axis

- Gravity gradient can provide a restoring torque to stabilize an Earth-pointing satellite.

Stability with Gravity Gradient:

- Assume satellite is nominally rotating with the orbital frame but is subject to small perturbations.

- Perturbations are small rotations ${}^{O}Q^{B} \approx I + \hat{\phi}$

$$\Rightarrow {}^{B}\omega = {}^{B}Q^{O} \begin{bmatrix} 0 \\ 0 \\ \omega_n \end{bmatrix} + \dot{\phi} = (I - \hat{\phi}) \begin{bmatrix} 0 \\ 0 \\ \omega_n \end{bmatrix} + \dot{\phi}$$

$$= \boxed{{}^{B}\omega = \begin{bmatrix} \dot{\phi}_1 - \phi_2 \omega_n \\ \dot{\phi}_2 + \phi_1 \omega_n \\ \dot{\phi}_3 + \omega_n \end{bmatrix}}$$

- Gravity Gradient Torque in Body Frame:

$$^{B}\tau_g = 3\omega_n^2\, {}^{B}r \times {}^{B}J\, {}^{B}r = 3\omega_n^2 \left({}^{B}Q^{O} \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{{}^{O}r}\right) \times {}^{B}J \left({}^{B}Q^{O}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

* $${}^{B}Q^{O}\begin{bmatrix}1\\0\\0\end{bmatrix} \approx (I - \hat{\phi})\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\-\phi_3\\\phi_2\end{bmatrix}$$

* Assume ${}^{B}J$ is diagonal (but not necessarily in same order as ${}^{P}J$)

$$^{B}\tau_g = 3\omega_n^2 \begin{bmatrix}(J_{22}-J_{33})\phi_2\phi_3\\(J_{11}-J_{33})\phi_2\\(J_{11}-J_{22})\phi_3\end{bmatrix}^{B}$$

- Plug into Euler's Equation and Linearize:

$$^{B}J_{11}\,{}^{B}\dot{\omega}_1 + ({}^{B}J_{33} - {}^{B}J_{22})\,{}^{B}\omega_2\,{}^{B}\omega_3 = {}^{B}\tau_1$$

$$\Downarrow$$

$$J_{11}(\ddot{\phi}_1 - \dot{\phi}_2\omega_n) + (J_{33}-J_{22})(\dot{\phi}_2 + \phi_1\omega_n)(\dot{\phi}_3 + \omega_n) = 3\omega_n^2(J_{22}-J_{33})\phi_2\phi_3$$

$$\Downarrow$$

$$\boxed{J_{11}\ddot{\phi}_1 + (J_{33}-J_{22}-J_{11})\omega_n\dot{\phi}_2 + (J_{33}-J_{22})\omega_n^2\phi_1 \approx 0}$$

$$J_{22}\dot{\omega}_2 + (J_{11}-J_{33})\omega_1\omega_3 = \tau_2$$

$$\Downarrow$$

$$J_{22}(\ddot{\phi}_2 + \dot{\phi}_1\omega_n) + (J_{11}-J_{33})(\dot{\phi}_1 - \phi_2\omega_n)(\dot{\phi}_3 + \omega_n) = 3\omega_n^2(J_{11}-J_{33})\phi_2$$

$$\Downarrow$$

$$\boxed{J_{22}\ddot{\phi}_2 + (J_{11}+J_{22}-J_{33})\omega_n\dot{\phi}_1 + 4\omega_n^2(J_{33}-J_{11})\phi_2 \approx 0}$$

$$J_{33}\dot{\omega}_3 + (J_{22}-J_{11})\omega_1\omega_2 = \tau_3$$

$$\Downarrow$$

$$J_{33}\ddot{\phi}_3 + (J_{22}-J_{11})(\dot{\phi}_1 - \phi_2\omega_n)(\dot{\phi}_2 + \phi_1\omega_n) = 3\omega_n^2(J_{11}-J_{22})\phi_3$$

$$\Downarrow$$

$$\boxed{J_{33}\ddot{\phi}_3 + 3\omega_n^2(J_{22}-J_{11})\phi_3 \approx 0}$$

- Motion about $\underline{n}$ axis is a decoupled simple harmonic oscillator:

$$\ddot{\phi}_3 + \underbrace{\frac{3\omega_n^2(J_{22}-J_{11})}{J_{33}}}_{\omega_3^2}\phi_3 = 0$$

$$\boxed{* \text{ Stability about } \underline{n} \text{ axis} \Rightarrow {}^{B}J_{22} > {}^{B}J_{11}}$$

- Motion about $\underline{r}$ and $\underline{t}$ is coupled and more complicated

* Define $K_R = \dfrac{J_{33}-J_{22}}{J_{11}}$, $K_T = \dfrac{J_{33}-J_{11}}{J_{22}}$

- Using a Laplace Transform:

$$\begin{bmatrix} s^2 + \omega_n^2 K_R & s\omega_n(K_R - 1) \\ s\omega_n(1-K_T) & s^2 + 4\omega_n^2 K_T \end{bmatrix} \begin{bmatrix} \phi_1(s) \\ \phi_2(s) \end{bmatrix} = 0$$

* Stability $\Rightarrow$ $Re(s_i) \leq 0$, $\det(A(s)) = 0$

$$\det(A) = s^4 + s^2\omega_n^2(1 + 3K_T + K_R K_T) + 4\omega_n^4 K_R K_T = 0$$

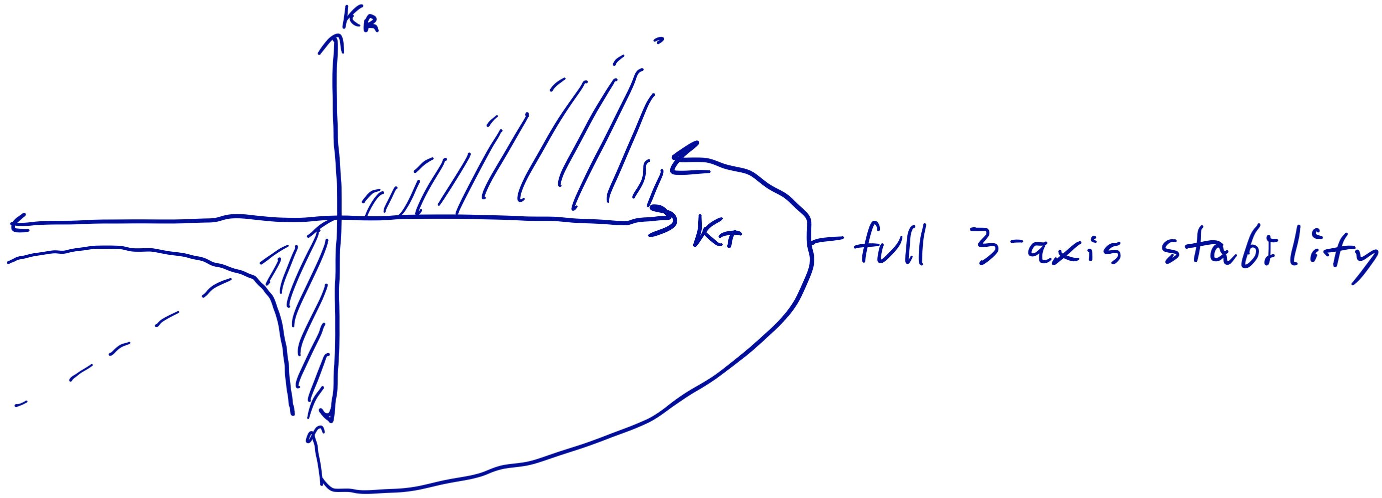
$$s^2 = \frac{-\omega_n^2(1 + 3K_T + K_R K_T) \pm \sqrt{\omega_n^4(1+3K_T+K_R K_T)^2 - 16\omega_n^4 K_R K_T}}{2}$$

* Best we can do is $s^2 \le 0$ $\Rightarrow$ s is pure imaginary

$$\Rightarrow \begin{cases} 1 + 3K_T + K_R K_T > 4 K_R K_T \\ K_R K_T > 0 \end{cases}$$

- Combine all stability conditions:

$$J_{22} > J_{11} \Rightarrow K_T > K_R$$



Take Away:

- It is possible to 3-axis stabilize an Earth pointing satellite by carefully shaping mass properties

- This is a very simple, passive, low-cost method of attitude stabilization that can work well for Earth-observation satellites.