Last Time:

Vectors, Reference Frames, basis, components

Today:

- Rotation Matrices
- A little group theory
- Quaternions

1) Shuster 1993
2) M+C Ch. 3
3) Hughes Ch. 2

"There is more to life than Vector spaces"

Rotation Matrices:

- Transform from body to inertial frame:

$$^N v = {}^N Q^B \, {}^B v$$

$$\underline{v} = \begin{bmatrix} \underline{n}_1 \\ \underline{n}_2 \\ \underline{n}_3 \end{bmatrix}^T \begin{bmatrix} ^N v_1 \\ ^N v_2 \\ ^N v_3 \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{bmatrix}^T \begin{bmatrix} ^B v_1 \\ ^B v_2 \\ ^B v_3 \end{bmatrix}$$

$$^N v = \begin{bmatrix} \underline{n}_1 \cdot \underline{v} \\ \underline{n}_2 \cdot \underline{v} \\ \underline{n}_3 \cdot \underline{v} \end{bmatrix} = \underline{n} \cdot \underline{v} = \underline{n} \cdot (\underline{b}^T \, {}^B v) = \underbrace{(\underline{n} \cdot \underline{b}^T)}_{^N Q^B} {}^B v$$

$$Q = \begin{bmatrix} \underline{n}_1 \\ \underline{n}_2 \\ \underline{n}_3 \end{bmatrix} \cdot \begin{bmatrix} \underline{b}_1 & \underline{b}_2 & \underline{b}_3 \end{bmatrix} = \begin{bmatrix} \underline{n}_1 \cdot \underline{b}_1 & \underline{n}_1 \cdot \underline{b}_2 & \underline{n}_1 \cdot \underline{b}_3 \\ \underline{n}_2 \cdot \underline{b}_1 & \underline{n}_2 \cdot \underline{b}_2 & \underline{n}_2 \cdot \underline{b}_3 \\ \underline{n}_3 \cdot \underline{b}_1 & \underline{n}_3 \cdot \underline{b}_2 & \underline{n}_3 \cdot \underline{b}_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} ^N b_1 & ^N b_2 & ^N b_3 \end{bmatrix}}_{\text{body basis in inertial components}} = \left.\begin{bmatrix} ^B n_1^T \\ ^B n_2^T \\ ^B n_3^T \end{bmatrix}\right\} \text{inertial basis in body components}$$

* What is the inverse of Q?

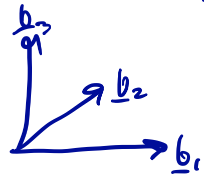
$$Q^T Q = \begin{bmatrix} \tilde{b}_1^T \\ \tilde{b}_2^T \\ \tilde{b}_3^T \end{bmatrix} \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$Q^{-1} Q = I \quad \Rightarrow \quad Q^T = Q^{-1}$$

"orthogonal matrix"

* What is the determinant of Q

- det(Q) measures "stretching" of vectors
- gives volume of parallelpiped formed by columns



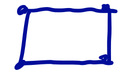
vol = 1

- A negative determinant implies a reflection

A Little Group Theory:

- A group has:

  1) An identity element
  2) An inverse
  3) Closed under multiplication

- Examples:
  - Positive reals
  - Discrete symmetry groups e.g.  C_4
  - $N \times N$ invertible matrices $GL(N)$
  - Rotations $SO(N)$
  - Rigid body motion $SE(3)$

- Continuous groups are called Lie Groups

* The group of 3D rotations is called $SO(3)$

S = "special" $\Rightarrow \det(Q) = 1$

O = "orthogonal" $\Rightarrow Q^T = Q^{-1}$

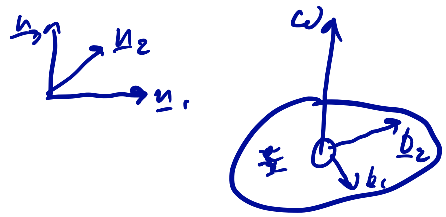
$3 = 3 \times 3$ matrix

Rotation Matrix Kinematics:

* How do we integrate $\omega(t)$ from a gyro?

$$\omega(t) \xrightarrow[?]{} \dot{Q}(t) \longrightarrow Q(t)$$

* Velocities in a rotating reference frame



$${}^N\dot{x} = Q\left({}^B\dot{x} + {}^B\omega \times {}^Bx\right)$$

$${}^B\dot{x} = Q^T\,{}^N\dot{x} - {}^B\omega \times {}^Bx$$

- sometimes called "kinematic transport theorem"
- often not written explicity in components

* Imagine a vector x fixed in the body frame

$${}^Nx = Q\,{}^Bx \Rightarrow {}^N\dot{x} = \dot{Q}\,{}^Bx + \cancel{Q\,{}^B\dot{x}}^{\,0}$$

$${}^N\dot{x} = Q\left({}^B\omega \times {}^Bx\right) = \dot{Q}\,{}^Bx$$

$$\Rightarrow \dot{Q} = Q\,{}^B\hat{\omega}, \quad \underbrace{\hat{\omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}}_{\text{skew symmetric}}$$

$$\underbrace{\hat{\omega}}_{\text{"hat opperator"}} v = \omega \times v$$

skew symmetric $\Rightarrow \hat{\omega}^T = -\hat{\omega}$

- Linear 1st order differential equation

$$\dot{Q} = Q\hat{\omega} \quad \longleftrightarrow \quad \dot{x} = Ax \;\Rightarrow\; x(t) = e^{At}x_0$$

- for constant ω

$$Q(t) = Q\underbrace{e^{\hat{\omega}t}}_{\text{matrix exponential}}$$

- ωt is an Axis-angle vector ϕ

$$Q = e^{\hat{\phi}} \approx I + \hat{\phi}$$

- Exponential gives mapping from axis-angle vectors to rotation matrices

- Useful for generating rotation matrices easily in Matlab (expm)

- You can also go the other way with logm

- Axis-angle vectors / skew-symmetric matrices are the Lie Algebra so(3)

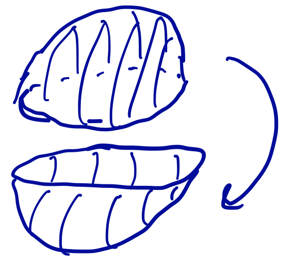
Quaternions:

- main advantage is in dynamics + numerical simulation

* Geometry:
  - Set of all possible axis-angle vectors $-\pi < \|\phi\| \leq \pi$
  - Visualize this as a disk in 2D:

  - Theres a discontinuous jump when we get to $\pm\pi$

- We want to get rid of the jump
- Stretch disk up out of the plane into a hemispher



- Make a copy, rotate it and glue it on underneath forming a sphere
- Now instead of jumping, we can continue smoothly onto the "southern hemisphere"
- Points on the unit sphere in 4D are given by:

$$q = \begin{bmatrix} r \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix}, \quad \left.\begin{array}{l} r = \text{Unit vector axis } (\mathbb{R}^3) \\ \theta = \text{angle of rotation} \end{array}\right\} r\theta = \phi$$

$$= \begin{bmatrix} V \\ s \end{bmatrix}, \quad \begin{array}{l} V = \text{"vector part"} \\ s = \text{"scalar part"} \end{array}$$