Last Time:

- TRIAD
- Wahba's Problem
- Convex Optimization Solution
- SVD Solution

Today

- Davenport q-Method

* M+C Ch. 5

* Bring laptop w/Matlab

* No class 4/30 (Berkeley Space Symposium)

---

q-Method:

- Wahba's Cost Function:

$$L(Q) = \sum_i w_i ({}^N r_i - Q\,{}^B r_i)^T ({}^N r_i - Q\,{}^B r_i)$$

- We expand this and eliminate constant terms to get:

$$L(Q) = -\sum 2 w_i ({}^N r_i^T Q\,{}^B r_i)$$

---

* More on the Trace trick

$$x^T y = \sum_i x_i y_i \qquad y x^T = \begin{bmatrix} y_1 x_1 & y_1 x_2 & \cdots & y_1 x_N \\ y_2 x_1 & y_2 x_2 & & \vdots \\ \vdots & & \ddots & \\ y_N x_1 & & & y_N x_N \end{bmatrix}$$

$$\Rightarrow \quad x^T y = \mathrm{Tr}(y x^T) = \mathrm{Tr}(x y^T)$$

- define $z^T = x^T A$

$$x^T A y = z^T y = \mathrm{Tr}(y z^T) = \mathrm{Tr}(y x^T A)$$

- Using the trace trick, we get:

$$L(Q) = \sum_i -w_i \left({}^N r_i^T \, Q \, {}^B r_i\right) = -\mathrm{Tr}\Big[\underbrace{\sum_i \left(w_i \, {}^B r_i \, {}^N r_i^T\right)}_{B = \text{"attitude profile matrix"}} Q\Big]$$

$$= \underbrace{-\mathrm{Tr}(BQ)}_{\text{linear function of } Q}$$

- Conversion from quaternion to Rotation Matrix

$$Q = \underbrace{I + 2\hat{v}(sI + \hat{v})}_{\text{This is a quadratic function of } q}$$

- Since $L(Q)$ is linear and $Q(q)$ is quadratic, we can write $L(q)$ as a quadratic function:

$$L(q) = L(Q(q)) = -q^T K q$$

- Now we have to figure out what K looks like...

- Remember the identity:

$$\hat{v}\hat{v} = vv^T - (v^T v) I$$

- Remember that $q^T q = s^2 + v^T v = 1$

$$Q = I + 2\hat{v}(sI + \hat{v}) = (s^2 + v^T v) I + 2s\hat{v} + 2vv^T - 2(v^T v) I$$

$$= (s^2 - v^T v) I + 2s\hat{v} + 2vv^T$$

- Plug $Q(q)$ into $L(Q)$:

$$L(Q) = -\sum_i w_i {}^N r_i^T Q {}^B r_i = -\sum_i w_i {}^N r_i^T \left[(s^2 - v^T v)I + 2s\hat{v} + 2vv^T\right] {}^B r_i$$

- The goal is to factor all of the "q stuff" out to the sides

$$L(q) = -\sum_i s \underbrace{(w_i {}^N r_i^T {}^B r_i)}_{Tr(B)} s - v^T \underbrace{(w_i {}^N r_i^T {}^B r_i)}_{Tr(B)} v + 2v^T \underbrace{(w_i {}^N r_i {}^B r_i^T)}_{B} v$$

$$+ 2s \underbrace{(w_i {}^N r_i^T {}^B \hat{r}_i^T)}_{z^T = \left[\sum_i w_i ({}^B r_i \times {}^N r_i)\right]^T} v$$

$$\Rightarrow L(q) = -\begin{bmatrix} v \\ s \end{bmatrix}^T \underbrace{\left[\begin{array}{c|c} B + B^T - Tr(B)I & z \\ \hline z^T & Tr(B) \end{array}\right]}_{K = \text{"Davenport Matrix"}} \begin{bmatrix} v \\ s \end{bmatrix}$$

---

* More on quadratic forms

  - Any matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix

$$M = \underbrace{V}_{\text{symmetric}} + \underbrace{S}_{\text{skew}}$$

  - The symmetric part of M is:

$$V = \frac{1}{2}(M + M^T)$$

  - The skew-symmetric part of M is:

$$S = \frac{1}{2}(M - M^T)$$

$$\Rightarrow \quad V + S = \tfrac{1}{2}(M + M^T) + \tfrac{1}{2}(M - M^T) = M$$

- In a quadratic form, the skew-symmetric part cancels out and only the symmetric part matters:

$$x^T S x = \tfrac{1}{2}\, x^T (M - M^T) x = \tfrac{1}{2} \underbrace{x^T M x}_{x^T y} - \tfrac{1}{2} \underbrace{x^T M^T x}_{y^T x} = 0$$

- define $y = Mx \Longrightarrow$

$$x^T M x = x^T V x + x^T \cancelto{0}{S} x = \tfrac{1}{2} x^T (M + M^T) x$$

- In terms of the quaternion, Wahba's problem is now:

$$\left.\begin{aligned} \min_q \quad & -q^T K q \\ \text{s.t.} \quad & q^T q = 1 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} \max_q \quad & q^T K q \\ \text{s.t.} \quad & q^T q = 1 \end{aligned}$$

- Since $\|q\| = 1$, we only get to pick its direction

* $q^T K q$ will be maximized when q is parallel to the eigenvector corresponding to the maximum eigenvalue of K

---

q-Method Algorithm:

1) Form $B = \sum_i w_i \, {}^B r_i \, {}^N r_i^T$ and $z = \sum_i w_i ({}^B r_i \times {}^N r_i)$

2) Construct $K = \left[\begin{array}{c|c} B + B^T - Tr(B) I & z \\ \hline z^T & Tr(B) \end{array}\right]$

3) Calculate $\max(\text{eig}(K))$ and corresponding eigenvector

4) Normalize the maximum eigenvector to obtain ${}^N q^B$

- Lots of other methods (QUEST, ESOQ, etc.) are derived from the q-Method by using tricks for solving step 3