Last Time:

- Rotation Matrices
- Groups
- Quaternions Pt. 1

* 1^{st} Assignment:

- Read SMAD ADCS section on Canvas
- Select spacecraft for your project
- Fill out Google Docs Form

By Wednesday 4/11

Today:

- Quaternions Pt. 2
- Rigid Body Dynamics Pt. 1

Quaternions:

$$q = \begin{bmatrix} r\sin(\theta/2) \\ \hline \cos(\theta/2) \end{bmatrix}, \quad \begin{matrix} r \in \mathbb{R}^3 = \text{axis of rotation} \\ \theta = \text{angle of rotation} \end{matrix}$$

$$r\theta = \phi$$

$$q = \begin{bmatrix} v \\ \hline s \end{bmatrix}, \quad \begin{matrix} v = \text{"vector part"} \\ s = \text{"scalar part"} \end{matrix}$$

* There is an Identity quaternion

$$q_I = \begin{bmatrix} r\sin(0) \\ \cos(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

* Quaternions have Inverses:

$$q^{-1} = \begin{bmatrix} r\sin(-\theta/2) \\ \cos(-\theta/2 \end{bmatrix} = \begin{bmatrix} -v \\ s \end{bmatrix} = q^{+} \quad \text{"conjugate"}$$

* There are 2 quaternions corresponding to every 3D rotation

$$\begin{bmatrix} r \sin\left(\frac{\theta + 2\pi}{2}\right) \\ \cos\left(\frac{\theta+2\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} -r \sin(\theta/2) \\ -\cos(\theta/2) \end{bmatrix} = -q$$

- quaternions "double cover" $SO(3)$

* Quaternions can be multiplied:

$$q_1 q_2 = \left[\begin{array}{c} s_1 v_2 + s_2 v_1 + v_1 \times v_2 \\ \hline s_1 s_2 - v_1^T v_2 \end{array}\right] = \left[\begin{array}{c|c} s_1 I + \hat{v}_1 & v_1 \\ \hline -v_1^T & s_1 \end{array}\right] \left[\begin{array}{c} v_2 \\ \hline s_2 \end{array}\right]$$

$$= \left[\begin{array}{c|c} s_2 I - \hat{v}_2 & v_2 \\ \hline -v_2^T & s_2 \end{array}\right] \left[\begin{array}{c} v_1 \\ \hline s_1 \end{array}\right]$$

* To rotate a vector:

$$\begin{bmatrix} {}^N x \\ 0 \end{bmatrix} = q \begin{bmatrix} {}^B x \\ 0 \end{bmatrix} q^+$$

* Quaternions work just like rotation matrices:

$$Q_3 = Q_2 Q_1 \longleftrightarrow q_3 = q_2 q_1$$

$$Q_1 = Q_2^T Q_3 \longleftrightarrow q_1 = q_2^+ q_3$$

$${}^N\hat{x} = Q\,{}^B\hat{x}\,Q^T \longleftrightarrow \begin{bmatrix} {}^N x \\ 0 \end{bmatrix} = q \begin{bmatrix} {}^B x \\ 0 \end{bmatrix} q^+$$

- Notice hat map $\longleftrightarrow$ "pure vector" quaternion

Quaternion Kinematics:

$$q_2 = q_1\,\delta q = q_1\begin{bmatrix} r\sin(\delta\theta/2) \\ \cos(\delta\theta/2)\end{bmatrix} \approx q_1\begin{bmatrix} r\,\delta\theta/2 \\ 1\end{bmatrix} = q_1 + q_1\begin{bmatrix} r\,\delta\theta/2 \\ 0\end{bmatrix}$$

$$\dot q = \frac{q_2 - q_1}{\delta t} = \frac{q_1\begin{bmatrix} r\,\delta\theta/2 \\ 0\end{bmatrix}}{\delta t} = \frac{1}{2} q_1 \begin{bmatrix}\omega \\ 0\end{bmatrix}$$

* Compare to rotation matrix:

$$\dot Q = Q\hat\omega \longleftrightarrow \dot q = \frac{1}{2} q \begin{bmatrix}\omega \\ 0\end{bmatrix}$$

- Factor of ½ comes from double cover

Converting From Quaternion to Rotation Matrix:

- Look at rotation of a vector:

$$q\hat x q^{+} = \left[\begin{array}{c|c} sI + \hat v & -v \\ \hline v^T & s\end{array}\right]\left[\begin{array}{c|c} sI + \hat v & v \\ \hline -v^T & s\end{array}\right]\left[\begin{array}{c} x \\ \hline 0\end{array}\right]$$

$$= \left[\begin{array}{c|c} (sI+\hat v)(sI+\hat v) + vv^T & 0 \\ \hline 0 & v^Tv + s^2\end{array}\right]\left[\begin{array}{c} x \\ \hline 0\end{array}\right]$$

$$\Rightarrow Q = (sI+\hat v)(sI+\hat v) + vv^T = s^2 I + 2s\hat v + \hat v\hat v + vv^T$$

- A different version of this expression is more common:
- note $\hat v\hat v = vv^T - (v^Tv)I \Rightarrow vv^T = \hat v\hat v + (v^Tv)I$
$$= \hat v\hat v + (1-s^2)I$$

$$\Rightarrow \boxed{Q = I + 2\hat v(sI + \hat v)}$$

- To go the other direction, invert this expression. It's messy. See Shepperd 1978 for details.