CATEGORY THEORY

ALEX SIMPSON

TSE-YU SU

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1 CATEGORY

1.1 *Definition*

A **Category** $\mathcal{C}$ is given by:

- A collection $|\mathcal{C}|$ or $\mathrm{obj}(\mathcal{C})$ of **objects**.

- For every $X, Y \in |\mathcal{C}|$, we have a collection $\mathcal{C}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of **morphisms** from X to Y.

- For $X \in |\mathcal{C}|$, we have $1_X \in \mathcal{C}(X, X)$, called the **identity morphism** on X.

- For any $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, we have a **composite morphism** $g \circ f \in \mathcal{C}(X, Z)$.

- These compositions must satisfy:

    - (**Identity law**) $1_Y \circ f = f = f \circ 1_X$.
    - (**Associative law**) $h \circ (g \circ f) = (h \circ g) \circ f$, for any $h \in \mathcal{C}(Z, W)$.

1.2 *Examples*

- **Set**: The category of sets.

    - Objects |**Set**|: The "collection"(can't talk about "set of all sets") of all sets.
    - Morphisms **Set**(X, Y): The set of all functions from X to Y.
    - Identities 1_X: The identity function on X.
    - Compositions: Compositions of functions.

    We say a category $\mathcal{C}$ is **locally small** if $\mathcal{C}(X, Y)$ is a set for all X, Y; moreover we say $\mathcal{C}$ is **small** if the collection of objects $|\mathcal{C}|$ is a set.

    Set is locally small but not small.

- **Top**: The category of topological spaces(and continuous functions).

    - Objects |**Top**|: The collection of all topological spaces.
    - Morphisms **Top**(X, Y): The set of all continuous functions from X to Y.

    And the obvious identities and associative law. Again, **Top** is locally small but not small.

- **Grp**: The category of groups with group homomorphism as morphisms.

- **Vect**$_K$: The category of vectors spaces over **K**, with the **K**-linear transformations as morphisms.

- **Rel**: The category of sets, with all binary relations as morphisms.

    (A binary relation between X, Y is a function $R : X \times Y \rightarrow \{\text{True}, \text{False}\}$. If $R(x, y) = $ True, we say $x R y$)

- Objects |**Rel**|: The collection of all sets.

- Morphisms **Rel**(X, Y): The set of all relations between X and Y.

- Identities $1_X : (a, b) \mapsto \begin{cases} \text{True} & \text{if } a = b \\ \text{False} & \text{if } a \neq b \end{cases}$

- Compositions: Define the "composition of two relations R, S" by

$$(R; S) : X \times Z \to \{\text{True,False}\}$$

$$(x, z) \mapsto \begin{cases} \text{True} & \text{if exists } y \text{ such that } xRy \text{ and } ySz \\ \text{False} & \text{otherwise} \end{cases}$$

All of above categories are locally small but not small, not will give some examples of small categories.

- **G**: Viewing a group G as a category.

  - Objects: Has only one objects $*$.

  - Morphisms: $\mathbf{G}(*, *) = G$.

  - Identities: $1_* = e$, the unity element of group G.

  - Compositions: Compositions as elements in group G.

In fact, one can do the same construction for monoids, monoids are equivalent to categories with only one object.

- **$\underline{P}$**: The poset P as a category.

  - Objects: $|\underline{P}| = P$.

  - Morphisms: $\underline{P}(x, y) = \begin{cases} \{x \leq y\} & \text{if } x \leq y \text{ (The first "}x \leq y\text{" here is a morphism)} \\ \varnothing & \text{otherwise} \end{cases}$

Again, we don't need all the axioms of poset here to construct a category, we never use the anti-symmetry axiom ($x \leq y$ and $y \leq x$ implies $x = y$). More generally, we can define categories for any **preorder set**, which is equivalent to category with a set as objects, and has at most one morphism for each homset, and poset are preorder set which the only isomorphisms are identities.

1.3 *Isomorphism, monomorphisms, epimorphisms*

There are three special kinds of morphisms, **isomorphisms, monomorphisms** and **epimorphisms**.

A morphism $X \xrightarrow{f} Y$ is said to be an **isomorphism** if there exists $Y \xrightarrow{f^{-1}} X$ such that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$.

f is said to be a **monomorphism** if for any $g, h \in \text{Hom}(Z, X)$, $f \circ g = f \circ h$ implies $f = g$.

Every isomorphisms is also a monomorphism(and epimorphism).

Categories	Isomorphisms	Monomorphisms	Epimorphisms
Set	bijections	injections	surjections
Top	homeomorphisms	injective continuous maps	surjective continuous maps
Grp	group isomorphisms	group monomorphisms	group epimorphisms
Vect$_K$	$\mathbb{K}$ − linear iso	inj $\mathbb{K}$ − linear transf	surj $\mathbb{K}$ − linear transf
G(a group)	all maps	all maps	all maps
P(a poset)	identities	all maps	all maps

Note that a morphism which is both a monomorphism and a epimorphism does not necessarily imply that it is an isomorphism. For example, in category of rings **Rng**, the embedding $\mathbb{Z} \to \mathbb{Q}$ is mono and epi(since $\mathbb{Q}$ is the fraction ring of $\mathbb{Z}$, and any ring homomorphism from it is unique determined by the image of $\mathbb{Z}$, hence determined by image of 1), but it has no inverse, so is not iso.

2 FUNCTOR

2.1 *Definition*

A **functor** F is a "transformation" between two categories $\mathcal{X}, \mathcal{Y}$, and is given by:

- **Objectation** A mapping of objects $|\mathcal{X}| \to |\mathcal{Y}|$.

- **Morphsim actions** For any $a, b \in |\mathcal{X}|$, and any $f \in \mathrm{Hom}(a, b)$, there is a $\mathrm{F}(f) \in \mathrm{Hom}(\mathrm{F}(a), \mathrm{F}(b))$.

- **Preserving identities** For any $a \in |\mathcal{X}|$, $\mathrm{F}(1_a) = 1_a$.

- **Preserving compositions** For any $a \xrightarrow{f} b \xrightarrow{g} c$, $\mathrm{F}(g \circ f) = \mathrm{F}(g) \circ \mathrm{F}(f)$.

2.2 *Examples*

- **Forgetful functor: Grp$\xrightarrow{\mathrm{F}}$Set**

  F sends each group to the set of group element, and sends group homomorphism ϕ to the same function but as a function between sets.

- **Homomorphism functor between two group categories**

  Since group category has only one object, the functor just need to preserve the morphisms, which is equivalent to preserving group operations, so the functors between 2 groups are just group homomorphisms between them.

- **Cat: Category of categories**

  We can form a category whose objects are categories and morphisms are functor. But this raises some problems, for example, can the category of all categories **Cal** be an object of itself?

We will circumvent these problems by considering **Cal** to be the category of **small** categories.

Exercise: Between 2 small categories, the functors between form a set. (So **Cal** is locally small but now small).

There a 3 types of categories, we have discuss the first and second types:

- Categories whose objects are mathematical structures and morphisms between them are transformations or relations between each two individual structures.

  Set, Grp, Top, Rel, Cat, Vect$_{\mathbb{K}}$.

- Categories which are categorifications of individual mathematical structures.

  G, M, P (category constructed by a single group, monoid, poset, respectively).

- Categories formed by category-theoretic constructions via existing categories.

  $\mathcal{C}^{\mathrm{op}}$: For any category $\mathcal{C}$, we define its opposite category $\mathcal{C}^{\mathrm{op}}$ by: $|\mathcal{C}^{\mathrm{op}}| = |\mathcal{C}|$, and $\mathcal{C}^{\mathrm{op}}(X, Y) = \mathcal{C}(Y, X)$. (same objects, but "reverses" the morphisms)

  An important usage of notion of opposite categories is that we can define the notion of "dual". (later)

  Facts:

  - $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.
  - For $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$, $(g \circ f)^{\mathrm{op}} = f^{\mathrm{op}} \circ g^{\mathrm{op}}$.
  - Functors $\mathcal{C} \xrightarrow{\mathrm{F}} \mathcal{D}$ and functors $\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{F}^{opp}} \mathcal{D}^{\mathrm{op}}$ are in one-to-one correspondence with the obvious way.

2.3 *Contravariant Functors*

An important reason for considering opposite categories is that it's extremely common that we need to consider functors from an opposite category to a non-opposite category: $\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{F}} \mathcal{D}$, such functors are called **Contravariant Functors** from $\mathcal{C}$ to $\mathcal{D}$.

 If we want to clarify $\mathcal{C} \xrightarrow{\mathrm{G}} \mathcal{D}$ is not contravariant from $\mathcal{C}$ to $\mathcal{D}$, we say G is a **Covariant Functor** from $\mathcal{C}$ to $\mathcal{D}$.

EXAMPLE. The construction of dual vector space V^* via given vector space V is actually

functorial in the contravariant (on itself) sense.

$$
\begin{array}{ccc}
\mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}} & & \mathbf{Vect}_{\mathbb{K}} \\
V^{\mathrm{op}} & & V^* \\
f^{\mathrm{op}} \uparrow & \xrightarrow{\ (\cdot)^*\ } & \uparrow f^* \\
W^{\mathrm{op}} & & W^*
\end{array}
$$

$$(f^*(w^*))(v) = w^*(f(v))$$

2.4 *Product Category*

The product $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}, \mathcal{D}$ is defined by:

- $|\mathcal{C} \times \mathcal{D}| = |\mathcal{C}| \times |\mathcal{D}|$.

- $(\mathcal{C} \times \mathcal{D})((X, Y), (X', Y')) = \mathcal{C}(X, X') \times \mathcal{D}(Y, Y')$.

Can easily check this define a category.

There are evident **projection functors**:

$$
\begin{aligned}
\pi_1 &: \mathcal{C} \times \mathcal{D} \quad \to \mathcal{C} \\
\pi_2 &: \mathcal{C} \times \mathcal{D} \quad \to \mathcal{D}
\end{aligned}
$$

Can define the product category of finite many categories.

More generally, for index set I, can define the product category $\prod\limits_{i \in \mathrm{I}} \mathcal{C}_i$. It's now clear what is meant by a **multi-argument** functor:

$$\mathcal{C}_1 \times \mathcal{C}_2 \times ... \times \mathcal{C}_n \xrightarrow{\mathrm{F}} \mathcal{D}$$

2.5 *Hom Functor*

The **Hom Functor** for a category $\mathcal{C}$ is:

$$
\begin{aligned}
\mathcal{C}(-, -) \ &: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \quad \longrightarrow \quad \mathbf{Set} \\
&\ (X, Y) \quad \longmapsto \quad \mathcal{C}(X, Y)
\end{aligned}
$$

$$
\begin{array}{ccccc}
& & & & \mathcal{C}(X_1, Y_1) \\
& & & & \\
X_1 & Y_1 & & & \\
f \uparrow & \downarrow g & \xrightarrow{\ \mathcal{C}(-,-)\ } & & \Big\downarrow \mathcal{C}(f, g) \\
X_2 & Y_2 & & & \\
& & & & \\
& & & & \mathcal{C}(X_2, Y_2)
\end{array}
$$

The hom functor $\mathcal{C}(-,-)$ takes tuple (f,g) to a function (between sets) from $\mathcal{C}(X_1,Y_1)$ to $\mathcal{C}(X_2,Y_2)$.

$\mathcal{C}(f,g)$ is defined by simply "join" $\alpha \in \mathcal{C}(X_1,Y_2)$ into the diagram:

$$
\begin{array}{ccc}
X_1 & \overset{\alpha}{\dashrightarrow} & Y_1 \\
f\uparrow & & \downarrow g \\
X_2 & \underset{g\,\circ\,\alpha\,\circ\,f}{\longrightarrow} & Y_2
\end{array}
$$

2.6   *Duality*

When we have a new category concept, we automatically get another category concept by interpreting the original category concept in dual category.

Dual notion of monomorphism:

DEFINITION. $X \overset{f}{\to} Y$ is an **epimorphism (epi)** if:

$$X^{\mathrm{op}} \overset{f^{\mathrm{op}}}{\longrightarrow} Y^{\mathrm{op}}$$

is a monomorphism.

2.7   *Exercise*

1. **(Powerset Functors)**

   (a) Find a **contravariant** functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$, whose object action is $S \mapsto \mathcal{P}S$: the powerset of S.

   (b) Find a **covariant** functor $\mathbf{Set} \to \mathbf{Set}$, whose object action is $S \mapsto \mathcal{P}S$.

   (c) Do Exercise 2 again, find a different one.

2. (a) Does functor $\mathbf{Set} \overset{F}{\to} \mathbf{Set}$ preserve monomorphisms?

   (b) Does functor $\mathbf{Set} \overset{F}{\to} \mathbf{Set}$ preserve epimorphisms?

Answers:

1. (a) Define the "preimage functor" Pre, for $f \in \mathbf{Set}(X,Y)$, and $B \subseteq Y$, $(\mathrm{Pre}(f))(B) = f^{-1}(B) \subseteq X$.

   (b) The "image functor" Im.

   (c) Define the "Unique image functor" U:

   For $A \subseteq X$, $(U(f))(A) = \{b \in Y | \exists! a \in A \text{ s.t. } f(a) = y\}$.

2. (a) Yes.

Since if $X \xrightarrow{f} Y$ is injective, then exists some $Y \xrightarrow{g} X$ s.t. $g \circ f = 1_X$. (g maps image of f to its preimage, and sends other elements in Y to arbitrary element of X)

So for any functor $\mathbf{Set} \xrightarrow{F} \mathbf{Set}$, $F(g \circ f) = F(g) \circ F(f) = 1_X$, which implies $F(f)$ is a monomorphism.

(b) Yes.

Similarly, if $X \xrightarrow{f} Y$ is surjective, then exists $Y \xrightarrow{g} X$ s.t. $f \circ g = 1_Y$.

3 NATURAL TRANSFORMATION

What transformations are **Natural** in a sense?

To answer this question, which lies at the very foundation of category theory, we need notion of **Functor**, and in order to define functor, we need the notion of **Category**.

EXAMPLE. Let V be a finite dimensional vector space over $\mathbf{K}$, then V^* is also a finite dimensional vector space of the same dimension over $\mathbf{K}$, i.e. there exists isomorphism:

$$V \stackrel{\phi}{\cong} V^*$$

But to build an isomorphism ϕ, there's an arbitrary (non-natural) choice involved, we need to choose a basis β for V, and then use β to define a corresponding basis β^* for V^*, and define ϕ via these two basis:

$$\beta = \{v_1, v_2, ...\}, \ \beta^* = \{v_1^*, v_2^*, ...\}$$

$$v_i^* : \sum_i a_i v_i \mapsto a_i$$

$$\phi : \sum_i a_i v_i \mapsto \sum_i a_i v_i^*$$

We also have $V \cong V^{**}$, but in this case we can find an isomorphism independent of choices of basis:

$$v \mapsto v^{**}, v^{**} \text{ maps elements in } V^* \text{ to } \mathbf{K} \text{ by } v^{**}(u^*) := u^*(v)$$

Is there a mathematical definition which characterize this naturality?

3.1 *Definition*

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, a **Natural Transformation** $\alpha : F \Rightarrow G$ is given by a family of components indexed by objects of $\mathcal{C}$:

$$(FX \xrightarrow{\alpha_X} GX)_{X \in |\mathcal{C}|}$$

that satisfies the following **naturality condition:**

$$\forall X \xrightarrow{f} Y \in \mathcal{C}(X, Y), \text{ the following diagram commutes:}$$

$$
\begin{array}{ccc}
FX & \xrightarrow{\alpha_X} & GX \\
Ff \downarrow & & \downarrow Gf \\
FY & \xrightarrow{\alpha_Y} & GY
\end{array}
$$

3.2 *Examples*

- $\epsilon : 1_{\mathbf{Vect_K}} \Rightarrow (\cdot)^{**}$:

$$\epsilon_V : V \longrightarrow V^{**}$$

$$v \longmapsto \left(v^{**} : f \mapsto f(v)\right)$$

- For any set X, we have:

$$\{\cdot\}_X : X \longrightarrow \mathcal{P}X$$

$$x \longmapsto \{x\}$$

So this defines a natural transformation from $1_{\mathbf{Set}}$ to $\mathcal{P}$: the (image) powerset functor.

- We also have:

$$\bigcup_X : \mathcal{P}\mathcal{P}X \longrightarrow \mathcal{P}X$$

$$\{X_\alpha \mid \alpha \in A\} \longmapsto \bigcup_{\alpha \in A} X_\alpha$$

This defines a natural transformation from $\mathcal{P}\mathcal{P}$ to $\mathcal{P}$.

3.3 *Functor category*

Let F, G, H be functors from $\mathcal{C}$ to $\mathcal{D}$, and $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$ be natural transformations.

$$
\mathcal{C} \;
\begin{array}{c}
F \\
\Downarrow \alpha \\
\longrightarrow \\
\Downarrow \beta \quad G \\
H
\end{array}
\; \mathcal{D}
$$

Obviously, there is a composite natural transformation of α, β from F to G.

We therefore have a **Functor Category** $[\mathcal{C}, \mathcal{D}]$, whose objects are functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$, and morphisms are natural transformations.

Size issues: If $\mathcal{C}, \mathcal{D}$ are both large, then $[\mathcal{C}, \mathcal{D}]$ is "very large". However, if $\mathcal{C}$ is small, and $\mathcal{D}$ is locally small, then $[\mathcal{C}, \mathcal{D}]$ is locally small; moreover, if $\mathcal{C}, \mathcal{D}$ are both samll, then$[\mathcal{C}, \mathcal{D}]$ is also small.

The above composition is called **Horizontal** composition. There is also **Vertical** composition, defined as follows:

DEFINITION (Whiskering). Let:

$$\mathcal{B} \xrightarrow{\ F\ } \mathcal{C} \quad \overset{G_1}{\underset{G_2}{\Downarrow \alpha}} \quad \mathcal{D} \xrightarrow{\ H\ } \mathcal{E}$$

Define:

$$H\alpha : HG_1 \Rightarrow HG_2$$

$$(H\alpha)_X = H\alpha_X \text{ for } X \in |\mathcal{C}|.$$

Note: $H\alpha_X$ is the morphism obtained by applying functor H to morphism $\alpha_X \in \mathcal{D}(G_1 X, G_2 X)$ for $X \in |\mathcal{C}|$.

Similarly, there is also:

$$\alpha F : G_1 F \Rightarrow G_2 F$$

$$(\alpha F)_Y = \alpha_{FY} \text{ for } Y \in |\mathcal{B}|.$$

DEFINITION (Horizontal Composition of functors). Let:

$$\mathcal{C} \quad \overset{F_1}{\underset{F_2}{\Downarrow \alpha}} \quad \mathcal{D} \quad \overset{G_1}{\underset{G_2}{\Downarrow \beta}} \quad \mathcal{E}$$

Define:

$$\beta * \alpha : G_1 F_1 \Rightarrow G_2 F_2$$

$$= (\beta F_2) \circ (G_1 \alpha)$$

$$= (G_2 \alpha) \circ (\beta F_1)$$

This above operation gives us that horizontal functor composition actually defines a functor:

$$[\mathcal{D}, \mathcal{E}] \times [\mathcal{C}, \mathcal{D}] \longrightarrow [\mathcal{C}, \mathcal{E}]$$

$$(G, F) \longmapsto G \circ F$$

$$(\alpha, \beta) \longmapsto \beta * \alpha$$

It is useful to know which morphisms(natural transformations) in functor category

$[\mathcal{C}, \mathcal{D}]$ are isomorphisms.

PROPOSITION.

The following are equivalent for $\alpha : F \Rightarrow G$, F, G are functors from $\mathcal{C}$ to $\mathcal{D}$.

- For all $X \in |\mathcal{C}|$, $FX \xrightarrow{\alpha_X} GX$ is an isomorphism in $\mathcal{D}$.

- α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$.

Such natural transformations are called **Natural Isomorphisms**.

Now we will step away from natural transformation and look at another construction of new category via existing category.

3.4 *Slice Category, I-indexed families of sets*

DEFINITION (Slice Category). Given a category $\mathcal{C}$ and an object $I \in |\mathcal{C}|$.

The **Slice Category** $\mathcal{C}/I$ of $\mathcal{C}$ over I is defined as:

- **Objects:** All morphisms of the form $X \xrightarrow{p} I$ in $\mathcal{C}$.

- **Morphisms(from $X \xrightarrow{p} I$ to $Y \xrightarrow{q} I$):** All morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$ such that the diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ f\ \ } & Y \\
& {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\
& I &
\end{array}
$$

- The identities and compositions are the same as in $\mathcal{C}$.

There is also **Co-slice Category** $I/\mathcal{C}$ of $\mathcal{C}$ under I. The objects are all morphisms $I \xrightarrow{\iota} X$ in $\mathcal{C}$, and similarly the morphisms are $X \xrightarrow{f} Y$ such that the corresponding diagram commutes.

Can also define the co-slice category $I/\mathcal{C}$ as $(\mathcal{C}^{\mathrm{op}}/I)^{\mathrm{op}}$.

EXAMPLE ((Slice Category in **Set**)). Let I be a set. Consider **Set**$/I$, an object in **Set**$/I$ is a function $X \xrightarrow{p} I$. We can mark the points in X according to where it maps in I by p.

Now consider $X \xrightarrow{p} I$ and $Y \xrightarrow{q} I$, $X \xrightarrow{f} Y$ makes the diagram commutes iff it preserves shapes:
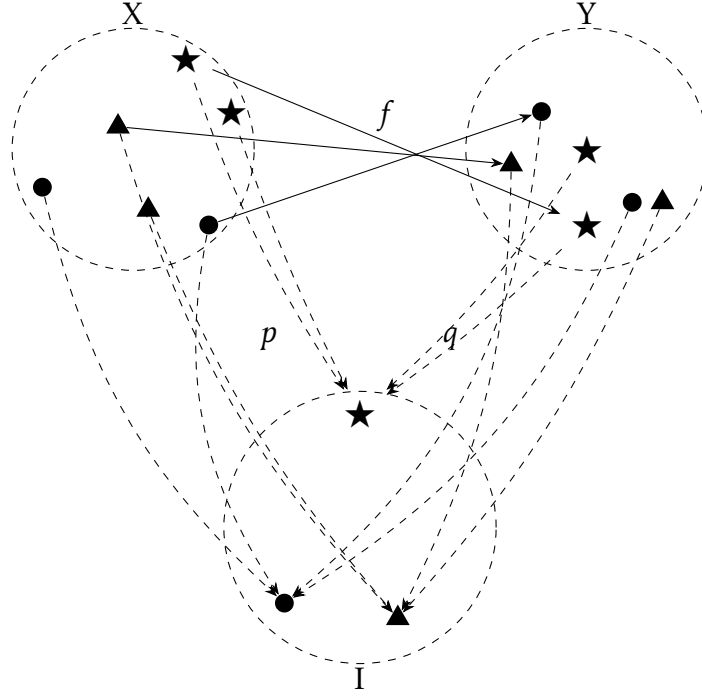
A morphism $f \in \mathbf{Set}/\mathrm{I}(p, q)$ is equivalent to a family of functions:

$$\left(f : p^{-1}(i) \longrightarrow q^{-1}(i)\right)_{i \in \mathrm{I}}$$

The slice category has a very strong connection to another naturally defined category which is the category of high index families of sets, whose morphisms are functions which preserve index.

DEFINITION ((I-indexed families of sets $\mathbf{Fam}_{\mathrm{I}}$)). Let I be a set, we define $\mathbf{I}$ to be the **discrete category** of I, whose objects are elements in I, and the only morphisms are the identities.

Now we define:

$$\mathbf{Fam}_{\mathrm{I}} = [\mathbf{I}, \mathbf{Set}]$$

i.e. the functor category from $\mathbf{I}$ to $\mathbf{Set}$.

Equivalently, the objects of $\mathbf{Fam}_{\mathrm{I}}$ are I-indexed families of set $(X_i)_{i \in \mathrm{1I}}$, and the morphisms $\mathbf{Fam}_{\mathrm{I}}((X_i), (Y_i))$ are equivalent to all families $(X_i \xrightarrow{f_i} Y_i)_{i \in \mathrm{I}}$.

The identities are the I-indexed families of identities on each i, and the compositions are compositions index-wise.

3.5 *Equivalence of Categories*

Now we will show that the slice category is "basically" the same as $\mathbf{Fam}_{\mathrm{I}}$, but one needs to be careful in how to formulate it. For example, in slice category, we can view the object p as for any $i \in \mathrm{I}$, we can maps i to $p^{-1}(i)$, and take this as any object in $\mathbf{Fam}_{\mathrm{I}}$. But in this

case, each $p^{-1}(i)$ is disjoint to each other, where objects in $\mathbf{Fam}_I$ are arbitrary families of sets. What we can do is formulate functors between each of two categories(one for each direction), which show 2 categories are very close related in a way called **Equivalence of Categories**.

EXAMPLE. Define two functors:

$$F : \mathbf{Set}/I \longrightarrow \mathbf{Fam}_I$$

$$(X \xrightarrow{p} I) \longmapsto (p^{-1}(i))_{i \in I}$$

$$S : \mathbf{Fam}_I \longrightarrow \mathbf{Set}/I$$

$$(X_i)_{i \in I} \longmapsto \Sigma_{i \in I} X_i = \{(i, x) \mid x \in X_i, \ i \in I\} \text{ (This is called the disjoint union)}$$

The mapping of morphisms is obvious.

Now we have two natural isomorphisms:

$$\Phi : 1_{\mathbf{Set}/I} \Rightarrow SF$$

$$\Psi : 1_{\mathbf{Fam}_I} \Rightarrow FS$$

Can easily verify there are indeed such natural isomorphisms.

DEFINITION ((Equivalence of Categories)). An equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$ is given by (F, G, α, β) where:

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

$$G : \mathcal{D} \longrightarrow \mathcal{C}$$

$$\alpha : 1_{\mathcal{C}} \Rightarrow SF$$

$$\beta : 1_{\mathcal{D}} \Rightarrow FS$$

Where α, β are natural isomorphisms.

Equivalently, the following diagrams holds:

$$\mathcal{C} \overset{1_{\mathcal{C}}}{\underset{GF}{\rightleftarrows}} \mathcal{C} \quad \alpha \Downarrow \cong \qquad \mathcal{D} \overset{1_{\mathcal{D}}}{\underset{FG}{\rightleftarrows}} \mathcal{D} \quad \beta \Downarrow \cong$$

Notice this is not an isomorphism between categories because FG, GF are not **equall to** the identity functors, instead we get functors naturally isomorphic to identities functors.

There are lots of nice equivalence in mathematics, for example, a famous one is the category of compact Hausdorff space with continuous functions is equivalent to the opposite of the category of commutative C^* algebras.

Now we're going to introduce an example of two categories that's a little bit weaker than a equivalence, but are still very similar. Fist we introduce a new category:

Mat$_K$:

- **Objects:** $\mathbb{N}$: all natural number.

- **Morphisms: Mat$_K$**$(n, m) = M_{m \times n}(\mathbb{K})$: all $m \times n$ matrices with entries in $\mathbb{K}$.

- **Identities & Composition:** Identity matrices and matrices composition.

Normally we name a category with its objects, but in this case the matrices are the morphisms of category.

Matrices play a central role in linear algebra. For example, to represent a transformation between two finite-dimensional vector spaces, we typically begin by choosing suitable bases for each space and then express the transformation as a matrix. There is a strong relationship between the category of matrices and the category of vector spaces.

There is a functor:

$$J : \textbf{Mat}_{\mathbb{K}} \longrightarrow \textbf{Vect}_{\mathbb{K}}$$

$$n \longmapsto \mathbb{K}^n$$

$$(n \overset{A}{\rightarrow} m) \longmapsto T_A, \text{ where } [T_A]_\beta = A, \beta \text{ is the standard basis of } \mathbb{K}^n.$$

This functor J is **full** and **faithful**, with the following definition:

DEFINITION ((Full & Faithful Functor)). A functor $\mathcal{C} \overset{F}{\rightarrow} \mathcal{D}$ is called:

- **full** if for any objects X, Y in $\mathcal{C}$, and for any $g \in \mathcal{D}(FX, FY)$, there exists $f \in \mathcal{C}(X, Y)$ such that $Ff = g$.

- **faithful** if for any morphisms $f \neq g$ in $\mathcal{C}(X, Y)$, $Ff \neq Fg$.

A full and faithful functor is called **fully faithful** or an **embedding** of categories.

Consider **FDVect**$_\mathbb{K}$, the category of finite-dimensional vector space, and **Mat**$_\mathbb{K} \xrightarrow{J}$ **FDVect**$_\mathbb{K}$ is again full and faithful. But in this case J satisfies another property, described as follow:

It's not true that every finite dimensional vector space **equals** to $\mathbb{K}^n$, although they may share the same dimension, they can still be distinct as objects. For example, $\mathbb{K}^n$ and its dual $(\mathbb{K}^n)^*$ are not the same vector space from category perspective, even though they have the same dimension. But what is true is that for any n-dimensional vector space V, there is an isomorphism between V and $\mathbb{K}^n$.

LEMMA. If $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is fully faithful, then F reflects isomorphisms, i.e. for any morphism f, if Ff is an isomorphism, then f is also an isomorphism. Sometimes we called such F is **conservative**.

DEFINITION. We say $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is **essentially surjective on objects** if:

$$\forall Y \in |\mathcal{D}|, \ \exists X \in |\mathcal{C}| \text{ s.t. } FX \cong Y$$

DEFINITION. A functor F is called a **weak equivalence** if F is essential surjective on objects and fully faithful.

THEOREM.

1. If F arises as part of an equivalence (F, G, α, β), then F is a weak equivalence.

2. If F is a weak equivalence, then assuming a suitable version of axiom of choice, F arise as part of an equivalence (F, G, α, β).

In the case of **Mat**$_\mathbb{K} \xrightarrow{J}$ **FDVect**$_\mathbb{K}$, the "choice" is to find a basis for each finite dimensional vector space in **FDVect**$_\mathbb{K}$.

(of 2).

Using the axiom of choice, define G : **FDVect**$_\mathbb{K}$ → **Mat**$_\mathbb{K}$ by choosing, for each Y ∈ |**FDVect**$_\mathbb{K}$|, an object X ∈ |**Mat**$_\mathbb{K}$| such that $Y \xrightarrow[\beta_Y]{\sim} FX$, and set GY = X (possible since F is essentially surjective).

We also need action of G on morphisms. Given $Y \xrightarrow{g} Y'$, since we want β to be a natural transformation, the following diagram must commute:

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \beta_Y\ } & FGY \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle FGg} \\
Y' & \xrightarrow[\beta_{Y'}]{\ } & FGY'
\end{array}
$$

Necessarily, FG$g = \beta_{Y'} \circ g \circ \beta_Y^{-1}$. Since F is fully faithful, Gg is uniquely defined. Finally, we need to verify such G is well-defined, i.e. it preserves identities and compositions. (easily verified by drawing a diagram)

To build an equivalence, we also need to define a natural isomorphism $\alpha : 1 \Rightarrow GF$. Since $FX \xrightarrow{\beta_{FX}} FGFX$ is an isomorphism, and by the lemma above F reflects isomorphisms, so exists an isomorphism $X \xrightarrow{F^{-1}\beta_{FX}} GFX$. $\square$

What is the "suitable version" of axiom of choice?

For locally small categories, we need the **axiom of global choice**.

For small categories, the ordinary axiom of choice suffices.

There is a nice category-theoretic formulation of axiom of choice:

In **Set**, every epimorphism splits.

In **Set**, suppose $X \xrightarrow{s} Y \xrightarrow{r} X$ and $r \circ s = 1_X$, then r is epi and s is mono.

In any category, an epi $Y \xrightarrow{r} X$ is **split** if there exists $X \xrightarrow{s}$ s.t. $r \circ s = 1_X$.

Similarly, a mono $X \xrightarrow{s} Y$ is **split** if there exists $Y \xrightarrow{r} X$ s.t. $r \circ s = 1_X$.

If one has such an $X \xrightarrow{s} Y \xrightarrow{r} X$, then we say Y is a **retract** of X, and the epi r is called a **retraction**, the mono s is called a **section**, and the composite $t = Y \xrightarrow{s \circ r} Y$ is an idempotent, that is $t = t \circ t$.

How does this connect to axiom of choice?

We can view a surjective function $Y \xrightarrow{g} X$ as an indexed set:

$$(Y_\alpha)_{\alpha \in X}, \text{ where } Y_\alpha = \{y \in Y | g(y) = \alpha\}$$

Since g is surjective, Y_α is non-empty, and since it splits, there exists a **selection** $X \xrightarrow{f} Y$ such that $g \circ f = 1_X$.

For each $x \in X$, such f **choose** an $f(x) \in Y$ such that $f(x) \in Y_x$.

<div align="center">

4 CONSTRUCTIONS WITHIN CATEGORIES

</div>

4.1 *Pullbacks*

In a category $\mathcal{C}$, a **pullback** of a pair of maps with common codomain (such pair of maps is called a **cospan**) $X \xrightarrow{f} Z \xleftarrow{g} Y$ is given by a **span** $X \xleftarrow{p} P \xrightarrow{q} Y$ for which $f \circ p = g \circ q$, and such that for every $X \xleftarrow{\alpha} W \xrightarrow{\beta} Y$ with $f \circ \alpha = g \circ \beta$, there exists a unique $W \xrightarrow{w} P$ such that $p \circ w = \alpha$ and $q \circ w = \beta$.
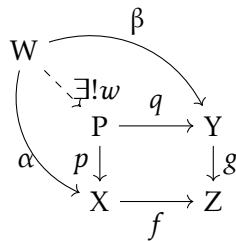
Given a cospan, it may or may not have a pullback. For some nice categories, the pullback always exists. For example, **Set** always has pullbacks:

For $X \xrightarrow{f} Z \xleftarrow{g} Y$, $P = \{(x, y) \in X \times Y | f(x) = g(y)\}$, with the projections π_1, π_2

$$
\begin{array}{ccc}
P & \xrightarrow{\pi_2} & Y \\
{\scriptstyle \pi_1}\downarrow\ \lrcorner & & \downarrow {\scriptstyle g} \\
X & \xrightarrow[f]{} & Z
\end{array}
$$

We place a $\lrcorner$ symbol at the upper left corner to indicate that the diagram is a pullback.

Again in **Set** can view a cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$ as a pair of Z–indexed set $(X_z)_{z \in Z}, (Y_z)_{z \in Z}$, and the pullback of f, g is the **fiber-product** $(X_z \times Y_z)_{z \in Z}$.

EXAMPLE.

- Pullback as inverse-image of function

$$
\begin{array}{ccc}
P & \xrightarrow{\iota_X} & X \\
{\scriptstyle f|_P}\downarrow\ \lrcorner & & \downarrow {\scriptstyle f} \\
Y' & \xrightarrow[\iota_Y]{} & Y
\end{array}
$$

  Where $P = f^{-1}(Y')$, and ι_X, ι_Y are the inclusion maps.

- Pullback as intersection of sets

$$
\begin{array}{ccc}
P & \xrightarrow{\iota_{P,1}} & X_1 \\
{\scriptstyle \iota_{P,2}}\downarrow\ \lrcorner & & \downarrow {\scriptstyle \iota_1} \\
X_2 & \xrightarrow[\iota_2]{} & X
\end{array}
$$

  Where $P = X_1 \cap X_2$, and the ι's are inclusions with corresponding domain and codomain.

Propositions:

- In any category $\mathcal{C}$, pullbacks of the same cospan are unique up to isomorphism.

- The maps $X \xleftarrow{p} P \xrightarrow{q} Y$ in a pullback are **jointly monomorphic**, i.e. given any $P \xleftarrow{v} W \xrightarrow{u} P$, if $p \circ u = p \circ v$ and $q \circ u = q \circ v$, then $u = v$.

- Pullback preserves monomorphisms. Given a pullback:

$$
\begin{array}{ccc}
P & \xrightarrow{\ p\ } & X \\
{\scriptstyle q}\downarrow & & \downarrow {\scriptstyle f} \\
Y & \xrightarrow{\ g\ } & Z
\end{array}
$$

If f is a monomorphism, then q is also a monomorphism, similarly, if g is a mono, then p is also a mono.

LEMMA ((Pullback lemma)). For a diagram of two commuting square sharing a common edge:

$$
\begin{array}{ccc}
\xrightarrow{\quad} & \xrightarrow{\quad} & \\
\downarrow \quad (a) \quad \downarrow \quad (b) \quad \downarrow & & \\
\xrightarrow{\quad} & \xrightarrow{\quad} &
\end{array}
$$

Where (a) is the left-square diagram, (b) is the right-square diagram, and let (a)+(b) denote the outer square diagram by composing the upper and lower two arrow.

- If (a) and (b) are both pullbacks, then (a)+(b) is also a pullback.

- If (a)+(b) is a pullback, and (b) is a pullback, then (a) is a pullback.

Will prove the generalization of the second statement:

For a diagram:

$$
\begin{array}{ccccc}
X'' & \xrightarrow{\ h'\ } & X' & \xrightarrow{\ h\ } & X \\
{\scriptstyle f''}\downarrow & (a) & \downarrow {\scriptstyle f'} \ (b) & & \downarrow {\scriptstyle f} \\
Y'' & \xrightarrow{\ g'\ } & Y' & \xrightarrow{\ g\ } & Y
\end{array}
$$

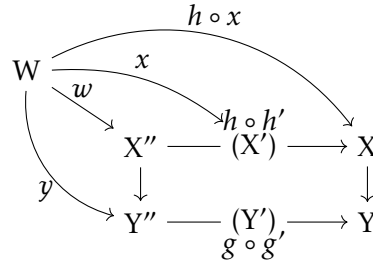- If (a)+(b) is a pullback, and h, f' are jointly monomorphic, then (a) is a pullback.

Proof. We need to prove for every $Y'' \xleftarrow{y} W \xrightarrow{x} X'$, there exists an unique $W \xrightarrow{w} X''$ such that the following diagram commutes:

$$
\begin{array}{ccc}
W & & \\
& \searrow^{x} & \\
\downarrow^{w} & & \\
& X'' \xrightarrow{h'} X' & \\
{\scriptstyle y}\swarrow \quad {\scriptstyle f''}\downarrow & & \downarrow {\scriptstyle f'} \\
& Y'' \xrightarrow{g'} Y' &
\end{array}
$$

Since (a)+(b) is a pullback, for $W \xrightarrow{h\circ x} X$ and $W \xrightarrow{y} Y''$, there exist an unique w such that

the following diagram commute:

To make the first diagram (a) commute, we need to check that $h' \circ w = x$. Since $h \circ x = h \circ (h' \circ w)$ and $f' \circ x = g' \circ y = g' \circ f'' \circ w = f' \circ h' \circ w$, by jointly monomorphic, we obtain that $x = h' \circ w$. $\qquad\square$

4.2 *Limit*

Pullback is a special case of a general construction in category called **limit**.

DEFINITION (Cone). A cone (O, π) of a diagram is an **object** together with a **collection of morphisms**, such that every triangle commutes.
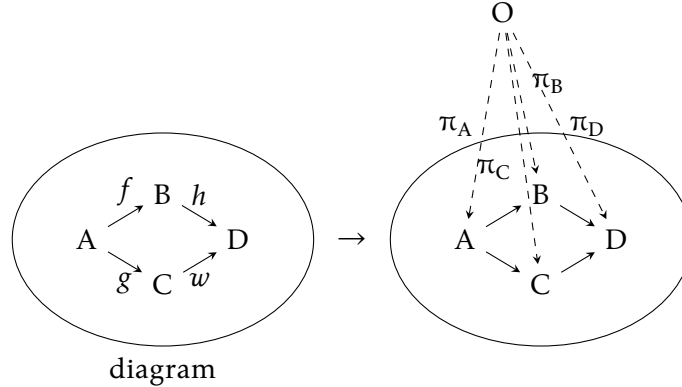
diagram

- $g \circ \pi_A = \pi_C$

- $f \circ \pi_A = \pi_B$

- $h \circ \pi_B = \pi_D$

- $w \circ \pi_C = \pi_D$

DEFINITION (limit). A limit of a diagram is a cone (L, π), which satisfies that for any other cone (O, π'), there exists an **unique** morphism $O \xrightarrow{\phi} L$ such that the diagram of two cones together with ϕ commutes.

 Can view the limit as the core, essence cone that any cone uniquely collapse into the limit.
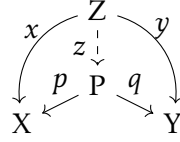
Fact: Limits are unique up to isomorphism.

Pullback is actually the limit of a diagram with the form $\to\leftarrow$. Now we will give some general results about limit:

EXAMPLE.   • Binary product

A **binary product** for X, Y is a span $X \xleftarrow{p} P \xrightarrow{q} Y$ such that for any $X \xleftarrow{x} Z \xrightarrow{y} Y$, there exists a unique $Z \xrightarrow{z} P$ such that $p \circ z = x$ and $q \circ z = y$. Binary product is actually the limit of a two-point diagram with no arrows.

A category $\mathcal{C}$ is said to have binary products if, for every $X, Y \in |\mathcal{C}|$, the binary product of X, Y exists.

$$
\begin{array}{ccc}
 & Z & \\
x \nearrow & \downarrow z & \searrow y \\
 & P & \\
\downarrow & p \swarrow \quad \searrow q & \downarrow \\
X & \leftarrow \qquad \rightarrow & Y
\end{array}
$$

– **Set** has products, the cartesian products.
– **Vect$_K$** has products, the cartesian products of vector spaces.
– **Grp** has products, the direct products.
– **Top** has products, the topological products of spaces.

We often write $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ for a chosen product in any category $\mathcal{C}$.

• I-indexed product in $\mathcal{C}$

$I \in |\mathbf{Set}|$ and consist of objects in $\mathcal{C}$.

A product of a pamily $(X_i)_{i \in I}$ is an object P and a family $(P \xrightarrow{p_i} X_i)_{i \in I}$ such that for any Z and family $(Z \xrightarrow{x_i} X_i)_{i \in I}$ there exists a unique $Z \xrightarrow{z} P$ such that for any $i \in I$, $p_i \circ z = x_i$.

$$
\begin{array}{ccc}
Z & \xrightarrow{z} & P \\
 & x_i \searrow & \downarrow p_i \\
 & & X_i
\end{array}
$$

A special case of indexed product is which the indexed set is empty. In this case the product is simply an object T, and for any other object Z there exists a unique map $Z \xrightarrow{u} T$. Such a T is called a **terminal object**. The usual notation for such unique morphism is $Z \xrightarrow{!_Z} T$.

We say a category $\mathcal{C}$ has product if for every index family, the product exists.

PROPOSITION. For a category $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has finite product.

2. $\mathcal{C}$ has a terminal object and binary product.

- Equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$

  An equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a map $E \xrightarrow{e} X$ such that $f \circ e = g \circ e$ and for any $Z \xrightarrow{z} X$ such that $f \circ z = g \circ z$, there exists a unique $Z \xrightarrow{w} E$ such that $e \circ w = z$.

  – **Set** has equalizers. $E = \{x \in X | f(x) = g(x)\}$, e is the inclusion map.

In any category, the equalizer $E \xrightarrow{e} X$ is a monomorphism.(exercise)

It's not true in general that the monomorphisms airse as an equalizer. Those monomorphisms which are also equalizers are called **regular monomorphisms**.

In **Set**, every monomorphism is regular, however in **Top** not every monomorphism is regular.

4.3 *Graph*

Diagram shapes are graphs.

Pullback		$\bullet \to \bullet \leftarrow \bullet$
Binary Product		$\bullet \quad \bullet$
I-indexed Product		$\bullet \to \bullet \to \bullet \to \bullet \to \cdots \to \bullet$
Terminal Object	is the limit for	empty diagram
Equalizer		$\bullet \overset{\frown}{\underset{\smile}{}} \bullet$
Projective limit		$\cdots \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet$

(extended to infinity to the left)

Specifically **directed multigraphs** (will just call them graph instead in this lecture), also called **quivers**. In this lecture a graph G is given by a collection |G| of **vertices** and for every $u, v \in |G|$ a collection $G(u, v)$ of edges with source u and target v.

The definition is similar with category except graph don't require identities and compositions.

We say a graph is **small/locally small/finite/locally finite** in the obvious way.

Just like graph generalize category, there is also generalization of functor which is **graph morphism**.

A graph morphism $G \xrightarrow{F} G'$ is given by a map of vertices $|G| \xrightarrow{F_o} |G'|$ and for every $u, v \in |G|$ a map of edges $G(u, v) \xrightarrow{F_{u,v}} G'(F_o(u), F_o(v))$.

Now we can define a graph category **Graph**, whose objects are all small graphs and morphisms are graph morphisms.

A diagram over a graph G in a category $\mathcal{C}$ is a graph morphism from G to $F_G(\mathcal{C})$, where F_G is the forgetful functor from **Cal** to **Graph**.

> A graph serves as a carrier; once its vertices are assigned objects and its arrows are assigned morphisms, then it becomes a diagram.

The limit for a graph is defined in the obvious way.

Note: Can constitute the category of cones of a specific diagram, then the limit is the terminal object in this category of cones.

DEFINITION. A category $\mathcal{C}$ is (**complete/finitely complete**) if every (**small/finite**) diagram has a limit.

THEOREM. The following are equivalent:

(1) $\mathcal{C}$ is complete.

(2) $\mathcal{C}$ has products and equalizers.

THEOREM. The following are equivalent:

(1) $\mathcal{C}$ is finitely complete.

(2) $\mathcal{C}$ has finite products and equalizers.

(3) $\mathcal{C}$ has a terminal object and pullbacks.

4.4 *Exercise*

1. What are products in the category **Rel** of relations?

2. In any category, the equalizer $E \xrightarrow{e} X$ is a monomorphism.

3. Prove that every monomorphism in **Set** is regular.

4. In **Top**, regular monomorphisms coincide with embeddings (continuous functions that preserves the topological structures).

5. Prove that the second and third statement in the last theorem are equivalent.

SCRATCH OF PROOF. Consider the following diagram:

$$
\begin{array}{c}
E \\
\downarrow \\
X \times Y \\
\swarrow \qquad \searrow \\
X \longrightarrow Z \longleftarrow Y
\end{array}
$$
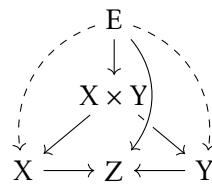
Where E is the equalizer, X × Y is the product.