Modern Algebra I Homework 10

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Question 1.

Consider

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & A^i \arrow[r, "f^i"] \arrow[d, "a_i"'] & B^i \arrow[r, "g^i"] \arrow[d, "b_i"'] & C^i \arrow[r] \arrow[d, "c_i"'] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d] & B^{i+1} \arrow[r, "g^{i+1}"] \arrow[d] & C^{i+1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Then we have:

$$\begin{tikzcd}
 & A^i/\mathrm{im}(a_{i-1}) \arrow[r] \arrow[d, "\bar{a}_i"'] & B^i/\mathrm{im}(b_{i-1}) \arrow[r] \arrow[d, "\bar{b_i}"'] & C^i/\mathrm{im}(c_{i-1}) \arrow[r] \arrow[d, "\bar{c}_i"'] & 0 \\
0 \arrow[r] & \ker(a_{i+1}) \arrow[r] & \ker(b_{i+1}) \arrow[r] & \ker(c_{i+1}) &
\end{tikzcd}$$

Observe that $\ker(\bar{a}_i) = \ker(a_i)/\mathrm{im}(a_{i-1}) = H^i(A)$, and $\mathrm{coker}(\bar{a}_i) = \mathrm{im}(a_i)/\ker(a_{i+1}) = H^{i+1}(A)$. Apply Snake Lemma, done.

Question 2.

$(a) \Rightarrow (b)$ has been proved in class.
$(b) \Rightarrow (a)$: Let M be projective, then it is a direct summand of some free $A-$module F, hence a submodule of F. But submodules of a free module over PID is also free, hence M is free.
$(a) \Rightarrow (c)$: It suffices to show A as a $A-$mod is trosion-free. Since A is a

domain, done.
$(c) \Rightarrow (a)$: Need an additional assumption: M is finitely generated. ($\mathbb{Q}$ is torsion-free but not free)
Assume M is f.g., then by Fundamental Theorem of Finitely Generated Module over PID, $M \cong A^n \oplus A/(a_1) \oplus A/(d_2) \oplus ... \oplus A/(d_t)$. Since M is torsion-free, the torsion part $A/(a_1) \oplus A/(d_2) \oplus ... \oplus A/(d_t)$ is zero hence $M \cong A^n$ is free.

Question 3.

(a) Let

$$0 \to I^0 \to I^1 \to ...$$

be an injective resolution of N, then there's an exact sequence:

$$0 \to N \xrightarrow{\iota} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} ...$$

Apply Hom functor $\mathrm{Hom}_A(M, -)$ to the above chains:

$$0 \to \mathrm{Hom}(M, I^0) \xrightarrow{\mathrm{Hom}(M,d^0)} \mathrm{Hom}(M, I^1) \xrightarrow{\mathrm{Hom}(M,d^1)} ...$$

$$0 \to \mathrm{Hom}(M, N) \xrightarrow{\mathrm{Hom}(M,\iota)} \mathrm{Hom}(M, I^0) \xrightarrow{\mathrm{Hom}(M,d^0)} \mathrm{Hom}(M, I^1) \xrightarrow{\mathrm{Hom}(M,d^1)} ...$$

The second chain is exact. To find $\mathrm{Ext}^i_A(M, N)$, need to find the i-th cohomology of the first chain.

$\mathrm{Ext}^0_A(M, N) = \ker(\mathrm{Hom}(M, d^0))$. Since hom functor $\mathrm{Hom}(M, -)$ preserve injection, $\ker(\mathrm{Hom}(M, d^0)) = \mathrm{im}(\mathrm{Hom}(M, \iota)) \cong \mathrm{Hom}(M, N)$.

(b) Consider two injective resolutions $I^\bullet, J^\bullet$ of N, and identity map id_N. There is a chain map $f^\bullet$ from $I^\bullet$ to $J^\bullet$, and similarly, a chain map $g^\bullet$ from $J\bullet$ to $I^\bullet$.
Apply hom functor to both chain, then $f^\bullet, g^\bullet$ maps the n-th cohomology into n-th cohomology and $fg^\bullet = id^\bullet$, the identity in cohomology, hence a quasi-isomorphism. So the Ext^i_A derivied in both injective resolution is isomorphic.