Modern Algebra I Homework 10

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Question 1.

(a) Consider the following commuting diagram:

$$\begin{tikzcd}
0 \arrow[r] & A^{k_3} \arrow[r] \arrow[d] & A^{k_3} \oplus A^{k_1} \arrow[r] \arrow[d] & A^{k_1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^n \arrow[r] \arrow[d, "\pi_3"'] \arrow[dr, dashed, "p_1"] & A^n \oplus A^m \arrow[r] \arrow[d, "p_1 \oplus p_2"] & A^m \arrow[r] \arrow[d, "\pi_1"] \arrow[dl, dashed, "p_2"'] & 0 \\
0 \arrow[r] & M_3 \arrow[r, "f"'] \arrow[d] & M_2 \arrow[r, "g"'] & M_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

Where p_1 is defined to be $f \circ \pi_3$, and since A^m is free (hence projective), exists $p_2 : A^m \to M_2$ s.t. $g \circ p_2 = \pi_1$.
Now it remains to show $p_1 \oplus p_2$ is surjective.
By five lemma, $p_1 \oplus p_2$ is surjective, hence M_2 can be generated by $n + m$ elements, and the kernel of $p_1 \oplus p_2$ is an image of $A^{k_3} \oplus A^{k_1}$, which is also finitely generated. So M_2 is finitely presented.

(b) Let $A^k \xrightarrow{\phi} A^m \xrightarrow{\pi} M_2 \to 0$ and $A^n \to M_3 \to 0$, and $M_3 \xrightarrow{f} M_2 \xrightarrow{g} M_1$
Since $g \circ \pi : A^m \to M_1$ is surjective, M_1 is finitely generated. Consider $K = \ker(g \circ \pi) = \pi^{-1}(M_3)$ (view M_3 as a submodule of M_2).
Let $x \in K$, then $\pi(x) \in M_3$, so is a finite sum $\pi(x) = \sum_{i=1}^{n} a_i v_i$.
Choose $y_i \in A^m$ s.t. $\pi(y_i) = v_i$, then $x - \sum_{i=1}^{n} a_i y_i \in \ker(\pi) = \mathrm{im}(\phi)$, which is finitely generated. Let $z_1, ..., z_k \in A^m$ generates $\ker(\pi)$, then $\{z_1, ..., z_k, y_1, ..., y_n\}$ generates K. Hence M_1 is finitely presented.

Question 2.

(a) Let $G \neq 0$ and $n = |G| < \infty$ be the order of G, choose $a \neq 0$, then no $g \in G$ satisfies $n \cdot g = a$.

(b) Let F_X be free over X, and choose a fixed $x \in X$. $F_{\{x\}} \cong \mathbb{Z}$ is a subgroup of F_X, let 1_x be the generator.
Assume F_X is divisible, and let $n = 2$, then there is some $g = \sum_{i=1}^{n} a_i 1_{x_i}$, $a_i \in \mathbb{Z}$, s.t. $2g = 1_x$.
Define a mapping:

$$X \xrightarrow{\phi} \mathbb{Z}$$
$$a \mapsto \begin{cases} 1 \text{ if } a = x, \\ 0 \text{ otherwise.} \end{cases}$$

, then there is an unique extension $\tilde{\phi} : F_X \to \mathbb{Z}$, and $\tilde{\phi}(2g) = \tilde{\phi}(1_x) = 1$. Therefore $x_j = x$ for some j and $2 \cdot a_j = 1$, but this is impossible in $\mathbb{Z}$, so F_X is not divisible.

(c) Let D be divisible, and consider quotient group D/K.
For $x + K \in D/K$ and $n \in \mathbb{N} - \{0\}$, there exists some $d \in D$ s.t. $n \cdot d = x$, hence $n \cdot (d + K) = n \cdot d + K = x + K$.

Question 3.

$\mathbb{Z}$ is a free $\mathbb{Z}-$mod, hence is projective.
To prove $\mathbb{Z}$ is not injective, consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \mathbb{Q} \\ & & \downarrow id & & \\ & & \mathbb{Z} & & \end{array}$$

If $\mathbb{Z}$ is injective, then there is some $f : \mathbb{Q} \to \mathbb{Z}$ making the diagram commute. Clearly $f(z) = z$ for integer z, and suppose $f(\frac{1}{2}) = z_1 \in \mathbb{Z}$, then $2f(\frac{1}{2}) = 1 = 2z_1$, which is impossible. So $\mathbb{Z}$ is not injective.
Since $\mathbb{Q}$ is divisible, hence is injective.
Since $\mathbb{Z}$ is a PID, every submodule of a free module is also free, suppose

$\mathbb{Q}$ is projective, then it is a direct summand of some free $\mathbb{Z}-$mod, hence a $\mathbb{Z}-$submod of free module, hence is free.
Since a $\mathbb{Z}-$mod homomorphism $\mathbb{Q} \to \mathbb{Z}$ is uniquely determined by the image of a single non-zero element, so $\mathbb{Q}$ can only be free of rank 1.($\mathbb{Z}$ is PID, so rank is well-defined), but then $\mathbb{Q} \cong \mathbb{Z}$, a contradiction.

Question 4.

Need to show for any short exact sequence of $B-$mod:

$$0 \to N_3 \to N_2 \to N_1 \to 0$$

, the sequence:

$$0 \to N_3 \otimes_B (B \otimes_A M) \to N_2 \otimes_B (B \otimes_A M) \to N_1 \otimes_B (B \otimes_A M) \to 0$$

is also exact.
Since tensor product is associative:

$$N_i \otimes_B (B \otimes_A M) = (N_i \otimes_B B) \otimes_A M = N_i \otimes_A M$$

We have the following diagram:

$$\begin{tikzcd}
N_i \otimes_B (B \otimes_A M) \arrow[r, "f_i \otimes_B 1_{(B\otimes_A M)}", "\text{B-mod hom}"'] \arrow[d, dash] & N_{i-1} \otimes_B (B \otimes_A M) \arrow[d, dash] \\
N_i \otimes_A M \arrow[r, "f_i \otimes_A 1_M", "\text{A-mod hom}"'] & N_{i-1} \otimes_A M
\end{tikzcd}$$

Since M is a flat $A-$mod, the following diagram commute:

$$0 \to N_3 \otimes_A M \to N_2 \otimes_A M \to N_1 \otimes_A M \to 0$$

So Im$(f_i \otimes_A 1_M)$ =Ker$(f_{i-1} \otimes_A 1_M)$, hence Im$(f_i \otimes_B 1_{(B\otimes_A M)})$ =Ker$(f_{i-1} \otimes_B 1_{(B\otimes_A M)})$, hence the functor $\bullet \otimes_B M_B$ is exact.