Modern Algebra I Homework 10

蘇則宇 (B11201005)

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Question 1.

First check $L(M,N;P)$ is a $A-$mod. Let $f,g \in L(M,N;P)$ and $a \in A$, then $(a\cdot f+g) : M\times N \to P$, $(a\cdot f+g)(x,y) = af(x,y)+g(x,y)$ is a $A-$bilinear map.

(a) For every $f \in L(M,N;P)$, exists a unique $A-$mod homomorphism $M\otimes_A N \xrightarrow{\tilde{f}} P$ s.t. $\tilde{f}(x\otimes y) = f(x,y)$.
Conversely, any $\phi \in \mathrm{Hom}_A(M\otimes N, P)$ defines a bilinear map $M \times N \to P$, a routine verification check that such relation is a $A-$mod isomorphism.

(b) Will show an isomorphism:

$$\mathrm{Hom}(M\otimes N, P) \xrightarrow{\phi} \mathrm{Hom}(M, \mathrm{Hom}(N,P))$$

Let $f \in \mathrm{Hom}(M\otimes N, P)$, then define $[\phi(f)(x)](y) = f(x\otimes y) \in P$. $A-$linear is trivial.
Let $M \xrightarrow{g} \mathrm{Hom}(N,P)$, $\bar{g}(x,y) = g(x)(y) \in P, x\in M, y \in N$ defines a $A-$bilinear map, hence exists a unique $A-$mod homomorphism
$M\otimes N \xrightarrow{\tilde{g}} P$ s.t. $\tilde{g}(x\otimes y) = \bar{g}(x,y)$.
The existence shows onto, and the uniqueness shows one-to-one.

(c) Same as (b) since $N\otimes M \cong N\otimes M$.

Question 2.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+2}"] & M_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & M_i \arrow[r, "d_i"] \arrow[d, "f_i"] & M_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r, "d'_{i+2}"] & N_{i+1} \arrow[r, "d'_{i+1}"] & N_i \arrow[r, "d'_i"] & N_{i-1} \arrow[r, "d'_{i-1}"] & \cdots
\end{tikzcd}$$

Question 3.

Can see if the diagram commute, then f_i maps $\ker(d_i)$ to $\ker(d_i')$, and maps maps $\operatorname{Im}(d_i)$ to $\operatorname{Im}(d_i')$ for all i.