CATEGORY THEORY

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1 CATEGORY

1.1 *Definition*

A **Category** $\mathcal{C}$ is given by:

- A collection $|\mathcal{C}|$ or obj($\mathcal{C}$) of **objects**.
- For every $X, Y \in |\mathcal{C}|$, we have a collection $\mathcal{C}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of **morphisms** from X to Y.
- For $X \in |\mathcal{C}|$, we have $1_X \in \mathcal{C}(X, X)$, called the **identity morphism** on X.
- For any $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, we have a **composite morphism** $g \circ f \in \mathcal{C}(X, Z)$.
- These compositions must satisfy:
  - **(Identity law)** $1_Y \circ f = f = f \circ 1_X$.
  - **(Associative law)** $h \circ (g \circ f) = (h \circ g) \circ f$, for any $h \in \mathcal{C}(Z, W)$.

1.2 *Examples*

- **Set**: The category of sets.
  - Objects |**Set**|: The "collection"(can't talk about "set of all sets") of all sets.
  - Morphisms **Set**(X, Y): The set of all functions from X to Y.
  - Identities 1_X: The identity function on X.
  - Compositions: Compositions of functions.

  We say a category $\mathcal{C}$ is **locally small** if $\mathcal{C}(X, Y)$ is a set for all X, Y; moreover we say $\mathcal{C}$ is **small** if the collection of objects $|\mathcal{C}|$ is a set.

  Set is locally small but not small.

- **Top**: The category of topological spaces(and continuous functions).
  - Objects |**Top**|: The collection of all topological spaces.
  - Morphisms **Top**(X, Y): The set of all continuous functions from X to Y.

  And the obvious identities and associative law. Again, **Top** is locally small but not small.

- **Grp**: The category of groups with group homomorphism as morphisms.
- $\mathbf{Vect}_{\mathbf{K}}$: The category of vectors spaces over **K**, with the **K**-linear transformations as morphisms.
- **Rel**: The category of sets, with all binary relations as morphisms.

  (A binary relation between X, Y is a function $R : X \times Y \to \{\mathrm{True}, \mathrm{False}\}$. If $R(x, y) = \mathrm{True}$, we say xRy)

- Objects $|\mathbf{Rel}|$: The collection of all sets.
- Morphisms $\mathbf{Rel}(X,Y)$: The set of all relations between X and Y.
- Identities $1_X : (a,b) \mapsto \begin{cases} \text{True} & \text{if } a = b \\ \text{False} & \text{if } a \neq b \end{cases}$
- Compositions: Define the "composition of two relations R, S" by

$$(R;S) : X \times Z \to \{\text{True,False}\}$$

$$(x,z) \mapsto \begin{cases} \text{True} & \text{if exists } y \text{ such that } xRy \text{ and } ySz \\ \text{False} & \text{otherwise} \end{cases}$$

All of above categories are locally small but not small, not will give some examples of small categories.

- **G**: Viewing a group G as a category.
  - Objects: Has only one objects $*$.
  - Morphisms: $\mathbf{G}(*,*) = G$.
  - Identities: $1_* = e$, the unity element of group G.
  - Compositions: Compositions as elements in group G.

  In fact, one can do the same construction for monoids, monoids are equivalent to categories with only one object.

- $\underline{\mathbf{P}}$: The poset P as a category.
  - Objects: $|\underline{\mathbf{P}}| = P$.
  - Morphisms: $\underline{\mathbf{P}}(x,y) = \begin{cases} \{x \leq y\} & \text{if } x \leq y \text{ (The first "}x \leq y\text{" here is a morphism)} \\ \varnothing & \text{otherwise} \end{cases}$

  Again, we don't need all the axioms of poset here to construct a category, we never use the anti-symmetry axiom ($x \leq y$ and $y \leq x$ implies $x = y$). More generally, we can define categories for any **preorder set**, which is equivalent to category with a set as objects, and has at most one morphism for each homset, and poset are preorder set which the only isomorphisms are identities.

1.3 *Isomorphism, monomorphisms, epimorphisms*

There are three special kinds of morphisms, **isomorphisms, monomorphisms** and **epimorphisms**.

A morphism $X \xrightarrow{f} Y$ is said to be an **isomorphism** if there exists $Y \xrightarrow{f^{-1}} X$ such that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$.

f is said to be a **monomorphism** if for any $g,h \in \mathrm{Hom}(Z,X)$, $f \circ g = f \circ h$ implies $f = g$.

Every isomorphisms is also a monomorphism(and epimorphism).

Categories	Isomorphisms	Monomorphisms	Epimorphisms
Set	bijections	injections	surjections
Top	homeomorphisms	injective continuous maps	surjective continuous maps
Grp	group isomorphisms	group monomorphisms	group epimorphisms
$\mathbf{Vect}_{\mathbb{K}}$	$\mathbb{K}$ – linear iso	inj $\mathbb{K}$ – linear transf	surj $\mathbb{K}$ – linear transf
G(a group)	all maps	all maps	all maps
$\underline{\mathbf{P}}$(a poset)	identities	all maps	all maps

Note that a morphism which is both a monomorphism and a epimorphism does not necessarily imply that it is an isomorphism. For example, in category of rings **Rng**, the embedding $\mathbb{Z} \to \mathbb{Q}$ is mono and epi(since $\mathbb{Q}$ is the fraction ring of $\mathbb{Z}$, and any ring homomorphism from it is unique determined by the image of $\mathbb{Z}$, hence determined by image of 1), but it has no inverse, so is not iso.

2 FUNCTOR

2.1 *Definition*

A **functor** F is a "transformation" between two categories $\mathcal{X}, \mathcal{Y}$, and is given by:

- **Objectation** A mapping of objects $|\mathcal{X}| \to |\mathcal{Y}|$.
- **Morphsim actions** For any $a, b \in |\mathcal{X}|$, and any $f \in \mathrm{Hom}(a,b)$, there is a $\mathrm{F}(f) \in \mathrm{Hom}(\mathrm{F}(a), \mathrm{F}(b))$.
- **Preserving identities** For any $a \in |\mathcal{X}|$, $\mathrm{F}(1_a) = 1_a$.
- **Preserving compositions** For any $a \xrightarrow{f} b \xrightarrow{g} c$, $\mathrm{F}(g \circ f) = \mathrm{F}(g) \circ \mathrm{F}(f)$.

2.2 *Examples*

- **Forgetful functor: Grp$\xrightarrow{\mathrm{F}}$Set**

  F sends each group to the set of group element, and sends group homomorphism ϕ to the same function but as a function between sets.

- **Homomorphism functor between two group categories**

  Since group category has only one object, the functor just need to preserve the morphisms, which is equivalent to preserving group operations, so the functors between 2 groups are just group homomorphisms between them.

- **Cat: Category of categories**

  We can form a category whose objects are categories and morphisms are functor. But this raises some problems, for example, can the category of all categories **Cal** be an object of itself?

We will circumvent these problems by considering **Cal** to be the category of **small** categories.

Exercise: Between 2 small categories, the functors between form a set. (So **Cal** is locally small but now small).

There a 3 types of categories, we have discuss the first and second types:

- Categories whose objects are mathematical structures and morphisms between them are transformations or relations between each two individual structures.

  Set, Grp, Top, Rel, Cat, Vect$_{\mathbb{K}}$.

- Categories which are categorifications of individual mathematical structures.

  G, M, P (category constructed by a single group, monoid, poset, respectively).

- Categories formed by category-theoretic constructions via existing categories.

  $\mathcal{C}^{\mathrm{op}}$: For any category $\mathcal{C}$, we define its opposite category $\mathcal{C}^{\mathrm{op}}$ by: $|\mathcal{C}^{\mathrm{op}}| = |\mathcal{C}|$, and $\mathcal{C}^{\mathrm{op}}(X, Y) = \mathcal{C}(Y, X)$. (same objects, but "reverses" the morphisms)

  An important usage of notion of opposite categories is that we can define the notion of "dual". (later)

  Facts:

  - $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.
  - For $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$, $(g \circ f)^{\mathrm{op}} = f^{\mathrm{op}} \circ g^{\mathrm{op}}$.
  - Functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and functors $\mathcal{C}^{\mathrm{op}} \xrightarrow{F^{opp}} \mathcal{D}^{\mathrm{op}}$ are in one-to-one correspondence with the obvious way.

2.3 *Contravariant Functors*

An important reason for considering opposite categories is that it's extremely common that we need to consider functors from an opposite category to a non-opposite category: $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D}$, such functors are called **Contravariant Functors** from $\mathcal{C}$ to $\mathcal{D}$.

If we want to clarify $\mathcal{C} \xrightarrow{G} \mathcal{D}$ is not contravariant from $\mathcal{C}$ to $\mathcal{D}$, we say G is a **Covariant Functor** from $\mathcal{C}$ to $\mathcal{D}$.

EXAMPLE. The construction of dual vector space V^* via given vector space V is actually functorial in the contravariant (on itself) sense.

$$\begin{array}{ccc} \mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}} & & \mathbf{Vect}_{\mathbb{K}} \\ V^{\mathrm{op}} & & V^* \\ f^{\mathrm{op}} \Big\uparrow & \xrightarrow{(\cdot)^*} & \Big\uparrow f^* \\ W^{\mathrm{op}} & & W^* \end{array}$$

$$(f^*(w^*))(v) = w^*(f(v))$$

2.4 *Product Category*

The product $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}, \mathcal{D}$ is defined by:

- $|\mathcal{C} \times \mathcal{D}| = |\mathcal{C}| \times |\mathcal{D}|$.
- $(\mathcal{C} \times \mathcal{D})((X, Y), (X', Y')) = \mathcal{C}(X, X') \times \mathcal{D}(Y, Y')$.

Can easily check this define a category.

There are evident **projection functors**:

$$\pi_1 : \mathcal{C} \times \mathcal{D} \quad \to \mathcal{C}$$
$$\pi_2 : \mathcal{C} \times \mathcal{D} \quad \to \mathcal{D}$$

Can define the product category of finite many categories.

More generally, for index set I, can define the product category $\prod_{i \in I} \mathcal{C}_i$. It's now clear what is meant by a **multi-argument** functor:

$$\mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n \xrightarrow{F} \mathcal{D}$$

2.5 *Hom Functor*

The **Hom Functor** for a category $\mathcal{C}$ is:

$$\begin{aligned} \mathcal{C}(-,-) \ : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \quad &\longrightarrow \quad \mathbf{Set} \\ (X, Y) \quad &\longmapsto \quad \mathcal{C}(X, Y) \end{aligned}$$

$$\begin{array}{ccccc} & & & & \mathcal{C}(X_1, Y_1) \\ X_1 & & Y_1 & & \\ f\uparrow & & \downarrow g & \xrightarrow{\mathcal{C}(-,-)} & \Big\downarrow \mathcal{C}(f, g) \\ X_2 & & Y_2 & & \\ & & & & \mathcal{C}(X_2, Y_2) \end{array}$$

The hom functor $\mathcal{C}(-,-)$ takes tuple (f,g) to a function (between sets) from $\mathcal{C}(X_1,Y_1)$ to $\mathcal{C}(X_2,Y_2)$.

$\mathcal{C}(f,g)$ is defined by simply "join" $\alpha \in \mathcal{C}(X_1,Y_2)$ into the diagram:

$$\begin{array}{ccc} X_1 & \overset{\alpha}{\dashrightarrow} & Y_1 \\ f\uparrow & & \downarrow g \\ X_2 & \xrightarrow[g\circ\alpha\circ f]{} & Y_2 \end{array}$$

2.6 *Duality*

When we have a new category concept, we automatically get another category concept by interpreting the original category concept in dual category.

Dual notion of monomorphism:

DEFINITION. $X \xrightarrow{f} Y$ is an **epimorphism (epi)** if:

$$X^{op} \xrightarrow{f^{op}} Y^{op}$$

is a monomorphism.

2.7 *Exercise*

1. (**Powerset Functors**)
   (a) Find a **contravariant** functor $\mathbf{Set}^{op} \to \mathbf{Set}$, whose object action is $S \mapsto \mathcal{P}S$: the powerset of S.
   (b) Find a **covariant** functor $\mathbf{Set} \to \mathbf{Set}$, whose object action is $S \mapsto \mathcal{P}S$.
   (c) Do Exercise 2 again, find a different one.

2. (a) Does functor $\mathbf{Set} \xrightarrow{F} \mathbf{Set}$ preserve monomorphisms?
   (b) Does functor $\mathbf{Set} \xrightarrow{F} \mathbf{Set}$ preserve epimorphisms?

Answers:

1. (a) Define the "preimage functor" Pre, for $f \in \mathbf{Set}(X,Y)$, and $B \subseteq Y$, $(\text{Pre}(f))(B) = f^{-1}(B) \subseteq X$.
   (b) The "image functor" Im.
   (c) Define the "Unique image functor" U:

   For $A \subseteq X$, $(U(f))(A) = \{b \in Y | \exists! a \in A \text{ s.t. } f(a) = y\}$.

2. (a) Yes.

Since if $X \xrightarrow{f} Y$ is injective, then exists some $Y \xrightarrow{g} X$ s.t. $g \circ f = 1_X$. (g maps image of f to its preimage, and sends other elements in Y to arbitrary element of X)

So for any functor $\mathbf{Set} \xrightarrow{F} \mathbf{Set}$, $F(g \circ f) = F(g) \circ F(f) = 1_X$, which implies $F(f)$ is a monomorphism.

(b) Yes.

Similarly, if $X \xrightarrow{f} Y$ is surjective, then exists $Y \xrightarrow{g} X$ s.t. $f \circ g = 1_Y$.

3 NATURAL TRANSFORMATION

What transformations are **Natural** in a sense?

To answer this question, which lies at the very foundation of category theory, we need notion of **Functor**, and in order to define functor, we need the notion of **Category**.

EXAMPLE. Let V be a finite dimensional vector space over $\mathbf{K}$, then V^* is also a finite dimensional vector space of the same dimension over $\mathbf{K}$, i.e. there exists isomorphism:

$$V \overset{\phi}{\cong} V^*$$

But to build an isomorphism ϕ, there's an arbitrary (non-natural) choice involved, we need to choose a basis β for V, and then use β to define a corresponding basis β^* for V^*, and define ϕ via these two basis:

$$\beta = \{v_1, v_2, ...\},\ \beta^* = \{v_1^*, v_2^*, ...\}$$

$$v_i^* : \sum_i a_i v_i \mapsto a_i$$

$$\phi : \sum_i a_i v_i \mapsto \sum_i a_i v_i^*$$

We also have $V \cong V^{**}$, but in this case we can find an isomorphism independent of choices of basis:

$$v \mapsto v^{**}, v^{**} \text{ maps elements in } V^* \text{ to } \mathbf{K} \text{ by } v^{**}(u^*) := u^*(v)$$

Is there a mathematical definition which characterize this naturality?

3.1 *Definition*

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, a **Natural Transformation** $\alpha : F \Rightarrow G$ is given by a family of components indexed by objects of $\mathcal{C}$:

$$(FX \xrightarrow{\alpha_X} GX)_{X \in |\mathcal{C}|}$$

that satisfies the following **naturality condition:**

$$\forall X \xrightarrow{f} Y \in \mathcal{C}(X,Y), \text{ the following diagram commutes:}$$

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}$$

3.2 *Examples*

- $\epsilon : 1_{\mathbf{Vect_K}} \Rightarrow (\cdot)^{**}$:

$$\epsilon_V : V \longrightarrow V^{**}$$
$$v \longmapsto \left(v^{**} : f \mapsto f(v)\right)$$

- For any set X, we have:

$$\{\cdot\}_X : X \longrightarrow \mathcal{P}X$$
$$x \longmapsto \{x\}$$

  So this defines a natural transformation from $1_{\mathbf{Set}}$ to $\mathcal{P}$: the (image) powerset functor.

- We also have:

$$\begin{array}{rcl} \bigcup_X : \mathcal{P}\mathcal{P}X & \longrightarrow & \mathcal{P}X \\ \left\{ X_\alpha \mid \alpha \in A \right\} & \longmapsto & \bigcup_{\alpha \in A} X_\alpha \end{array}$$

  This defines a natural transformation from $\mathcal{P}\mathcal{P}$ to $\mathcal{P}$.

3.3 *Functor category*

Let F, G, H be functors from $\mathcal{C}$ to $\mathcal{D}$, and $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$ be natural transformations.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, "G"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=2ex, "\Downarrow \alpha"] \arrow[r, phantom, shift right=2ex, "\Downarrow \beta"] & \mathcal{D}
\end{tikzcd}$$

Obviously, there is a composite natural transformation of α, β from F to G.

We therefore have a **Functor Category** $[\mathcal{C}, \mathcal{D}]$, whose objects are functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$, and morphisms are natural transformations.

Size issues: If $\mathcal{C}, \mathcal{D}$ are both large, then $[\mathcal{C}, \mathcal{D}]$ is "very large". However, if $\mathcal{C}$ is small, and $\mathcal{D}$ is locally small, then $[\mathcal{C}, \mathcal{D}]$ is locally small; moreover, if $\mathcal{C}, \mathcal{D}$ are both samll, then$[\mathcal{C}, \mathcal{D}]$ is also small.

The above composition is called **Horizontal** composition. There is also **Vertical** composition, defined as follows:

Definition (Whiskering). Let:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] & \mathcal{C} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E} \end{tikzcd}$$

Define:

$$H\alpha : HG_1 \Rightarrow HG_2$$
$$(H\alpha)_X = H\alpha_X \text{ for } X \in |\mathcal{C}|.$$

Note: $H\alpha_X$ is the morphism obtained by applying functor H to morphism $\alpha_X \in \mathcal{D}(G_1X, G_2X)$ for $X \in |\mathcal{C}|$.

Similarly, there is also:

$$\alpha F : G_1F \Rightarrow G_2F$$
$$(\alpha F)_Y = \alpha_{FY} \text{ for } Y \in |\mathcal{B}|.$$

Definition (Horizontal Composition of functors). Let:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

Define:

$$\begin{aligned} \beta * \alpha : G_1F_1 &\Rightarrow G_2F_2 \\ &= (\beta F_2) \circ (G_1\alpha) \\ &= (G_2\alpha) \circ (\beta F_1) \end{aligned}$$

This above operation gives us that horizontal functor composition actually defines a functor:

$$\begin{aligned} [\mathcal{D}, \mathcal{E}] \times [\mathcal{C}, \mathcal{D}] &\longrightarrow [\mathcal{C}, \mathcal{E}] \\ (G, F) &\longmapsto G \circ F \\ (\alpha, \beta) &\longmapsto \beta * \alpha \end{aligned}$$

It is useful to know which morphisms(natural transformations) in functor category

$[\mathcal{C}, \mathcal{D}]$ are isomorphisms.

PROPOSITION.

The following are equivalent for $\alpha : F \Rightarrow G$, F, G are functors from $\mathcal{C}$ to $\mathcal{D}$.

- For all $X \in |\mathcal{C}|$, $FX \xrightarrow{\alpha_X} GX$ is an isomorphism in $\mathcal{D}$.
- α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$.

Such natural transformations are called **Natural Isomorphisms**.

Now we will step away from natural transformation and look at another construction of new category via existing category.

3.4 *Slice Category, I-indexed families of sets*

DEFINITION. Slice Category Given a category $\mathcal{C}$ and an object $I \in |\mathcal{C}|$.

The **Slice Category** $\mathcal{C}/I$ of $\mathcal{C}$ over I is defined as:

- **Objects:** All morphisms of the form $X \xrightarrow{p} I$ in *mathcal*C.
- **Morphisms(from** $X \xrightarrow{p} I$ **to** $Y \xrightarrow{q} I$**):** All morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$ such that the diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & I & \end{tikzcd}$$

- The identities and compositions are the same as in $\mathcal{C}$.

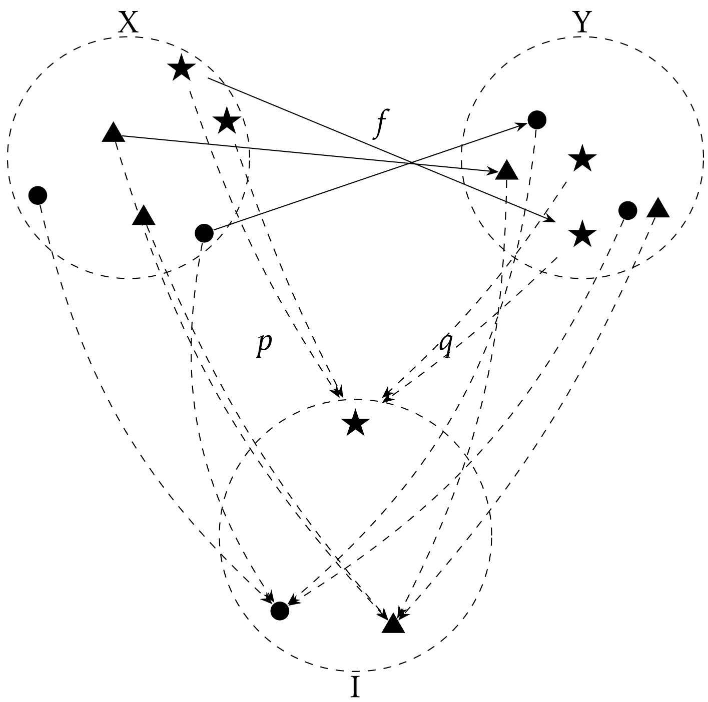
There is also **Co-slice Category** $I/\mathcal{C}$ of $\mathcal{C}$ under I. The objects are all morphisms $I \xrightarrow{\iota} X$ in $\mathcal{C}$, and similarly the morphisms are $X \xrightarrow{f} Y$ such that the corresponding diagram commutes.

Can also define the co-slice category $I/\mathcal{C}$ as $(\mathcal{C}^{op}/I)^{op}$.

EXAMPLE. (Slice Category in **Set**)

Let I be a set. Consider **Set**/I, an object in **Set**/I is a function $X \xrightarrow{p} I$. We can mark the points in X according to where it maps in I by p.

Now consider $X \xrightarrow{p} I$ and $Y \xrightarrow{q} I$, $X \xrightarrow{f} Y$ makes the diagram commutes iff it preserves shapes:



A morphism $f \in \mathbf{Set}/\mathrm{I}(p,q)$ is equivalent to a family of functions:

$$\left(f : p^{-1}(i) \longrightarrow q^{-1}(i)\right)_{i\in\mathrm{I}}$$

The slice category has a very strong connection to another naturally defined category which is the category of high index families of sets, whose morphisms are functions which preserve index.

DEFINITION. (I-indexed families of sets $\mathbf{Fam}_\mathrm{I}$)

Let I be a set, we define **I** to be the **discrete category** of I, whose objects are elements in I, and the only morphisms are the identities.

Now we define:

$$\mathbf{Fam}_\mathrm{I} = [\mathbf{I}, \mathbf{Set}]$$

i.e. the functor category from **I** to **Set**.

Equivalently, the objects of $\mathbf{Fam}_\mathrm{I}$ are I-indexed families of set $(\mathrm{X}_i)_{i\in 1\mathrm{I}}$, and the morphisms $\mathbf{Fam}_\mathrm{I}((\mathrm{X}_i),(\mathrm{Y}_i))$ are equivalent to all families $(\mathrm{X}_i \xrightarrow{f_i} \mathrm{Y}_i)_{i\in\mathrm{I}}$.

The identities are the I-indexed families of identities on each i, and the compositions are compositions index-wise.

3.5 *Equivalence of Categories*

Now we will show that the slice category is "basically" the same as $\mathbf{Fam}_\mathrm{I}$, but one needs to be careful in how to formulate it. For example, in slice category, we can view the object p as for any $i \in \mathrm{I}$, we can maps i to $p^{-1}(i)$, and take this as any object in $\mathbf{Fam}_\mathrm{I}$. But in this case, each $p^{-1}(i)$ is disjoint to each other, where objects in $\mathbf{Fam}_I$ are arbitrary families of sets. What we can do is formulate functors between each of two categories(one for each direction), which show 2 categories are very close related in a way called **Equivalence of Categories**.

EXAMPLE. Define two functors:

$$F : \mathbf{Set}/I \longrightarrow \mathbf{Fam}_I$$
$$(X \xrightarrow{p} I) \longmapsto (p^{-1}(i))_{i \in I}$$

$$S : \mathbf{Fam}_I \longrightarrow \mathbf{Set}/I$$
$$(X_i)_{i \in I} \longmapsto \Sigma_{i \in I} X_i = \{(i, x) \mid x \in X_i,\ i \in I\} \text{ (This is called the disjoint union)}$$

The mapping of morphisms is obvious.

Now we have two natural isomorphisms:

$$\Phi : 1_{\mathbf{Set}/I} \Rightarrow SF$$
$$\Psi : 1_{\mathbf{Fam}_I} \Rightarrow FS$$

Can easily verify there are indeed such natural isomorphisms.

DEFINITION. (Equivalence of Categories)

An equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$ is given by (F, G, α, β) where:

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$
$$G : \mathcal{D} \longrightarrow \mathcal{C}$$
$$\alpha : 1_{\mathcal{C}} \Rightarrow SF$$
$$\beta : 1_{\mathcal{D}} \Rightarrow FS$$

Where α, β are natural isomorphisms.

Equivalently, the following diagrams holds:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "1_{\mathcal{C}}"] \arrow[r, phantom, "\alpha \Downarrow\cong"] & \mathcal{C} \arrow[l, bend left, "GF"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "1_{\mathcal{D}}"] \arrow[r, phantom, "\beta \Downarrow\cong"] & \mathcal{D} \arrow[l, bend left, "FG"]
\end{tikzcd}
$$

Notice this is not an isomorphism between categories because FG, GF are not **equall to** the identity functors, instead we get functors naturally isomorphic to identities functors.

There are lots of nice equivalence in mathematics, for example, a famous one is the category of compact Hausdorff space with continuous functions is equivalent to the opposite of the category of commutative C^* algebras.

Now we're going to introduce an example of two categories that's a little bit weaker than a equivalence, but are still very similar. Fist we introduce a new category:

$\mathbf{Mat_K}$:

- **Objects:** $\mathbb{N}$: all natural number.
- **Morphisms:** $\mathbf{Mat_K}(n, m) = M_{m\times n}(\mathbb{K})$: all $m \times n$ matrices with entries in $\mathbb{K}$.
- **Identities & Composition:** Identity matrices and matrices composition.

Normally we name a category with its objects, but in this case the matrices are the morphisms of category.

Matrices play a central role in linear algebra. For example, to represent a transformation between two finite-dimensional vector spaces, we typically begin by choosing suitable bases for each space and then express the transformation as a matrix. There is a strong relationship between the category of matrices and the category of vector spaces.

There is a functor:

$$
\begin{aligned}
J : \mathbf{Mat}_{\mathbb{K}} &\longrightarrow \mathbf{Vect}_{\mathbb{K}} \\
n &\longmapsto \mathbb{K}^n \\
(n \xrightarrow{A} m) &\longmapsto T_A, \text{ where } [T_A]_\beta = A, \beta \text{ is the standard basis of } \mathbb{K}^n.
\end{aligned}
$$

This functor J is **full** and **faithful**, with the following definition:

DEFINITION. (Full & Faithful Functor)

A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is called:

- **full** if for any objects X, Y in $\mathcal{C}$, and for any $g \in \mathcal{D}(FX, FY)$, there exists $f \in \mathcal{C}(X, Y)$ such that $Ff = g$.
- **faithful** if for any morphisms $f \neq g$ in $\mathcal{C}(X, Y)$, $Ff \neq Fg$.

A full and faithful functor is called **fully faithful** or an **embedding** of categories.

Consider $\mathbf{FDVect}_{\mathbb{K}}$, the category of finite-dimensional vector space, and $\mathbf{Mat}_{\mathbb{K}} \xrightarrow{J} \mathbf{FDVect}_{\mathbb{K}}$ is again full and faithful. But in this case J satisfies another property, described as follow:

It's not true that every finite dimensional vector space **equals** to $\mathbb{K}^n$, although they may share the same dimension, they can still be distinct as objects. For example, $\mathbb{K}^n$ and its dual $(\mathbb{K}^n)^*$ are not the same vector space from category perspective, even though they have the same dimension. But what is true is that for any n-dimensional vector space V, there is an isomorphism between V and $\mathbb{K}^n$.

LEMMA. If $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is fully faithful, then F reflects isomorphisms, i.e. for any morphism f, if Ff is an isomorphism, then f is also an isomorphism. Sometimes we called such F is **conservative**.

DEFINITION. We say $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is **essentially surjective on objects** if:

$$\forall Y \in |\mathcal{D}|,\ \exists X \in |\mathcal{C}| \text{ s.t. } FX \cong Y$$

DEFINITION. A functor F is called a **weak equivalence** if F is essential surjective on objects and fully faithful.

THEOREM.

1. If F arises as part of an equivalence (F, G, α, β), then F is a weak equivalence.

2. If F is a weak equivalence, then assuming a suitable version of axiom of choice, F arise as part of an equivalence (F, G, α, β).

In the case of $\mathbf{Mat}_{\mathbb{K}} \xrightarrow{J} \mathbf{FDVect}_{\mathbb{K}}$, the "choice" is to find a basis for each finite dimensional vector space in $\mathbf{FDVect}_{\mathbb{K}}$.

Proof. (of 2)

Using the axiom of choice, define $G : \mathbf{FDVect}_{\mathbb{K}} \to \mathbf{Mat}_{\mathbb{K}}$ by choosing, for each $Y \in |\mathbf{FDVect}_{\mathbb{K}}|$, an object $X \in |\mathbf{Mat}_{\mathbb{K}}|$ such that $Y \underset{\beta_Y}{\xrightarrow{\sim}} FX$, and set $GY = X$ (possible since F is essentially surjective).

We also need action of G on morphisms. Given $Y \xrightarrow{g} Y'$, since we want β to be a natural transformation, the following diagram must commute:

$$\begin{tikzcd} Y \arrow[r, "\beta_Y"] \arrow[d, "g"'] & FGY \arrow[d, "FGg"] \\ Y' \arrow[r, "\beta_{Y'}"'] & FGY' \end{tikzcd}$$

Necessarily, $FGg = \beta_{Y'} \circ g \circ \beta_Y^{-1}$. Since F is fully faithful, Gg is uniquely defined. Finally, we need to verify such G is well-defined, i.e. it preserves identities and compositions. (easily verified by drawing a diagram)

To build an equivalence, we also need to define a natural isomorphism $\alpha : 1 \Rightarrow GF$. Since $FX \xrightarrow{\beta_{FX}} FGFX$ is an isomorphism, and by the lemma above F reflects isomorphisms, so exists an isomorphism $X \xrightarrow{F^{-1}\beta_{FX}} GFX$. □

What is the "suitable version" of axiom of choice?

For locally small categories, we need the **axiom of global choice**.

For small categories, the ordinary axiom of choice suffices.

There is a nice category-theoretic formulation of axiom of choice:

In **Set**, every epimorphism splits.

In **Set**, suppose $X \xrightarrow{s} Y \xrightarrow{r} X$ and $r \circ s = 1_X$, then r is epi and s is mono.

In any category, an epi $Y \xrightarrow{r} X$ is **split** if there exists $X \xrightarrow{s}$ s.t. $r \circ s = 1_X$.

Similarly, a mono $X \xrightarrow{s} Y$ is **split** if there exists $Y \xrightarrow{r} X$ s.t. $r \circ s = 1_X$.

If one has such an $X \xrightarrow{s} Y \xrightarrow{r} X$, then we say Y is a **retract** of X, and the epi r is called a **retraction**, the mono s is called a **section**, and the composite $t = Y \xrightarrow{s \circ r} Y$ is an idempotent, that is $t = t \circ t$.

How does this connect to axiom of choice?

We can view a surjective function $Y \xrightarrow{g} X$ as an indexed set:

$$(Y_\alpha)_{\alpha \in X}, \text{ where } Y_\alpha = \{y \in Y | g(y) = \alpha\}$$

Since g is surjective, Y_α is non-empty, and since it splits, there exists a **selection** $X \xrightarrow{f} Y$ such that $g \circ f = 1_X$.

For each $x \in X$, such f **choose** an $f(x) \in Y$ such that $f(x) \in Y_x$.

4 CONSTRUCTIONS WITHIN CATEGORIES

4.1 *Pullbacks*

In a category $\mathcal{C}$, a **pullback** of a pair of maps with common codomain (such pair of maps is called a **cospan**) $X \xrightarrow{f} Z \xleftarrow{g} Y$ is given by a **span** $X \xleftarrow{p} P \xrightarrow{q} Y$ for which $f \circ p = g \circ q$, and such that for every $X \xleftarrow{\alpha} W \xrightarrow{\beta} Y$ with $f \circ \alpha = g \circ \beta$, there exists a unique $W \xrightarrow{w} P$ such that $p \circ w = \alpha$ and $q \circ w = \beta$.

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists! w"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& P \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Given a cospan, it may or may not have a pullback. For some nice categories, the pullback always exists. For example, **Set** always has pullbacks:

$$\text{For } X \xrightarrow{f} Z \xleftarrow{g} Y,\ P = \{(x, y) \in X \times Y | f(x) = g(y)\}, \text{ with the projections } \pi_1, \pi_2$$

$$\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

We place a $\lrcorner$ symbol at the upper left corner to indicate that the diagram is a pullback.

Again in **Set** can view a cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$ as a pair of Z–indexed set $(X_z)_{z \in Z}, (Y_z)_{z \in Z}$, and the pullback of f, g is the **fiber-product** $(X_z \times Y_z)_{z \in Z}$.

Example.

- Pullback as inverse-image of function

$$\begin{tikzcd}
P \arrow[r, "\iota_X"] \arrow[d, "f|_P"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y' \arrow[r, "\iota_Y"'] & Y
\end{tikzcd}$$

  Where $P = f^{-1}(Y')$, and ι_X, ι_Y are the inclusion maps.

- Pullback as intersection of sets

$$\begin{tikzcd}
P \arrow[r, "\iota_{P,1}"] \arrow[d, "\iota_{P,2}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "\iota_1"] \\
X_2 \arrow[r, "\iota_2"'] & X
\end{tikzcd}$$

  Where $P = X_1 \cap X_2$, and the ι's are inclusions with corresponding domain and codomain.

Propositions:

- In any category $\mathcal{C}$, pullbacks of the same cospan are unique up to isomorphism.
- The maps $X \xleftarrow{p} P \xrightarrow{q} Y$ in a pullback are **jointly monomorphic**, i.e. given any $P \xleftarrow{v} W \xrightarrow{u} P$, if $p \circ u = p \circ v$ and $q \circ u = q \circ v$, then $u = v$.
- Pullback preserves monomorphisms. Given a pullback:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

If f is a monomorphism, then q is also a monomorphism, similarly, if g is a mono, then p is also a mono.

LEMMA. (Pullback lemma)

For a diagram of two commuting square sharing a common edge:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

Where (a) is the left-square diagram, (b) is the right-square diagram, and let (a)+(b) denote the outer square diagram by composing the upper and lower two arrow.

- If (a) and (b) are both pullbacks, then (a)+(b) is also a pullback.
- If (a)+(b) is a pullback, and (b) is a pullback, then (a) is a pullback.

Will prove the generalization of the second statement:

For a diagram:

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"'] \arrow[dr, phantom, "(a)"] & X' \arrow[r, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "(b)"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "g'"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

- If (a)+(b) is a pullback, and h, f' are jointly monomorphic, then (a) is a pullback.

Proof. We need to prove for every $Y'' \xleftarrow{y} W \xrightarrow{x} X'$, there exists an unique $W \xrightarrow{w} X''$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[dr, "w"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\ & X'' \arrow[r, "h'"] \arrow[d, "f''"'] & X' \arrow[d, "f'"] \\ & Y'' \arrow[r, "g'"'] & Y' \end{tikzcd}$$

Since (a)+(b) is a pullback, for $W \xrightarrow{h \circ x} X$ and $W \xrightarrow{y} Y''$, there exist an unique w such that the following diagram commute:

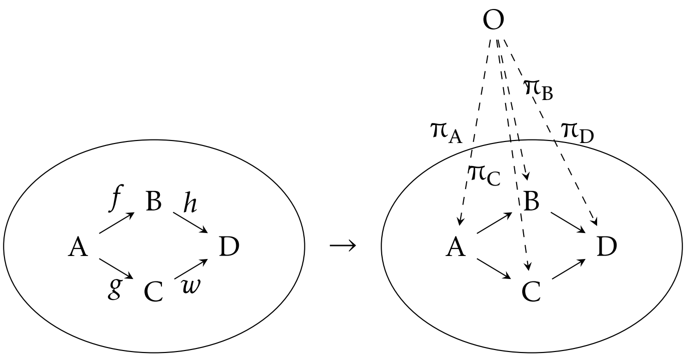
$$\begin{tikzcd}
W \arrow[dr, "w"] \arrow[drr, bend left, "x"] \arrow[drrr, bend left=40, "h \circ x"] \arrow[ddr, bend right, "y"'] & & & \\
& X'' \arrow[r, dash, "h \circ h'"] \arrow[d] & (X') \arrow[r] & X \arrow[d] \\
& Y'' \arrow[r, dash] & (Y') \arrow[r, "g \circ g'"'] & Y
\end{tikzcd}$$

To make the first diagram (a) commute, we need to check that $h' \circ w = x$. Since $h \circ x = h \circ (h' \circ w)$ and $f' \circ x = g' \circ y = g' \circ f'' \circ w = f' \circ h' \circ w$, by jointly monomorphic, we obtain that $x = h' \circ w$. $\square$

4.2 *Limit*

Pullback is a special case of a general construction in category called **limit**.

DEFINITION (Cone). A cone (O, π) of a diagram is an **object** together with a **collection of morphisms**, such that every triangle commutes.



diagram

- $g \circ \pi_A = \pi_C$
- $f \circ \pi_A = \pi_B$
- $h \circ \pi_B = \pi_D$
- $w \circ \pi_C = \pi_D$

DEFINITION (limit). A limit of a diagram is a cone (L, π), which satisfies that for any other cone (O, π'), there exists an **unique** morphism $O \xrightarrow{\phi} L$ such that the diagram of two cones together with ϕ commutes.

Can view the limit as the core, essence cone that any cone uniquely collapse into the limit.

Fact: Limits are unique up to isomorphism.

Pullback is actually the limit of a diagram with the form $\rightarrow\leftarrow$. Now we will give some general results about limit:

EXAMPLE.

- Binary product

  A **binary product** for X, Y is a span $X \xleftarrow{p} P \xrightarrow{q} Y$ such that for any $X \xleftarrow{x} Z \xrightarrow{y} Y$, there exists a unique $Z \xrightarrow{z} P$ such that $p \circ z = x$ and $q \circ z = y$. Binary product is actually the limit of a two-point diagram with no arrows.

  A category $\mathcal{C}$ is said to have binary products if, for every $X, Y \in |\mathcal{C}|$, the binary product of X, Y exists.

$$\begin{tikzcd} & Z \arrow[d, dashed, "z"'] \arrow[ddl, bend right, "x"'] \arrow[ddr, bend left, "y"] & \\ & P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

  - **Set** has products, the cartesian products.
  - $\mathbf{Vect_K}$ has products, the cartesian products of vector spaces.
  - **Grp** has products, the direct products.
  - **Top** has products, the topological products of spaces.

  We often write $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ for a chosen product in any category $\mathcal{C}$.

- I-indexed product in $\mathcal{C}$

  $I \in |\mathbf{Set}|$ and consist of objects in $\mathcal{C}$.

  A product of a pamily $(X_i)_{i\in I}$ is an object P and a family $(P \xrightarrow{p_i} X_i)_{i\in I}$ such that for any Z and family $(Z \xrightarrow{x_i} X_i)_{i\in I}$ there exists a unique $Z \xrightarrow{z} P$ such that for any $i \in I$, $p_i \circ z = x_i$.

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[dr, "x_i"'] & P \arrow[d, "p_i"] \\ & X_i \end{tikzcd}$$

  A special case of indexed product is which the indexed set is empty. In this case the product is simply an object T, and for any other object Z there exists a unique map $Z \xrightarrow{u} T$. Such a T is called a **terminal object**. The usual notation for such unique morphism is $Z \xrightarrow{1_Z} T$.

  We say a category $\mathcal{C}$ has product if for every index family, the product exists.

Proposition. For a category $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has finite product.
2. $\mathcal{C}$ has a terminal object and binary product.

- Equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$

  An equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a map $E \xrightarrow{e} X$ such that $f \circ e = g \circ e$ and for any $Z \xrightarrow{z} X$ such that $f \circ z = g \circ z$, there exists a unique $Z \xrightarrow{w} E$ such that $e \circ w = z$.

  - **Set** has equalizers. $E = \{x \in X | f(x) = g(x)\}$, e is the inclusion map.

In any category, the equalizer $E \xrightarrow{e} X$ is a monomorphism.(exercise)

It's not true in general that the monomorphisms airse as an equalizer. Those monomorphisms which are also equalizers are called **regular monomorphisms**.

In **Set**, every monomorphism is regular, however in **Top** not every monomorphism is regular.

4.3 *Graph*

Diagram shapes are graphs.

Pullback		$\bullet \to \bullet \leftarrow \bullet$
Binary Product		$\bullet \quad \bullet$
I-indexed Product		$\bullet \to \bullet \to \bullet \to \bullet \to \cdots \to \bullet$
	is the limit for	
Terminal Object		empty diagram
Equalizer		$\bullet \rightrightarrows \bullet$
Projective limit		$\cdots \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet$ (extended to infinity to the left)

Specifically **directed multigraphs** (will just call them graph instead in this lecture), also called **quivers**. In this lecture a graph G is given by a collection $|G|$ of **vertices** and for every $u, v \in |G|$ a collection $G(u, v)$ of edges with source u and target v.

The definition is similar with category except graph don't require identities and compositions.

We say a graph is **small/locally small/finite/locally finite** in the obvious way.

Just like graph generalize category, there is also generalization of functor which is **graph morphism**.

A graph morphism $G \xrightarrow{F} G'$ is given by a map of vertices $|G| \xrightarrow{F_o} |G'|$ and for every $u, v \in |G|$ a map of edges $G(u,v) \xrightarrow{F_{u,v}} G'(F_o(u), F_o(v))$.

Now we can define a graph category **Graph**, whose objects are all small graphs and morphisms are graph morphisms.

A diagram over a graph G in a category $\mathcal{C}$ is a graph morphism from G to $F_G(\mathcal{C})$, where F_G is the forgetful functor from **Cal** to **Graph**.

The limit for a graph is defined in the obvious way.

Definition. A category $\mathcal{C}$ is (**complete**/**finitely complete**) if every (**small**/**finite**) diagram has a limit.

Theorem. The following are equivalent:

(1) $\mathcal{C}$ is complete.

(2) $\mathcal{C}$ has products and equalizers.

Theorem. The following are equivalent:

(1) $\mathcal{C}$ is finitely complete.

(2) $\mathcal{C}$ has finite products and equalizers.

(3) $\mathcal{C}$ has a terminal object and pullbacks.

4.4 *Exercise*

1. What are products in the category **Rel** of relations?
2. In any category, the equalizer $E \xrightarrow{e} X$ is a monomorphism.
3. Prove that every monomorphism in **Set** is regular.
4. In **Top**, regular monomorphisms coincide with embeddings (continuous functions that preserves the topological structures).
5. Prove that the second and third statement in the last theorem are equivalent.