Solution

Let

$$(1+\sqrt{2})^n = a_n + b_n\sqrt{2} \in \mathbb{Z}(\sqrt{2})$$

Since

$$(\sqrt{2}+1)^{2026} + (\sqrt{2}-1)^{2026} = 2a_n, \text{ and } \sqrt{2}-1 < 1$$

Hence

$$\lfloor (1+\sqrt{2})^{2026} \rfloor = \lfloor 2a_n - (\sqrt{2}-1)^{2026} \rfloor = 2a_n - 1$$

There is a recursion relation:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

Consider the vecotr $(a_n, b_n)^T$ mod 10:

$$\begin{aligned} (1,0)^T &\to (1,1)^T \\ &\to (3,2)^T \\ &\to (7,5)^T \\ &\to (17,12)^T \equiv (7,2) \\ &\to (11,9) \equiv (1,9) \\ &\to (19,10) \equiv (-1,0) \end{aligned}$$

So the cycle is of length 12. Let $\tilde{a}_n \coloneqq a \pmod{10}$.
Sincd $2026 = 12 \cdot 168 + 10$, $a_{2026} = \tilde{a}_{10} \equiv -7 \equiv 3$, the answer is $2 \cdot 3 - 1 = 5$.