CS5319 Advanced Discrete Structure

Exam 2 – December 03, 2019 (2 hours)

Answer all questions. Total marks = 105. Maximum score = 105 (out of 100).

1. Solve the following recurrence relations:

   (a) (10%) $a_n = 2a_{n-1} + 4^n$ for $n \geq 1$, and $a_0 = 1$.

   (b) (15%) $a_n + 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$, and $a_0 = 1$, $a_1 = 9$.

2. Let a_n be the number of length-n bitstrings that end with 0 but have no consecutive 0s. Let b_n be the number of length-n bitstrings that have exactly one pair of consecutive 0s.

   For example, $a_3 = 2$ and $b_3 = 2$.

   (a) (15%) Express a_n as a recurrence relation, and get the closed form of a_n.

   (b) (5%) Express b_n in terms of $a_1, a_2, \ldots, a_n$.

   (c) (5%) Let $A(x)$ and $B(x)$ be the generating function for the sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$, respectively. Express $B(x)$ in terms of $A(x)$.

   (d) (5%) Find the value of b_7. (No explanation is needed for this part.)

   Hint: Guess the sequence $(a_0, a_1, a_2, \ldots)$ and use the result of (b).

3. (15%) Let $\Delta(G)$ denote the maximum degree among all the vertices in an undirected graph G. Show that for any undirected graph G, we can color its vertices with at most $\Delta(G) + 1$ colors such that no two adjacent vertices share the same color.

   Hint: Induction on the number of vertices.

4. In the 17th century, Pierre de Fermat claimed that "the equation $x^n + y^n = z^n$ does not have positive integral solution (x, y, z) when n is an integer with $n \geq 3$." This claim is commonly known as "Fermat's last theorem, as mentioned in the class. Despite efforts by many mathematicians, this theorem is only proven by Andrew Wiles in 1994, which was 329 years after Fermat's death.

   (15%) Use this theorem (or otherwise) to show that $\sqrt[n]{2}$ (the nth root of 2) is not a rational number for all integers $n \geq 3$.

5. (10%) Show that there exists a polynomial $x^3 + ax^2 + bx + c$, where $a, b, c = \pm 1$, such that its roots are all real.

6. (10%) Show that for any set S of ten distinct integers chosen from $[1, 100]$, we can always find two disjoint subsets of S, such that the sum of the integers in these subsets are equal.

   Example: Suppose $S = \{1, 4, 15, 26, 28, 49, 60, 83, 94, 100\}$. Then, we may find subsets

   $$S_1 = \{28, 60\} \qquad S_2 = \{1, 4, 83\}$$

   such that the sum of integers of these two disjoint subsets are equal.

   Hint: First, forget about the constraint "disjoint".