1080 62135 呂佳恩

1. if we have a derangement with len "n-1". We want to add another component in the derangement. This element can only be in the first n-1 spaces. Suppose this element is in the i th position, for the element originally in the i th position there are two situations.

1. the original element at i th position swap places at n th then we have to handle the n-2 elements left. The permutation is D_{n-2}
2. the i th element is not at the n th position. it is just like n-1 objects permutations. being D_{n-1}

$$\Rightarrow D_n = (n-1)(D_{n-1} + D_{n-2})$$

(b) $D_n = (n-1)(D_{n-1} + D_{n-2})$

$$= nD_{n-1} + nD_{n-2} - D_{n-1} - D_{n-2}$$

$$D_n - nD_{n-1} = -(D_{n-1} - (n-1)D_{n-2})$$

Denote $P_n = D_n - nD_{n-1}$

$$P_{n-1} = D_{n-1} - (n-1)D_{n-2}$$

$$P_n = -P_{n-1}$$

$$P_{n-1} = -P_{n-2}$$

$$\vdots$$

$$\times\ P_3 = -P_2$$

$$P_n = (-1)^n$$

$$D_n - nD_{n-1} = (-1)^n$$

$$D_n = nD_{n-1} + (-1)^n$$

2. (a) $a_r - 6a_{r-1} + 8a_{r-2} = 2^r$

$x^2 - 6x + 8 = 0.$ $\quad (x-4)(x-2) = 0$

$a_n = A \cdot 4^n + B\,2^n$

Homogeneous solution : $a_n = A \cdot 4^n + B \cdot 2^n$

Particular solution: Guess $a_n = C \cdot n \cdot 2^n$

$a_n - 6a_{n-1} + 8a_{n-2} = 2^n$

$Cn - 3C(n-1) + 2C(n-2) = 1$

$3C - 4C = 1 \quad C = -1$

$a_n = -n \cdot 2^n + A \cdot 4^n + B \cdot 2^n$

$$\begin{cases} a_0 = 2 = A + B \\ a_1 = 10 = -2 + 4A + 2B \end{cases} \Rightarrow \begin{matrix} A = 4 \\ B = -2 \end{matrix}$$

$\Rightarrow a_n = 4 \cdot 4^n - 2 \cdot 2^n - n \cdot 2^n$

$= 4^{n+1} - 2^n(n+2)$ ※

(b) $a_n - 5a_{n-1} + 3a_{n-2} + 9a_{n-3} = 0$

$a_0 = 7.$
$a_1 = 16.$ $\Rightarrow$ $x^3 - 5x^2 + 3x + 9 = 0.$
$a_2 = 65$ $\quad (x+1)(x-3)^2 = 0$

$\Rightarrow a_3 = 5a_2 - 3a_1 - 9a_0 = 325 - 48 - 63 = 214$

$a_n = (A + Bn)(3)^n + C \cdot (-1)^n$

$$\begin{matrix} A + C = 7. \\ 3A + 3B - C = 16. \\ 9A + 18B + C = 65 \end{matrix} \Rightarrow \begin{matrix} A = 5 \\ B = 1 \\ C = 2 \end{matrix} \Rightarrow a_n = (5+n)\,3^n + 2(-1)^n$$ ※

3. $b_r = a_r^2$

$b_r - 2b_{r-1} = 1 \qquad b_0 = 4$

Suppose $\quad b_r = 2^r \cdot A - 1$

$b_0 = A - 1 \quad \Rightarrow A = 5$

$b_r = 5 \cdot 2^r - 1$

$a_r = \sqrt{5 \cdot 2^r - 1}$

(b)

$b_r = \lg a_r$

$a_r^2 - 2a_{r-1} = 0$

$2b_r - (1 + b_{r-1}) = 0 \qquad b_0 = 2$

$2b_r - b_{r-1} = 1$

suppose $\quad b_r = A(2)^{-r} + 1$

$b_0 = A + 1 = 2 \qquad A = 1$

$b_r = 1 + (2)^{-r}$

$a_r = 2^{1+(2)^{-r}}$

$= 2^{(2^{-r}+1)}$

4. $$\sum_{r=2}^{\infty}(a_r - 5a_{r-1} + 6a_{r-2})x^r = 0$$

$$A(x) - a_1 x - a_0 - 5x(A(x) - a_0) + 6x^2 A(x) = 0$$

$$A(x) = \frac{6.}{1-5x+6x^2}$$

(b)

$$\sum_{r=2}^{\infty}(a_r - 2a_{r-1} - 3a_{r-2})x^r$$

$$\Rightarrow \sum_{r=2}^{\infty}(4^r + 6)x^r$$

$$A(x) - a_1 x - a_0 - 2x(A(x) - a_0) - 3x^2 A(x)$$

$$= \frac{16x^2}{1-4x} + \frac{6x^2}{1-x}$$

$$A(x) = \frac{\frac{6x^2}{1-x} + \frac{16x^2}{1-4x} + a_1 x + a_0 - 2a_0 x}{3x^2 - 2x + 1}$$

$$= \frac{20x + \frac{16x^2}{1-4x} + \frac{6x^2}{1-x} - 20}{3x^2 - 2x + 1}$$

5. Let $b_n :=$ # of binary strings with 01001 occurring at position $n \Rightarrow 2^{n-5} - b_{n-3}$

$$\underline{\quad}\ \underline{\quad}\ \underline{\quad} \cdots \underline{\quad}\ \underset{n-3}{\underline{0}}\ \underset{n-2}{\underline{1}}\ \underset{n-1}{\underline{0}}\ \underset{n}{\underline{0}}\ \underline{1}$$

$b_1 = b_2 = b_3 = b_4 = 0 \quad b_5 = 1 \quad b_0 = 1$

$x^5 b_5 = 2^0 \cdot x^5 - b_2 x^5$

$x^6 b_6 = 2^1 \cdot x^6 - b_3 x^6$

$\vdots$

+)

$$B(x) - 1 = x^5 \left(\frac{1}{1-2x} - x^3 (B(x) - 1)\right)$$

$$B(x)(x^3+1) = \frac{x^5}{1-2x} + x^3 + 1$$

$$B(x) = \frac{x^5}{(1-2x)(x^3+1)} + \cancel{\frac{x^3+1}{x^3+1}}\ 1 \quad ※$$

(b) C_n be # of n digit string with 01001 occurring at position n for the first time.

$$b_n = C_n + C_{n-5} b_5 + C_{n-6} b_6 + \cdots \quad C_5 b_{n-5} + C_0 b_n$$

$b_1 = b_2 = b_3 = b_4 = 0$

$C_0 = C_1 = C_2 = C_3 = C_4 = 0$

$$x b_1 = x b_1 c_0 + x b_0 c_1$$

$$x^2 b_2 = x^2 b_2 c_0 + x^2 b_1 c_1 + x^2 b_0 c_2$$

$$\vdots$$

$$+)$$

$$B(x) - \underset{1}{\cancel{b_0}} = B(x)\, C(x)$$

$$B(x)\, C(x) = (b_1 c_0 + b_0 c_1) x + (b_2 c_0 + b_1 c_1 + b_0 c_2) x^2 + \cdots$$

$$C(x) = \frac{\cancel{B(x)} - \frac{1}{B(x)}}{\cancel{B(x)}} = 1 - \frac{(1-2x)(1+x^3)}{x^5 + (1-2x)(1+x^3)}$$