1.

(a) Show that the total number of permutations of p red balls and 0, or 1, or 2, …, or q white balls is

$$\frac{p!}{p!}+\frac{(p+1)!}{p!1!}+\frac{(p+2)!}{p!2!}+\dots+\frac{(p+q)!}{p!q!}.$$

assume that there are p red balls and k white balls

The permutation could be denoted as $\frac{(p+k)!}{p!\,k!}$

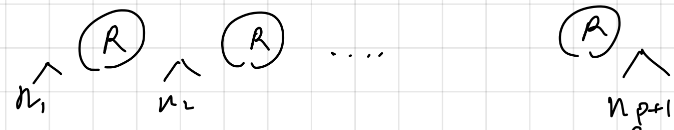
since there are $0 \sim q$ white balls

$$\Rightarrow \sum_{k=0}^{q}\frac{(p+k)!}{p!\,k!} = \frac{p!}{p!}+\frac{(p+1)!}{p!1!}+\frac{(p+2)!}{p!2!}+\dots+\frac{(p+q)!}{p!q!}$$

(b) Show that the sum in part (a) is

$$\frac{(p+q+1)!}{(p+1)!q!}.$$

There are p red balls, we have to put $0 \sim q$ white among them. there are $p+1$ spots among all the balls.



suppose $n_{(i)}$ denote the number of white balls in the ith spot

$$0 \le n_1 + n_2 + \dots\dots n_{p+1} \le q$$

$$H^{p+1}_{0} + H^{p+1}_{1} + H^{p+1}_{2} + \dots \; H^{p+1}_{q} = H^{p+2}_{q} = \binom{p+1}{q} = \frac{(p+q+1)!}{q!\,(p+1)!}$$

(c) Show that the total number of permutations of 0, or 1, or 2, …, or p red balls with 0, or 1, or 2, …, or q white balls is

$$\frac{(p+q+2)!}{(p+1)!(q+1)!}-1.$$

Num of p red with $0 \sim q$ white is $\frac{(p+q+1)!}{q!\,(p+1)!}$

we need to calculate $\sum_{k=0}^{p}\frac{(k+q+1)!}{q!\,(k+1)!}$

$$\frac{(q+1)!}{1!\,q!}+\frac{(q+2)!}{q!\,2!}+\frac{(q+3)!}{q!\,3!}+\dots\frac{(p+q+1)!}{q!\,(p+1)!}+\frac{q!}{q!}-\frac{q!}{q!}$$

This is similar to question (b)

fill this for ease to calculate

$$\Rightarrow \frac{(p+q+2)!}{(p+1)!\,(q+1)!}-\frac{q!}{q!}=\frac{(p+q+2)!}{(p+1)!\,(q+1)!}-1$$

2、

2. (a) Prove the identity using combinatorial arguments

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1$$

$k \cdot k! \Rightarrow$ longest continuous rising sequence starting from 1 is $1, 2, 3, \ldots (n+1-k)$

arrange, $(n-k+2) \ldots$ $n+1$ first $\Rightarrow k!$ ways

insert $1, 2, 3 \ldots (n-k+1)$

k ways (cannot insert before $n-k+2$)

LHS: All cases of possible rising sequence from 1, excluding from $1, 2, \ldots (n+1)$

Therefore It equals to $(n+1)! - 1$

(b) Show that any non-negative integer m can be expressed uniquely in the form:

$$m = a_1 \times 1! + a_2 \times 2! + a_3 \times 3! + \cdots + a_i \times i! + \cdots$$

where each a_i is a non-negative integer, with $0 \le a_i \le i$ for $i = 1, 2, \ldots$.

Base case: $n=1$ is true $1 = 1 \times 1!$
$0 = 0 \times 1!$

Induction: Assume that any integer $m' \le n! - 1$ can be expressed uniquely as $m' = a_1 \times 1! + a_2 \times 2! + a_3 \times 3! + a_4 \times 4! + \ldots + a_{n-1}(n-1)!$
Where $0 \le a_i \le i$ for $i = 1, 2, 3 \ldots (n-1)$

Any Integer between $0 \sim n! - 1$ can be expressed uniquely as $m = a_1 \times 1! + a_2 \times 2! + \ldots \; 0 \times n!$

Any integer between $n! \sim 2n! - 1$ can be expressed uniquely as $m = a_1 \times 1! + a_2 \times 2! + \ldots \; 1 \times n!$

Any integer between $k\,n! \sim (k+1)\,n! - 1$ can be expressed uniquely as $m = a_1 \times 1! + a_2 \times 2! + \ldots \ldots + k \times n!$

⋮

From the above, for any integer n, any integer $m \le (n+1)! - 1$ can be expressed uniquely as

$m = a_1 \times 1! + a_2 \times 2! + a_3 \times 3! + \ldots \; a_n \times n!$

where $0 \le a_i \le i$ for $i = 1, 2, 3 \ldots n$

3.

From n distinct integers, we select two groups of integers such that the smallest integer in the first group is larger than the largest integer in the second group. Furthermore, we require that the first group has m integers, and the second group has k integers. How many possible ways to do the selection? Express your answer in terms of n, m, and k.

1st group has m while second group has k

$m + k \leq n$ (total number of integers)

pick $m+k$ integers, the numbers greater than the mth largest number go into group 1. The rest go into group 2.

$\Rightarrow \binom{n}{m+k}$

4.

How many different five-letter sequences can be made using the letters A, B, C, D such that the sequence does not include the word BAD — that is, sequences such as ABADD, or BADAA, are excluded.

$1 \times 2 \times 3 \times 5$

Total permutation _ _ _ _ _ $\Rightarrow 4^5$

$\boxed{BAD}$ _ _
4×4

_ $\boxed{BAD}$ _

_ _ $\boxed{BAD}$

$4^5 - 4^2 \times 3$

5.

Consider an n-sided polygon where no three diagonals meet at a point. Among all the triangles formed by the polygon's sides and diagonals, how many of them are there such that

(a) all three of its vertices are on the polygon?
(b) two of its vertices are on the polygon and the remaining one is an intersection of two diagonals?
(c) only one of its vertices is on the polygon (and each of the remaining vertices is an intersecton of two diagonals)?
(d) none of its vertices is on the polygon (i.e., all vertices are intersections of diagonals)?

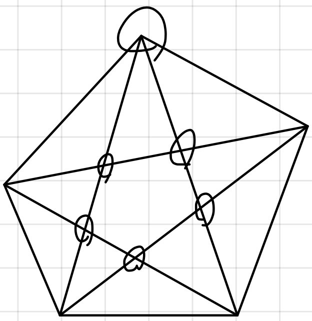
(a) pick three vertices $\binom{n}{3}$

(b) $\binom{n}{2} - n$ diagonals

with $\binom{n}{4}$ intersections

$4 \cdot \binom{n}{4}$ since one intersection creates 4 triagles

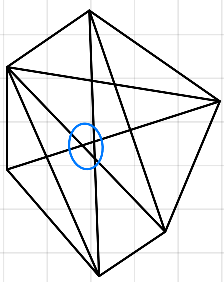
(c) we need 5 vertices



$\binom{n}{5} \cdot \left[\binom{5}{2} - 5\right]$

$= \binom{n}{5} \cdot 5$

(d) 6 vertices could meet the requirements



$\binom{n}{6}$