CS5319 Advanced Discrete Structure

Homework 1
Due: October 12, 2021 (11:59pm)
Exam 1: November 2, 2021

1. (a) Show that the total number of permutations of p red balls and 0, or 1, or 2, ..., or q white balls is
$$\frac{p!}{p!} + \frac{(p+1)!}{p!1!} + \frac{(p+2)!}{p!2!} + \ldots + \frac{(p+q)!}{p!q!}.$$
   (b) Show that the sum in part (a) is
$$\frac{(p+q+1)!}{(p+1)!q!}.$$
   Hint: Recall the "hockey stick" identity from Tutorial 1?
   (c) Show that the total number of permutations of 0, or 1, or 2, ..., or p red balls with 0, or 1, or 2, ..., or q white balls is
$$\frac{(p+q+2)!}{(p+1)!(q+1)!} - 1.$$

2. (a) Prove the identity using combinatorial arguments
$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1$$
   (b) Show that any non-negative integer m can be expressed uniquely in the form:
$$m = a_1 \times 1! + a_2 \times 2! + a_3 \times 3! + \cdots + a_i \times i! + \cdots$$
   where each a_i is a non-negative integer, with $0 \le a_i \le i$ for $i = 1, 2, \ldots$.

3. From n distinct integers, we select two groups of integers such that the smallest integer in the first group is larger than the largest integer in the second group. Furthermore, we require that the first group has m integers, and the second group has k integers. How many possible ways to do the selection? Express your answer in terms of n, m, and k.

4. How many different five-letter sequences can be made using the letters A, B, C, D such that the sequence does not include the word BAD — that is, sequences such as A<u>BAD</u>D, or <u>BAD</u>AA, are excluded.

5. Consider an n-sided polygon where no three diagonals meet at a point. Among all the triangles formed by the polygon's sides and diagonals, how many of them are there such that

   (a) all three of its vertices are on the polygon?
   (b) two of its vertices are on the polygon and the remaining one is an intersection of two diagonals?
   (c) only one of its vertices is on the polygon (and each of the remaining vertices is an intersecton of two diagonals)?
   (d) none of its vertices is on the polygon (i.e., all vertices are intersections of diagonals)?

6. (Lecture Review: No marks) In the lecture, we have shown that the ratio between

$$n! \text{ and } \sqrt{n}\left(\frac{n}{e}\right)^n$$

is equal to some constant K as n tends to infinity, where $K = \sqrt{2\pi}$.[†] The following is a review of what we have done.

First, define

$$I_n = \int_0^{\pi/2} \sin^n x \, dx, \quad \text{for } n = 0, 1, 2, 3, \ldots$$

(a) Compute I_0 and I_1.

(b) Show that $I_{n+1} \leq I_n$, for every n. *Hint:* Use physical meaning.

(c) Show that for all $n \geq 2$,

$$I_n = \frac{n-1}{n} I_{n-2}.$$

Hint: $\int \sin^n x \, dx = \int \sin^{n-1} x \, d(-\cos x)$.

(d) Conclude from (c) that

$$\frac{I_{2n+1}}{I_{2n-1}} = \frac{2n}{2n+1}.$$

(e) Use (b) and (d) to conclude that

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

(f) Use (a), (c), and (d) to show that

$$\frac{I_{2n+1}}{I_{2n}} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2n-1) \cdot (2n-1) \cdot (2n+1)} \cdot \frac{2}{\pi}$$

(g) Taking square root on both sides of (f), and then multiplying $2 \cdot 4 \cdot 6 \cdots (2n)$ to the numerator and the denomiator, show that

$$\sqrt{\frac{I_{2n+1}}{I_{2n}}} = \frac{(2^n n!)^2}{(2n)!\sqrt{2n+1}} \cdot \sqrt{\frac{2}{\pi}}.$$

(h) Substitute $n! \approx K\sqrt{n}\,(n/e)^n$ into (g), and show that $K = \sqrt{2\pi}$ as n tends to infinity.

[†]This proof comes from the SOS Math website, `http://www.sosmath.com/calculus/integration/powerproduct/problem/problem.html`