CS5319 Advanced Discrete Structure

Homework 4
Due: November 29, 2021 (11:59pm)
Exam 2: December 07, 2021

1. Fermat once conjectured that for $n \geq 0$, all numbers $F_n = 2^{2^n} + 1$ are primes. Indeed, the numbers
$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$$
are all primes.

   This conjecture was later disproved by Euler in 1732, who showed that $F_5 = 4294967297 = 641 \times 6700417$. In spite of this, Fermat numbers have many interesting properties. In this question, we shall base on it to give an alternative proof (due to Goldbach) that there are infinitely many primes.

   (a) Show that for all $n \geq 1$, $F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2$.

   (b) Using the result of (a), argue that Fermat numbers are pairwise relatively prime.

   (c) Show that if we pick one prime factor from each Fermat number, they must be all distinct.

   (d) Using the result of (c), conclude that there are infinitely many primes.

2. Consider a 3×9 grid with each grid point either colored red or white.

   Show that no matter how we color the points, there is always a rectangle in the grid whose four corners have the same color. (Refer to Figure 1 for an example.)

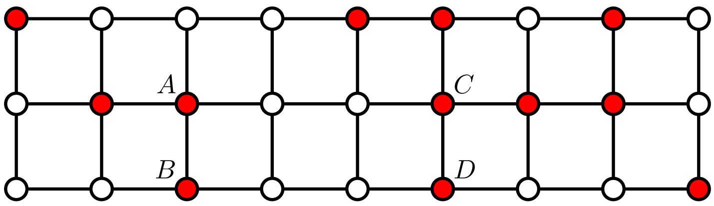

Figure 1: Here, the corners of the rectangle $ABCD$ are all red.

   (Follow-up: No marks) Show that if we replace the grid by a 3×7 grid, the above result still holds. In contrast, show that if we use a 3×6 grid, it is possible that under some coloring, no rectangle in the grid has four corners colored by the same color.

3. Consider a round table with n position, where there is a piece of delicious cake at each position. Magnus and Derek are friends, and they want to play the following game. At the beginning, Magnus starts at a particular position (and takes the cake there). In each round Magnus chooses a positive integer m as the distance of the targeted seat from his current position, while Derek decides a direction, clockwise or counterclockwise, for Magnus to move. The goal of Magnus is to visit as many different positions (and thus takes as many pieces of cake) as possible, while the goal of Derek is to counteract Magnus's goal, by minimizing the number of positions Magnus can move to.

Show that when n is a power of 2, no matter what Derek does, Magnus can always visit all the positions.

Example: When $n = 4$, Magnus can first choose 2. No matter what Derek does, Magnus can always visit the opposite position. Then, by choosing 1, Magnus can move to one of the unvisited positions, and then visit the other by choosing 2 again.

4. Suppose we have a path P with n vertices, $P = (v_1, v_2, \ldots, v_n)$. Show that we can assign each vertex v_i a distinct integer $f(i)$ chosen from 1 to n, such that $|f(i) - f(i+1)|$ for $i = 1, 2, \ldots n-1$ are all distinct.

   For instance, when $n = 5$, the following is a possible assignment satisfying the above requirement: $f(1) = 2$, $f(2) = 5$, $f(3) = 1$, $f(4) = 3$, $f(5) = 4$.

5. After learning the diagonalization technique, Peter has come up with the following proof, showing that the set

$$X = \{\ x \mid x \in (0,1) \ \text{ and } \ x \text{ has } k \text{ decimal places } \ \text{and} \ \ k \in \mathbb{N}\ \}$$

   is uncountable:

   > *We prove this by contradiction. Assume to the contrary that there is a one-to-one correspondence between items in X and items in $\mathbb{N}$. Then, we can list the items in X one by one, say $x_1, x_2, x_3, \ldots$. Now, consider the number x such that its digit in the first decimal place is different from x_1, its digit in the second decimal place is different from x_2, and in general, its digit in the jth decimal palce is different from x_j for all j. Then, x is not listed by the correspondence, and a contradiction occurs as desired.*

   However, each number in X is a rational number; for instance, $0.33215 = 33215/100000$. Thus, $X \subseteq \mathbb{Q}$ (where $\mathbb{Q}$ is countable), which implies X must be countable.

   So, what's wrong with Peter's proof?