CS5319 Advanced Discrete Structure

Exam 2 – December 07, 2021 (13:20–15:10)

Answer all questions. Total marks = 100. Maximum score = 100 (out of 100).

1. Solve the following recurrence relations:

   (a) (15%) $a_n = 2a_{n-1} + 3a_{n-2} + 25 \times 4^{n-2}$ for $n \geq 2$, and $a_0 = 1$, $a_1 = 32$.

   (b) (15%) $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$, and $a_0 = 1$, $a_1 = 15$.

2. (20%) Solve the following recurrence relation for $n \geq 0$:

$$a_n a_{n-2} = (a_{n-1})^2 + 2a_{n-1}a_{n-2} \qquad \text{for } n \geq 2$$

   with initial conditions

$$a_0 = 2, \quad a_1 = 4.$$

3. (20%) Show that for any positive integer k, $2020^{2k} - 1$ is a multiple of 2021.

4. (20%) Show that when 16 distinct numbers are selected from 1 to 100, we must be able to find four distinct numbers w, x, y, z such that they can form two pairs, say (w, x) and (y, z), such that for each pair, either the sum or the difference, is a multiple of 25.

5. (10%) Find four different ways to select five distinct integers from 1 to 9, such that the sum of their squares is a square number.

   Hint: You may write a program if you have time.

4. if we divide any number by 25, there are 25 possible remainders

So we divide 1~100 into 25 groups, with their remainder of divided by 25 being the classifier

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74
75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99
100																								

we pair up the groups so that the remainders add up to 25

(1, 24)

(2, 23)

⋮

etc

we now have

0 (1, 24) (2, 23) (3, 22) ....

if we pick one from the pairs.

e.g 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

then we have picked 13 elements

the remaining 3 elements will either come from existing picked groups, or their paired group is already picked. (pigeon hole)

① if the group is already picked, the difference will be a multiplier of 25

② if their paired group is picked, the sum will be a multiplier of 25

3、 $2020^{2k}-1 = 2021 \times n$

for $k=1$

$2020^{2k}-1 = 4080399 = 2021 \times 2019$

Suppose $k=n$ is correct.

$2020^{2n}-1 = 2021 \cdot p \Rightarrow 2020^{2n} = 2021p+1$

for $k=n+1$

$$2020^{2(n+1)}-1 = 2020^{2n+2}-1$$
$$= 2020^{2n} \cdot 4080400 - 1$$
$$= (2021p+1) \cdot 4080400 - 1$$
$$= 2021p \cdot 4080400 + 4080399$$
$$= 2021p \cdot 4080400 + 2021 \cdot 2019$$

by induction, we find out that $2020^{2k}-1$ for any positive integer k, is a multiple of 2021

5、 Smallest possible

$1^2+2^2+3^2+4^2+5^2 = 55$

largest possible

$5^2+6^2+7^2+8^2+9^2 = \frac{\cancel{9}^{3}\cdot \cancel{8}^{4}\cdot 17}{\cancel{6}} - 30 = 174$

We search for sum 64, 81, 121, 144. 169. 100

1 4 9 16 $\underline{25}$ 36 49 64 81

$169 = 64+49+36+16+4 = 2^2+4^2+6^2+7^2+8^2$

$144 = 1+4+9+49+81$

$100 = 1+9+16+25+49$

$121 = 1+4+16+36+64$

(.

make a smart guess $C\cdot 4^n$ like the proffesor.

$$C4^n = 2C4^{n-1} + 3\cdot C\cdot 4^{n-2} + 25\cdot 4^{n-2}$$

$$16C = 8C + 3C + 25$$

$$16C = 11C + 25$$

$$5C = 25 \qquad C = 5$$

$\Rightarrow$ homogenous solution $A\cdot 3^n + B(-1)^n$

$$x^2 - 2x - 3 = (x+1)(x-3)$$

$$-1 \qquad 3$$

$$a_0 = 1 \qquad a_1 = 32$$

$$1 = A + B + 5 \qquad -4 = A + B$$

$$\cancel{32} = 3A - B + \cancel{20}$$

$$12 = 3A - B$$

$$\Rightarrow a_n = 5\cdot 4^n + 2\cdot 3^n - 6\cdot(-1)^n$$

(b) Make a smart guess $a_n = 3^n(An + B)$

$$a_0 = 1 = B$$

$$a_1 = 3\cdot(A+B) = 15$$

$$A + B = 5 \qquad A = 4$$

Solution : $3^n(4n+1)$

2.

$a_0 = 2$

$a_1 = 4$) 2

$a_2 = 16$) 4

$a_3 = 96$) 6

$a_4 = 768$ 8

$a_n a_{n-2} = (a_{n-1})^2 + 2a_{n-1}a_{n-2}$

$a_{n-1}a_{n-3} = (a_{n-2})^2 + 2a_{n-2}a_{n-3}$

$$a_n = \frac{(a_{n-1})^2}{a_{n-2}} + 2a_{n-1}$$

$$= a_{n-1}\left(\frac{a_{n-1}}{a_{n-2}} + 2\right)$$

$\hookrightarrow 2\cdot(n-1)$

Guess:

$a_n = a_{n-1} \cdot 2n$

$a_{n-1} = a_{n-2} \cdot 2(n-1)$

$\vdots$

$a_1 = a_0 \cdot 2$

$a_0 = 2$

$a_n = 2^{n+1} \cdot n!$ ※

$a_0 = 2$ ✓

$a_1 = 4$ ✓

$a_2 = 16 = 2^3 \cdot 2$ ✓

$a_3 = 96 = 2^4 \cdot 6$ ✓

$a_4 = 768 = 2^5 \cdot 24$ ✓

Guess is correct