Advanced Discrete Structure

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1. (a)

$$F_n = 2^{2^n}+1$$

$$F_n - 2 = 2^{2^n}-1 = \left(2^{2^n/2}-1\right)\left(2^{2^n/2}+1\right)$$

$$= \left(2^{2^{n-1}}-1\right)\left(2^{2^{n-1}}+1\right)$$

$$= \left(2^{2^{n-1}}-1\right)\cdot F_{n-1}$$

$$= \left(F_{n-1}-2\right)\cdot F_{n-1}$$

$$= \left(F_{n-2}-2\right) F_{n-2}\cdot F_{n-1}$$

$$\vdots$$

$$= (\underbrace{F_0-2}_{1})\, F_0 \cdot F_1 \cdots\cdots F_{n-2}\cdot F_{n-1}$$

$$\Rightarrow F_n = F_0 \cdot F_1 \cdot F_2 \cdots F_n \cdot F_{n-1} \ \#$$

(b) since $F_n = F_0 \times F_1 \times \cdots F_{n-1}+2$, we can conclude

that $F_n \% F_0 = 2$, $F_n \% F_1 = 2$, $F_m \% F_1 = 2$.

$0 \le m \le n-1$

then we get $\gcd(F_n, F_m) = \gcd(F_n, 2) = \gcd(2^{2^n}+1, 2)$

$= 1$

we can conclude that Fermat numbers are

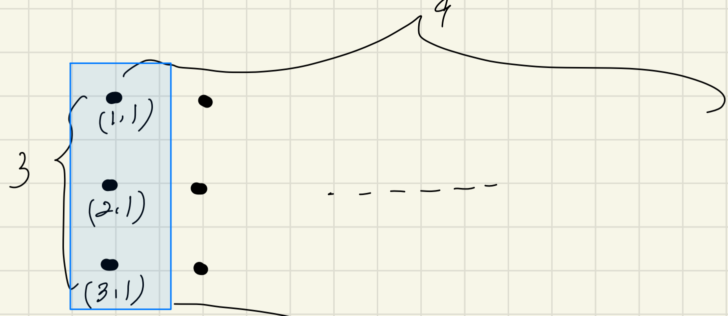
pairwise relatively prime.

(c) If they are not distinct, $\gcd(F_n, F_m) \neq 1$, but we have already proved $\gcd(F_n, F_m) = 1$ in (b).

(d) Since there are infinite ① Fermat numbers, and Fermat numbers are pairwise relative prime.

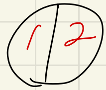
Moreover, by (c), we pick ② 1 distinct prime from each Fermat numbers, then by ①, ②. we get infinitely many primes.

2. 3×9 grid, then we have



and in (1,1), we can put either white or red $\Rightarrow$ 2 choices

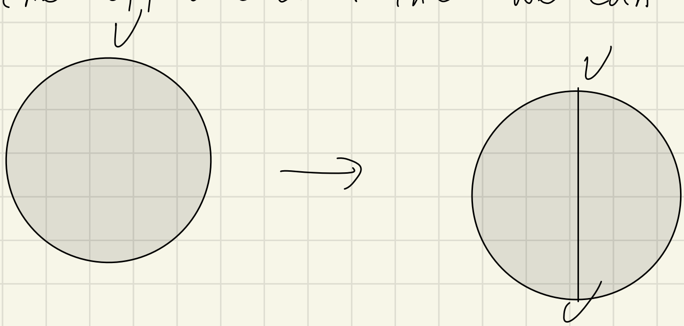
(2,1), (3,1) are all 2 choices then we have 8 choices in one column. However, there are 9 column. By Pigeonhole Principle, there are at least 2 column have same type, then there are four corners colored in same colors.

3. basic case: $n=2$ 

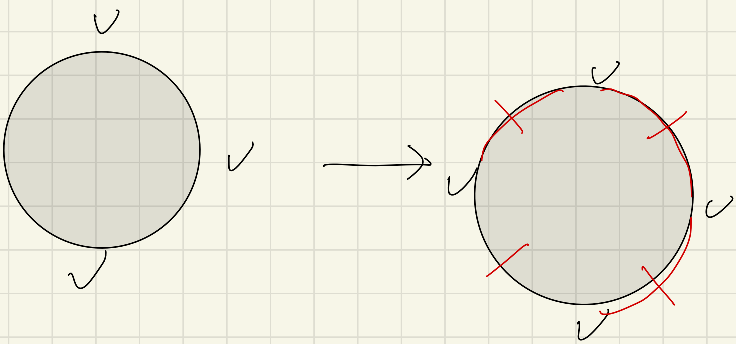
is true

inductive case

we first randomly select a position, and choose the opposite side. Then we can get



接下來我們取分割後的一半,然後再取其對角線



因為是 power of 2, 所以可以繼續沿上圖紅線切割,再取 opposite side, 即可獲得全部 cakes.

4. Since we need $|f(x)-f(x-1)|$ is distinct, then we suppose $f(1)=1$, $f(2)=n-1$

$f(3)=n-(n-1)+(n-2)$, ... then we can get

$|f(x)-f(x-1)|=n-x+1$, $1\le x\le n$, then we can prove that they are distinct.

Then