CS5319 Advanced Discrete Structure

The Last Homework
Due: January 05, 2022 (11:59pm)
Tutorial: January 06, 2022
Exam 3: January 11, 2022

1. Let $(A, *)$ be a semigroup. Show that, for a, b, c in A, if $a * c = c * a$ and $b * c = c * b$, then $(a * b) * c = c * (a * b)$.

2. Let $(A, *)$ be a monoid such that for every x in A, $x * x = e$, where e is the identity element. Show that $(A, *)$ is an abelian group.

3. Let p be a prime. Let $(G, \star)$ be a group. If the size of G is p^2, show that there exists a subgroup $(G', \star)$ of $(G, \star)$ such that $|G'|$ is p.

   Hint: Pick an element $g \in G$ that is not an identity. If g is a generator of $(G, \star)$, can you get some subgroup $(G', \star)$? If g is not a generator, what is the size of the subgroup generated by g?

4. (a) Determine the number of distinct 2×2 chessboards whose cells are painted white and black. Two chessboards are considered distinct if one cannot be obtained from another through rotation.

   (b) Repeat part (a) for 4×4 chessboards.

5. Consider a cube with each face colored by one of the n colors. In how many distinct ways can the cube be colored? (Two colorings are equal if one can be transformed to the other by rotating the cube.)