(1)

1. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ where a_r is the number of ways in which the sum r will show when two distinct dice are rolled, with the first one showing even and the second one showing odd.

For the first dice, the generating function is
$(x^2 + x^4 + x^6)$

For the second dice, the generating function is
$(x^1 + x^3 + x^5)$

$\Rightarrow$ Result : $(x^2+x^4+x^6)(x^1+x^3+x^5)$

$= x^3 + x^5 + x^7 + x^5 + x^7 + x^9 + x^7 + x^9 + x^{11}$

$= x^3 + 2x^5 + 3x^7 + 2x^9 + x^{11}$

(2)

2. How many different ways are there to color n distinct objects $(n \geq 3)$ using 3 colors if every color must be used at least once?

Three colors R, G, B.

For each color,

$EGF_R(x) = EGF_G(x) = EGF_B(x) = \left(\frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots \frac{1}{n!}x^n\right) = e^x - 1$

$EGF_{combined} = (e^x-1)(e^x-1)(e^x-1)$

$= e^{3x} - 3e^{2x} + 3e^x - 1$

Coeff of x^n in $EGF_{combined}$ $\times n!$

$= \cancel{n!}\left(\frac{1}{\cancel{n!}}(3x)^n - 3 \times \frac{1}{\cancel{n!}}(2x)^n + 3 \times \frac{1}{\cancel{n!}}x^n\right)$

$\Rightarrow 3^n - 3(2^n) + 3 \times 1^n$

$= 3^n - 3 \times 2^n + 3$ *

(3)

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3. Find the coefficient of x^n in the following expansion

$$\frac{x+1}{x^2 - x - 6}$$

$\frac{x+1}{x^2-x-6} = \frac{x+1}{5}\left(\frac{1}{x-3} - \frac{1}{x+2}\right)$

$= \frac{x+1}{5}\left[\left(\frac{-1}{3}\right)\frac{1}{1-\frac{x}{3}} - \left(\frac{1}{2}\right)\frac{1}{1+\frac{x}{2}}\right]$

$= \frac{x+1}{5}\left[\left(\sum_{j\geq 0}\left(-\frac{1}{3}\right)^{j+1}x^j\right) + \left(\sum_{j\geq 0}\left(\frac{-1}{2}\right)^{j+1}x^j\right)\right]$

Coeff of x^n

$\begin{cases} \frac{1}{5}\left[\left(-\frac{1}{3}\right)^n + \left(\frac{-1}{2}\right)^n + \left(-\frac{1}{3}\right)^{n+1} + \left(\frac{-1}{2}\right)^{n+1}\right] & (\text{when } n \geq 1) \\ \frac{1}{5}\left[\left(\frac{-1}{3}\right)^{n+1} + \left(\frac{-1}{2}\right)^{n+1}\right] & (\text{when } n < 1) \end{cases}$

(4)

4. Express the following sum in closed form:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}w^m\frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}w^m\frac{x^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{m=0}^{n}\binom{n}{m}w^m$$

$$=\sum_{n=0}^{\infty}\frac{x^n}{n!}(1+w)^n$$

$$=\sum_{n=0}^{\infty}\frac{[x(1+w)]^n}{n!} = e^{(1+w)x}$$

$$\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x\right)$$

(5)

5. Find the exponential generating function of the sequence

$$(1,\ 1\times 5,\ 1\times 5\times 9,\ 1\times 5\times 9\times 13,\ \ldots,\ 1\times 5\times\cdots\times(4r+1),\ \ldots)$$

EGF: $1+\sum_{n=1}^{\infty}\frac{5\times 9\times\cdots(4n+1)}{n!}x^n$

$$=1+\sum_{n=1}^{\infty}\frac{(\frac{1}{4}+1)(\frac{1}{4}+2)\cdots(n+\frac{1}{4})}{n!}(4x)^n$$

$$=1+\sum_{n=1}^{\infty}\frac{(-\frac{1}{4}-1)(-\frac{1}{4}-2)\cdots(-\frac{1}{4}-n)}{n!}(-4x)^n$$

$$=1+\sum_{n=1}^{\infty}\binom{-\frac{5}{4}}{n}(-4x)^n$$

$$\Rightarrow (1-4x)^{-\frac{5}{4}}$$