CS 5319
Advanced Discrete Structure

Lecture 4:
Generating Functions II

Outline

- Introduction
- Generating Functions for

  (1) Combinations

  (2) Permutations
- Distribution of Objects
- More Applications

This Lecture

Generating Functions for Permutations

GF for Permutations

- Suppose we have 2 objects: a, b
- There are 2 ways to arrange 1 object
  - We may describe this by:

$$\boxed{a + b}$$

- There are 2 ways to arrange 2 objects
  - We may describe this by:

$$\boxed{ab + ba}$$

GF for Permutations

- A possible GF for these terms could be :

$$1 + (a + b)\, x + (ab + ba)\, x^2$$

- However, after simplification, we get :

$$1 + (a + b)\, x + (2ab)\, x^2$$

so that the distinct permutations *ab* and *ba* cannot be recognized

GF for Permutations

- Similarly, when we are interested only in the number of permutations, we may want to define some GF like :

$$F(x) = P(n,0) + P(n,1)\,x + P(n,2)\,x^2 + \ldots$$

- Unfortunately, there is no simple closed-form expression for $F(x)$

GF for Permutations

- On the other hand, recall that :

$$
\begin{aligned}
(1 + x)^n = 1 + C(n,1)\, x + C(n,2)\, x^2 \\
+ C(n,3)\, x^3 + \ldots \\
+ C(n, n-1)\, x^{n-1} + x^n \\
= 1 + P(n,1)\, x / 1! + P(n,2)\, x^2 / 2! \\
+ P(n,3)\, x^3 / 3! + \ldots \\
+ P(n, n-1)\, x^n / (n-1)! + P(n,n)\, x^n / n!
\end{aligned}
$$

GF for Permutations

- This motivates us to study another type of GF
- Precisely, to represent a sequence $(a_0, a_1, a_2, \ldots)$, the GF is defined as follows :

$$F(x) = a_0 + a_1 x / 1! + a_2 x^2 / 2! + a_3 x^3 / 3! + \ldots$$

- This GF is called the exponential generating function (EGF) of the sequence
  - coefficient of $x^r = a_r / r!$

GF for Permutations

- Ex : $(1+x)^n$ is the EGF for

$$P(n,0),\ P(n,1),\ \ldots,\ P(n,n)$$

- Ex : e^x is the EGF for

$$1,\ 1,\ 1,\ 1,\ \ldots$$

- Ex : $(1-2x)^{-3/2}$ is the EGF for

$$1,\ \ 1\times 3,\ \ 1\times 3\times 5,\ \ 1\times 3\times 5\times 7,\ \ \ldots$$

GF for Permutations

- The EGF has interesting behavior
- Suppose we have one object
- The EGF for the number of permutations of this object is :

$$\boxed{1 + x}$$

- But when we have n distinct objects (without repetition), the EGF becomes :

$$\boxed{(1 + x)^n}$$

GF for Permutations

- Suppose we have p objects of the same kind
- The EGF for the number of permutations of this object is :

$$\boxed{1 + x + x^2/2! + x^3/3! + \ldots + x^p/p!}$$

- When we have 2 kinds of objects (with p and q repeats, respectively), the EGF is :

$$\boxed{\begin{gathered}\left(1 + x + x^2/2! + x^3/3! + \ldots + x^p/p!\right) \times \\ \left(1 + x + x^2/2! + x^3/3! + \ldots + x^q/q!\right)\end{gathered}}$$

GF for Permutations

- Ex : Suppose we have two objects of the first kind, and three objects of another kind

The EGF is :

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)$$

$$= 1+\left(\frac{1}{1!}+\frac{1}{1!}\right)x+\left(\frac{1}{1!1!}+\frac{1}{2!}+\frac{1}{2!}\right)x^2$$

$$+\left(\frac{1}{1!2!}+\frac{1}{2!1!}+\frac{1}{3!}\right)x^3+\left(\frac{1}{1!3!}+\frac{1}{2!2!}\right)x^4+\left(\frac{1}{2!3!}\right)x^5$$

GF for Permutations

- Ex : Suppose we have n kinds of objects, each with unlimited supply

The EGF is :

$$\boxed{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)^n = e^{nx} = \sum_{r=0}^{\infty}\frac{n^r}{r!}x^r}$$

- So how many ways to get r objects, when order is important?

GF for Permutations

- Ex : Consider all *r*-digit quaternary strings (with digits 0, 1, 2, or 3)

  How many of them contains at least one 1, one 2, and one 3 ?

- Hint: What is the EGF for each of the digits ?

GF for Permutations

- The EGF for digit 0 is: e^x
- The EGF for digit 1 is: $e^x - 1$
- The EGF for digit 2 is: $e^x - 1$
- The EGF for digit 3 is: $e^x - 1$

➔ The EGF for quaternary strings with at least one 1, one 2, and one 3 is :

$$e^x (e^x - 1)^3 = e^{4x} - 3e^{3x} + 3e^{2x} - e^x$$

➔ The desired answer is: $4^r - 3 \times 3^r + 3 \times 2^r - 1$

GF for Permutations

- Ex : Consider all r-digit quaternary strings

  How many contains even number of 0's ?

  How many contains even number of 0's and even number of 1's ?

Hint :

$$\boxed{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \frac{e^x + e^{-x}}{2}}$$

GF for Permutations

- Ex : Let $(a_0, a_1, a_2, \ldots)$ be the sequence such that a_r is the number of ways to choose r or less objects from r distinct objects and distribute them into n distinct cells, with objects in the cell ordered

- Show that EGF of the sequence is: $e^x / (1 - x)^n$

Distribution of Objects

Distribution of Objects

- In Lecture Notes 2, we have studied the case of distributing objects into *distinct positions*

- In the following, we shall focus on the case of distributing objects into *non-distinct positions*
- There are two cases :

1. when objects are distinct
2. when objects are non-distinct

Case 1: Distinct Objects

- Before we study non-distinct positions, let us revisit the case when positions are distinct
- Suppose we have r distinct objects and n cells
- Each cell can hold only *any number* of objects
- All r objects are used
- If ordering of objects within cell is not important, # of ways is :

$$\boxed{n^r}$$

Case 1: Distinct Objects

- Assume $r \geq n$, and each cell has at least 1 object
- All r objects are used
- If ordering of objects within cell is not important, what will be # of ways ?

This is equivalent to finding # of
r-permutation of the n distinct cells,
with each cell appearing at least once

Case 1: Distinct Objects

- The EGF for the first cell is :

$$\boxed{\left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right) = e^x - 1}$$

- EGF for permutation of *n* cells is :

$$\boxed{(e^x - 1)^n}$$

Case 1: Distinct Objects

- To find the coefficient of x^r, we see that :

$$(e^x - 1)^n = \sum_{j=0}^{n} C(n, j)\,(-1)^j\, e^{x(n-j)}$$

$$= \sum_{j=0}^{n} C(n, j)\,(-1)^j \sum_{r=0}^{\infty} \frac{1}{r!}\,(n-j)^r\, x^r$$

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{j=0}^{n} C(n, j)\,(-1)^j (n-j)^r$$

Case 1: Distinct Objects

- Thus # of r-permutations of n cells, each cell appearing at least once, is :

$$\boxed{a_r(n) = \sum_{j=0}^{n} C(n, j)\,(-1)^j\,(n-j)^r}$$

- This term is a multiple of $n!$ (Why?)
- Let $S(r,n) = a_r(n) / n!$
  - What is the physical meaning of $S(r,n)$?

Case 1: Distinct Objects

- $S(r, n)$ is called Stirling number of the 2nd kind
- The table below shows some of the $S(r, n)$ values :

r \ n	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	
7	1	63	301	350	140	21	1

Case 1: Distinct Objects

- Now, suppose we are distributing r objects into n non-distinct cells, each cell can contain any number of objects

  ➔ the # of ways is :

$$\boxed{S(r,1) + S(r,2) + \ldots + S(r, n)}$$

- Here, we assume that $S(i, j) = 0$ when $i < j$

Case 1: Distinct Objects

- Recall that

$$(e^x - 1)^n = \sum_{r=0}^{\infty} \frac{x^r}{r!} \, a_r(n)$$

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!} \, n! \, S(r, n)$$

- Then, we have : (see next page)

Case 1: Distinct Objects

- … the coefficient of $x^r / r!$ in

$$e^{e^x - 1} - 1 = \frac{(e^x - 1)}{1!} + \frac{(e^x - 1)^2}{2!} + \frac{(e^x - 1)^3}{3!} + \ldots$$

is equal to $S(r,1) + S(r,2) + \ldots + S(r, n) + \ldots$

which is exactly the # ways to distribute r non-distinct items into n non-distinct cells, for $r \le n$

Case 2: Non-Distinct Objects

- Next, we will discuss the case of distributing non-distinct objects into non-distinct cells
- In particular, we shall look at the partition of an integer into positive integral parts, in which order of these parts is not important
- Ex: There are five different partitions of 4 :

$$4\ ,\quad 1+3,\quad 2+2,\quad 1+1+2,\quad 1+1+1+1$$

Case 2: Non-Distinct Objects

- Observe that in the polynomial

$$1 + x + x^2 + x^3 + \ldots + x^n$$

the coefficient of x^k is # ways of having k 1's in a partition of integer n ; thus in

$$1 + x + x^2 + x^3 + \ldots$$

the coefficient of x^k is # ways of having k 1's in a partition of any integer at least k

Case 2: Non-Distinct Objects

- Similarly, in the polynomial

$$1 + x^2 + x^4 + \ldots + x^{[n/2]}$$

the coefficient of x^{2k} is # ways of having k 2's in a partition of integer n ; thus in

$$1 + x^2 + x^4 + x^6 + \ldots$$

the coefficient of x^{2k} is # ways of having k 2's in a partition of any integer at least $2k$

Case 2: Non-Distinct Objects

- What is so special about the following function?

$$
\begin{aligned}
F(x) = {} & (1 + x + x^2 + x^3 + \ldots) \times \\
& (1 + x^2 + x^4 + x^6 + \ldots) \times \\
& (1 + x^3 + x^6 + x^9 + \ldots) \times \\
& (1 + x^4 + x^8 + x^{12} + \ldots) \times \\
& \qquad \ldots \times (1 + x^n + x^{2n} + x^{3n} + \ldots)
\end{aligned}
$$

- It is the ordinary GF for the number of partitions of r, with no parts exceeding n

Case 2: Non-Distinct Objects

- Note that the previous $F(x)$ is equal to :

$$F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}$$

- Ex :

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + \ldots$$

Case 2: Non-Distinct Objects

- What is so special about the following function?

$$F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ \ldots}$$

- It is the ordinary generating function for the sequence $(p_0, p_1, p_2, \ldots)$ where p_i denotes the number of different partitions of an integer i

Case 2: Non-Distinct Objects

- What is so special about the following function?

$$F(x) = \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots(1-x^{2n+1})}$$

- It is the ordinary generating function for the number of different partitions of an integer i into odd parts, with no parts exceeding $2n+1$

Case 2: Non-Distinct Objects

- What is so special about the following function?

$$F(x) = \frac{1}{(1 - x)\,(1 - x^3)\,(1 - x^5)\,(1 - x^7)\,\ldots}$$

- It is the ordinary generating function for the number of different partitions of an integer *i* into odd parts

Case 2: Non-Distinct Objects

- What is so special about the following functions?

$$(1 + x)\ (1 + x^2)\ (1 + x^3)\ \ldots\ (1 + x^n)$$

$$(1 + x)\ (1 + x^2)\ (1 + x^3)\ (1 + x^4)\ldots$$

- The first one is the ordinary generating function for number of different partitions of an integer i into distinct parts, with no parts exceeding n
- How about the second one?

Case 2: Non-Distinct Objects

- What do you observe from the following ?

$$(1+x)\,(1+x^2)\,(1+x^3)\,(1+x^4)\,\ldots$$

$$= \frac{(1-x^2)}{(1-x)} \times \frac{(1-x^4)}{(1-x^2)} \times \frac{(1-x^6)}{(1-x^3)} \times \cdots$$

$$= \frac{1}{(1-x)\,(1-x^3)\,(1-x^5)\,\ldots}$$

Case 2: Non-Distinct Objects

- The previous equality indicates that

  # ways to partition i into distinct parts

  is exactly equal to

  # ways to partition i into odd parts

- Ex: To partition 6

  into distinct parts: 6 , 5 + 1, 4 + 2, 3 + 2 + 1

  into odd parts: 5 + 1, 3+3, 3+1+1+1, 1+1+1+1+1+1

Case 2: Non-Distinct Objects

- From the following

$$
\begin{aligned}
& (1 - x)(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\ \ldots \\
= \ & (1 - x^2)(1 + x^2)(1 + x^4)(1 + x^8)\ \ldots \\
= \ & (1 - x^4)(1 + x^4)(1 + x^8)\ \ldots \\
= \ & 1
\end{aligned}
$$

we can conclude that there is exactly one way to partition any integer into distinct 2 powers (How?)

Case 2: Non-Distinct Objects

- Directly from the previous identity, we see that

$$1 - x = \frac{1}{(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots}$$

$$= (1 - x + x^2 - x^4 + x^8 - \ldots)\times$$

$$(1 - x^2 + x^4 - x^6 + x^8 - \ldots)\times$$

$$(1 - x^4 + x^8 - x^{12} + x^{16} - \ldots)\times \cdots$$

Case 2: Non-Distinct Objects

- The previous equality shows that for any $i > 1$, if we partition i into 2 powers, then

  # ways when number of parts is odd

  is exactly equal to

  # ways when number of parts is even

- Ex: To partition 5 into 2 powers

  when # parts is odd: 2+2+1, 1+1+1+1+1

  when # parts is even: 4+1, 2+1+1+1