Chapter 16: Greedy Algorithm

About this lecture

- Introduce Greedy Algorithm
- Look at some problems solvable by Greedy Algorithm

Coin Changing

- Suppose that in a certain country, the coin dominations consist of:

$1, $2, $5, $10

- You want to design an algorithm such that you can make change of any x dollars using the fewest number of coins

Coin Changing

- An idea is as follows:

1. Create an empty bag
2. while ($x > 0$) {

    Find the largest coin c at most x;

    Put c in the bag;

    Set $x = x - c$;

   }
3. Return coins in the bag

Coin Changing

- It is easy to check that the algorithm always return coins whose sum is x
- At each step, the algorithm makes a greedy choice (by including the largest coin) which looks best to come up with an optimal solution (a change with fewest #coins)
- This is an example of Greedy Algorithm

Coin Changing

- Is Greedy Algorithm always working?
- No!
- Consider a new set of coin denominations:

  \$1, \$4, \$5, \$10

- Suppose we want a change of \$8
- Greedy algorithm: 4 coins (5,1,1,1)
- Optimal solution: 2 coins (4,4)

Greedy Algorithm

- We will look at some non-trivial examples where greedy algorithm works correctly

- Usually, to show a greedy algorithm works:
  - ✓ We show that some optimal solution includes the greedy choice

    ➔ selecting greedy choice is correct
  - ✓ We show optimal substructure property

    ➔ solve the subproblem recursively

Activity Selection

- Suppose you are a freshman in a school, and there are many welcoming activities
- There are n activities $A_1, A_2, \ldots, A_n$
- For each activity A_k, it has
  - a start time s_k, and
  - a finish time f_k

Target: Join as many as possible!

Activity Selection

- To join the activity A_k,
  - you must join at s_k ;
  - you must also stay until f_k
- Since we want as many activities as possible, should we choose the one with

  (1) Shortest duration time?

  (2) Earliest start time?

  (3) Earliest finish time?

  (4) Last start time? Last finish time?

Activity Selection

- Shortest duration time may not be good:
  A_1 : [4:50, 5:10),
  A_2 : [3:00, 5:00), A_3 : [5:05, 7:00),
- Though not optimal, #activities in this solution R (shortest duration first) is at least half #activities in an optimal solution O
  - ✓ One activity in R clashes with at most 2 in O
  - ✓ If $|O| > 2|R|$, R should have one more activity

Activity Selection

- Earliest start time may even be worse:

  A_1 : [3:00, 10:00),

  A_2 : [3:10, 3:20), A_3 : [3:20, 3:30),

  A_4 : [3:30, 3:40), A_5 : [3:40, 3:50) …

- In the worst-case, the solution contains 1 activity, while optimal has n-1 activities

Greedy Choice Property

- To our surprise, earliest finish time works!
- We actually have the following lemma:

- Lemma: For the activity selection problem, some optimal solution includes an activity with earliest finish time

How to prove?

Proof: (By "Cut-and-Paste" argument)

- Let OPT = an optimal solution
- Let A_j = activity with earliest finish time
- If OPT contains A_j, done!
- Else, let A' = earliest finish activity in OPT
  - ✓ Since A_j finishes no later than A', we can replace A' by A_j in OPT without conflicting other activities in OPT
  - ➔ an optimal solution containing A_j (since it has same #activities as OPT)

Optimal Substructure

- Let A_j = activity with earliest finish time
- Let S = the subset of original activities that do not conflict with A_j
- Let OPT = optimal solution containing A_j
- Lemma:

  OPT – { A_j } must be an optimal solution for the subproblem with input activities S

Proof: (By contradiction)

- First, OPT $- \{ A_j \}$ can contain only activities in S
- If it is not an optimal solution for input activities in S, let C be some optimal solution for input S

  ➔ C has more activities than OPT $- \{ A_j \}$

  ➔ C $\cup \{A_j\}$ has more activities than OPT
  ➔ Contradiction occurs

Greedy Algorithm

- The previous two lemmas implies the following correct greedy algorithm:

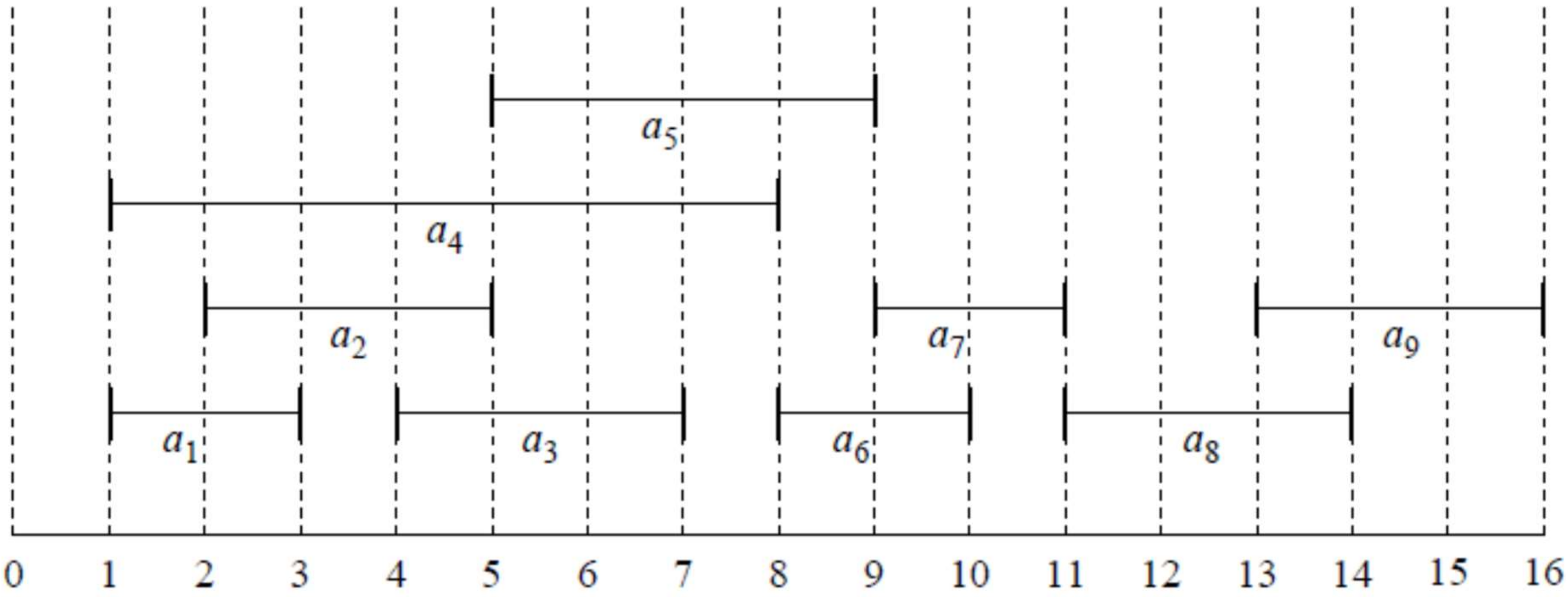
```
S = input set of activities ;
while (S is not empty) {
  A = activity in S with earliest finish time;
  Select A and update S by removing
  activities having conflicts with A;
}
```

If finish times are sorted in input, running time = O(n)

Example

S sorted by finish time: *[Leave on board]*

i	1	2	3	4	5	6	7	8	9
s_i	1	2	4	1	5	8	9	11	13
f_i	3	5	7	8	9	10	11	14	16



Maximum-size mutually compatible set: $\{a_1, a_3, a_6, a_8\}$.

Not unique: also $\{a_2, a_5, a_7, a_9\}$.

Pseudo code

Greedy-Activity-Selector(s, f)

1. $n = s.length$

2. $A = \{a_1\}$

3. $k = 1$

4. For $m = 2$ to n

5. if $s[m] \geq f[k]$

6. $A = A \cup \{a_m\}$

7. $k = m$

8. Return A

Designing a greedy algorithm

- **Greedy-choice property:** A global optimal solution can be achieved by making a local optimal (greedy) choice.

- **Optimal substructure:** An optimal solution to the problem contains its optimal solution to subproblem.

0-1 Knapsack Problem

- Suppose you are a thief, and you are now in a jewelry shop (nobody is around !)
- You have a big knapsack that you have "borrowed" from some shop before
  - ✓ Weight limit of knapsack: W
- There are n items, I_1, I_2, …, I_n
  - ✓ I_k has value v_k, weight w_k

Target: *Get items with total value as large as possible without exceeding weight limit*

0-1 Knapsack Problem

- We may think of some strategies like:
  (1) Take the most valuable item first
  (2) Take the densest item (with v_k/w_k is maximized) first
- Unfortunately, someone shows that this problem is very hard (NP-complete), so that it is unlikely to have a good strategy
- Let's change the problem a bit…

Fractional Knapsack Problem

- In the previous problem, for each item, we either take it all, or leave it there
  - ✓ Cannot take a fraction of an item
- Suppose we can allow taking fractions of the items; precisely, for a fraction c
  - ✓ c part of I_k has value cv_k, weight cw_k

Target: Get as valuable a load as possible, without exceeding weight limit

Fractional Knapsack Problem

- Suddenly, the following strategy works:

  Take as much of the densest item (with v_k/w_k is maximized) as possible

  ✓ The correctness of the above greedy-choice property can be shown by cut-and-paste argument

- Also, it is easy to see that this problem has optimal substructure property

➔ implies a correct greedy algorithm

Fractional Knapsack Problem

- However, the previous greedy algorithm (pick densest) does not work for 0-1 knapsack
- To see why, consider $W = 50$ and:

  $I_1 : v_1 = \$60, \quad w_1 = 10 \quad$ (density: 6)

  $I_2 : v_2 = \$100, \quad w_2 = 20 \quad$ (density: 5)

  $I_3 : v_3 = \$120, \quad w_3 = 30 \quad$ (density: 4)

- Greedy algorithm: \$160 (I_1, I_2)
- Optimal solution: \$220 (I_2, I_3)

Encoding Characters

- In ASCII, each character is encoded using the same number of bits (8 bits)
  - called fixed-length encoding
- However, in real-life English texts, not every character has the same frequency
- One way to encode the texts is:
  - ✓ **Encode frequent chars with few bits**
  - ✓ **Encode infrequent chars with more bits**

➔ called variable-length encoding

Encoding Characters

- Variable-length encoding may gain a lot in storage requirement

- Example:
  - ✓ Suppose we have a 100K-char file consisted of only chars a, b, c, d, e, f
  - ✓ Suppose we know a occurs 45K times, and other chars each 11K times

➔ Fixed-length encoding: 300K bits

Encoding Characters

- Example (cont.):

  Suppose we encode the chars as follows:

  a → 0, b → 100, c → 101,

  d → 110, e → 1110, f → 1111

- Storage with the above encoding:

  (45x1 + 33x3 + 22x4) x 1K

  = 232K bits (reduced by 25% !!)

Encoding Characters

- Thinking a step ahead, you may consider an even "better" encoding scheme:

  a → 0, b → 1, c → 00,

  d → 01, e → 10, f → 11

- This encoding requires less storage since each char is encoded in fewer bits …
- What's wrong with this encoding?

Prefix Code

- Suppose the encoded texts is: 0101
- We cannot tell if the original text is

  abab, dd, abd, aeb, or ...

- The problem comes from:

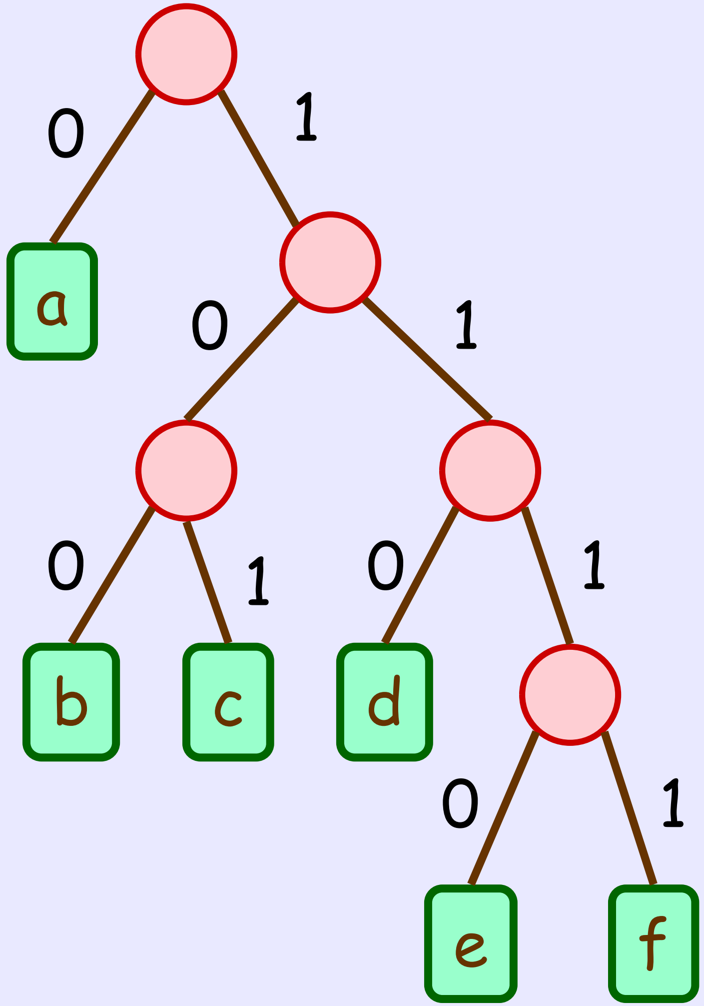
  one codeword is a prefix of another one

Prefix Code

- To avoid the problem, we generally want each codeword **not** a prefix of another
  - ✓ called **prefix** code, or **prefix-free** code
- Let **T** = text encoded by prefix code
- We can easily decode **T** back to original:
  - ✓ **Scan T from the beginning**
  - ✓ **Once we see a codeword, output the corresponding char**
  - ✓ **Then, recursively decode remaining**

Prefix Code Tree

- Naturally, a prefix code scheme corresponds to a prefix code tree
  - ✓ Each char → a leaf
  - ✓ Root-to-leaf path → codeword
- E.g., a → 0, b → 100, c → 101, d → 110, e → 1110, f → 1111



Optimal Prefix Code

- Question: Given frequencies of each char, how to find the optimal prefix code scheme (or optimal prefix code tree)?
- Precisely:

  Input: S = a set n chars, $c_1, c_2, \ldots, c_n$

  with c_k occurs f_{c_k} times

- Target: Find codeword w_k for each c_k such that $\Sigma_k |w_k| f_{c_k}$ is minimized

Huffman Code

- In 1952, David Huffman (then an MIT PhD student) thinks of a greedy algorithm to obtain the optimal prefix code tree
- Let c and c' be chars with least frequencies. He observed that:

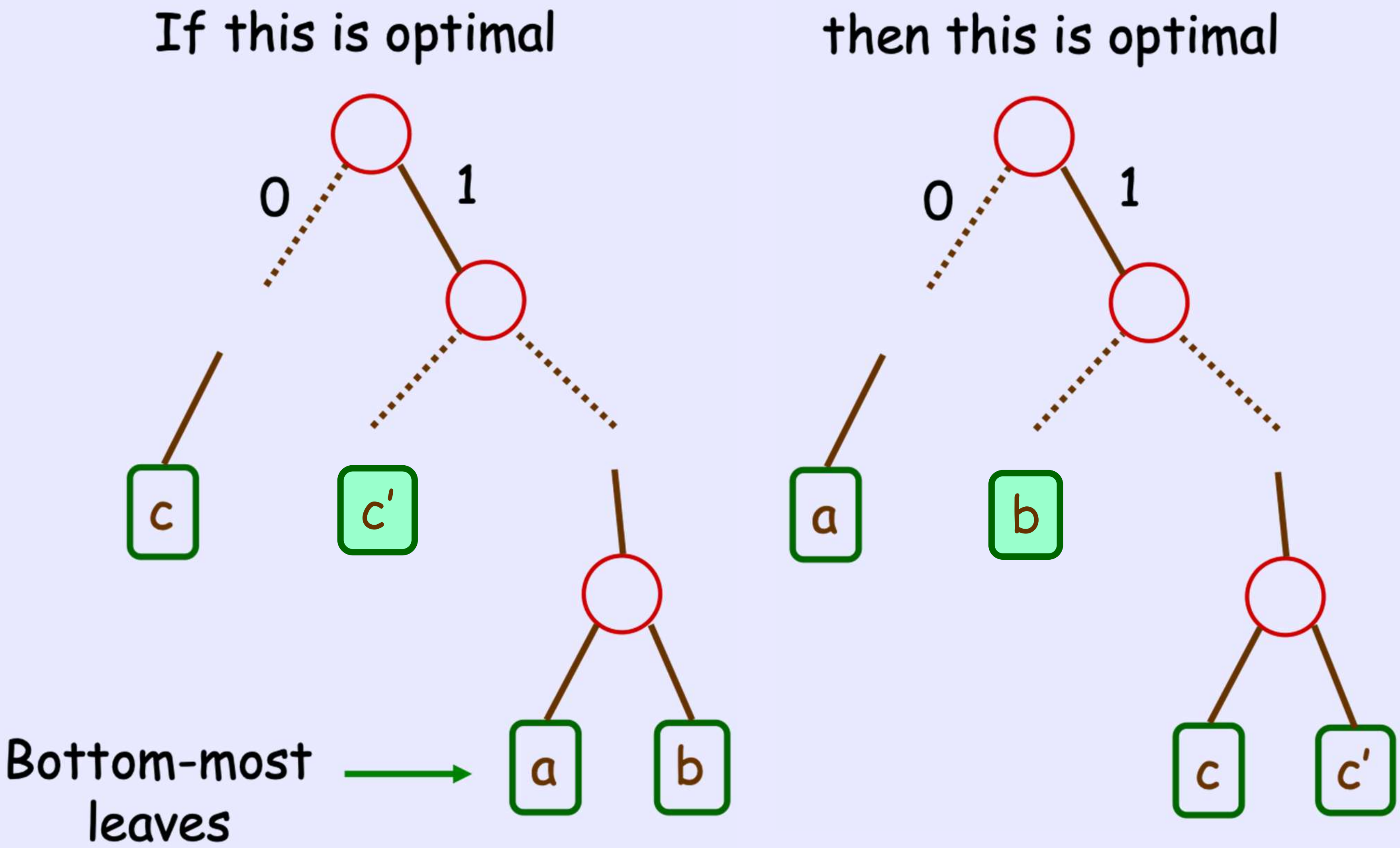
- Lemma: There is some optimal prefix code tree with c and c' sharing the same parent, and the two leaves are farthest from root

Proof: (By "Cut-and-Paste")

- Let OPT = some optimal solution
- If c and c′ as required, done!
- Else, let a and b be two bottom-most leaves sharing same parent (such leaves must exist… why??)
  - swap a with c, swap b with c′
  - an optimal solution as required

    (since it at most the same Σ_k $|w_k|$ f_k as OPT … why??)

Graphically:



Optimal Substructure

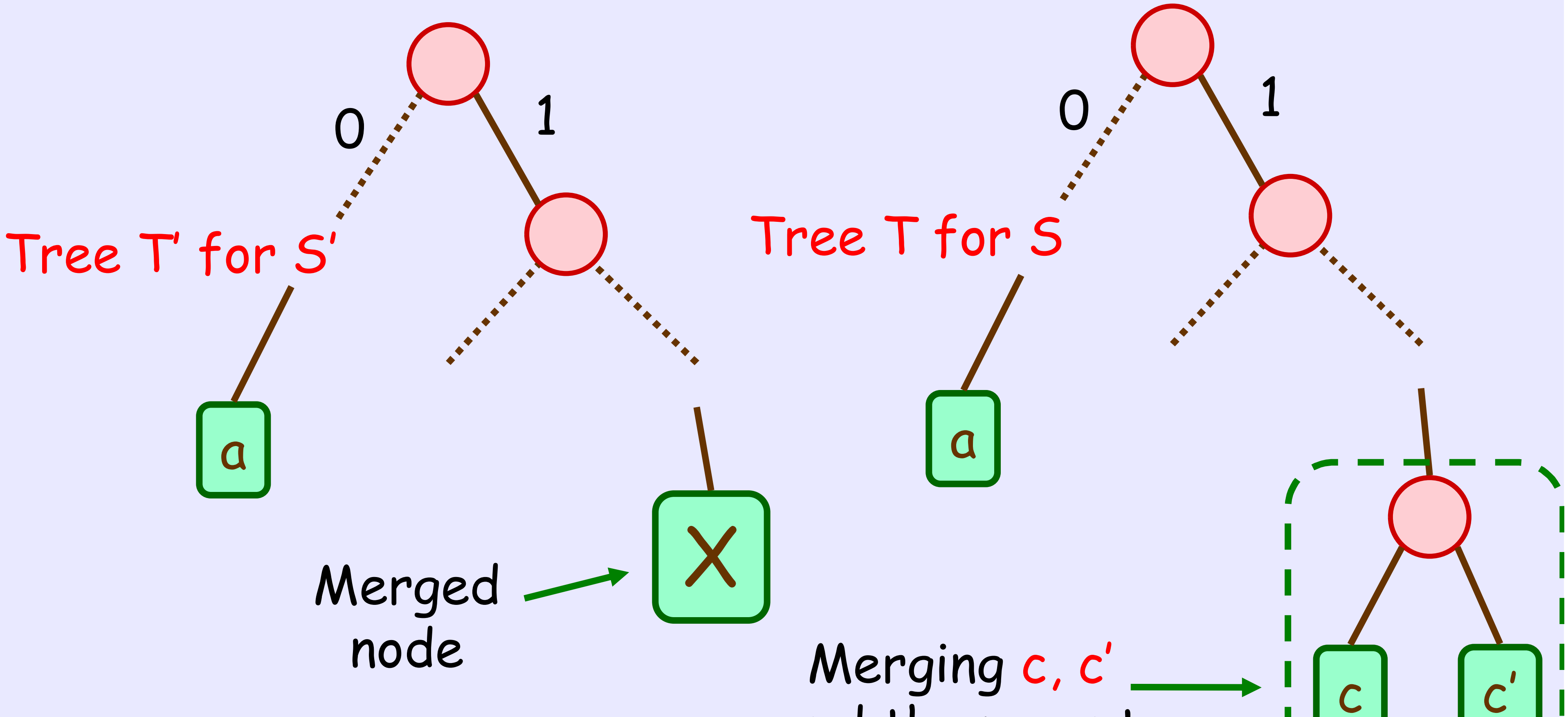
- Let OPT be an optimal prefix code tree with c and c′ as required
- Let T′ be a tree formed by merging c, c′, and their parent into one node
- Consider S′ = set formed by removing c and c′ from S, but adding X with $f_X = f_c + f_{c'}$
- **Lemma: If T′ is an optimal prefix code tree for S′, then T obtained from T′ by replacing the leaf node X with an internal node having c and c′ is an optimal prefix code tree for S.**

Proof

- If T is not optimal tree for S then there exists an optimal tree T'' for S and cost(T'') < cost(T).
- Let T''' be the tree T'' with the common parent of c and c' replaced by a leaf with X
- Then cost(T''') = cost (T'')-fc-fc'< cost(T) - fc- fc'= cost(T')
- Yielding a contradiction to the assumption that T' represent an optimal prefix code for S'

Graphically, the lemma says:

If this is optimal for S′ then this is optimal for S

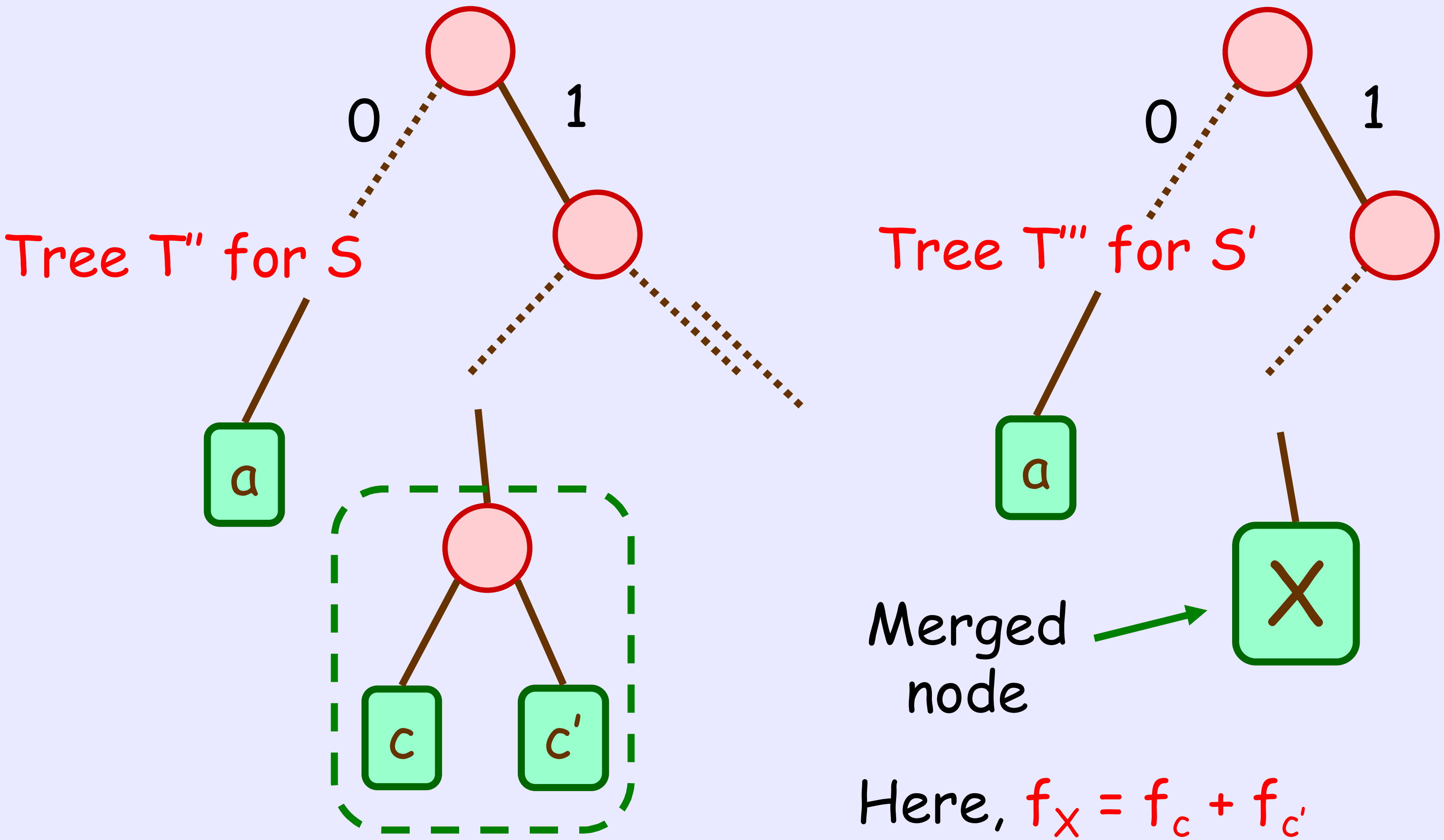


Here, $f_X = f_c + f_{c'}$

contradiction to the assumption that T′ represent an optimal prefix code for S′

If this is optimal for S

Then this is optimal for S′ and better than T′



Huffman Code

- Questions:
- Based on the previous lemmas, can you obtain Huffman's coding scheme?
- What is the running time?

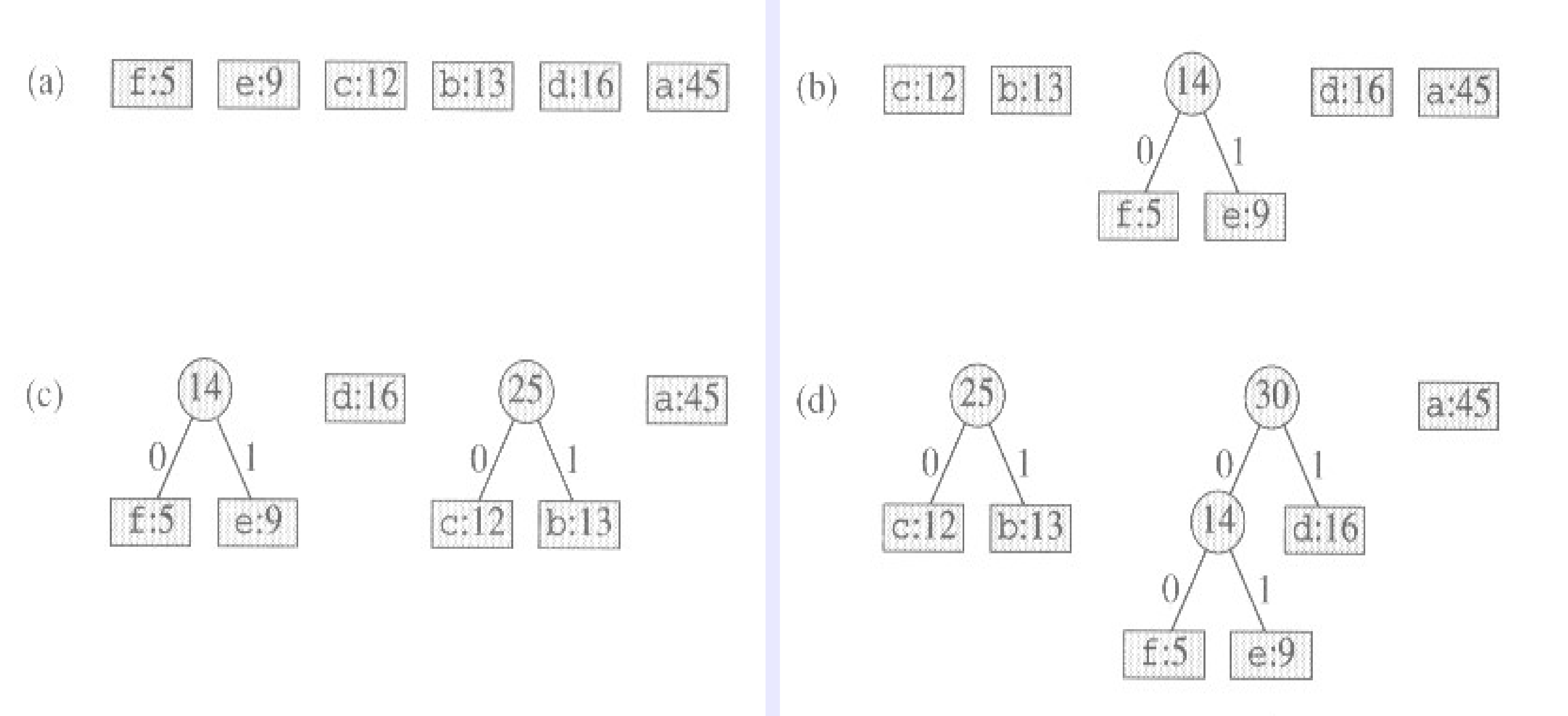
  O(n log n) time, using heap (how??)

Huffman(S) { // build Huffman code tree

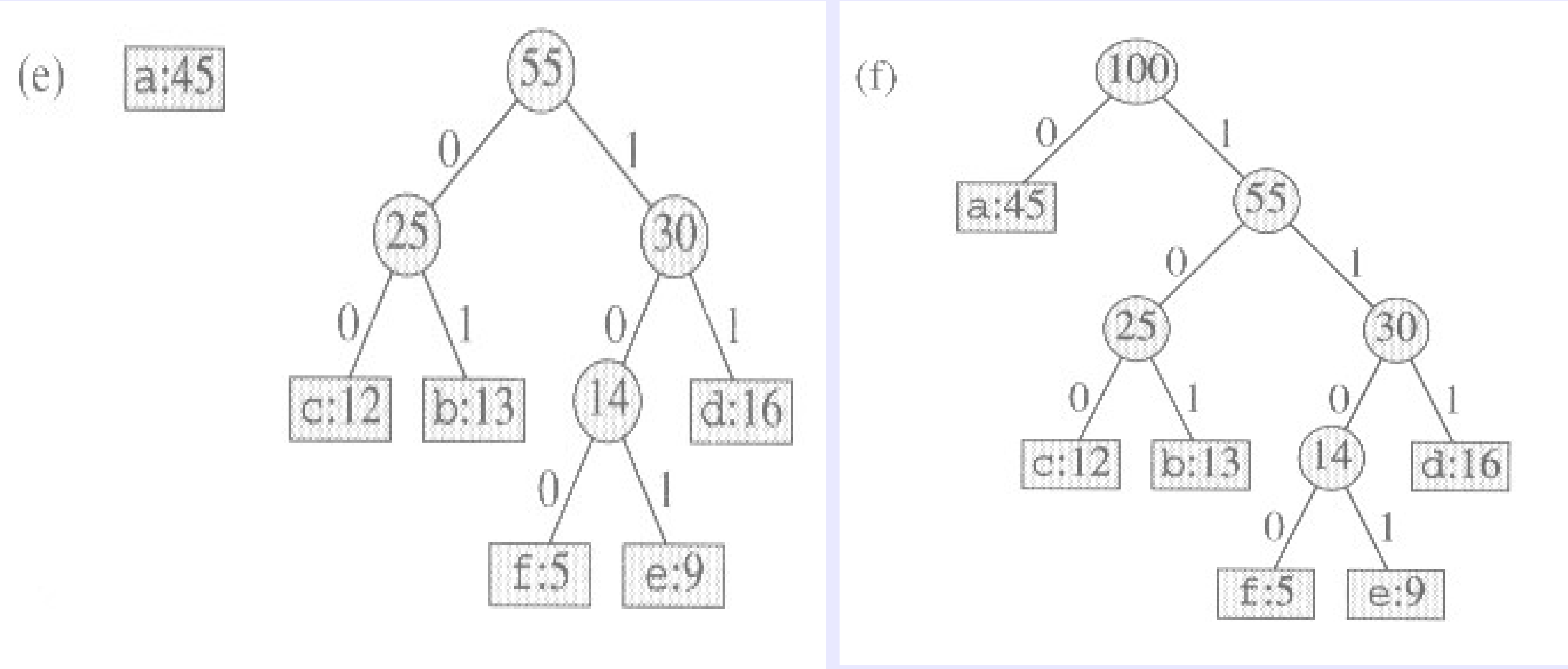
1. Find least frequent chars c and c'
2. S' = remove c and c' from S,
   but add char X with $f_X = f_c + f_{c'}$
3. T' = Huffman(S')
4. Make leaf X of T' an internal node by connecting two leaves c and c' to it
5. Return resulting tree

}

The steps of Huffman's algorithm



The steps of Huffman's algorithm



Constructing a Huffman code

HUFFMAN(C)

$n \leftarrow |C|$

$Q \leftarrow C$ /* initialize the min-priority queue with the character in C */

for $i \leftarrow 1$ **to** $n - 1$

  do allocate a new node z

    z.$left$ = x = EXTRACT-MIN(Q)

    z.$right$ = y = EXTRACT-MIN(Q)

    z.$freq$ = $x.freq + y.freq$

    INSERT(Q, z)

return EXTRACT-MIN(Q) // return the root of the tree

Complexity: $O(n \log n)$

Practice at home

- Exercises: 16.1-2, 16.1-5, 16.2-2, 16.2-3, 16.2-5, 16.3-2, 16.3-8

- What is an optimal Huffman code tree for the following set of frequencies?

  A:1 B:1 C:2 d:3 e:5