Chapter 35: Approximation Algorithms

About this Tutorial

- Decision vs Optimization
- NP-Hard Problems
- Dealing with NP-Hard Problems
  - Exact Algorithms
  - Heuristic Algorithms
    - Randomized Algorithms
    - Approximation Algorithms
    - …

Decision vs Optimization

- Last time, we have talked about decision problems, in which the answer is either YES or NO

  E.g., Peter gives us a map $G = (V,E)$, and he asks us if there is a path from A to B whose length is at most 100

Decision vs Optimization

- A more natural type of problem is called optimization problems, in which we want to obtain a best solution

  E.g., Peter gives us a map G = (V,E), and he asks what is the length of the shortest path from A to B

- Usually, the answer to an optimization problem is a number

Decision vs Optimization

- Two major types of optimization problems: minimization or maximization
  - Previous example is a minimization problem

- An example for a maximization problem:
  - Peter gives us a map $G = (V,E)$, and he asks what is the maximum number of edge-disjoint paths from A to B

Decision vs Optimization

- Decision problem and optimization problem are closely related :

(1) Peter gives us a map $G = (V,E)$, and he asks what is the length of the shortest path from A to B

(2) Peter gives us a map $G = (V,E)$, and he asks us if there is a path from A to B with length at most k

Decision vs Optimization

- We see that if Problem (1) can be solved, we can immediately solve Problem (2)
- In general, if the optimization version can be solved, the corresponding decision version can be solved !
  - What if its decision version is known to be NP-complete ??

Decision vs Optimization

- For example, the following is a famous optimization problem called Max-Clique :
  - ✓ Given an input graph G, what is the size of the largest clique in G ?
- Its decision version, Clique, is NP-complete:
  - ✓ Given an input graph G, is there a clique of size at least k ?

NP-Hard

- If the decision version is NP-complete, then it is unlikely that the optimization problem has a polynomial-time algorithm
  - We call such optimization problem an NP-hard problem
- So, perhaps no polynomial-time algorithm may exist… Should we give up solving the NP-hard problems?

Dealing with NP-Hard problems

- Although a problem is NP-hard, it does not mean that it cannot be solved
- At least, we can try naïve brute force search, only that it needs exponential time
- Other common strategies :
  - Exact Algorithms
  - Heuristic Algorithms
    - Randomized Algorithms
    - Approximation Algorithms
    - …

Exact Algorithm

- Given a graph G with n vertices,
  - ✓ a brute force approach to solve the Max-Clique problem is to select every subset of G, and test if it is a clique
  - ✓ Running time: $O(2^n n^2)$ time
- Though time is exponential, it works well when n is small, and we can improve it …
- Tarjan & Trojanowski [1977]: $O(1.26^n)$ time

Randomized Algorithm

- Use randomization to help
- Idea 1: Design an algorithm that answers correctly most of the time (but sometimes may give wrong answer), and it always run in polynomial time
- Idea 2: Design an algorithm that always give a correct answer, runs mostly in polynomial-time (but sometimes runs in exponential time)

Approximation Algorithm

- Target: runs in polynomial time
- Give-ups: may not find optimal solution …
  - Yet, we want to show that the solution we find is "close" to optimal
- E.g., in a maximization problem, we may have an algorithm that always returns a solution at least half the optimal
- How can we do that ??
  - (when we don't even know what optimal is ??)

Example : Min Vertex Cover

- Given a graph $G = (V,E)$, we want to select the minimum # of vertices such that each edge has at least one vertex selected
- Real-life example:
  - edge: road
  - vertex : road junction
  - selected vertex: guard
- This problem is NP-hard (Why?)

Example : Min Vertex Cover

- Let us consider the following algorithm:

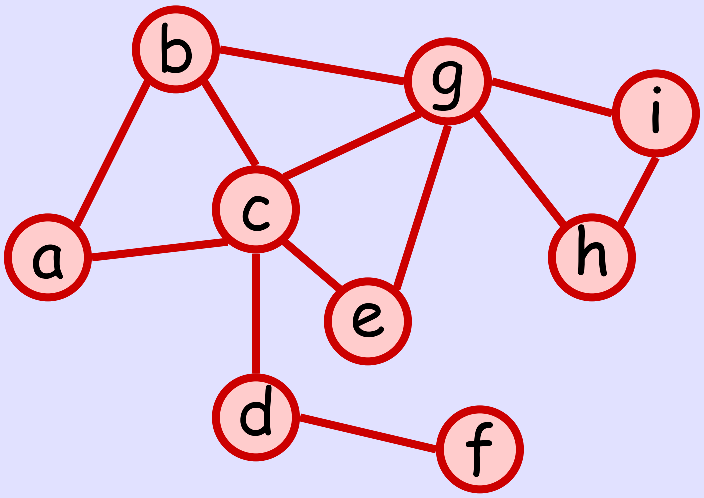
1. C = an empty set
2. while (there is edge in G) {
    3. Pick an edge, say (u,v) ;
    4. Put u and v into C ;
    5. Remove u, v, and all edges adjacent to u or v ;

   }

6. return C

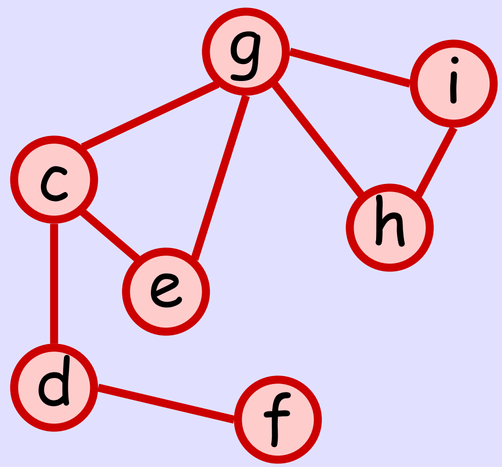
Example Run

original G



Example Run

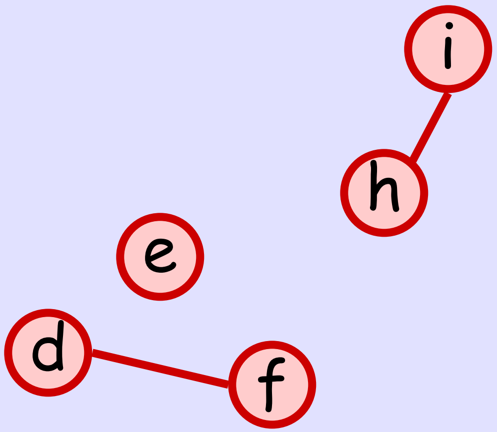
Picking (a,b)



C = { a, b }

Example Run

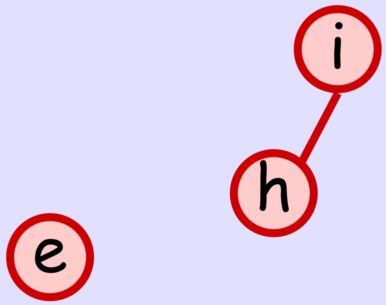
Picking (c,g)



C = { a, b, c, g }

Example Run

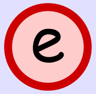
Picking (d,f)



C = { a, b, c, g, d, f }

Example Run

Picking (h,i)



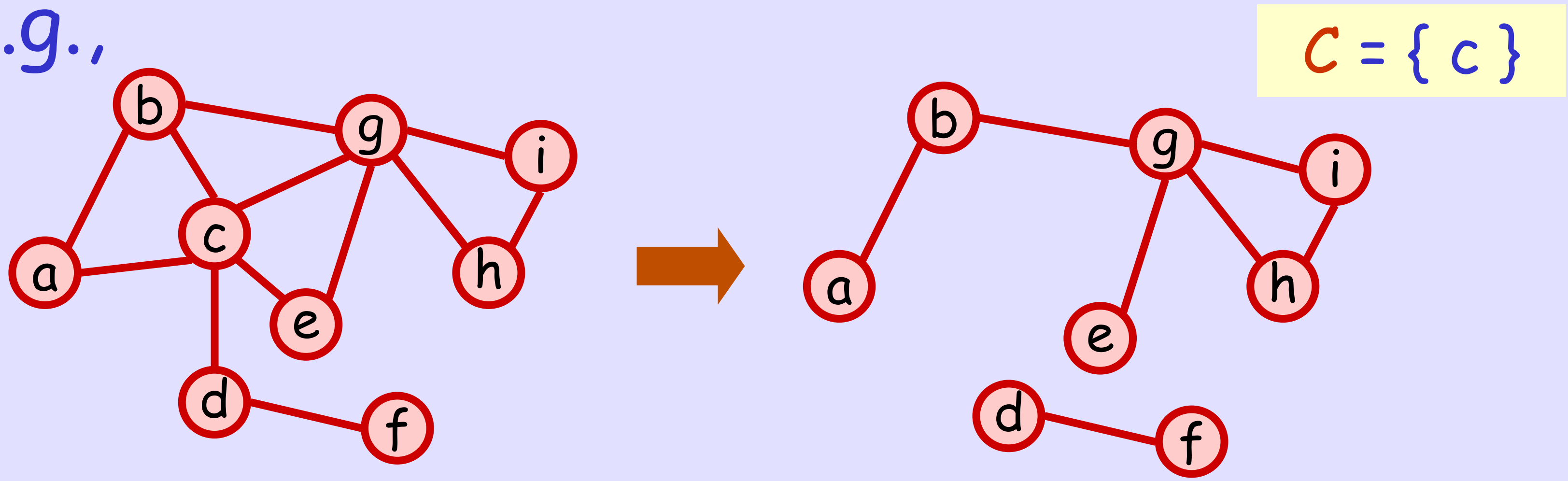
C = { a, b, c, g, d, f, h, i }

Example : Min Vertex Cover

- What is so special about *C* ?
  - Vertices in *C* must cover all edges !!
  - But … it may not be the smallest one
- How far is it from the optimal ?
  - At most 2 times (why??)
  - Because each edge in line 3 of the algorithm can only be covered by its endpoints → in each iteration, one of the selected vertex must be in the optimal vertex cover

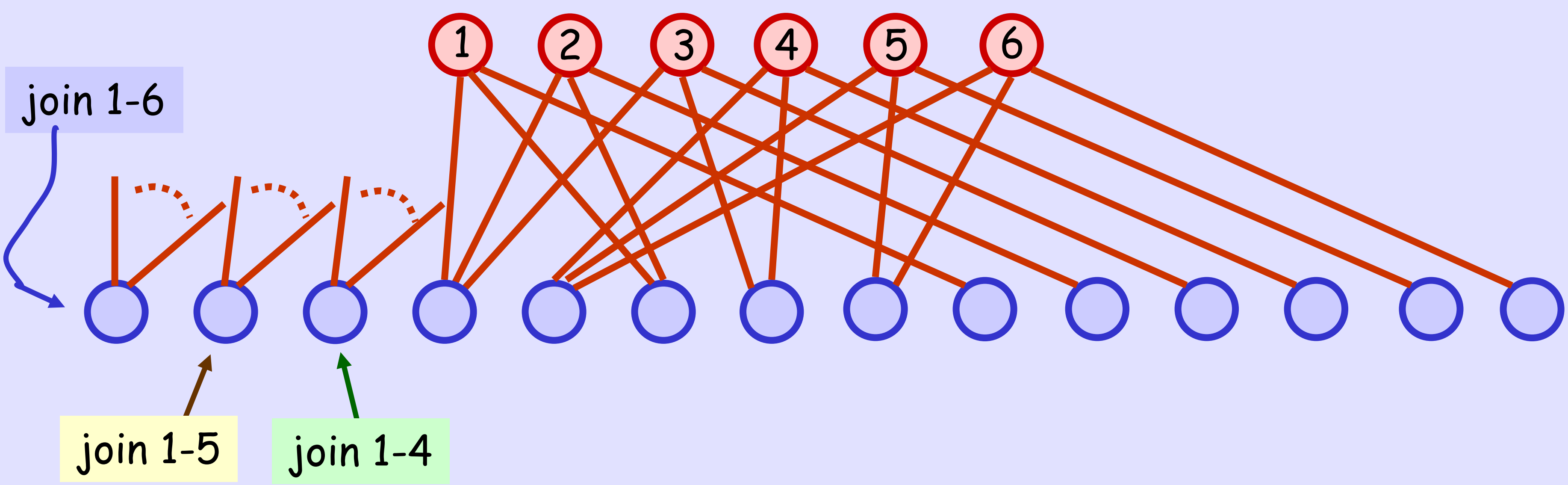
Example : Min Vertex Cover

- Another algorithm, perhaps a more natural one, is to select the vertex that covers most edges in each iteration
  - After the selection, we remove the vertex, and all its adjacent edges
- E.g.,



- Unfortunately, when the input graph has n vertices, this new algorithm can only guarantee a cover at most O(log n) times the optimal (instead of at most 2 times before)
- A worst-case scenario looks like :

Optimal : 6 nodes (red)   New algo : 14 nodes (blue)



The traveling-salesman problem

- Euclidean traveling-salesman problem: to find in a complete weighted undirected graph $G=(V,E)$ a hamiltonian cycle (a tour) with minimum cost. The edges weights $c(u,v)$ are nonnegative integers. And, the weight function satisfies the following triangle inequality:

$$c(u,w) \leq c(u,v) + c(v,w).$$

Euclidean traveling-salesman

APPROX_TSP_TOUR(G, c)

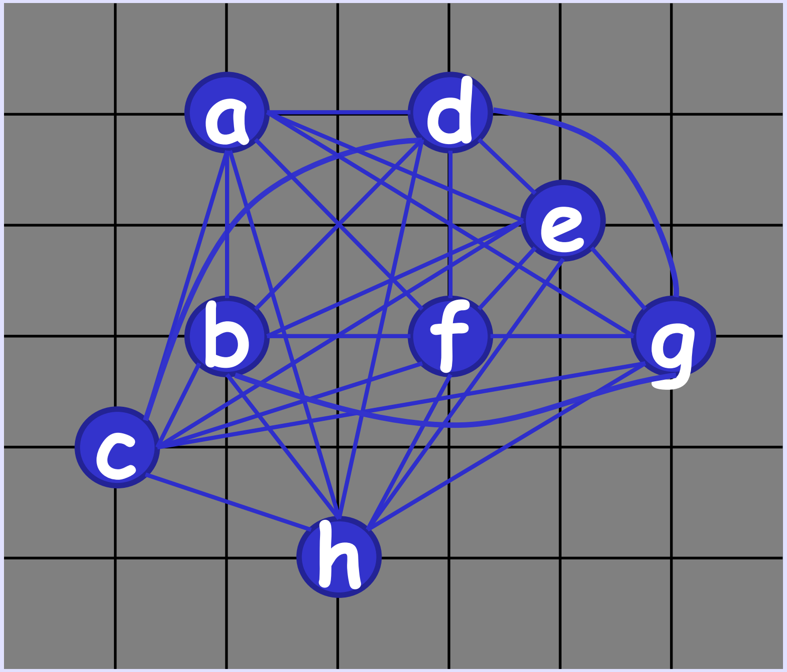
1 Select a vertex r$\in$ G.V to be a root vertex

2 grow a MST T for G from root r using MST_PRIM(G, c, r)

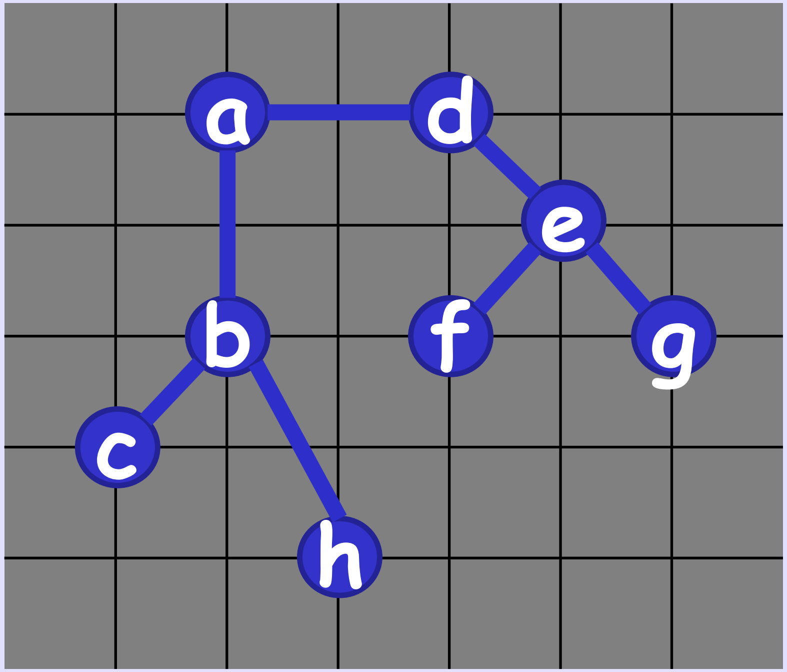
3 Let H be the list of vertices visited in a preorder walk of T

4 **return** the hamiltonian cycle H that visit the vertices in the order H

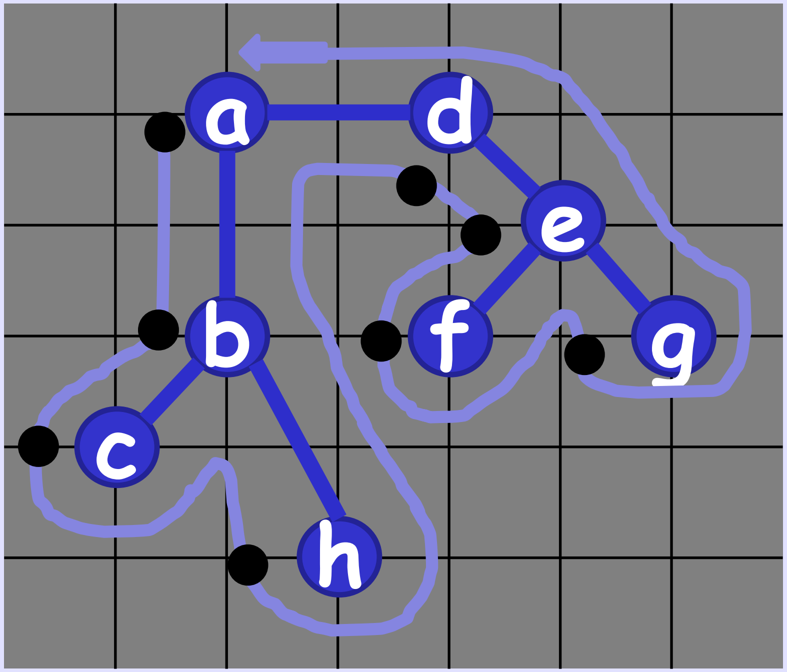
Time complexity: $O(E) = O(V^2)$



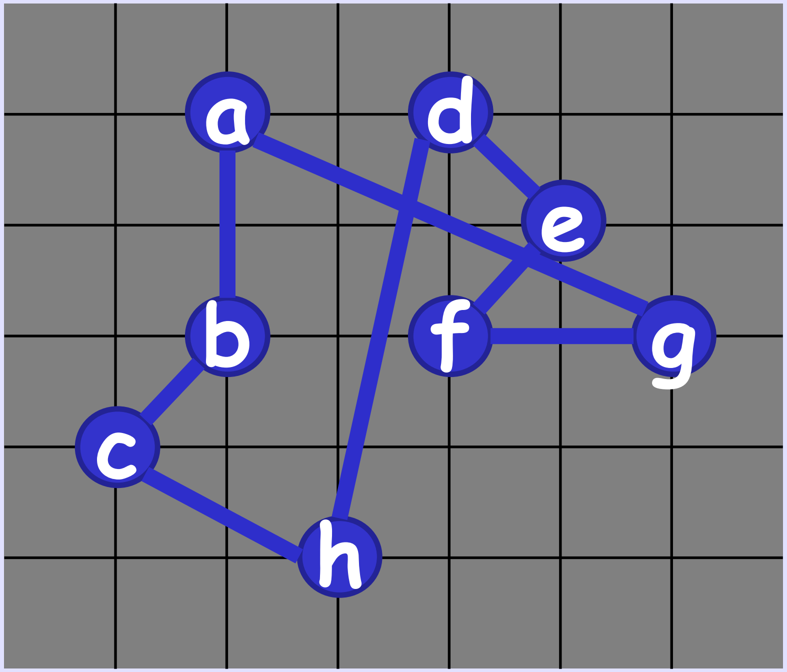
(a) A complete undirected graph. Vertices lie on intersections of integer grid lines



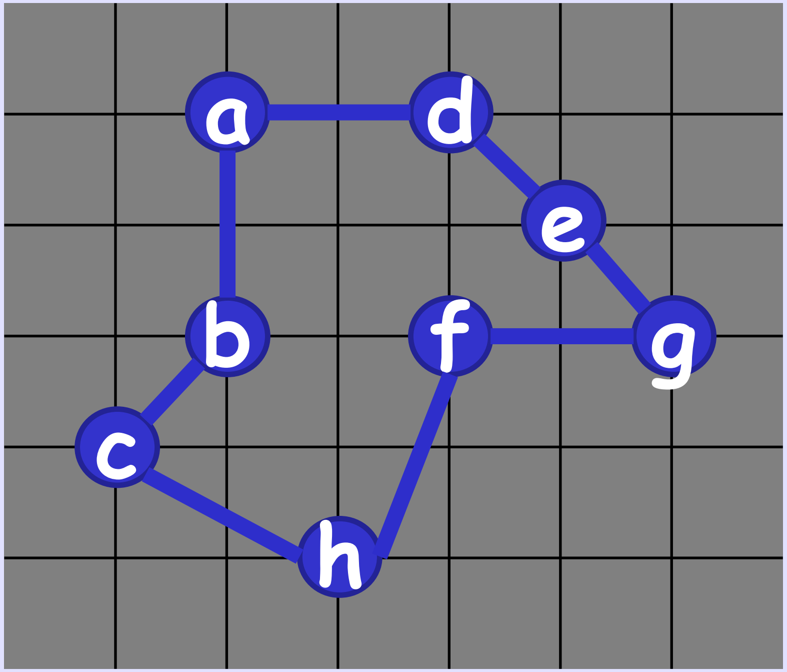
(b) A minimum spanning tree T of the complete graph, as computed by MST-PRIM.



(c) A preorder walk of T, starting at a. A full walk of the tree visits the vertices in the order a, b, c, b, h, b, a, d, e, f, e, g, e, d, a.



(d) A tour obtained by visiting the vertices in the order given by the preorder walk.



(e) An optimal tour H for the original complete graph.

Theorem

Approx-TSP-Tour is a polynomial -time 2 -approximation algorithm

Let:

T: a minimum spanning tree T

W: a full walk of T

H: a tour of length for Approx-TSP-Tour

H*: an optimal tour of length

Proof

- Let T be a minimum spanning tree. Deleting any edge from H*, we can obtain a spanning tree. Thus, $|T| \leq |H^*|$.
A full walk, denoted by W, of T lists the vertices when they are first visited and also whenever they are return to after a visit to a subtree. In our example, W=(a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)

Proof

- Clearly, $|W|=2|T|$. Thus, $|W| \le 2|H^*|$. Note that W is not a tour. It visits a vertex more than once. However, by triangle inequality, we can delete unnecessary visits to a vertex without increasing the cost to obtain H. In our example, H=(a, b, c, h, d, e, f, g) Thus, $|H| \le |W| \le 2|H^*|$. #
- Can we find an approximation algorithm for the general TSP Problem ?

Example : Max-Cut

- Given a graph $G = (V,E)$, we want to partition V into disjoint sets (V_1,V_2) such that #edges in-between them (i.e., with exactly one end-point in each set) is maximized
  - (V_1,V_2) is usually called a cut
  - target: find a cut with maximum #edges
- This problem is NP-hard

Example : Max-Cut

- Fact: If the graph has m edges, the maximum #edges in any cut is m

- Thus, if we can find a cut which has at least m/2 edges, this will be at least half of the optimal

- How to find this cut ?

- Let us consider the following algorithm:

1. $V_1 = V_2 =$ empty set ;
2. Label the vertices by $x_1, x_2, \ldots, x_n$
3. For ($k = 1$ to n) {

```
/* Fix location of x_k */
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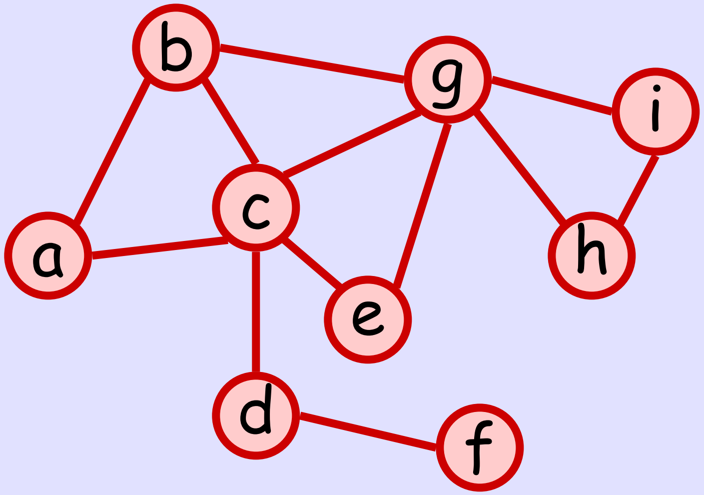
   Fix x_k to the set such that more in-between edges (with those already fixed vertices $x_1, x_2, \ldots, x_{k-1}$) are obtained ;

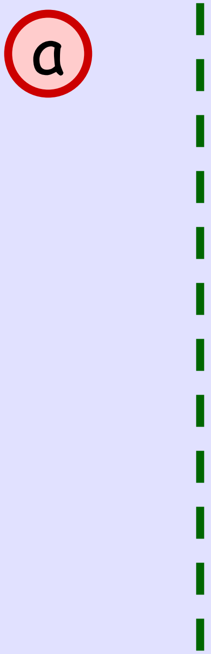
   }
4. return the cut (V_1, V_2) ;

Example Run

original G

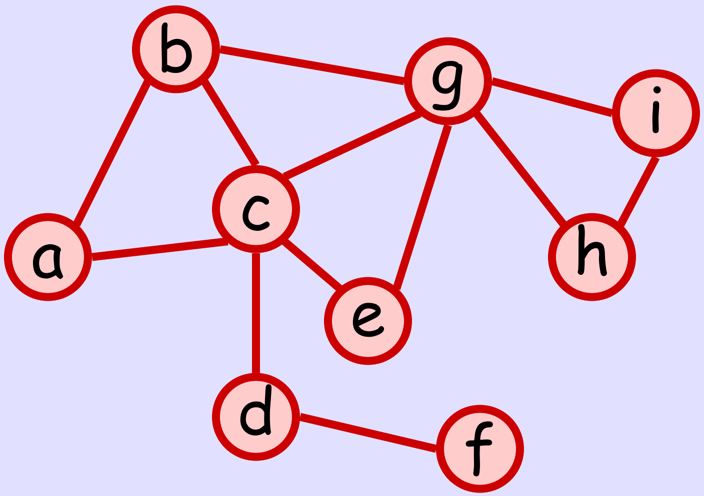


Fix vertex a

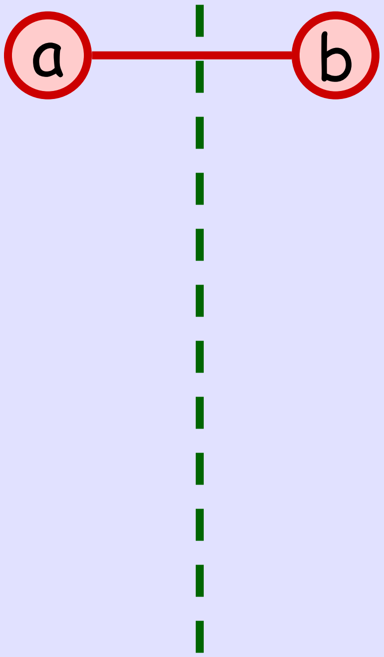


Example Run

original G

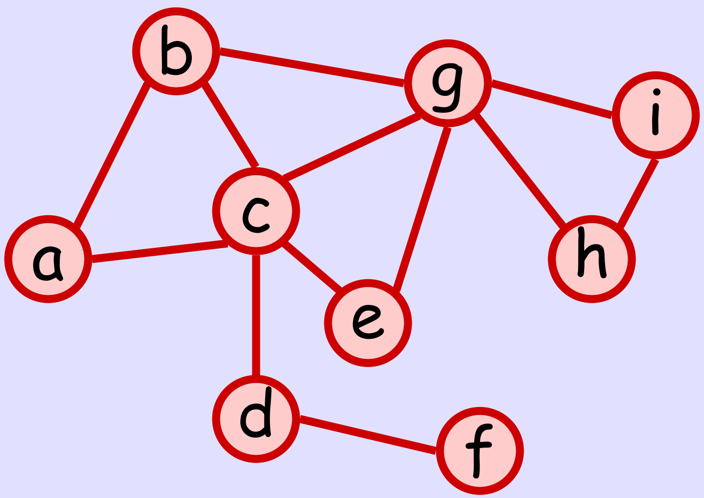


Fix vertex b

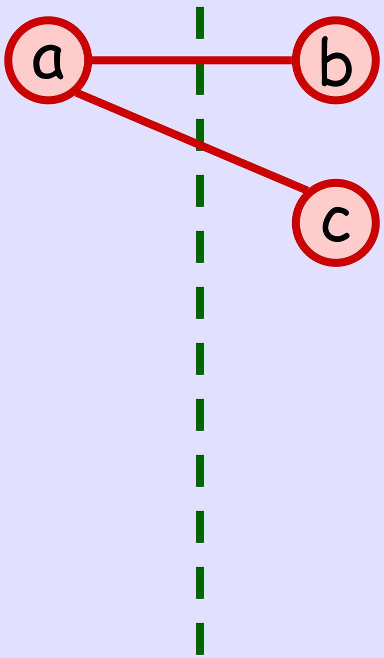


Example Run

original G



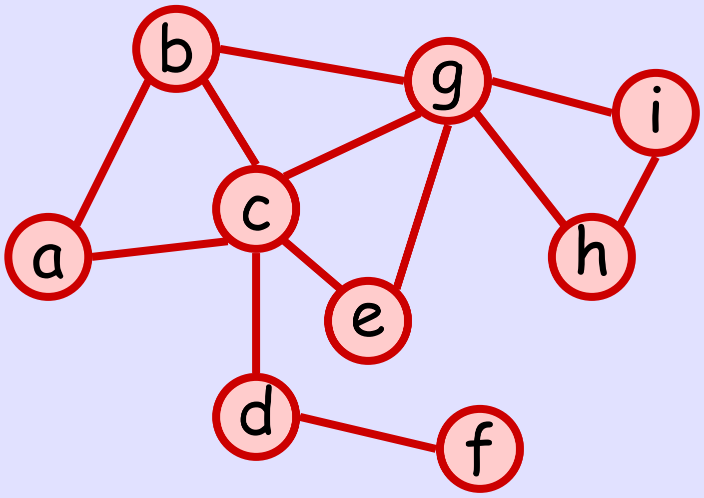
Fix vertex c



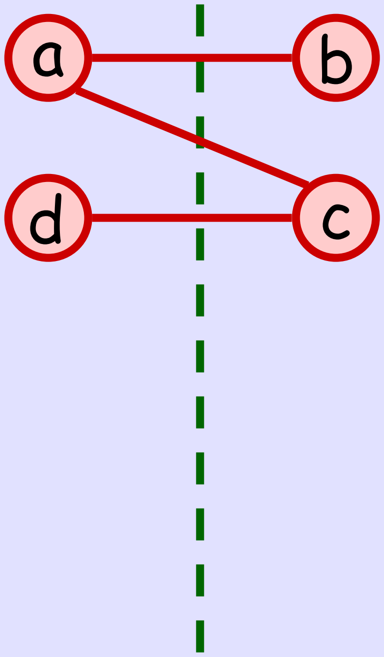
vertex c can be
added to either side

Example Run

original G

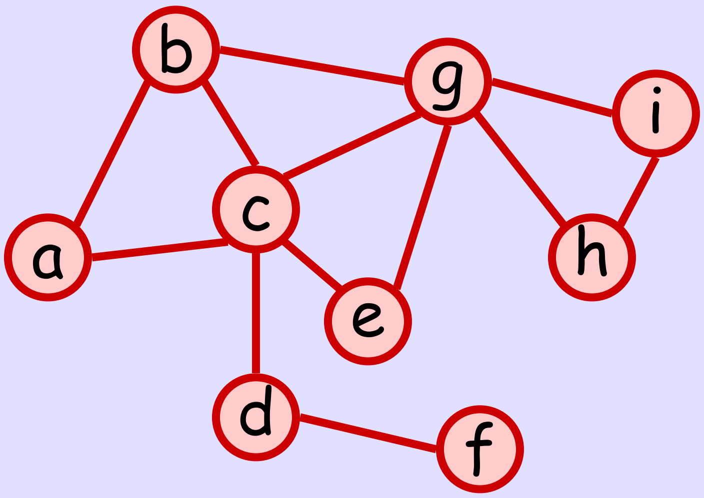


Fix vertex d

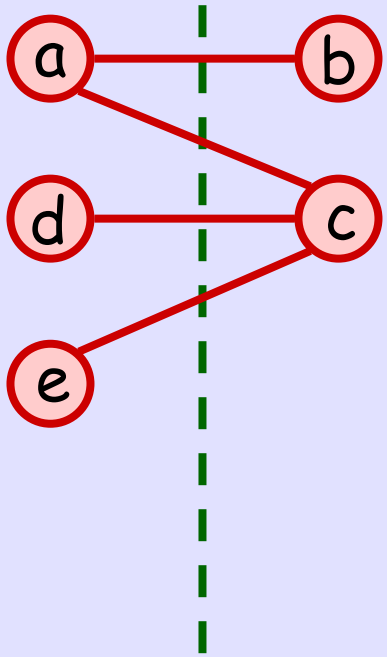


Example Run

original G

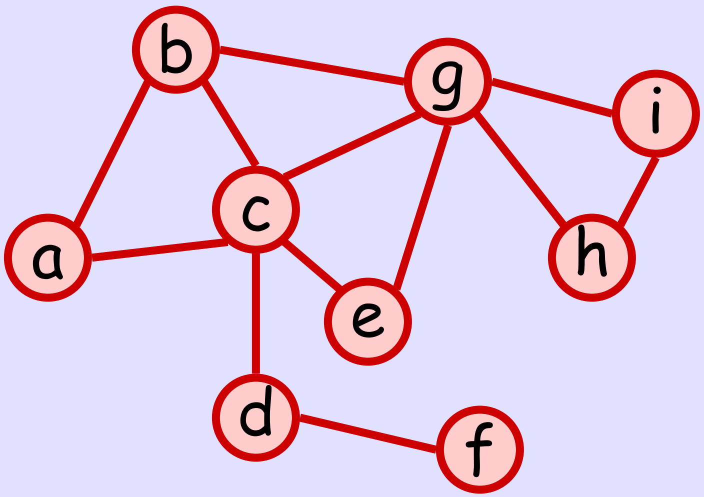


Fix vertex *e*

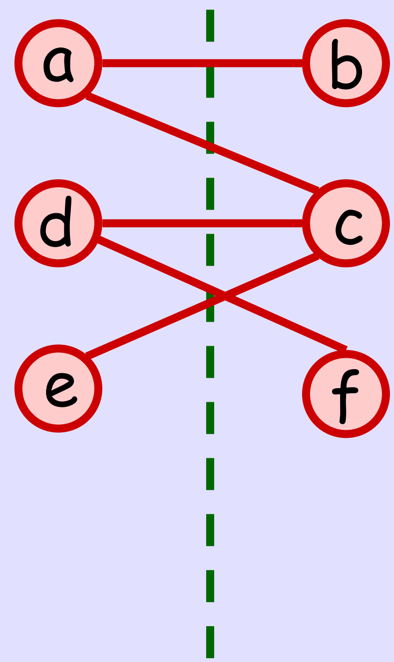


Example Run

original G

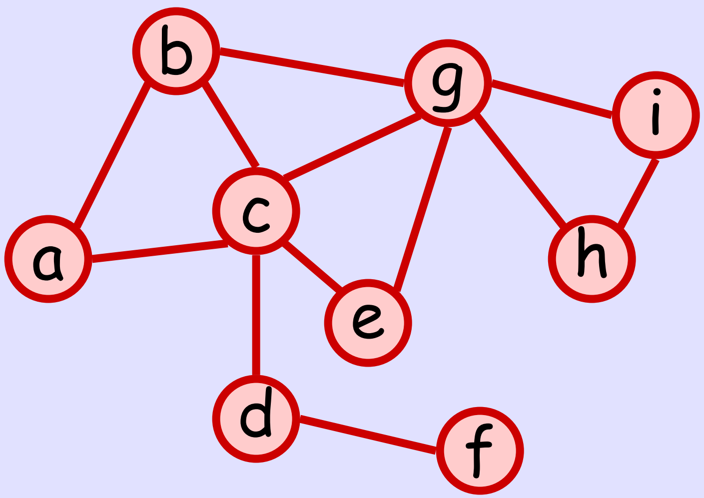


Fix vertex f

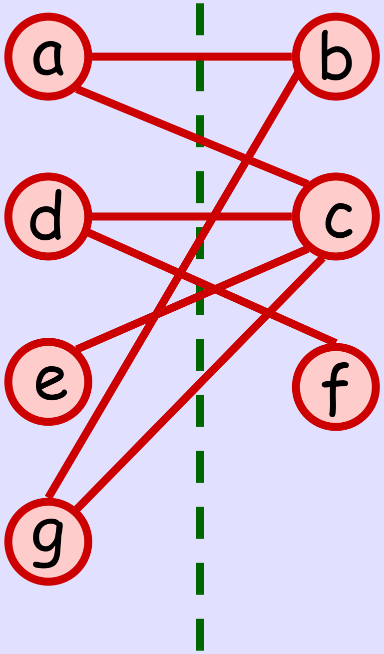


Example Run

original G

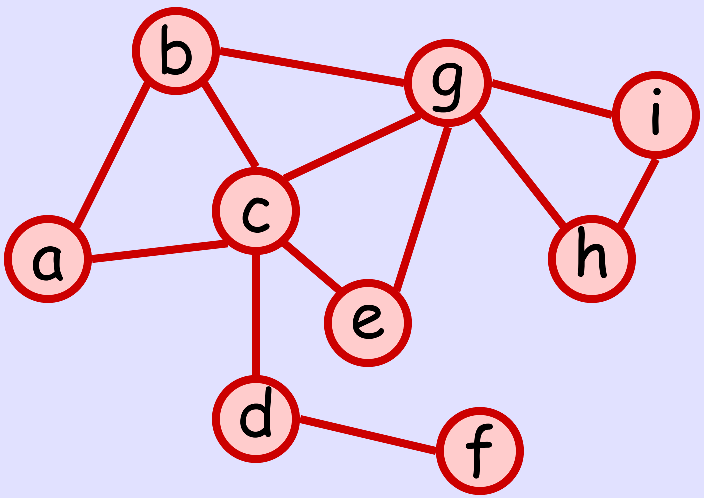


Fix vertex *g*

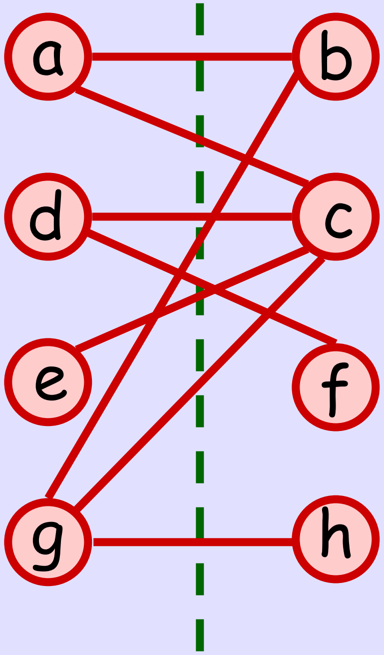


Example Run

original G

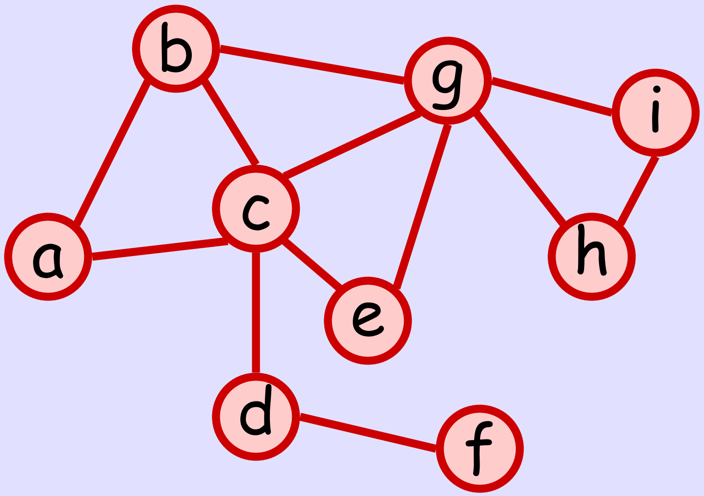


Fix vertex h

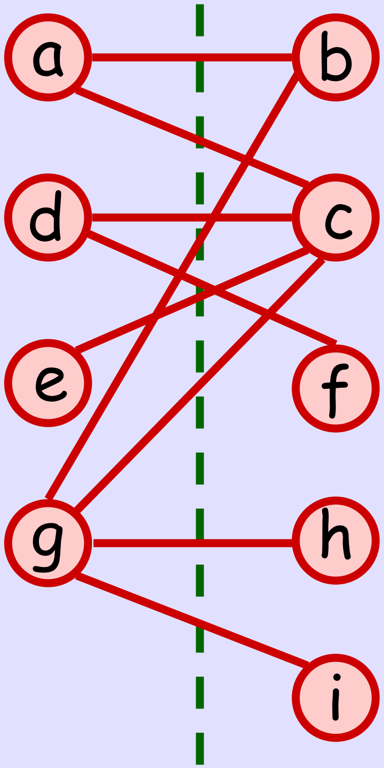


Example Run

original G



Fix vertex i



#in-between edges = 9

Example : Max-Cut

- How far is our cut from the optimal ?
  - At most 2 times (why??)
  - When a vertex v is fixed, we will add some edges into the cut, and discard some edges (u, v) if u is placed in the same set as v
  - But when each vertex is fixed :

    #edges added $\geq$ #edges discarded

  ➔ total #edges added $\geq$ m/2

Homework

- Exercises: 35.1-4, 35.1-5

True or False

- If we can prove that the lower bound of an NPC problem is exponential, then we have proved that NP $\neq$ P.
- Any NP-hard problem can be solved in polynomial time if there is an algorithm that can solve the satisfiability problem in polynomial time.

Thank You &

Happy New Year!