Chapter 22: Elementary Graph Algorithms II

About this lecture

- Depth First Search
  - DFS Tree and DFS Forest

- Properties of DFS
  - Parenthesis theorem (very important)
  - White-path theorem (very useful)

Depth First Search (DFS)

- An alternative algorithm to find all vertices reachable from a particular source vertex s
- Idea:

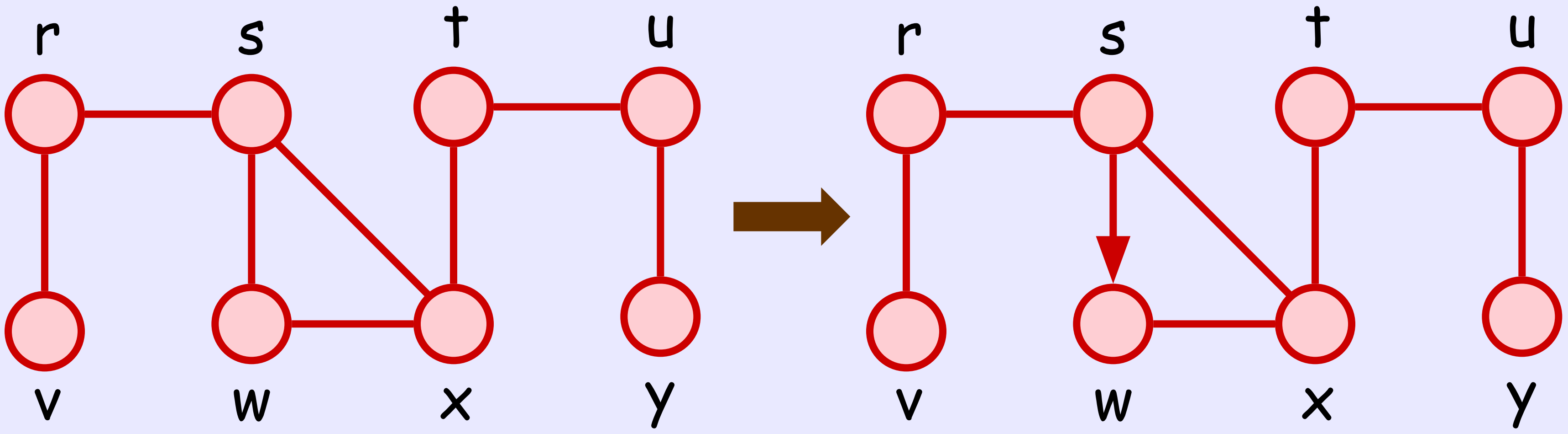
  Explore a branch as far as possible before exploring another branch
- Easily done by recursion or stack

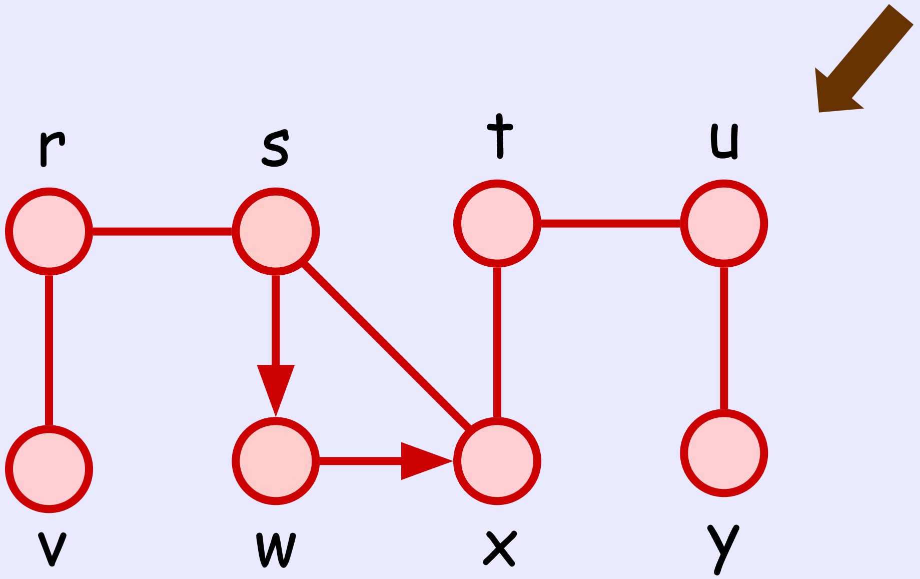
The DFS Algorithm

```
DFS(u)
{    Mark u as discovered ;
     while (u has unvisited neighbor v)
          DFS(v);
     Mark u as finished /*visited*/;
}
```

The while-loop explores a branch as far as possible before the next branch

Example (s = source)



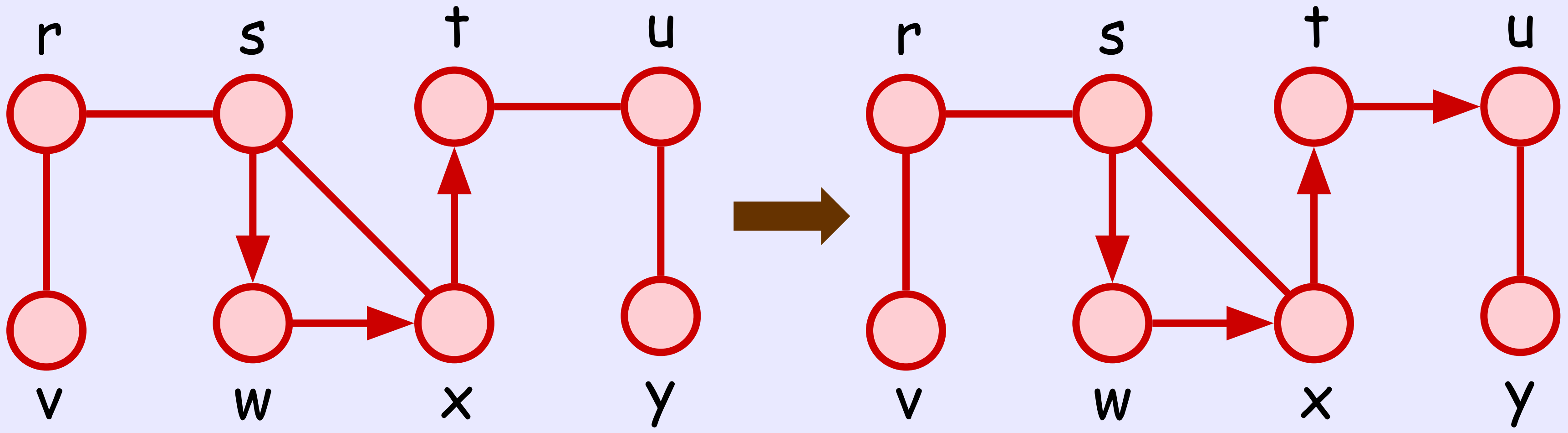


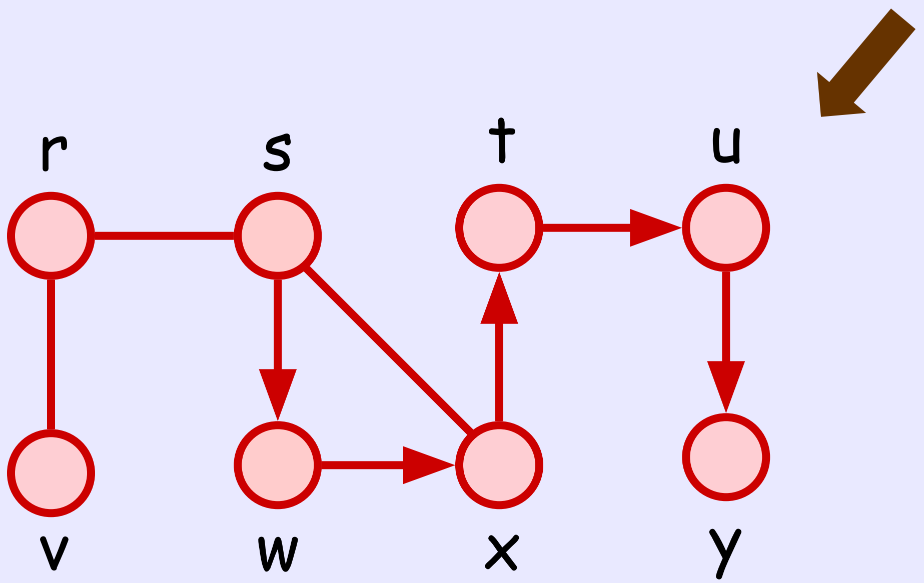
● finished

○ discovered

→ direction of edge when new node is discovered

Example (s = source)





● finished

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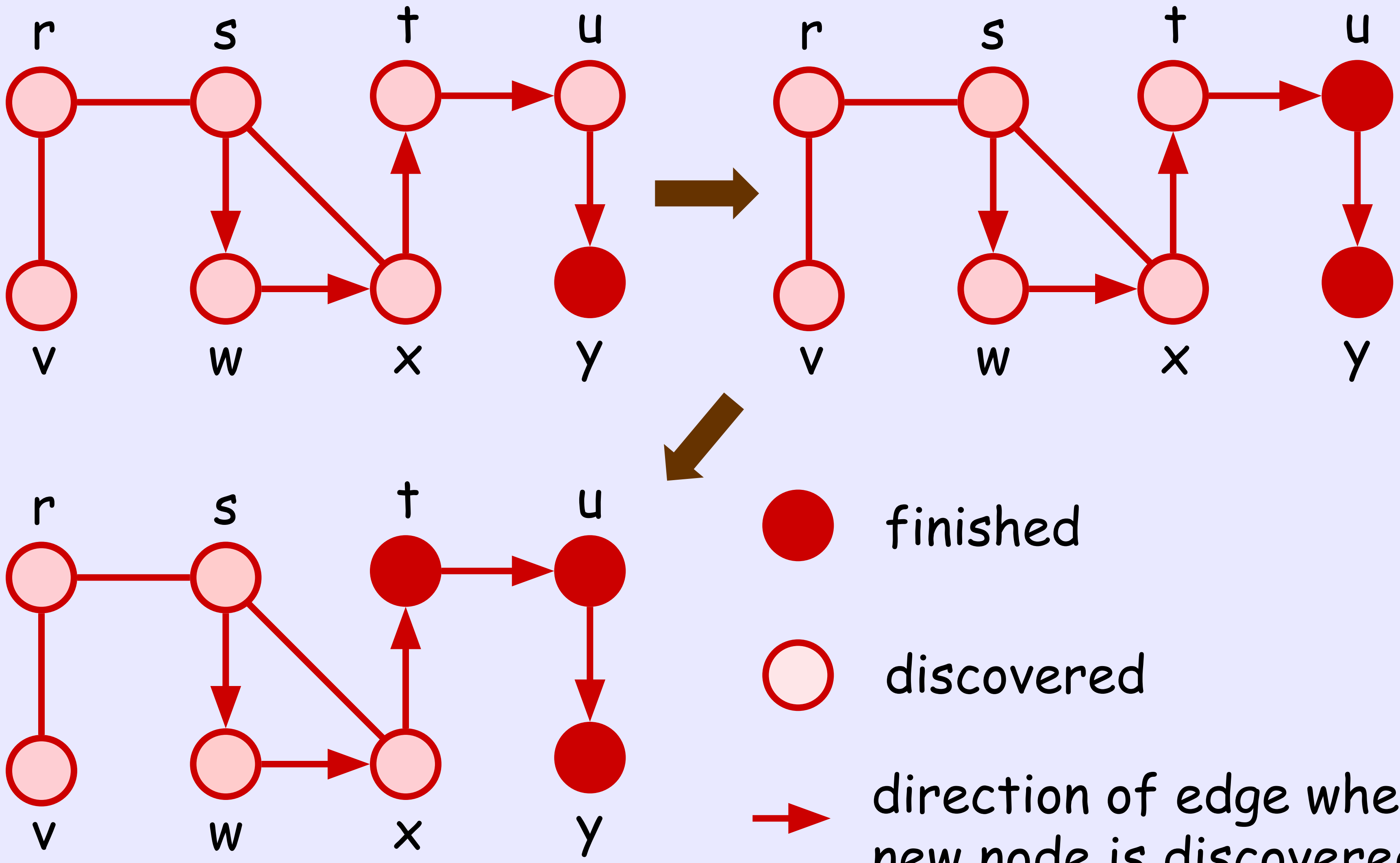
Example (s = source)

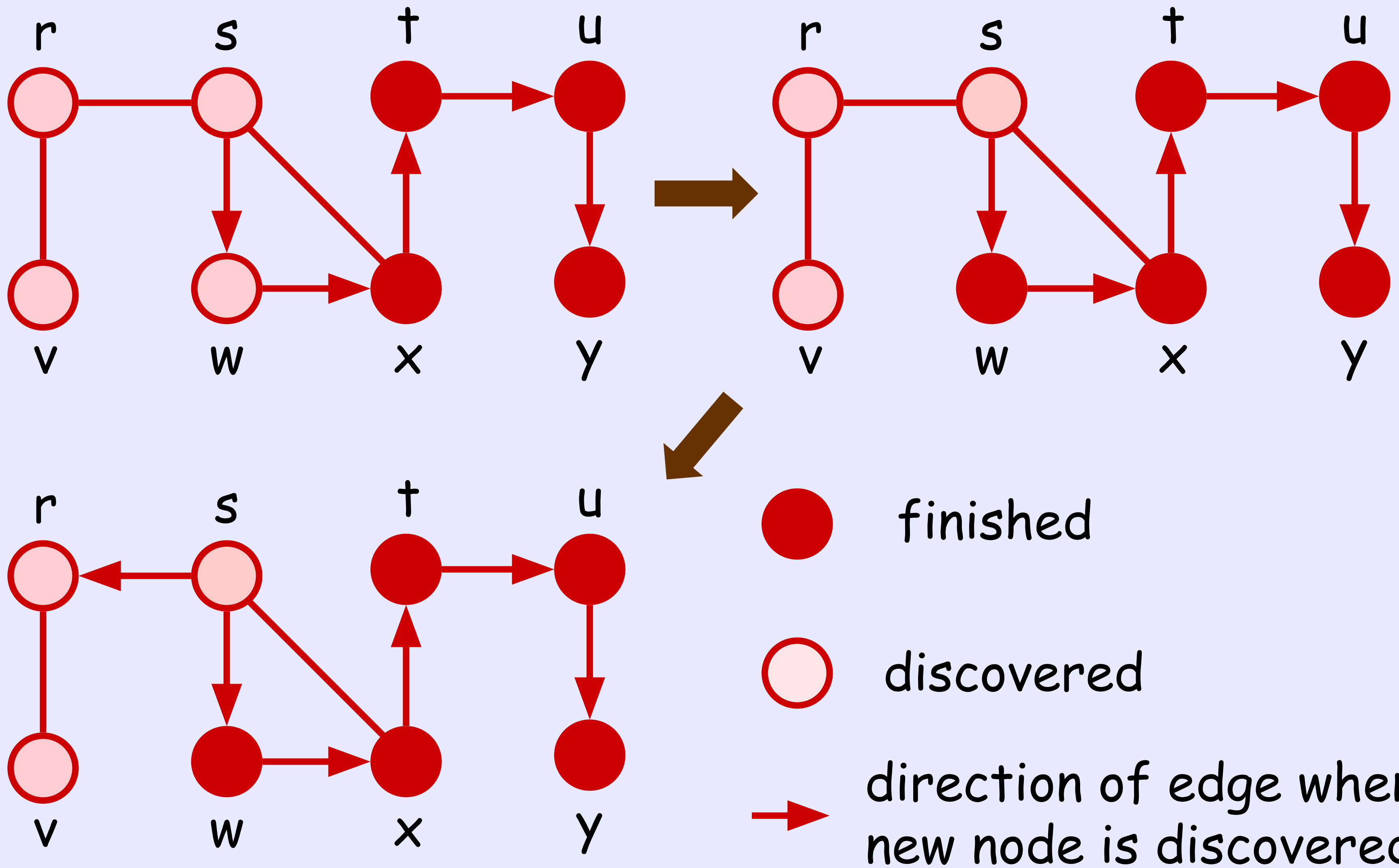


● finished

○ discovered

→ direction of edge when new node is discovered

Example (s = source)



● finished

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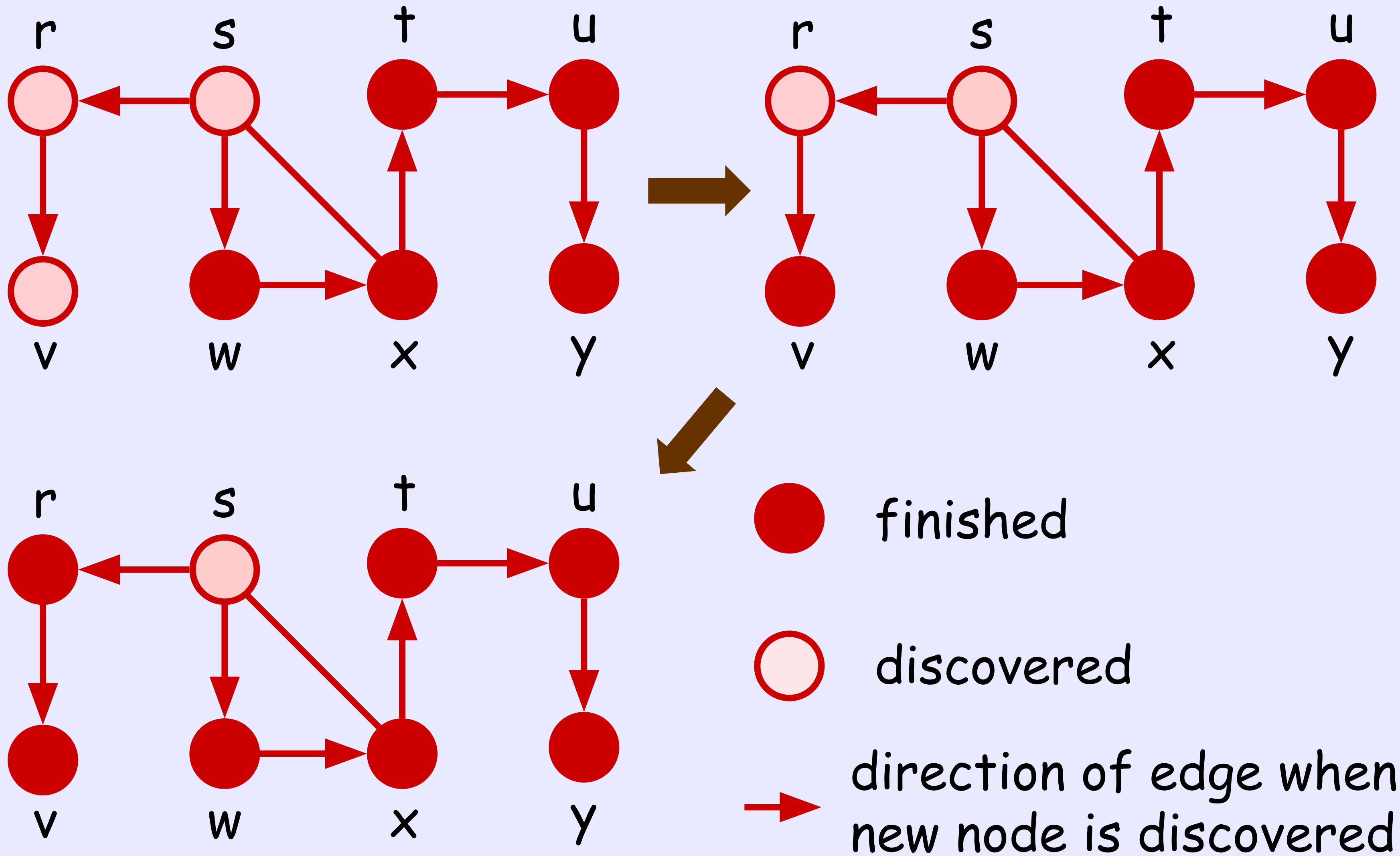
→ direction of edge when new node is discovered

Example (s = source)

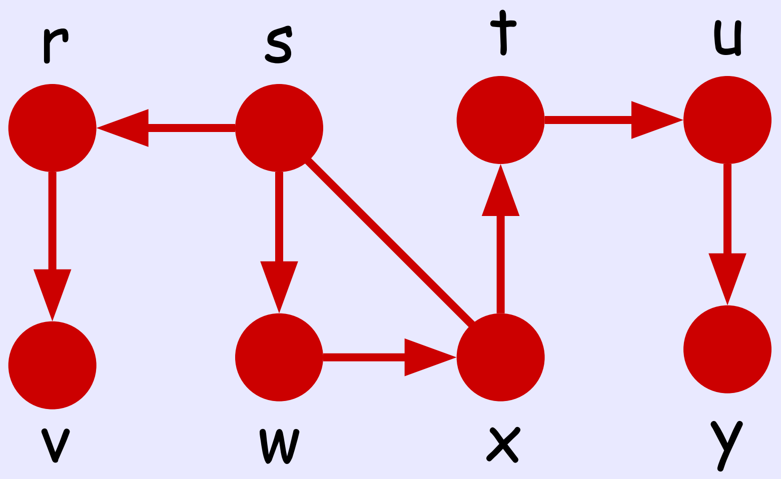


● finished

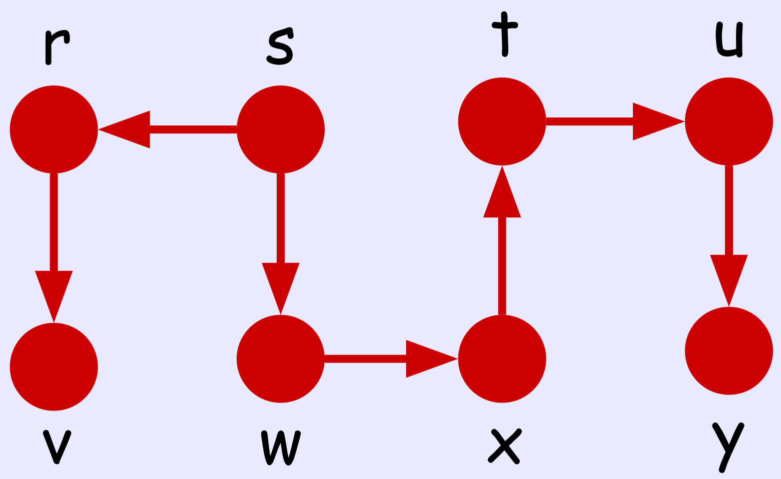
○ discovered

→ direction of edge when new node is discovered

Example (s = source)



Done when s is finished



The directed edges form a tree that contains all nodes reachable from s

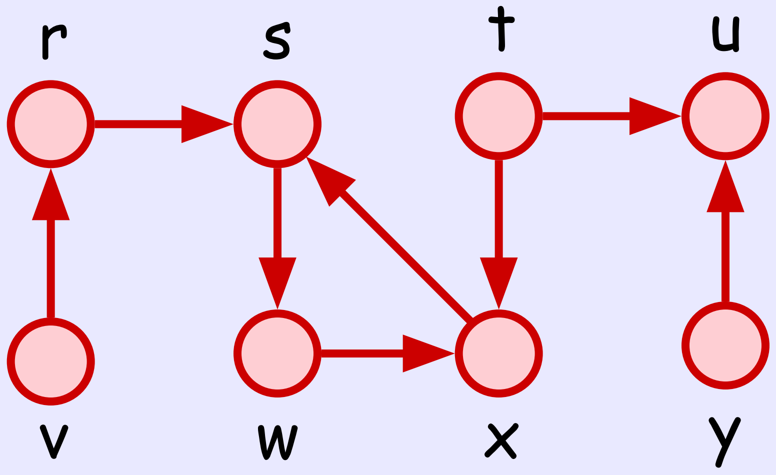
Called DFS tree of s

Generalization

- Just like BFS, DFS may not visit all the vertices of the input graph G, because :
  - G may be disconnected
  - G may be directed, and there is no directed path from s to some vertex
- In most application of DFS (as a subroutine) , once DFS tree of s is obtained, we will continue to apply DFS algorithm on any unvisited vertices …

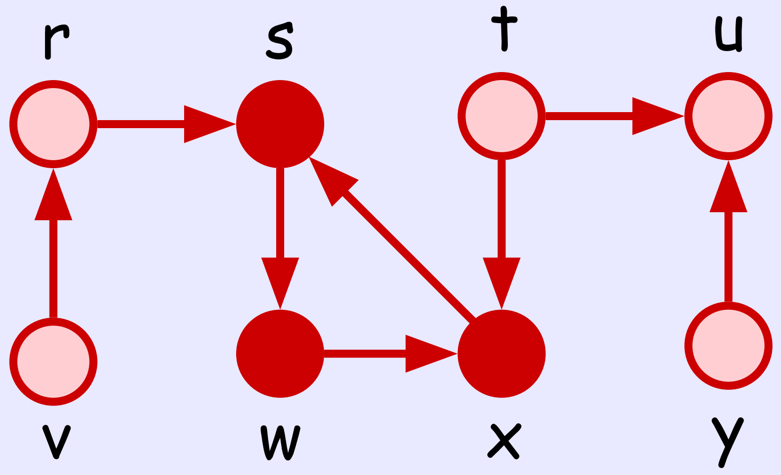
Generalization (Example)

Suppose the input graph is directed



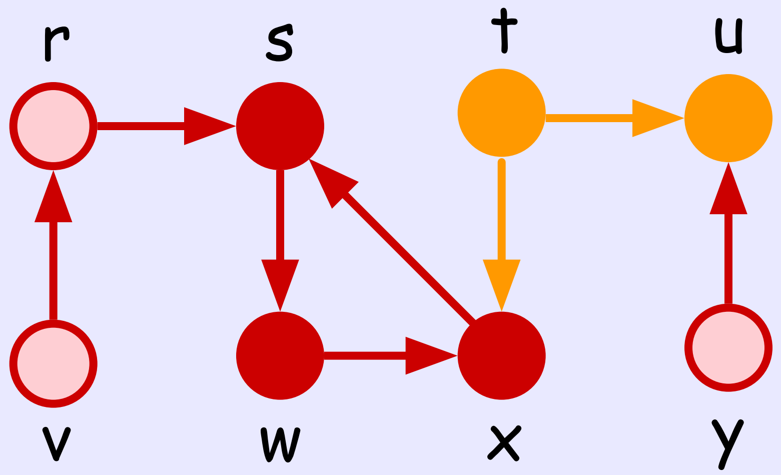
Generalization (Example)

1. After applying DFS on s



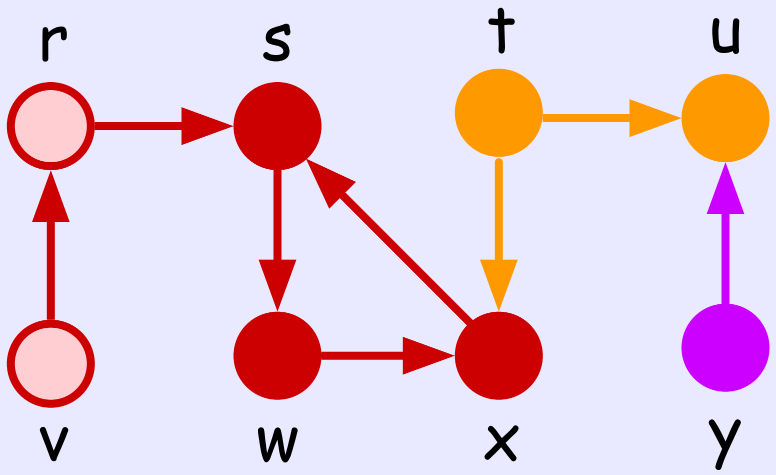
Generalization (Example)

2. Then, after applying DFS on t



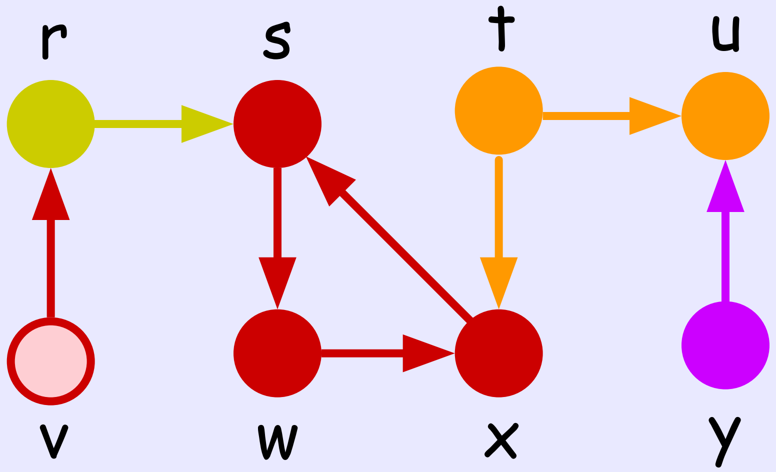
Generalization (Example)

3. Then, after applying DFS on y



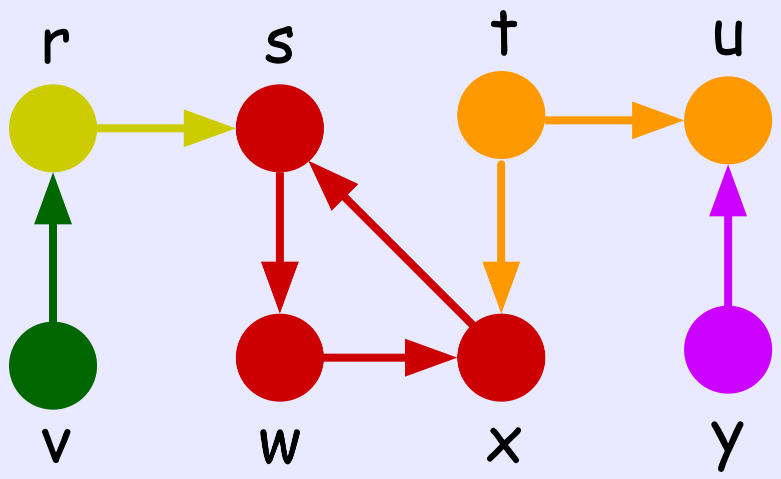
Generalization (Example)

4. Then, after applying DFS on r



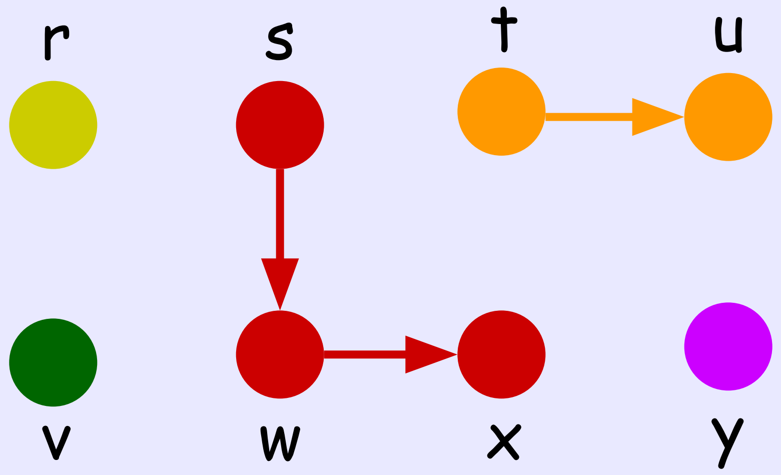
Generalization (Example)

5. Then, after applying DFS on v



Generalization (Example)

Result : a collection of rooted trees called DFS forest



Performance

- Since no vertex is discovered twice, and each edge is visited at most twice (why?)
  ⇒ Total time: $O(|V|+|E|)$

- As mentioned, apart from recursion, we can also perform DFS using a LIFO stack (Do you know how?)

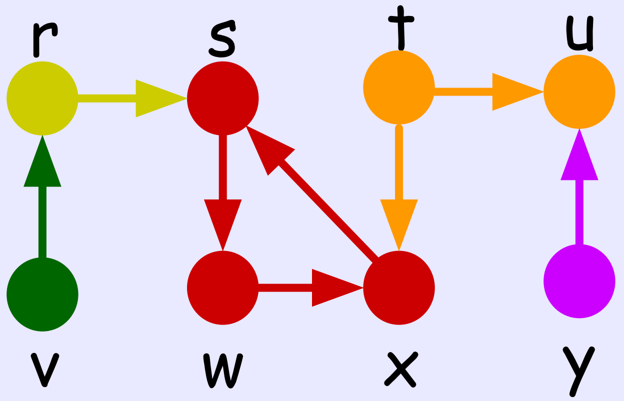
Who will be in the same tree ?

- Because we can only explore branches in an unvisited node

   DFS(u) may not contain all nodes reachable by u in its DFS tree

E.g, in the previous run,

v can reach r, s, w, x but v 's tree does not contain any of them



Can we determine who will be in the same tree ?

Who will be in the same tree ?

- Yes, we will soon show that by white-path theorem, we can determine who will be in the same tree as v at the time when DFS is performed on v

- Before that, we will define the discovery time and finishing time for each node, and show interesting properties of them

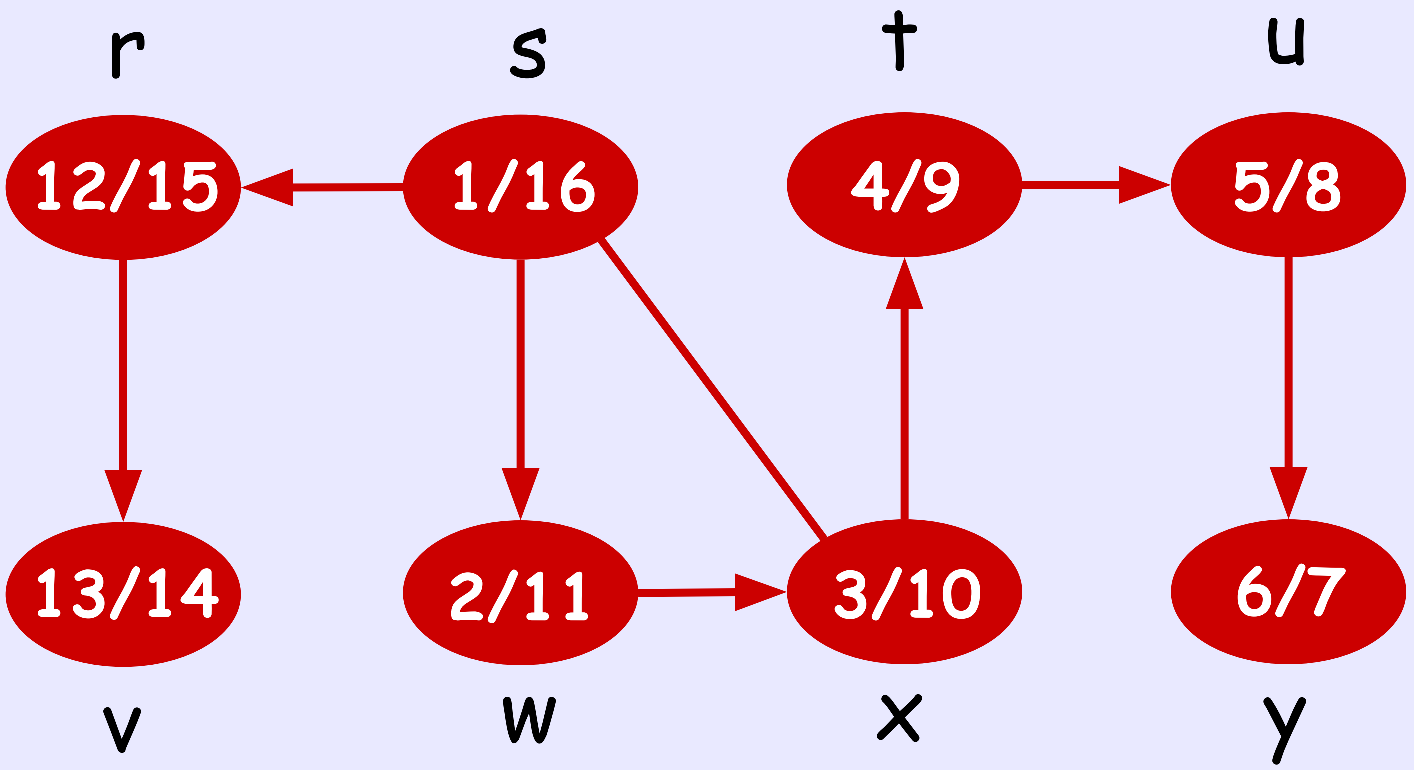
Discovery and Finishing Times

- When the DFS algorithm is run, let us consider a global time such that the time increases one unit :
  - when a node is discovered, or
  - when a node is finished

    (i.e., finished exploring all unvisited neighbors)
- Each node u records :

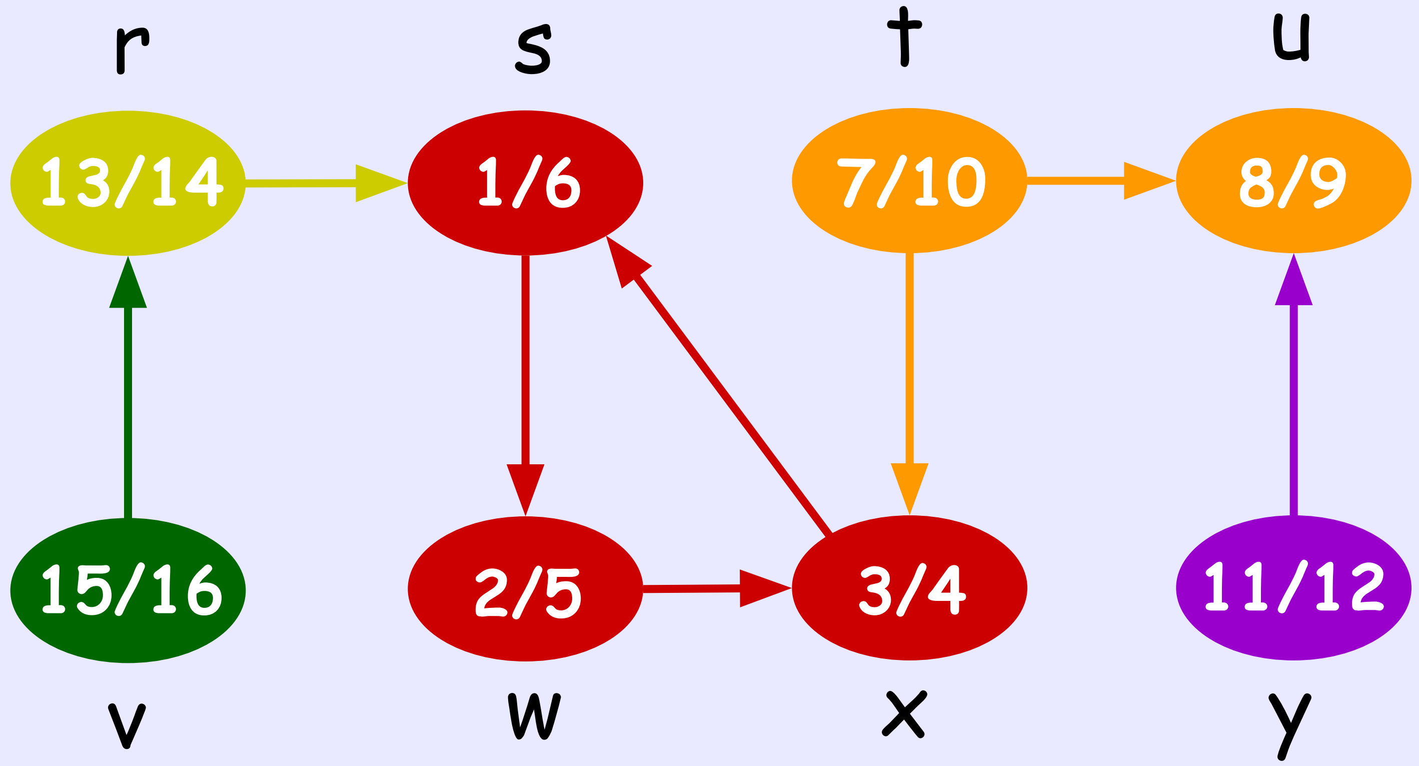
  d(u) = the time when u is discovered, and
  f(u) = the time when u is finished

Discovery and Finishing Times



In our first example (undirected graph)

Discovery and Finishing Times



In our second example
(directed graph)

Nice Properties

- Lemma: For any node u, $d(u) < f(u)$
- Lemma: For nodes u and v,
  $d(u)$, $d(v)$, $f(u)$, $f(v)$ are all distinct
- Theorem (Parenthesis Theorem):
  Let u and v be two nodes with $d(u) < d(v)$
  Then, either
  1. $d(u) < d(v) < f(v) < f(u)$ [contain], or
  2. $d(u) < f(u) < d(v) < f(v)$ [disjoint]

Proof of Parenthesis Theorem

- Consider the time when v is discovered
- Since u is discovered before v, there are two cases concerning the status of u :
  - Case 1: (u is not finished)
    This implies v is a descendant of u
    $\Rightarrow$ $f(v) < f(u)$ (why?)
  - Case 2: (u is finished)
    $\Rightarrow$ $f(u) < d(v)$

Corollary

- Corollary:

  v is a (proper) descendant of u

  if and only if

  $d(u) < d(v) < f(v) < f(u)$

- Proof: v is a (proper) descendant of u

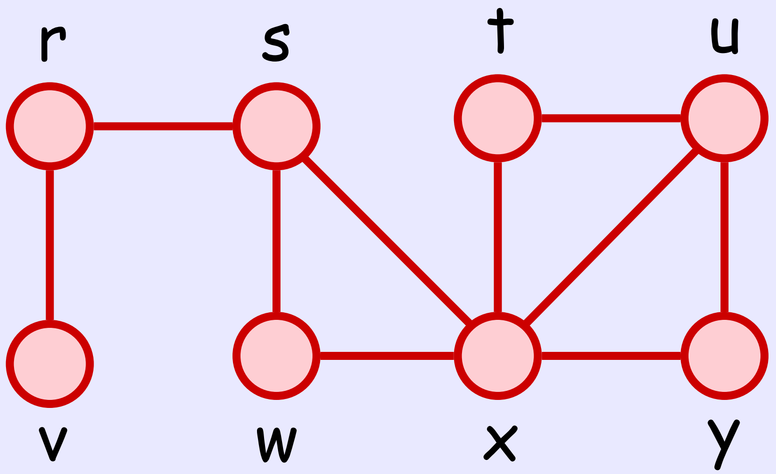
  $\Leftrightarrow$ $d(u) < d(v)$ and $f(v) < f(u)$

  $\Leftrightarrow$ $d(u) < d(v) < f(v) < f(u)$

White-Path Theorem

- Theorem: In a DF forest of a graph G = (V, E), vertex v is a descendant of vertex u if and only if at the time d(u) that the search discovers u, there is a path from u to v consisting entirely of white nodes (unvisited nodes) only

E.g.,



Proof (Part 1)

- => Suppose that v is a descendant of u
  Let P = (u, w_1, w_2, …, w_k, v) be the directed path from u to v in DFS tree of u

  Then, apart from u, each node on P must be discovered after u
  ⇒ They are all unvisited by the time we perform DFS on u
  ⇒ Thus, at this time, there exists a path from u to v with unvisited nodes only

Proof (Part 2)

- So, every descendant of u is reachable from u with unvisited nodes only

- To complete the proof, it remains to show the converse (<=) :

  ✔ Any node reachable from u with unvisited nodes only becomes u 's descendant is also true

  (We shall prove this by contradiction)

Proof (Part 2)

- Suppose on contrary the converse is false
- Then, there exists some v, reachable from u with unvisited nodes only, does not become u 's descendant
  - If more than one choice of v, let v be one such vertex closest to u

□ $d(u) < f(u) < d(v) < f(v)$ … EQ.1

Proof (Part 2)

- Let $P = (u, w_1, w_2, \ldots, w_k, v)$ be any path from u to v using unvisited nodes only
- By our choice of v (closest one), all $w_1, w_2, \ldots, w_k$ become u's descendants

Handle special case: when $u = w_k$

- This implies:

$$d(u) \leq d(w_k) < f(w_k) \leq f(u)$$

- Combining with EQ.1, we have

$$d(w_k) < f(w_k) < d(v) < f(v)$$

Proof (Part 2)

- However, since there is an edge (no matter undirected or directed) from w_k to v, if $d(w_k) < d(v)$, then we must have

  $d(v) < f(w_k)$ ... (why??)

- Consequently, it contradicts with :

  $d(w_k) < \underline{f(w_k) < d(v)} < f(v)$

 Proof completes

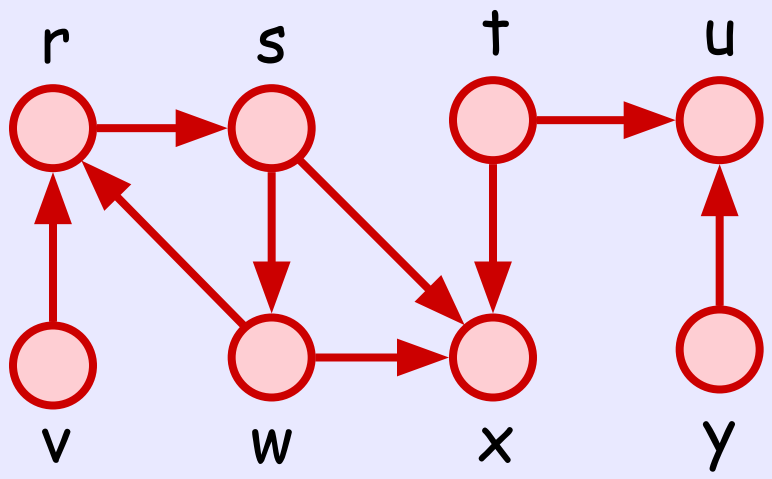
Classification of Tree Edges

- After a DFS process, we can classify the edges of a graph into four types :

1. Tree : Edges in the DFS forest
2. Back : From descendant to ancestor when explored (include self loop)
3. Forward : From ancestor to descendant when explored (exclude tree edge)
4. Cross : Others (no ancestor-descendant relation)

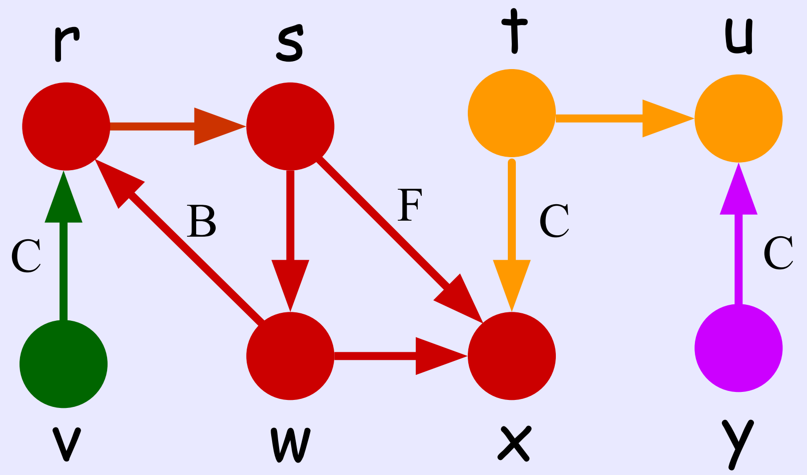
Example

Suppose the input graph is directed



Example

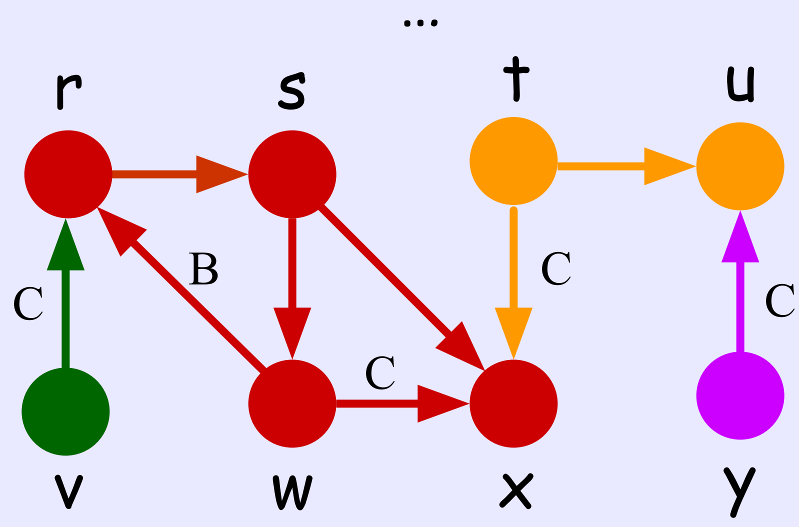
Suppose this is the DFS forest obtained



Can you classify the type of each edge ?

Example

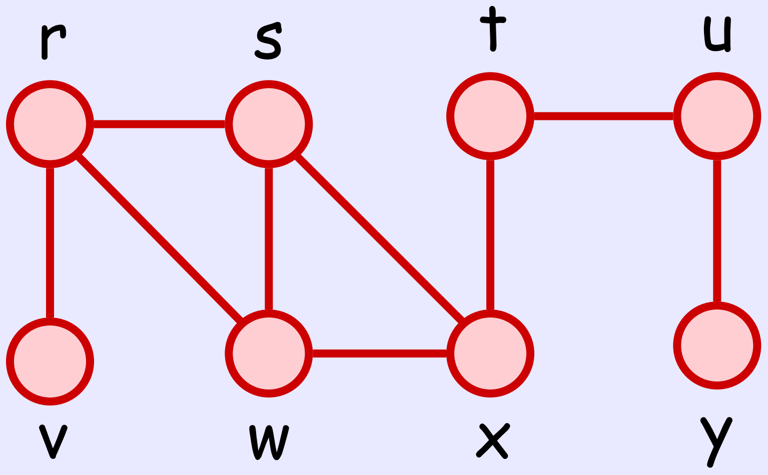
Suppose the DFS forest is different now



Can you classify the type of each edge ?

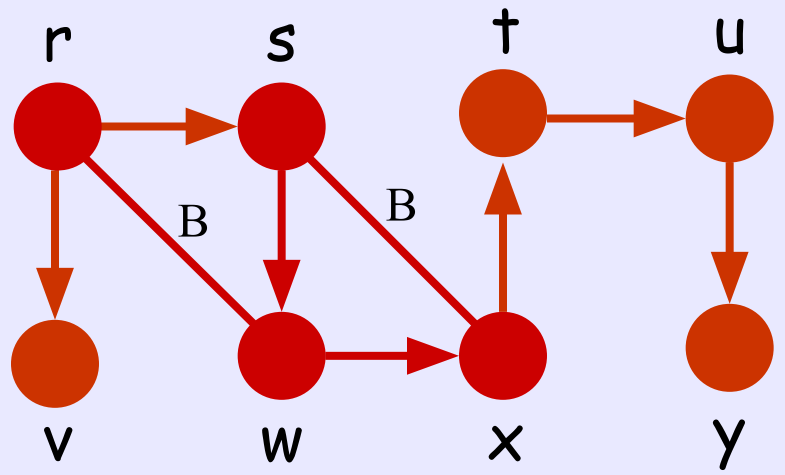
Example

Suppose the input graph is undirected



Example

Suppose this is the DFS forest obtained



Can you classify the type of each edge ?

Edges in Undirected Graph

- Theorem: After DFS of an undirected graph, every edge is either a tree edge or a back edge

- Proof: Let (u,v) be an edge. Suppose u is discovered first. Then, v will become u's descendent (white-path) so that f(v) < f(u)

□ If u discovers v □ (u,v) is tree edge

□ Else, (u,v) is explored after v discovered

Then, (u,v) must be explored from v because f(v) < f(u) □ (u,v) is back edge

Practice at Home

- Exercise: 22.3-5, 22.3-8, 22.3-9, 22.3-11