Chapter 6 Heapsort

About this lecture

- Introduce Heap
  - Shape Property and Heap Property
  - Heap Operations
- Heapsort: Use Heap to Sort
- Fixing heap property for all nodes
- Use Array to represent Heap
- Introduce Priority Queue

Heap

A heap (or binary heap) is a binary tree that satisfies both:

(1) Shape Property

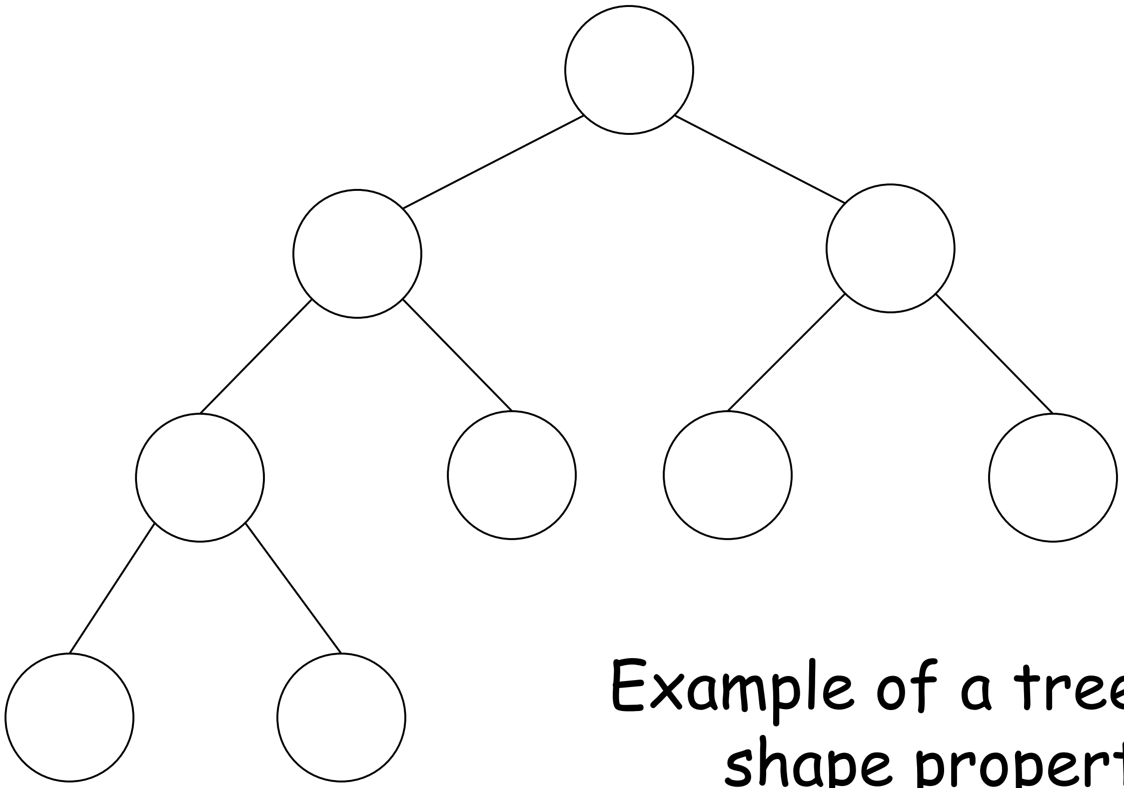
– All levels, except deepest, are fully filled

– Deepest level is filled from left to right

(2) Heap Property

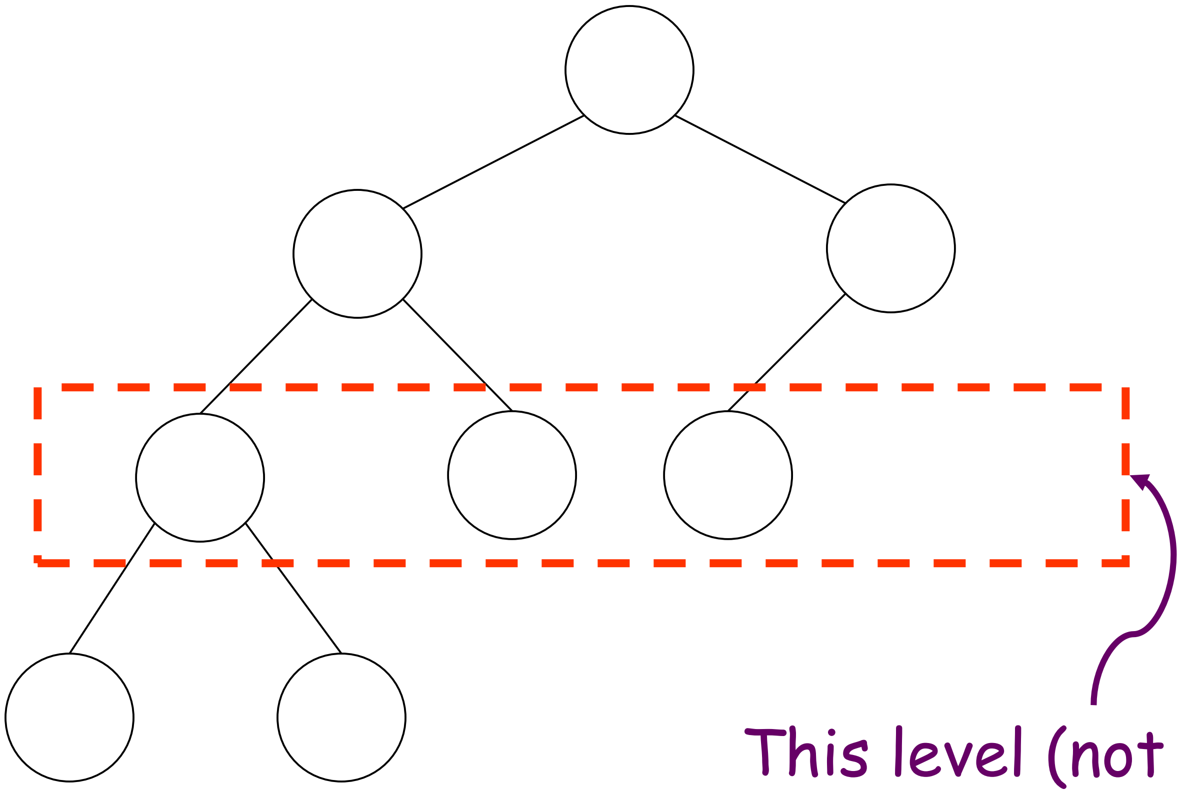
– Value of a node $\leq$ Value of its children

Satisfying Shape Property



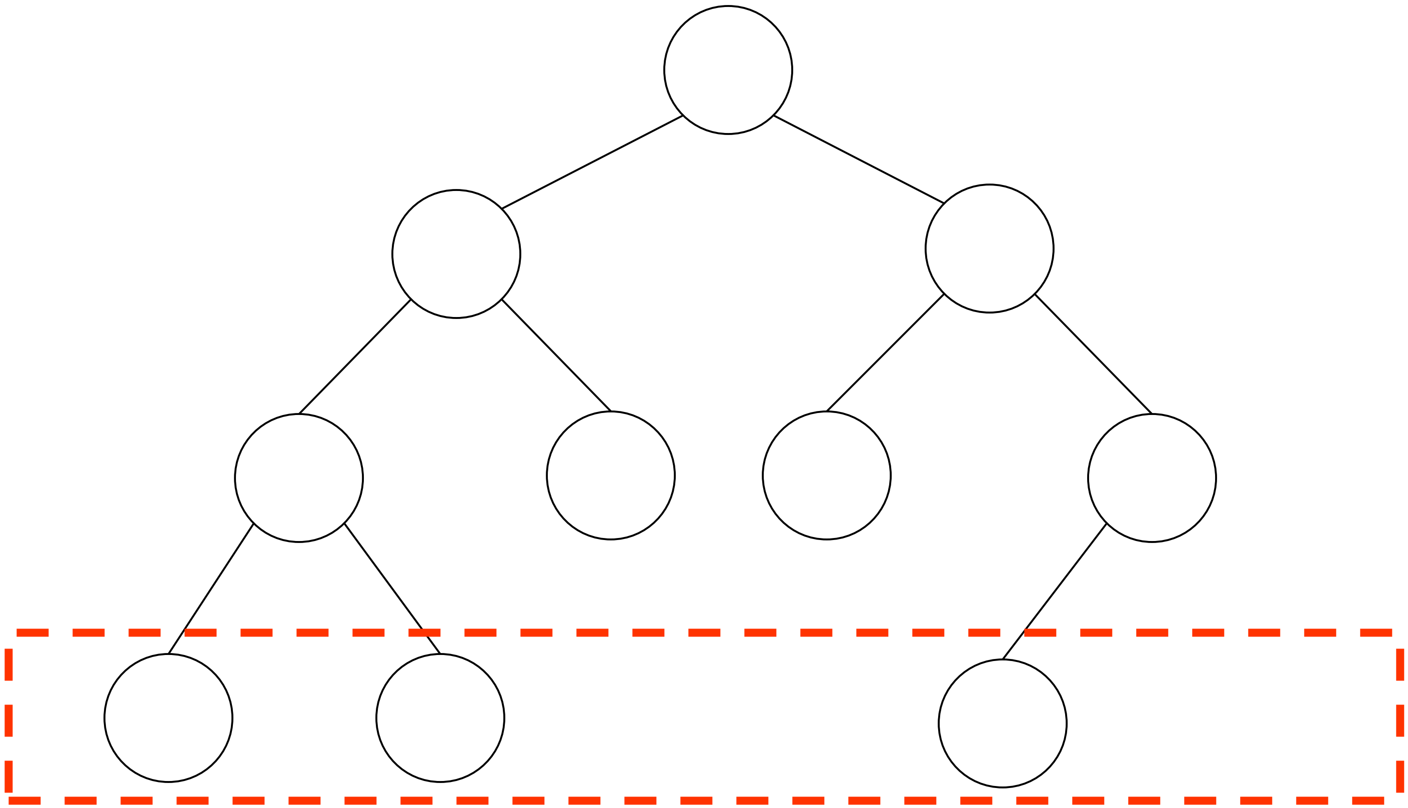
Example of a tree with shape property

Not Satisfying Shape Property



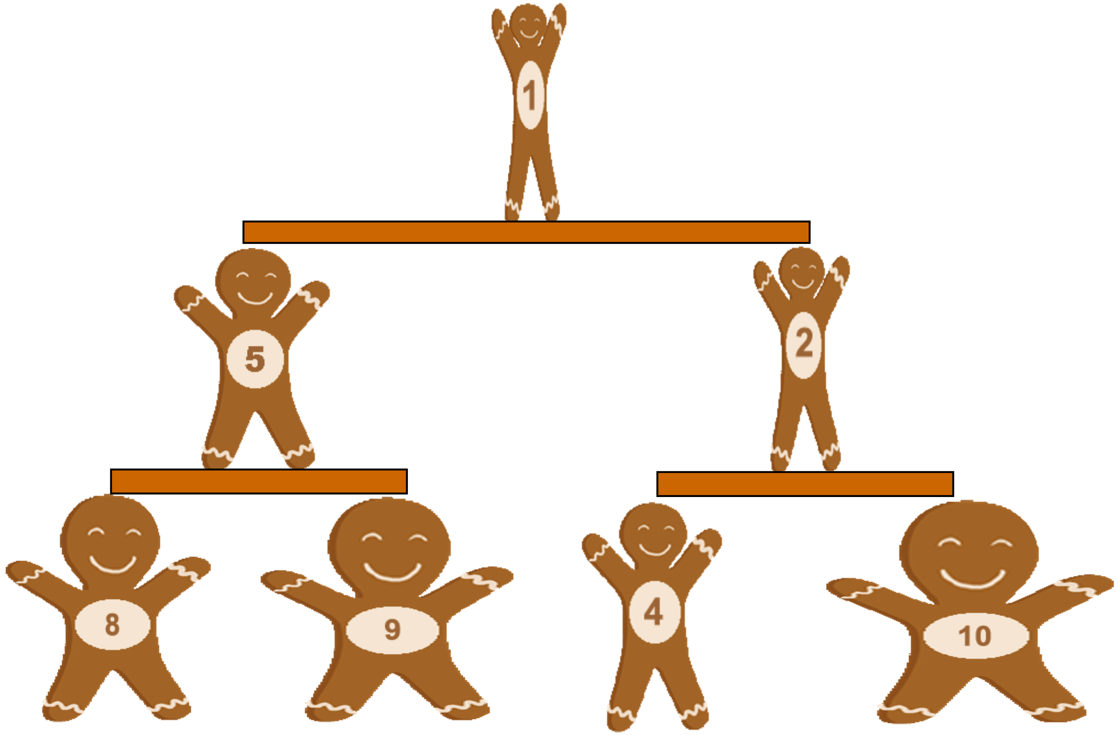
This level (not deepest)
is NOT fully filled

Not Satisfying Shape Property

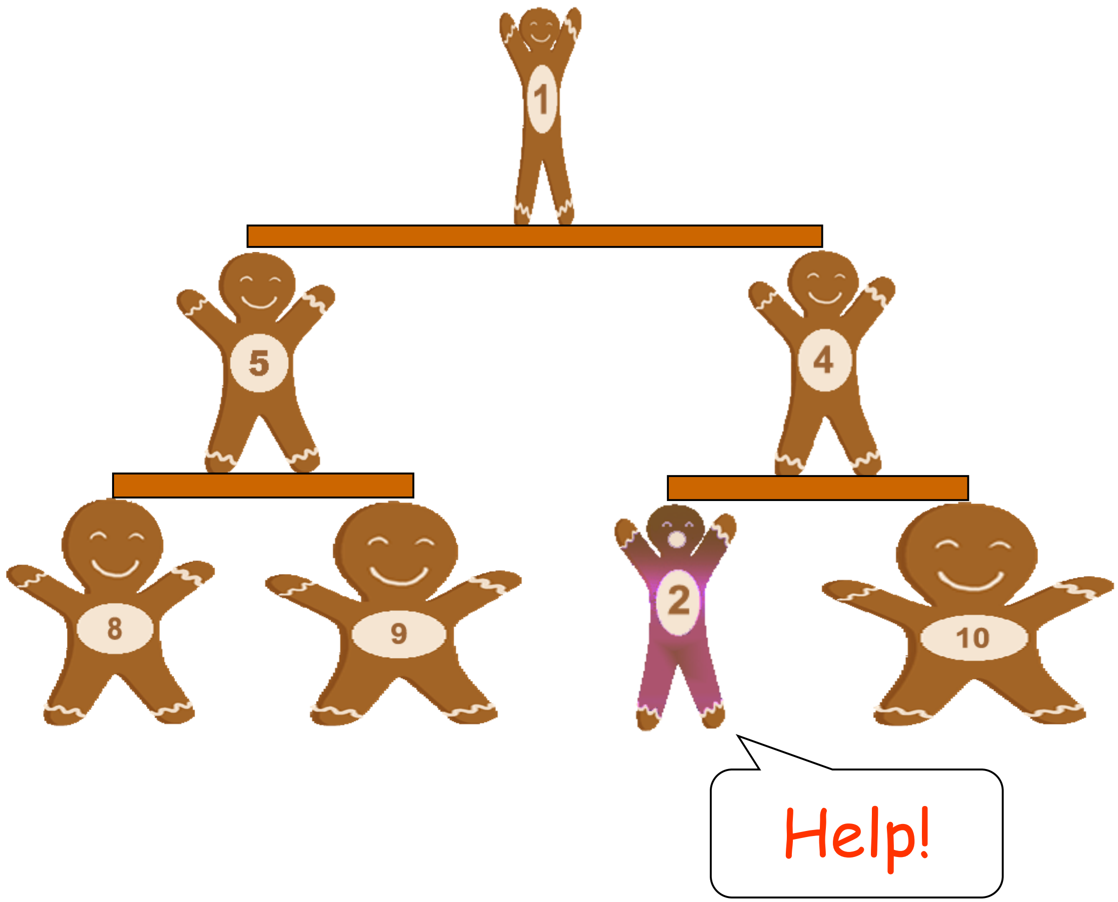


Deepest level NOT
filled from left to right

Satisfying Heap Property



Not Satisfying Heap Property



Min-Heap

Q. Given a heap, what is so special about the root's value?

A. … always the minimum

Because of this, the previous heap is also called a min-heap

Heap Operations

- **Find-Min** : find the minimum value

  ➔ $\Theta(1)$ time

- **Extract-Min** : delete the minimum value

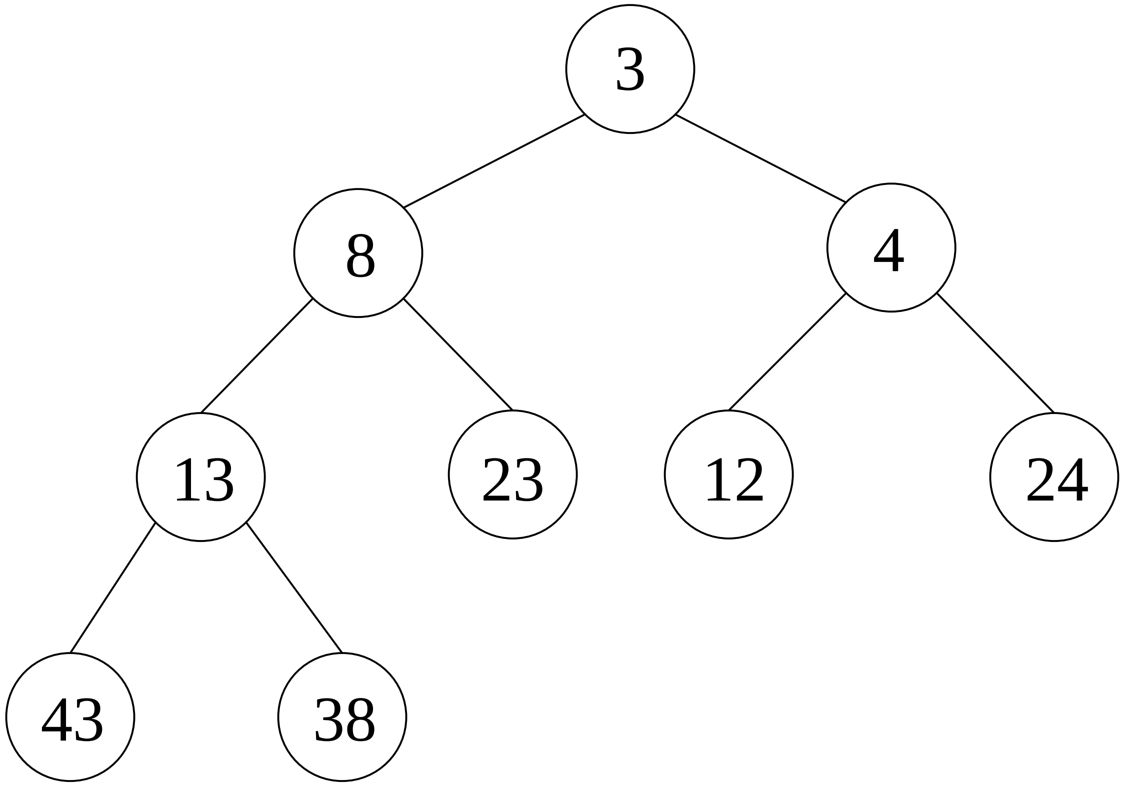
  ➔ $O(\log n)$ time (how??)

- **Insert** : insert a new value into heap

  ➔ $O(\log n)$ time (how??)

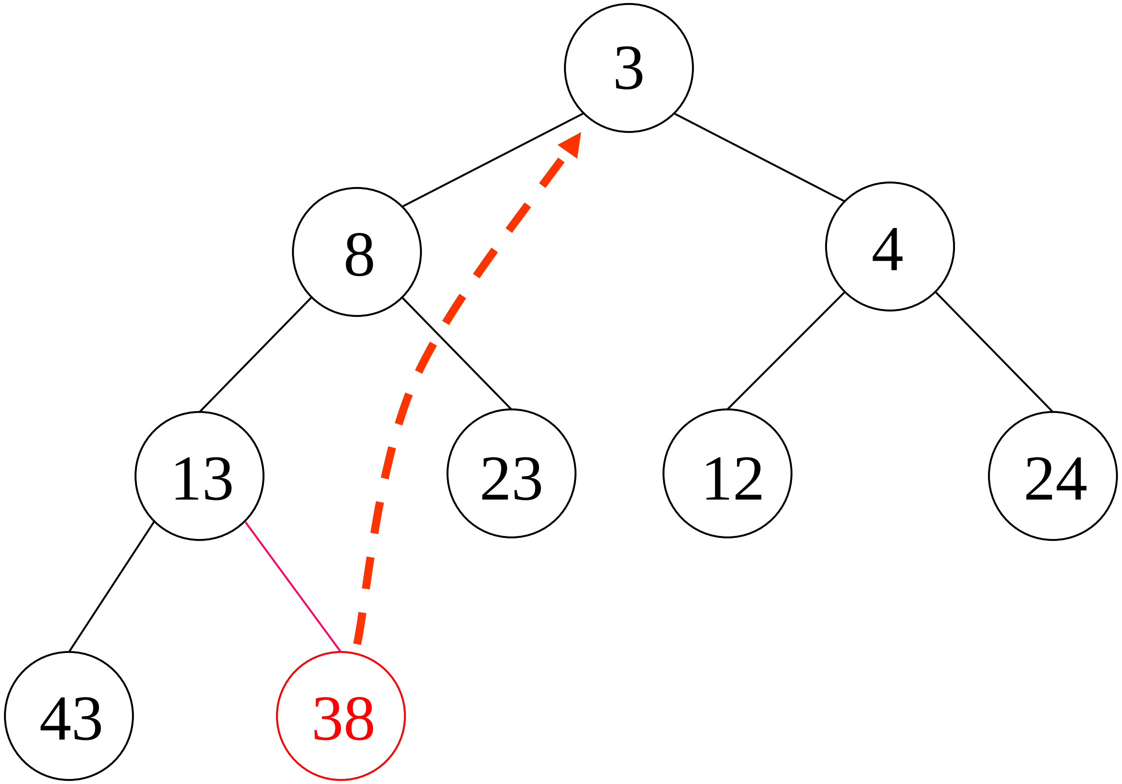
n = # nodes in the heap

How to do Extract-Min?



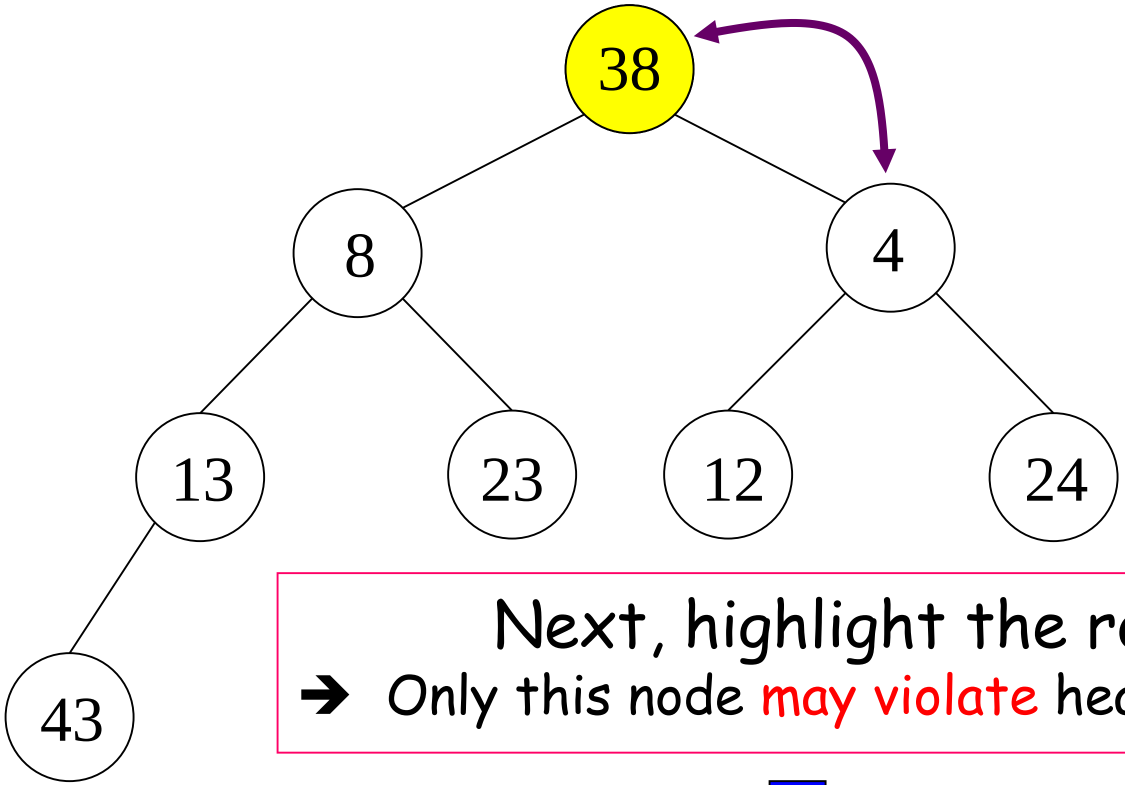
Heap before Extract-Min

Step 1: Restore Shape Property



Copy value of last node to root.
Next, remove last node

Step 2: Restore Heap Property

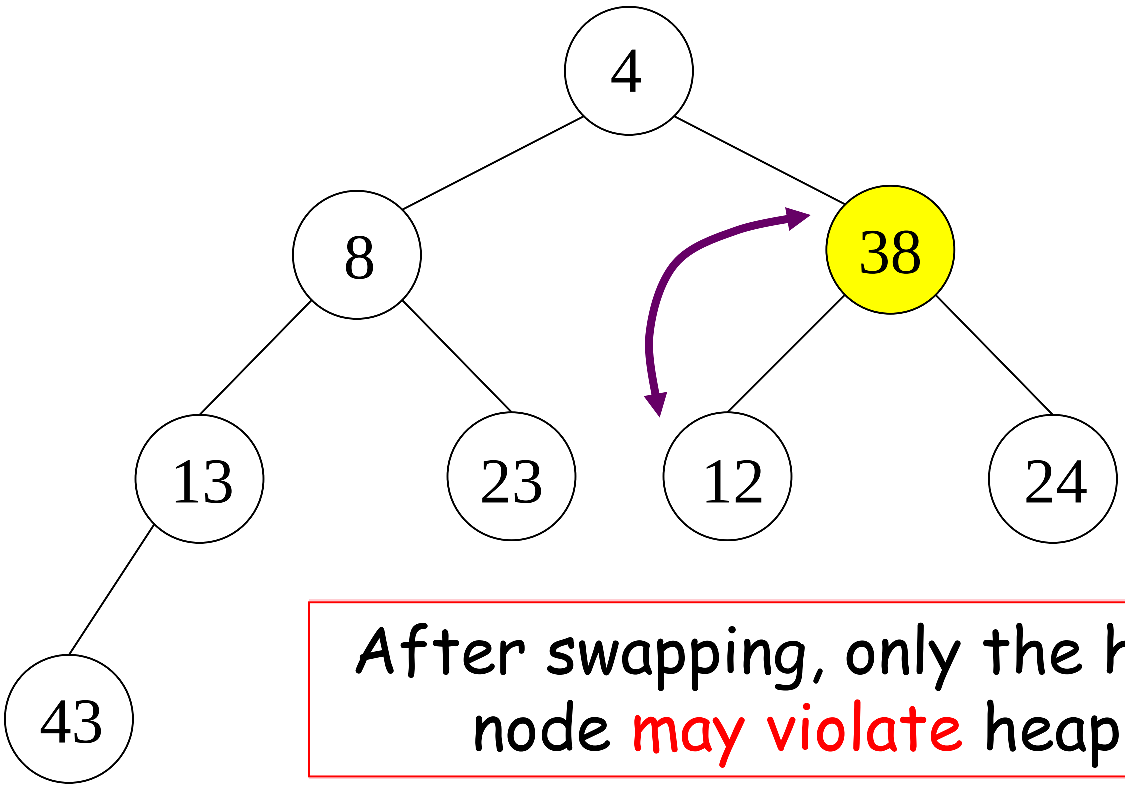


Next, highlight the root

➔ Only this node may violate heap property

If violates, swap highlighted node with "smaller" child
(if not, everything done)

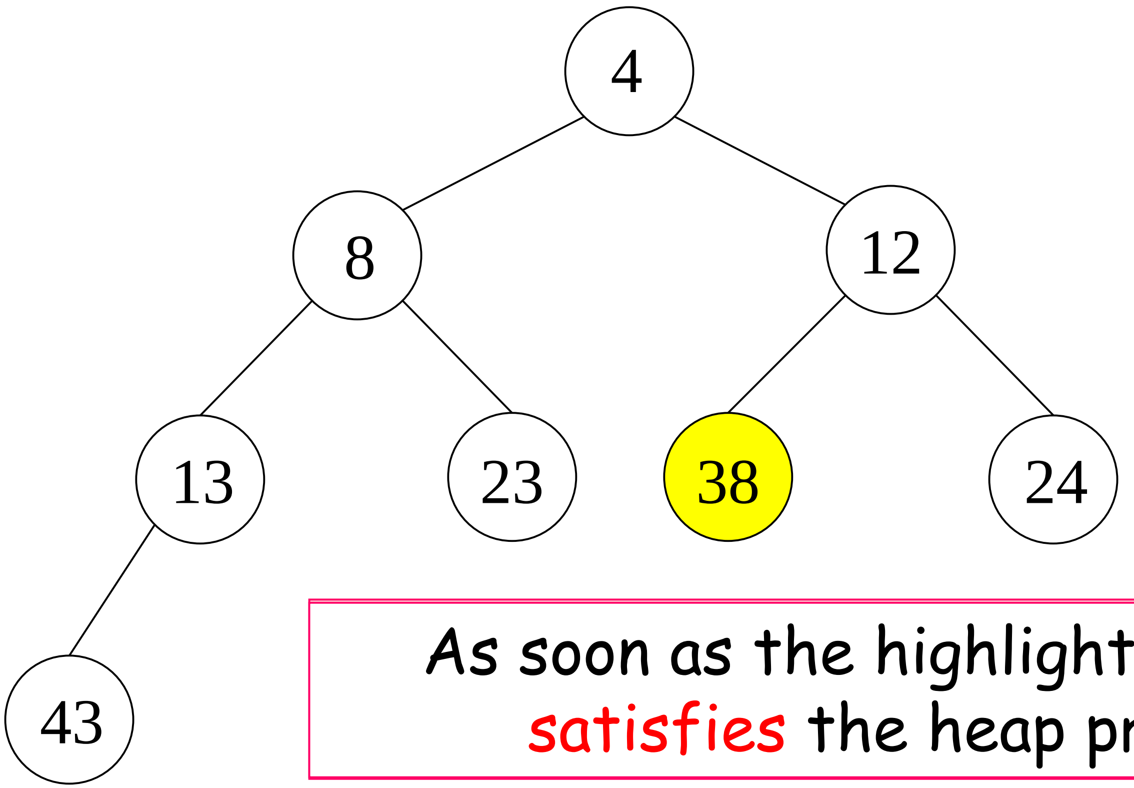
Step 2: Restore Heap Property



After swapping, only the highlighted node may violate heap property

If violates, swap highlighted node with "smaller" child (if not, everything done)

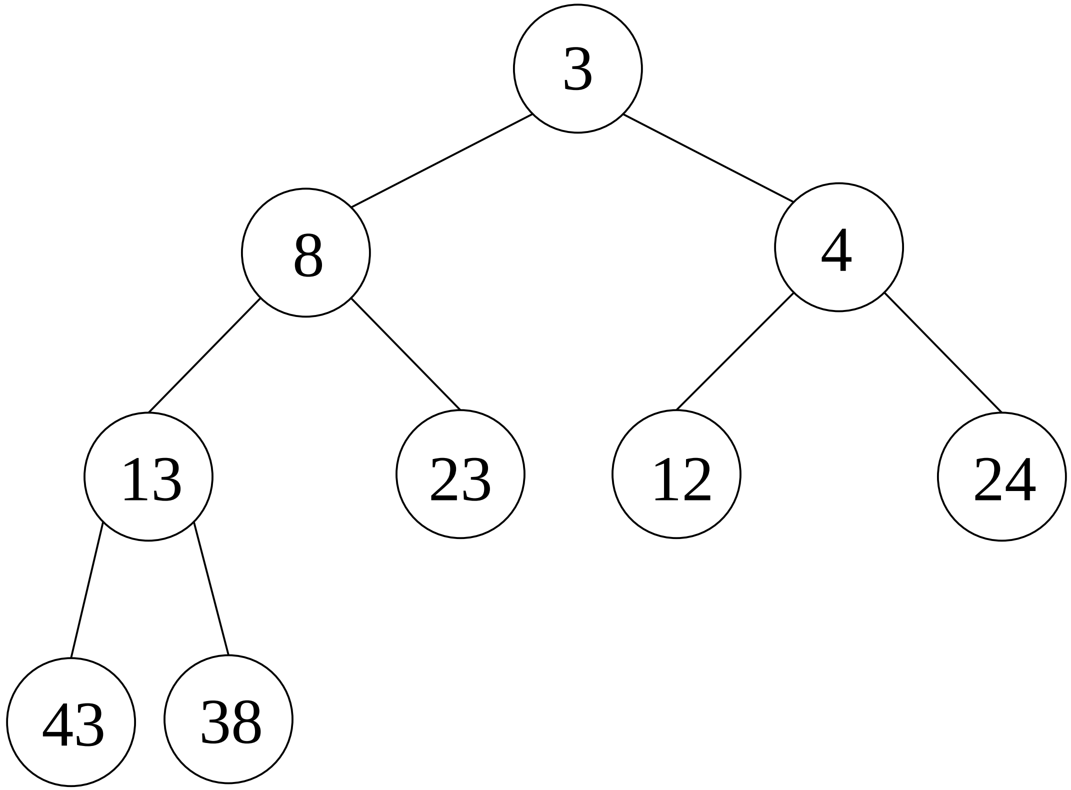
Step 2: Restore Heap Property



As soon as the highlighted node satisfies the heap property

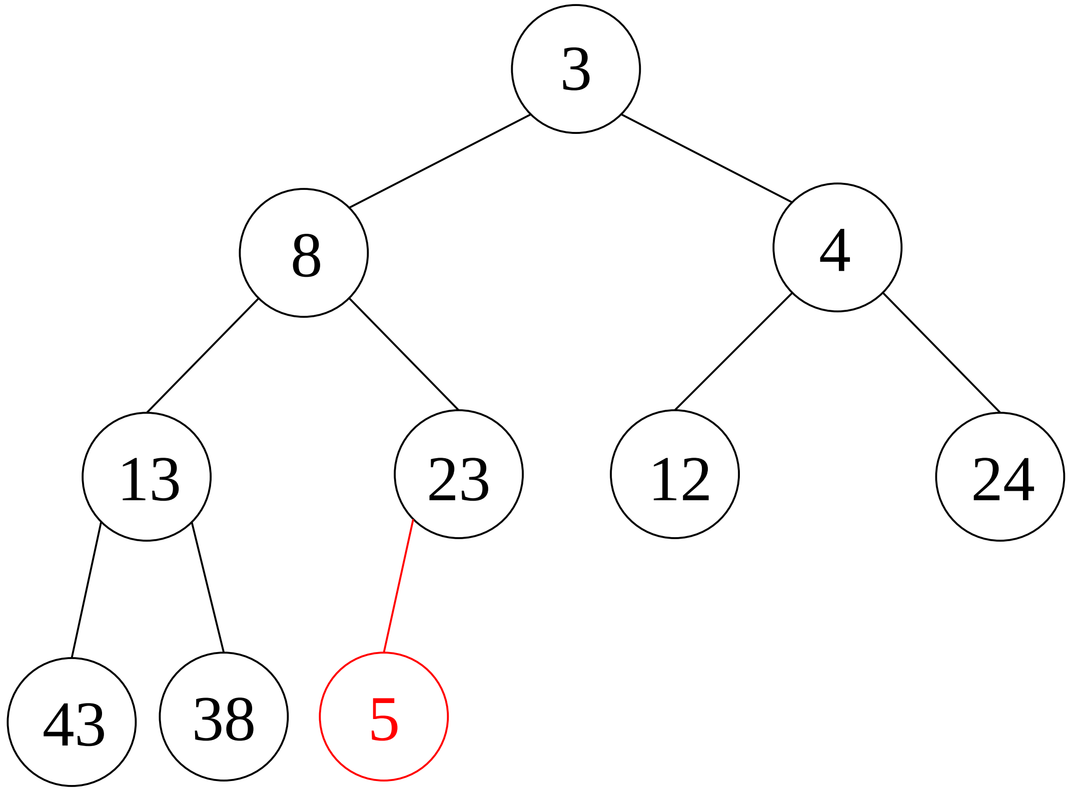
Everything done !!!

How to do Insert?



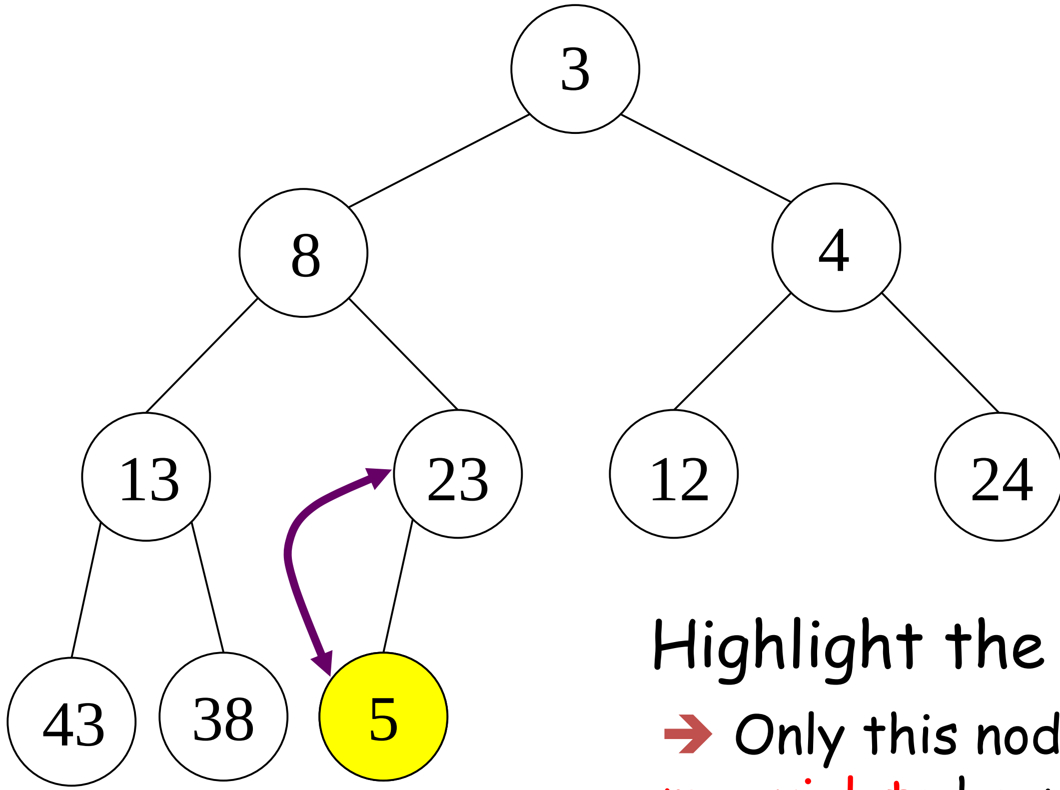
Heap before Insert

Step 1: Restore Shape Property



Create a new node with the new value.
Next, add it to the heap at correct position

Step 2: Restore Heap Property

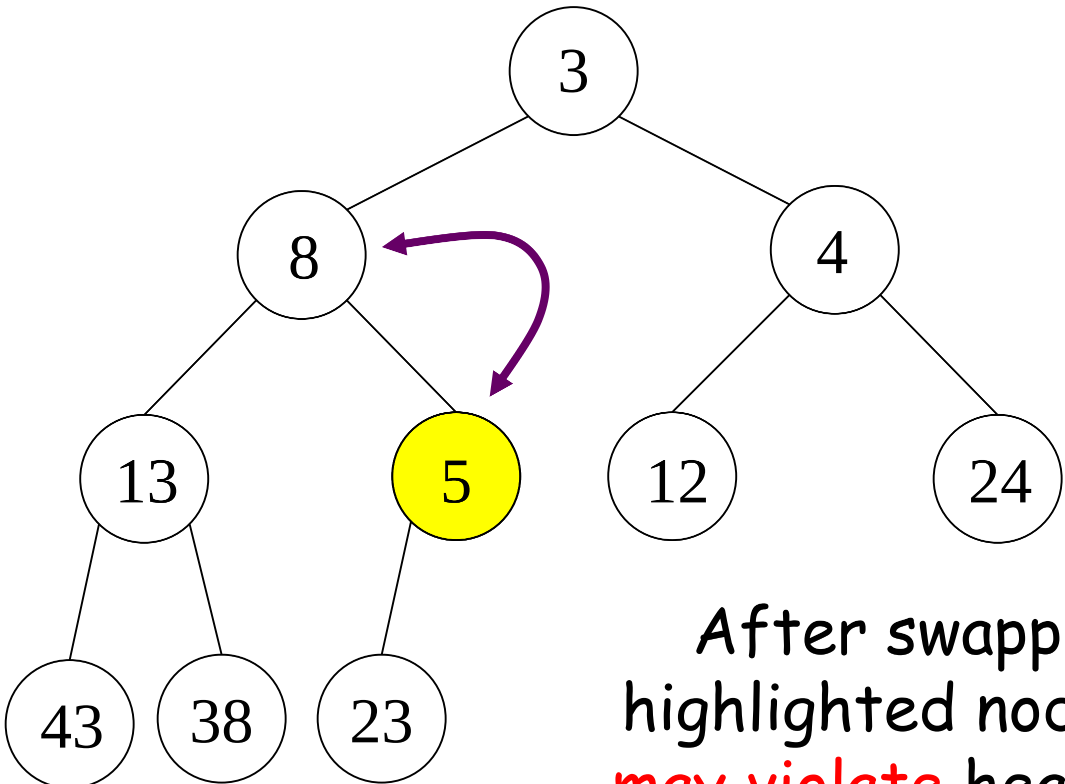


Highlight the new node

➔ Only this node's parent may violate heap property

If violates, swap highlighted node with parent (if not, everything done)

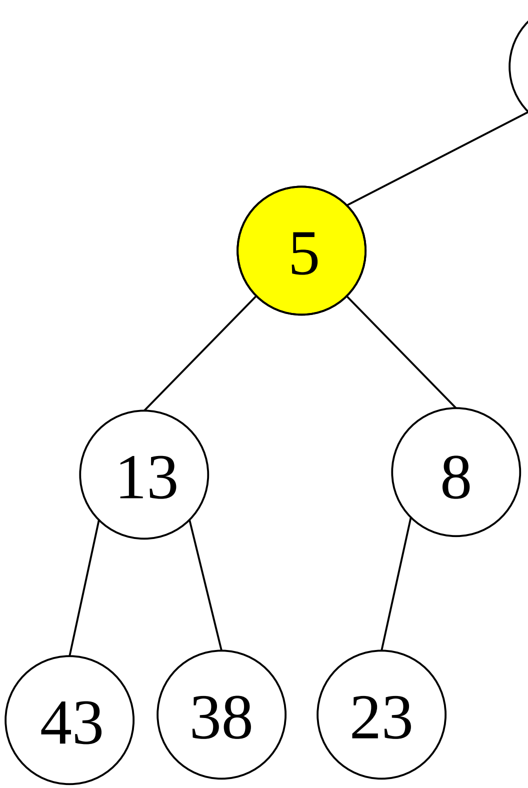
Step 2: Restore Heap Property



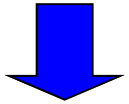
After swapping, only highlighted node's parent may violate heap property

If violates, swap highlighted node with parent (if not, everything done)

Step 2: Restore Heap Property



As soon as highlighted node's parent satisfies heap property



Everything done !!!

Running Time

Let h = node-height of heap

- Both Extract-Min and Insert require $O(h)$ time to perform

Since $h = \Theta(\log n)$ (why??)

➔ Both require $O(\log n)$ time

n = # nodes in the heap

Heapsort

Q. Given n numbers, can we use heap to sort them, say, in ascending order?

A. Yes, and extremely easy !!!

1. Call Insert to insert n numbers into heap
2. Call Extract-Min n times

   ➔ numbers are output in sorted order

Runtime: $n \times O(\log n) + n \times O(\log n) = O(n \log n)$

This sorting algorithm is called heapsort

Challenge

(Fixing heap property for all nodes)

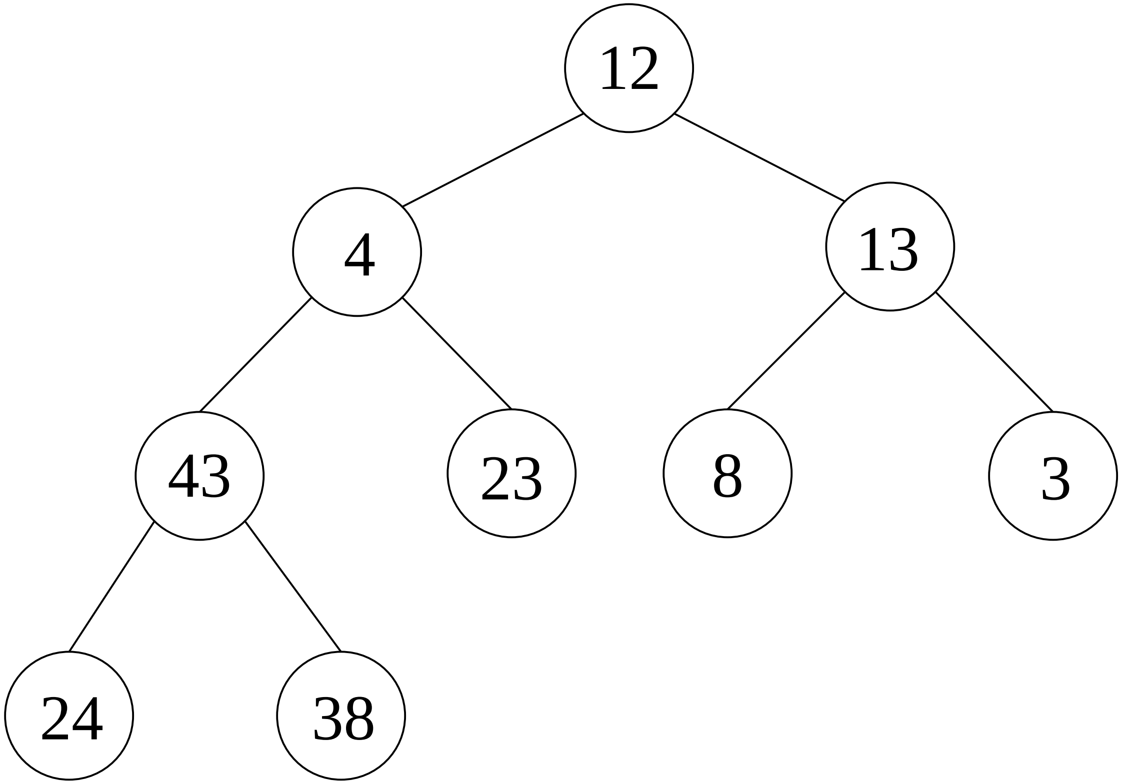
Suppose that we are given a binary tree which satisfies the shape property

However, the heap property of the nodes may not be satisfied …

Question: Can we make the tree into a heap in $O(n)$ time?

n = # nodes in the tree

How to make it a heap?

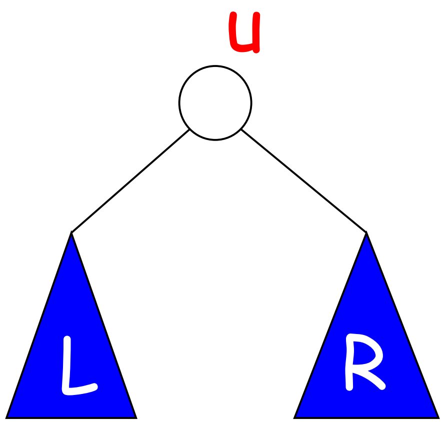


Observation

u = root of a binary tree

L = subtree rooted at u's left child

R = subtree rooted at u's right child



Obs: If L and R satisfy heap property, we can make the tree rooted at u satisfy heap property in $O(\max\{\text{height}(L), \text{height}(R)\})$ time.

We denote the above operation by Heapify(u)

Heapify

✓ Then, for any tree T, we can make T satisfy the heap property as follows:

Step 1. h = node_height(T) ;

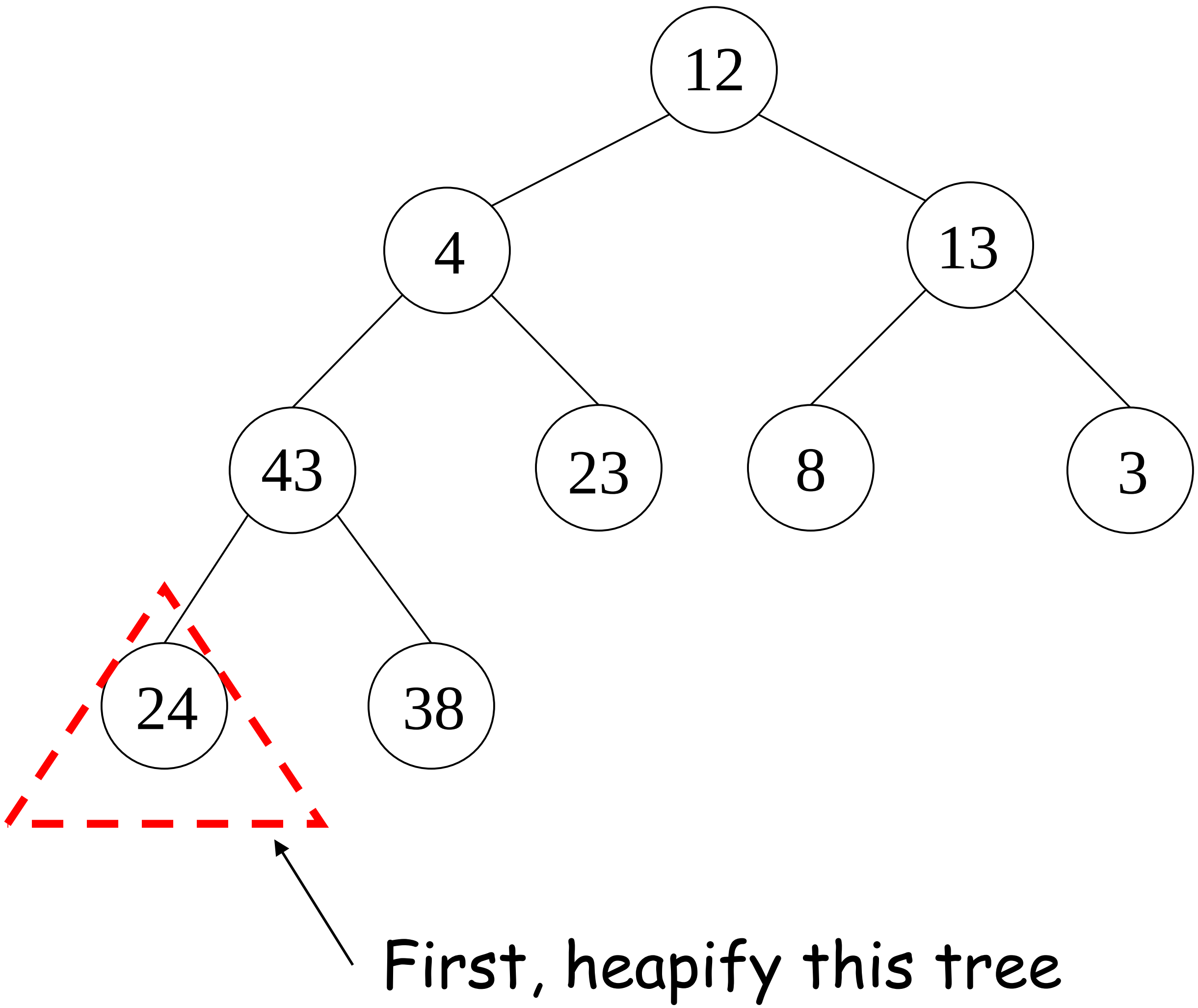
Step 2. for k = h, h-1, …, 1

for each node u at level k

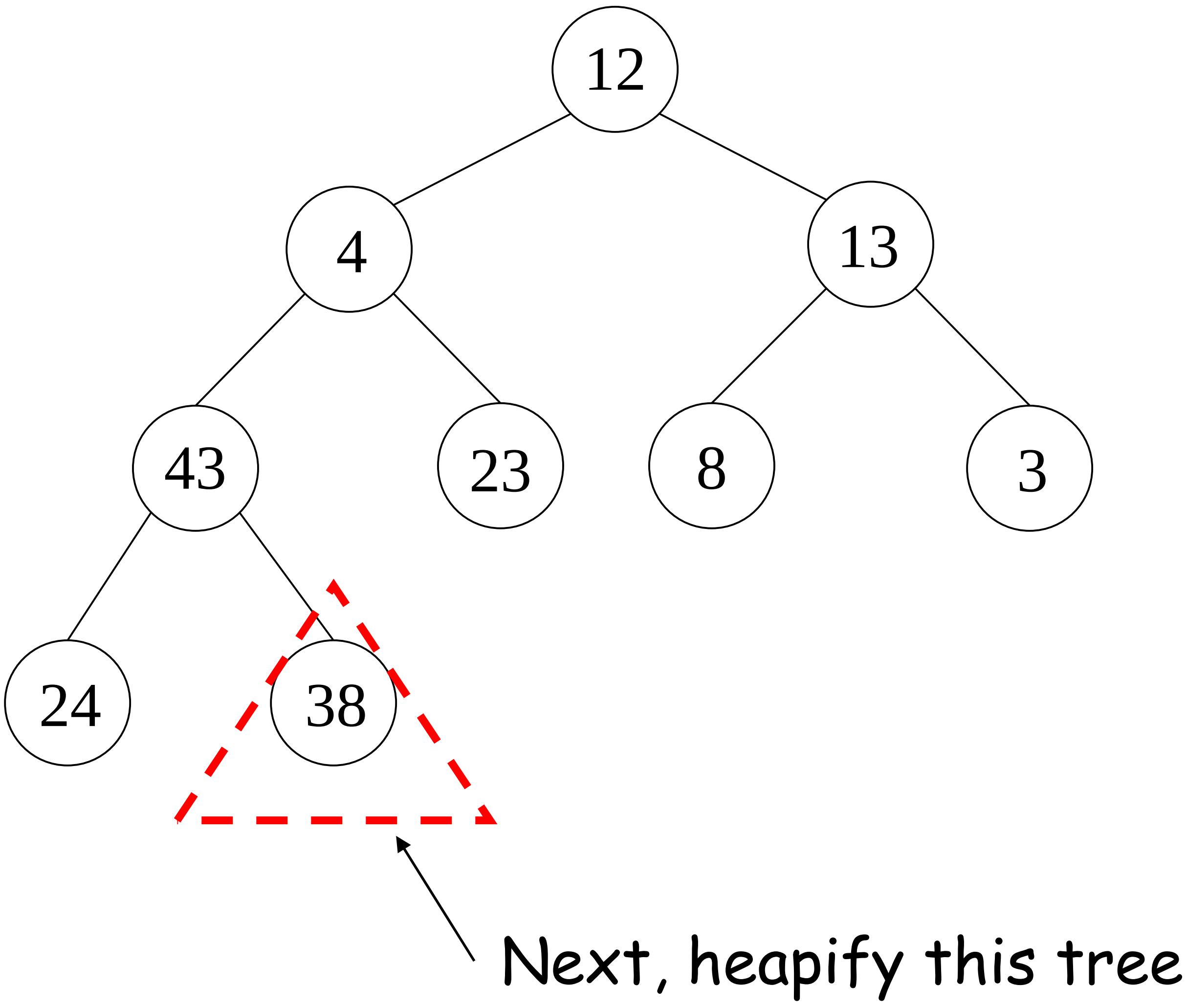
Heapify(u) ;

Why is the above algorithm correct?

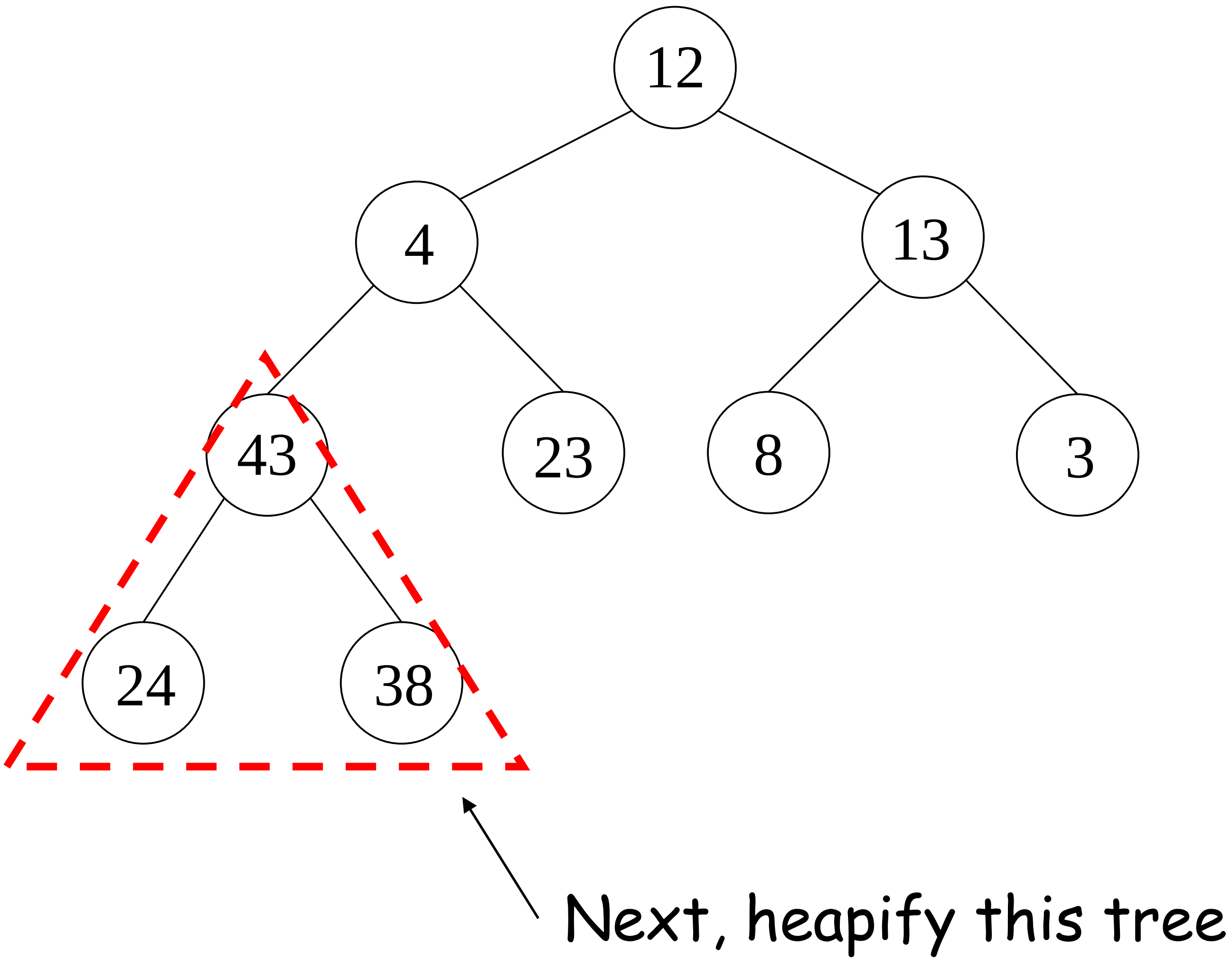
Example Run



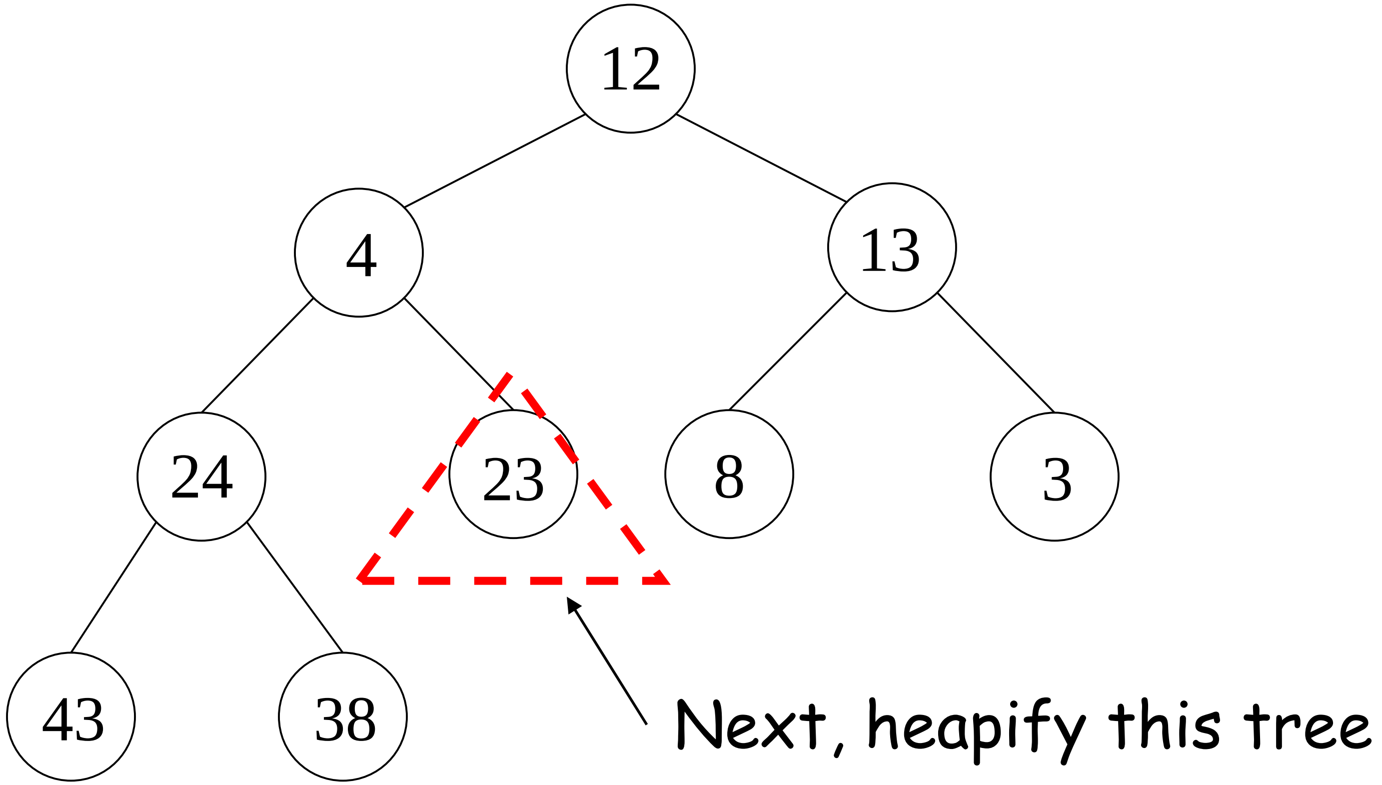
Example Run



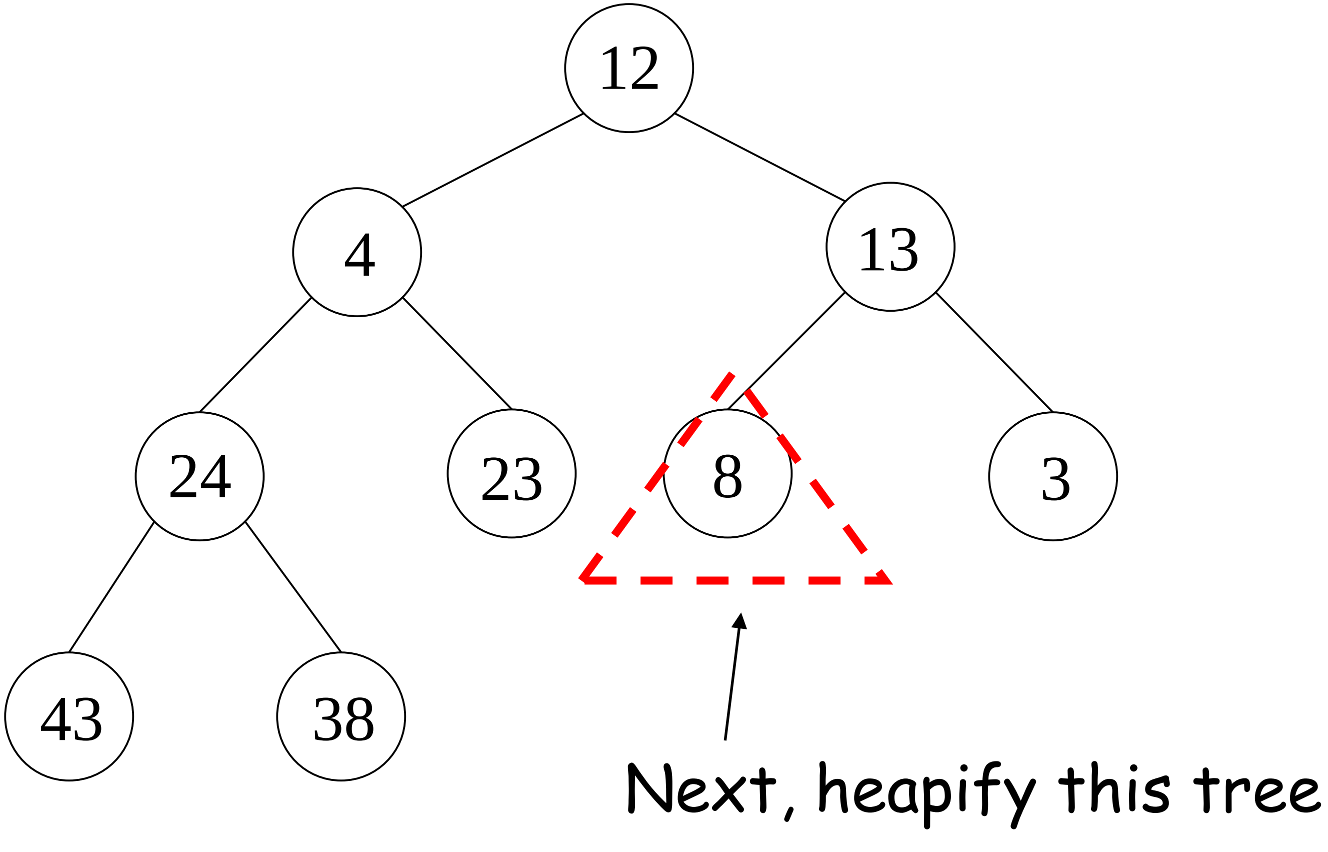
Example Run



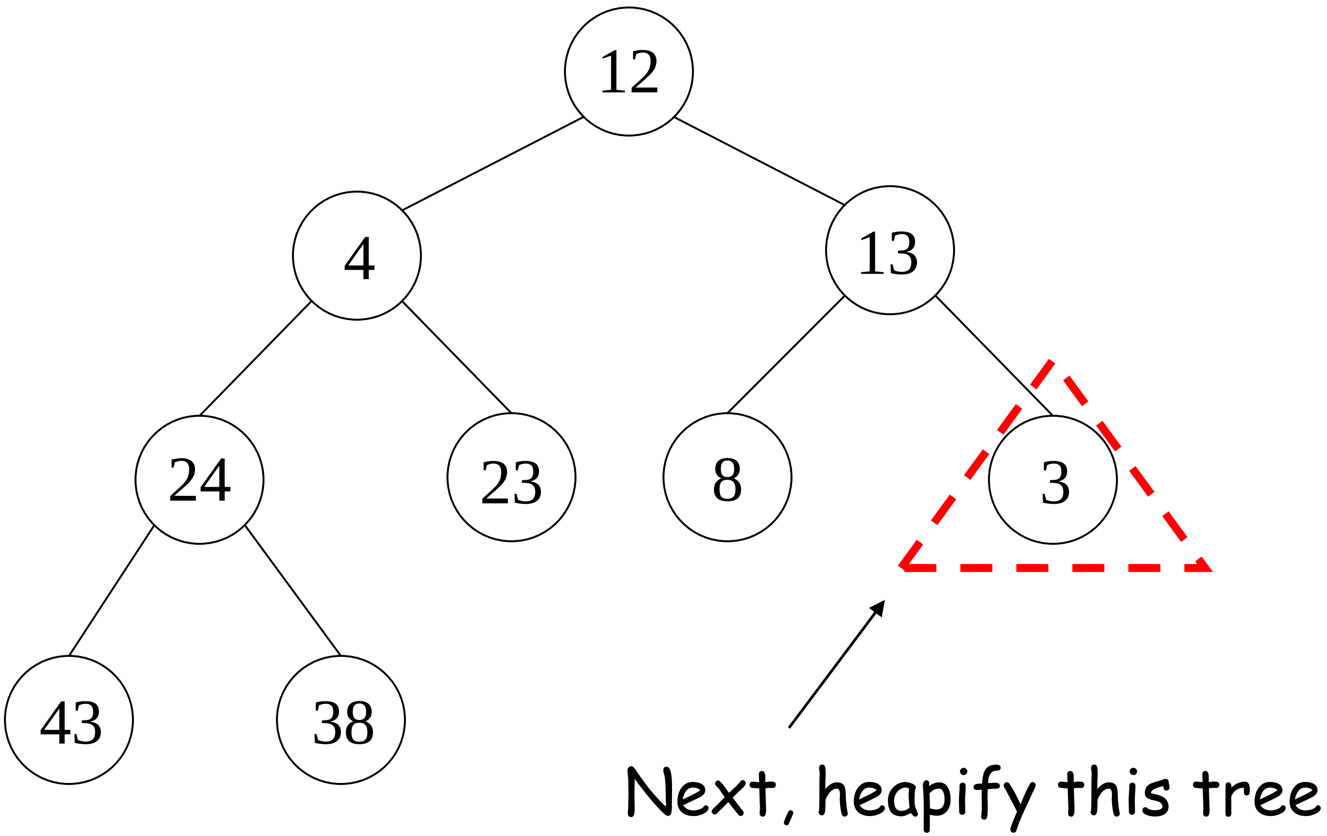
Example Run



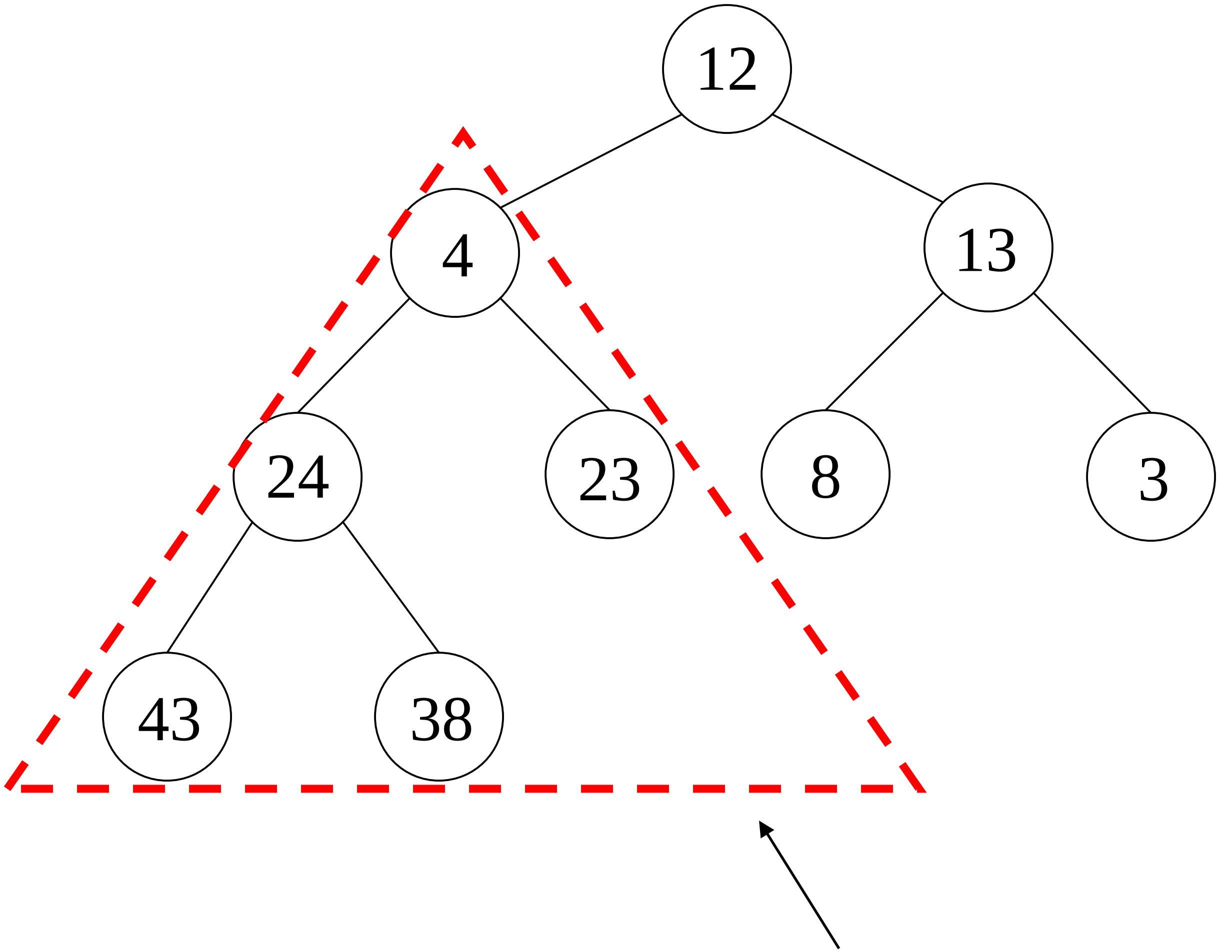
Example Run



Example Run

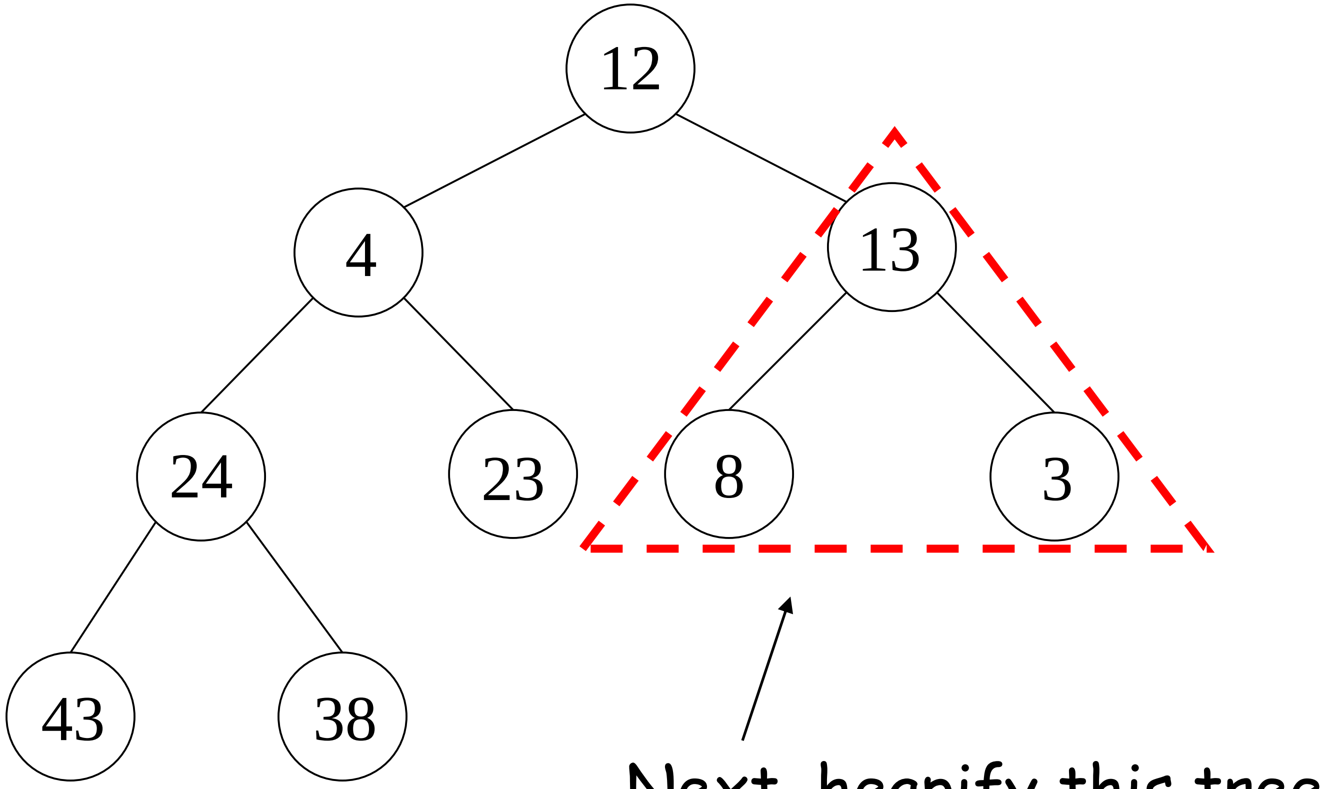


Example Run

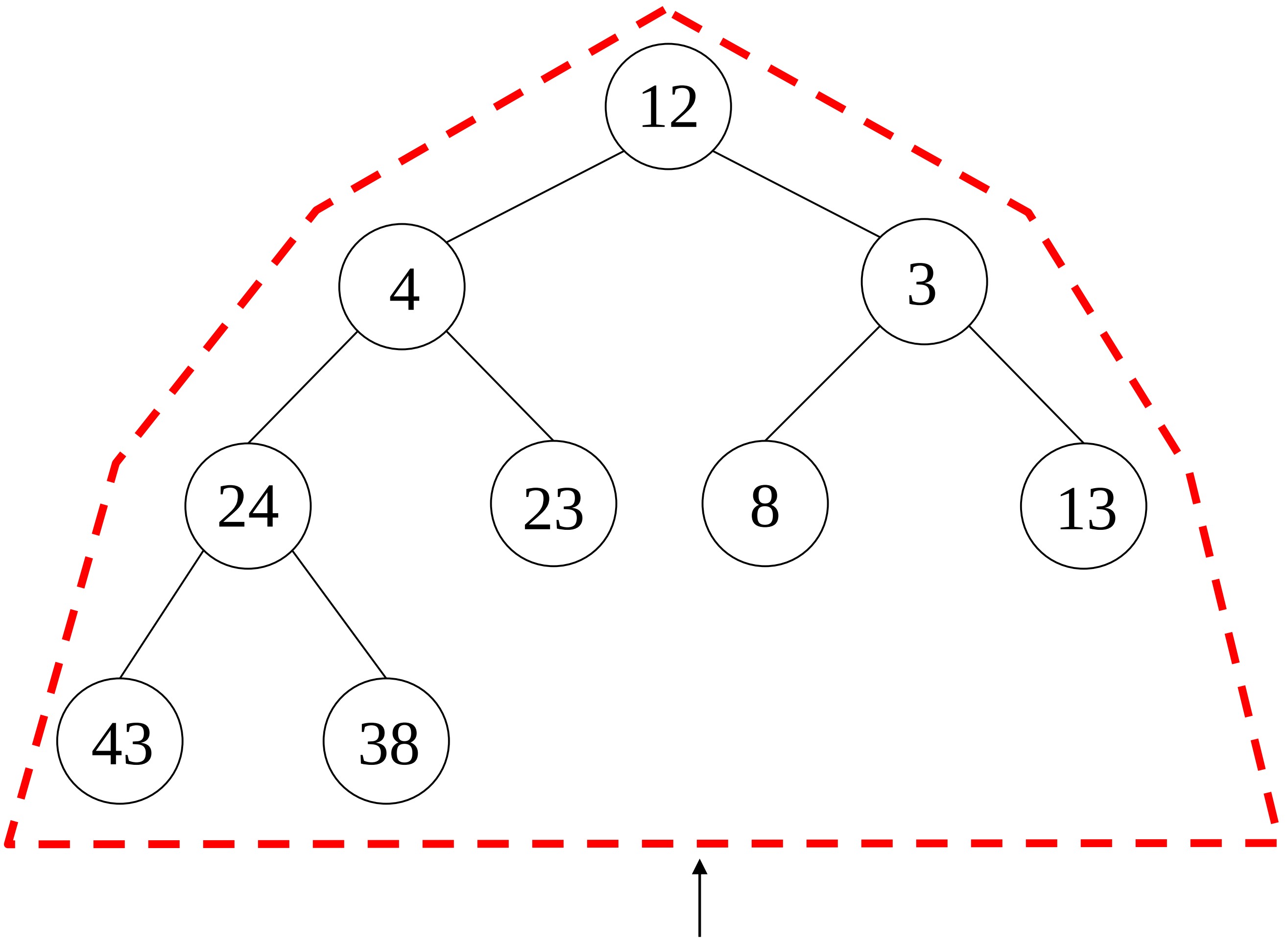


Next, heapify this tree

Example Run

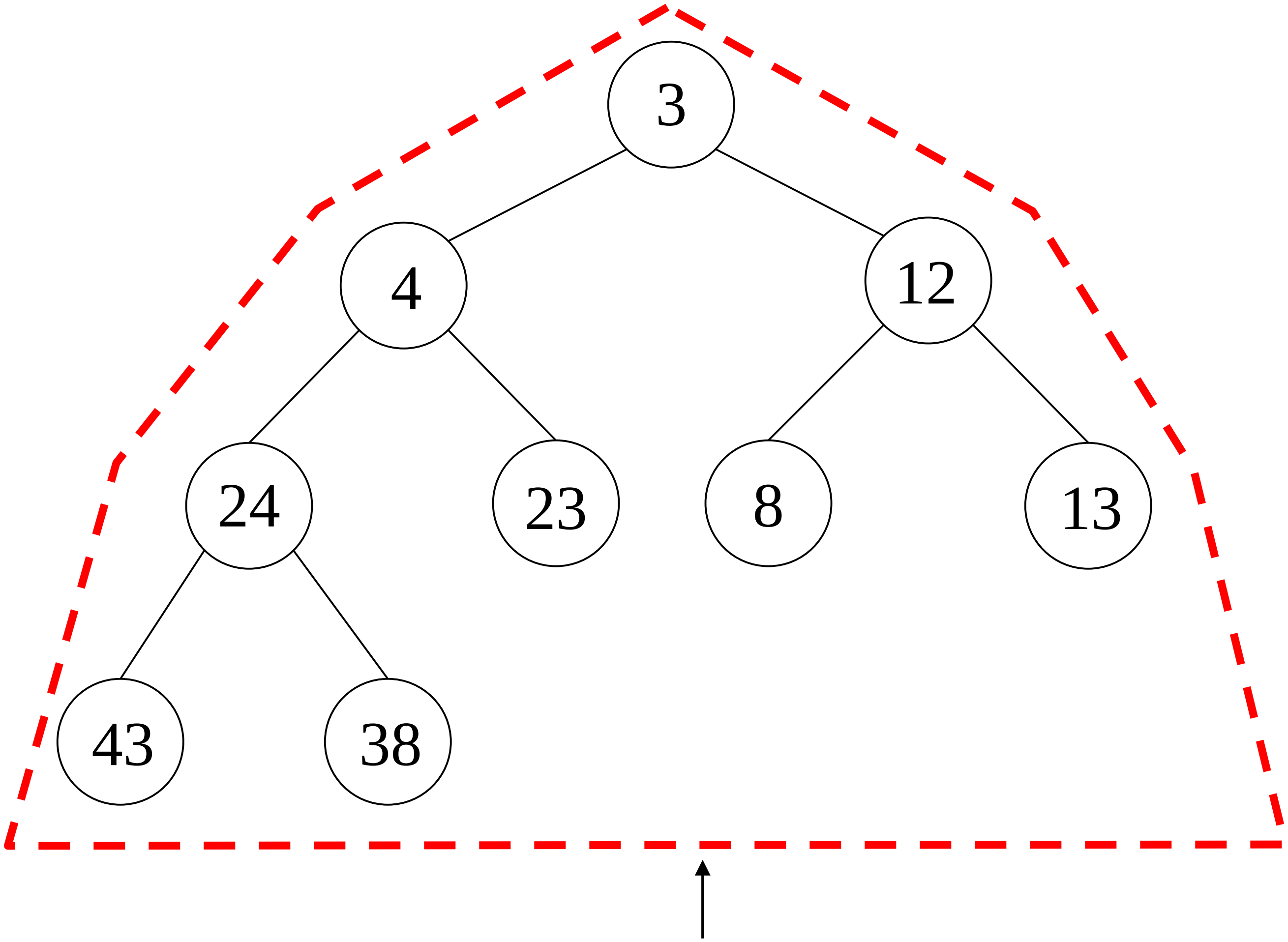


Example Run



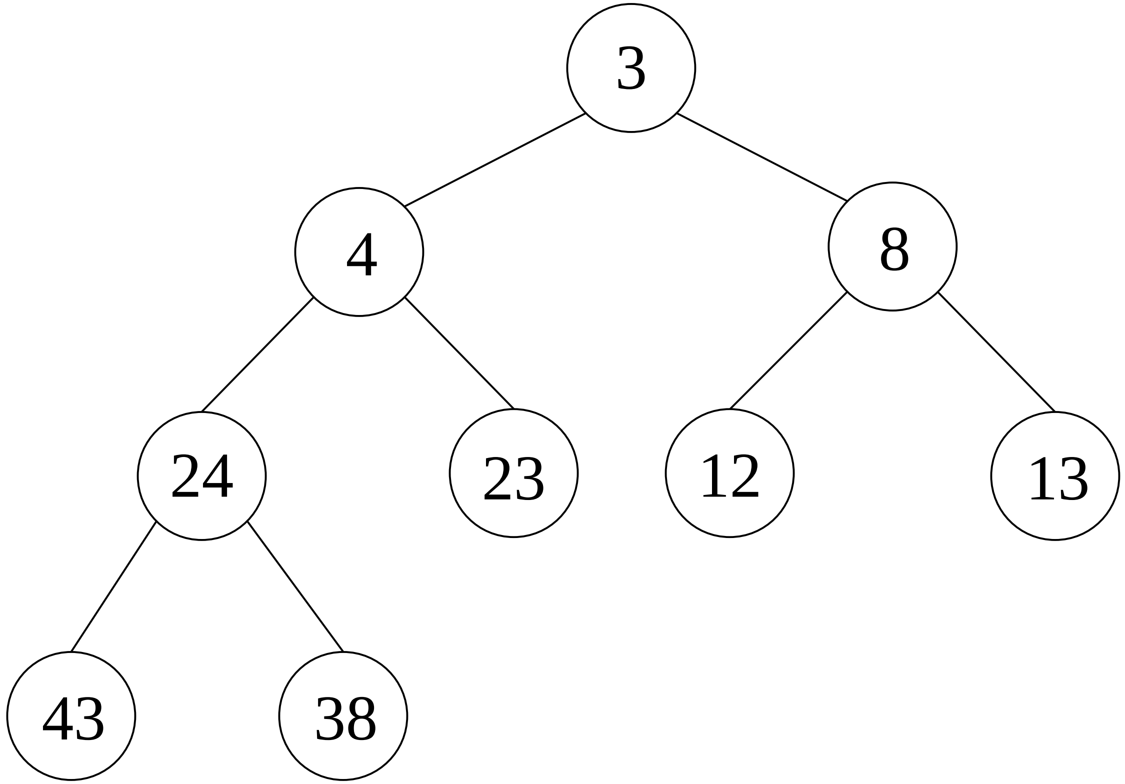
Finally, heapify the whole tree

Example Run



Finally, heapify the whole tree

Example Run



Everything Done !

Back to the Challenge

(Fixing heap property for all nodes)

Suppose that we are given a binary tree which satisfies the shape property

However, the heap property of the nodes may not be satisfied …

Question: Can we make the tree into a heap in $O(n)$ time?

n = # nodes in the tree

Back to the Challenge
(Fixing heap property for all nodes)

Let h = node-height of a tree

So, $2^{h-1} \leq n \leq 2^{h} - 1$ (why??)

For a tree with shape property,

at most 2^{h-1} nodes at level h,

exactly 2^{h-2} nodes at level h-1,

exactly 2^{h-3} nodes at level h-2, …

Back to the Challenge
(Fixing heap property for all nodes)

Using the previous algorithm to solve the challenge, the total time is at most

$2^{h-1} \times 1 + 2^{h-2} \times 2 + 2^{h-3} \times 3 + \ldots + 1 \times h$ [why??]

$= 2^h \left(1 \times ½ + 2 \times (½)^2 + 3 \times (½)^3 + \ldots + h \times (½)^h \right)$

$\leq 2^h \sum_{k=1 \text{ to } \infty} k \times (½)^k = 2^h \times 2 \leq 4n$

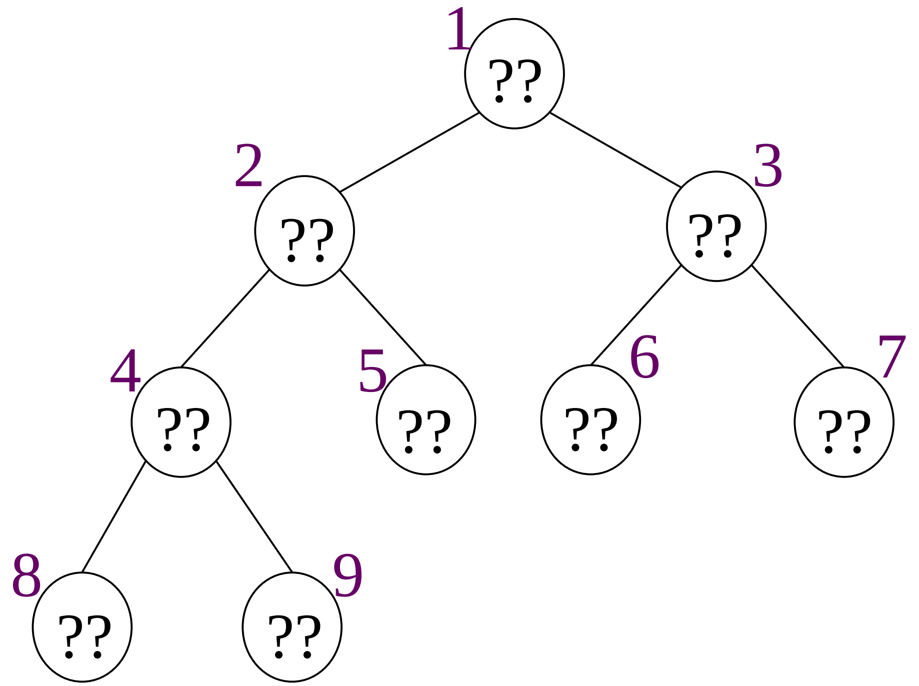
➔ Thus, total time is O(n)

Array Representation of Heap

Given a heap.

Suppose we mark the position of root as 1, and mark other nodes in a way as shown in the right figure. (BFS order)

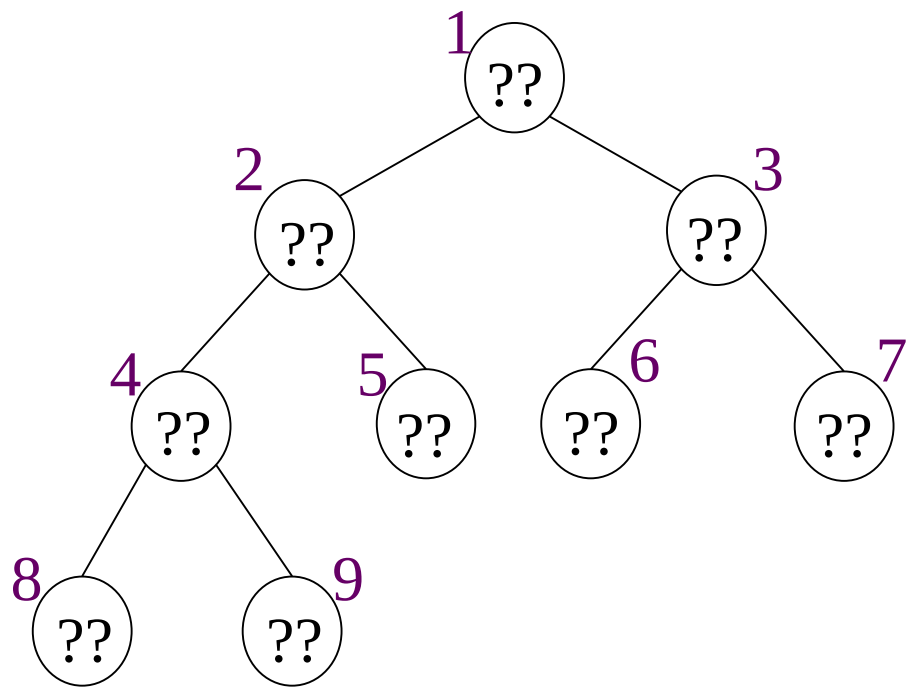
Anything special about this marking?



Array Representation of Heap

Something special:

1. If the heap has n nodes, the marks are from 1 to n
2. Children of x, if exist, are 2x and 2x+1
3. Parent of x is $\lfloor x/2 \rfloor$



Array Representation of Heap

- The special properties of the marking allow us to use an array A[1..n] to store a heap of size n

Advantage:
Avoid storing or using tree pointers !!

Max-Heap

We can also define a max-heap, by changing the heap property to:

Value of a node $\geq$ Value of its children

Max heap supports the following operations:

(1) Find Max, (2) Extract Max, (3) Insert

Do you know how to do these operations?

Priority Queue

Consider S = a set of items, each has a key

Priority queue on S supports:

Min(S): return item with min key

Extract-Min(S): remove item with min key

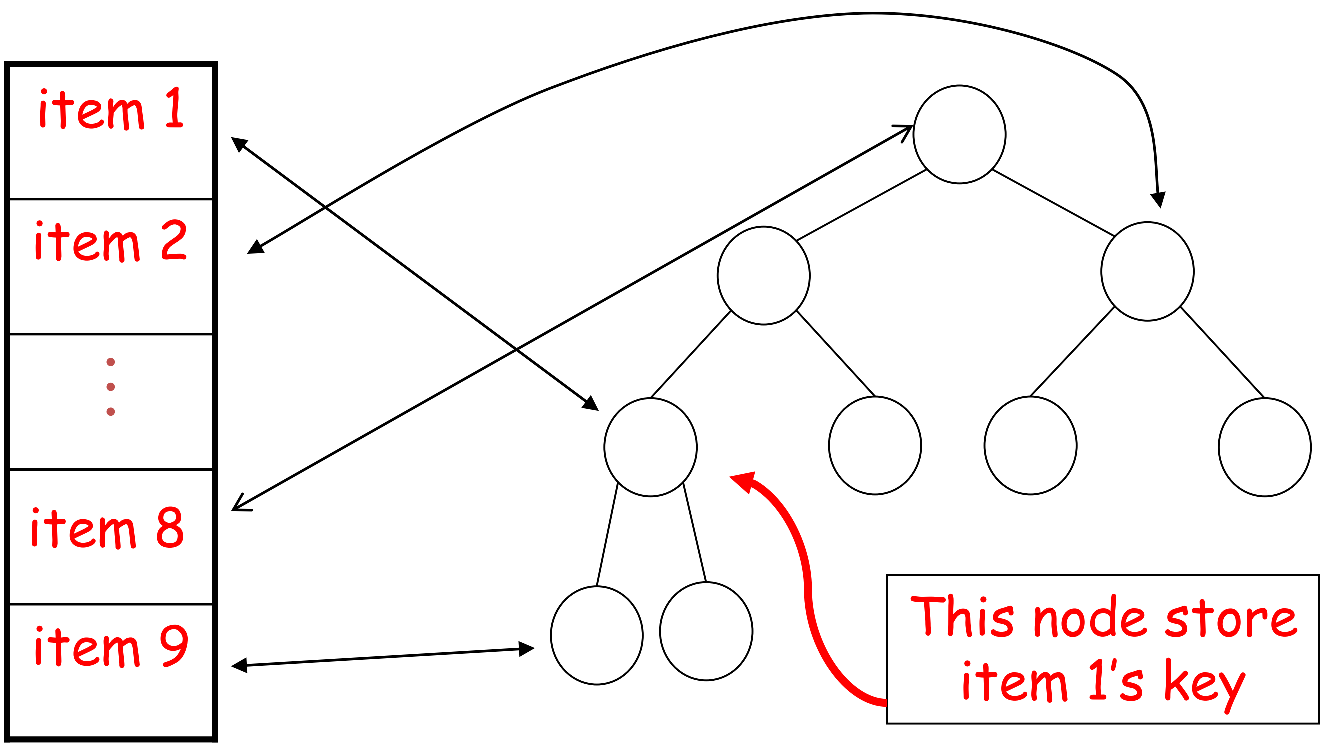
Insert(S, x, k): insert item x with key k

Decrease-Key(S, x, k): decrease key of item x to k

Using Heap as Priority Queue

1. Store the items in an array
2. Use a heap to store keys of the items
3. Store links between an item and its key

E.g.,



Using Heap as Priority Queue

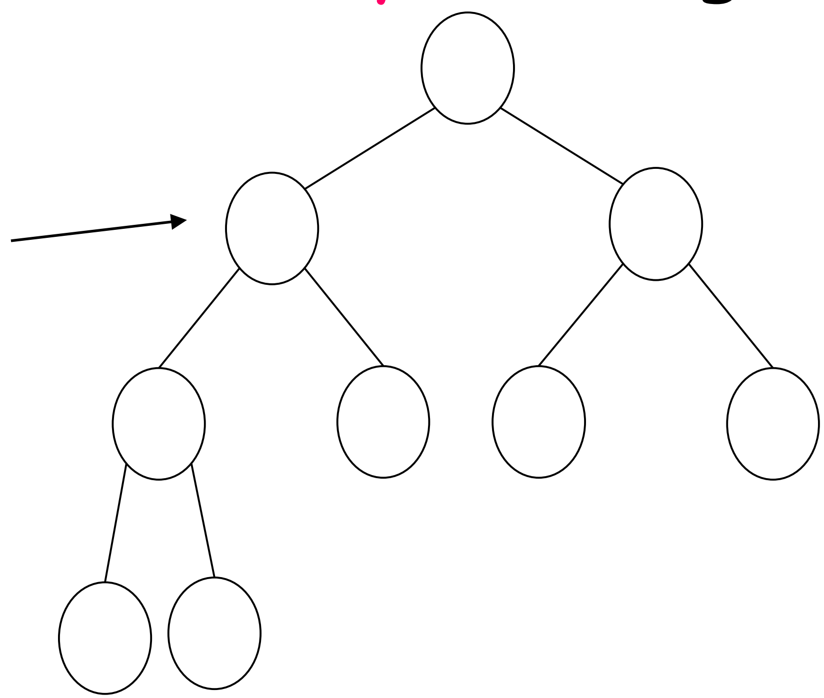
Previous scheme supports Min in O(1) time, Extract-Min and Insert in O(log n) time

It can support Decrease-Key in O(log n) time

e.g.,

Node storing key value of item x

How do we decrease the key to k ?



Practice at home

- Exercises: 6.2-5, 6.2-6, 6.3-3, 6.4-3,6.5-5, 6.5-7
- Problem 6-1