COM 5335 Network Security
Lecture 3
Finite Fields I

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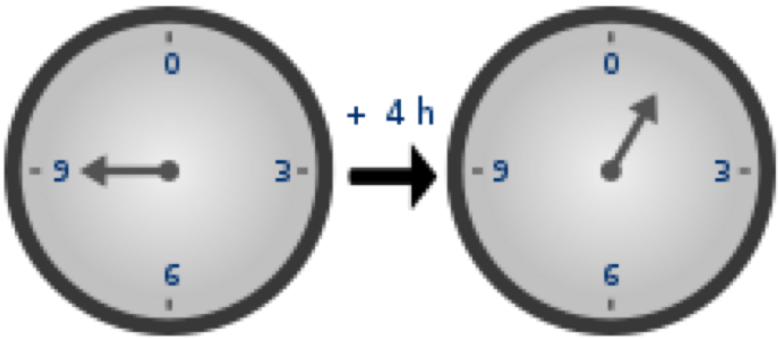
Modular Arithmetic

- It's sometimes called the 'clock arithmetic'.
- It uses a finite number of values and loops back from either end:
  - $a \pmod n \equiv a+n \pmod n \equiv a+2*n \pmod n$



Modulo 7 Example

...

-21	-20	-19	-18	-17	-16	-15
-14	-13	-12	-11	-10	-9	-8
-7	-6	-5	-4	-3	-2	-1
0	**1**	**2**	**3**	**4**	**5**	**6**
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34

...

Modular Arithmetic

- Define **modulo operator** $a \bmod n$ as the remainder when a is divided by n.
- We use the term **congruence** for '$a \equiv b \bmod n$'.
  - It reads "a is congruent to b modulo n".
  - When divided by n, a & b have same remainder
  - e.g. $100 \equiv 34 \bmod 11 \equiv 1 \bmod 11$
  - $-12 \bmod 7 \equiv -5 \bmod 7 \equiv 2 \bmod 7 \equiv 9 \bmod 7$
- b is also called the **residue** of $a \bmod n$

Modular Arithmetic Operations

- Include additions & multiplications
- Apply modulo to reduce answer within n.
- Basic properties
  - $a+b \bmod n \equiv (a \bmod n)+(b \bmod n) \bmod n$
  - $a*b \bmod n \equiv (a \bmod n)*(b \bmod n) \bmod n$

Modulo 5 Example

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Divisors

- A non-zero number b **divides** a if, for some m, we have $a=m*b$ (a,b,m all integers)
  - If we divide a by b, there's no remainder.
- Denoted by $b|a$
- b is called a ***divisor*** *of* a
  - e.g. each of $1,2,3,4,6,8,12,24$ divides 24.

Modular Arithmetic

- Modular arithmetic for integer n:
  - $Z_n = \{0, 1, \dots, n-1\}$
  - $(Z_n, +, *)$ forms a commutative ring (to be explained later)
- Some Remarks
  - If $(a+b) \equiv (a+c) \bmod n$ then $b \equiv c \bmod n$.
  - If $(a*b) \equiv (a*c) \bmod n$ then $b \equiv c \bmod n$ **only if** a is relatively prime to n.

Greatest Common Divisor (GCD)

- A.k.a. the highest common factor (HCF).
- An elementary concept in number theory.
- $GCD\ (a,b)$ of a and b is the largest number that divides both a and b .
  - e.g. $GCD(60,24) = 12$
- Numbers are **relatively prime** if their $GCD = 1$.
  - e.g. $GCD(8,15) = 1$; 8 & 15 are relatively prime.

Euclid's Algorithm (輾轉相除法)

- An efficient way to find the $GCD(a,b)$
- Based on the lemma that
  - $GCD(a,b) = GCD(b, a \bmod b)$
- Apply **Euclid's Algorithm** to compute $GCD(a,b)$:
  - A=a, B=b
  - while B>0
    - R = A mod B
    - A = B, B = R
  - return A

Example GCD(1970,1066)

- 1970 = 1 x 1066 + 904 gcd(1066, 904)
- 1066 = 1 x 904 + 162 gcd(904, 162)
- 904 = 5 x 162 + 94 gcd(162, 94)
- 162 = 1 x 94 + 68 gcd(94, 68)
- 94 = 1 x 68 + 26 gcd(68, 26)
- 68 = 2 x 26 + 16 gcd(26, 16)
- 26 = 1 x 16 + 10 gcd(16, 10)
- 16 = 1 x 10 + 6 gcd(10, 6)
- 10 = 1 x 6 + 4 gcd(6, 4)
- 6 = 1 x 4 + 2 gcd(4, 2)
- 4 = 2 x 2 + 0 gcd(2, 0)

Introduction to Finite Field

- Important in cryptography
  - AES, Elliptic Curve, IDEA, XTR
- Operations on “abstract elements”
  - What constitutes a “number” and the type of operations varies considerably
- Groups, rings, fields from abstract algebra

Groups

- $(G,*)$: a set G of elements with operation '$*$' satisfying
  - Closure: $a,b \in G \Rightarrow a*b \in G$
  - Associativity: $(a*b)*c = a*(b*c)$
  - Identity: $\exists e$ s.t. $e*a = a*e = a$
  - Inverse: $\forall a\ \exists\ a^{-1}$ s.t. $a*a^{-1} = a^{-1}*a = e$

- If commutativity also holds
  - i.e. $a*b = b*a$ then it is called an **abelian group**

Example 1

- $G = \{0,1,2,3\}$
- Operation: $+ \ (mod\ 4)$
- $(G,+)$ is an abelian group.

Example 2

- $G = \{0,1,2,3\}$
- Operator: * $(mod\ 4)$
- Is $(G,*)$ a group? If not, which condition fails?

Example 3

- $G = \{1,2,3,4\}$
- Operator: * $(mod\ 5)$
- $(G,*)$ is an abelian group.

Example 4

- $G = \{1,2,3\}$
- Operator: * $(mod\ 5)$
- Is $(G,*)$ a group? If not, which condition fails?

Example 5

- $G = \{1,2,3\}$
- Operator: * $(mod\ 4)$
- Is $(G,*)$ a group? If not, which condition fails?

Rings

- $(R, +, *)$ a set R of elements with two operations '+' and '*' satisfying the following conditions
  - $(R, +)$ is an abelian group.
  - $(R, *)$ is a semi-group, i.e.
    - Closure: $a, b \in R \Rightarrow a*b \in R$
    - Associativity: $(a*b)*c = a*(b*c)$
  - Distributivity: $a*(b+c) = a*b + a*c$, $(b+c)*a = b*a + c*a$,
- If '*' is also commutative, it's called a **commutative ring.**
- If the multiplicative identity exists, it's called a **ring with 1**.
- Exercise: Is $\{0,1,2,3; (+, *) \ (mod\ 4)\}$ a ring?

Example of Ring: Z_6

- $Z_6=\{0,1,2,3,4,5\}$
- $+$: *mod 6* addition
- $*$: *mod 6* multiplication
- Additive identity = 0
- Multiplicative identity = 1

Z_6

- Additive inverse of 5?
  - $5+1=0$, $-5=1$
- Multiplicative inverse of 5?
  - $5*5=1$, $5^{-1}=5$
- Multiplicative inverse of 3?
  - 3 has no multiplicative inverse.
- Elements of a ring may not have multiplicative inverse.

Fields

- A ring $(R, +, *)$ satisfying:
  - $(R, +)$ is an abelian group
  - $(R \backslash \{0\}, *)$ is an abelian group
- In short, a field is a **commutative division ring.**

- Exercise: Test if $\{0,1,2,3; (+, *) \ (mod\ 4)\}$ is a field.
- Exercise: Test if $\{0,1,2,3, 4; (+, *) \ (mod\ 5)\}$ is a field.

Galois Fields

- Finite fields play a key role in cryptography
- The number of elements in a finite field **must** be a power of a prime p^n (big theorem!)
- Known as Galois fields
- Denoted by $GF(p^n)$
- Most important finite fields:
  - $GF(p)$
  - $GF(2^n)$

Galois Fields GF(p)

- $GF(p)$ is the set of integers $Z_p=\{0,1, \ldots , p\text{-}1\}$ with arithmetic operations modulo a prime p
- $(Z_p, +, *)$ forms the finite field $GF(p)$.
  - Since each item has a multiplicative inverse
- Division is "well-behaved"
  - We can perform addition, subtraction, multiplication, and division in $GF(p)$.
- If p is prime, then Z_p is a field. $Z_p=GF(p)$.
- If n is not prime, then Z_n is not a field. Z_n is a commutative ring with 1.

Example GF(7)

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(b) Multiplication modulo 7

Multiplicative Inverse of GF(7)

0	*1*	*2*	*3*	*4*	*5*	*6*
-	*1*	*4*	*5*	*2*	*3*	*6*

Finding Multiplicative Inverses in Z_p

- Finding the mult. inverse of 337 in Z_{1021}
  - Run Euclid's algorithm
  - $1021-3*337=10$
  - $337-33*10=7$
  - $10-1*7=3$
  - $7-2*3=1$

Finding Multiplicative Inverses in Z_p

- Run extended Euclid's algorithm
  - $1=1*7+(-2)*3=1*7+(-2)(10-1*7)$
  - $=(-2)10+3(7)$
  - $=(-2)10+3(337-33*10)$
  - $=(3)337+(-101)10$
  - $=3(337)+(-101)(1021-3*337)$
  - $=(-101)1021+(306)337$
  - $337^{-1}=306 \bmod 1021$, multiplicative inverse.

Euclid’s Algorithm in C

```
//Precondition: a,b > 0
int gcd(int a, int b) {
    while (b != 0){
        t = b;
        b = a % b;
        a = t;
        return a ;
    }
}
```

Some Remarks

- If n is not prime, then Z_n is not a field.
- Given $x \in Z_n$, x^{-1} may not exist.
- Under what condition will x^{-1} exist?

Polynomial Arithmetic

- Consider $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$
- Both + and * can be performed on polynomials if they can be performed on $a_0, \ldots, a_n$.
- Suppose $a_0, \ldots, a_n \in R$, R is a ring. Denote the set of polynomials by $R[x]$.
- $R[x]$ forms a ring, usually called the **polynomial ring**.
- Is $\{f \in R[x] \mid deg(f) \leq n\}$ a ring?
- If F is a field. Is $F[x]$ a field?

Ordinary Polynomial Arithmetic

- $Z[x]$ arithmetics
  - Let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 - x + 1$
  - $f(x) + g(x) = x^3 + 2x^2 - x + 3$
  - $f(x) - g(x) = x^3 + x + 1$
  - $f(x) * g(x) = x^5 + 3x^2 - 2x + 2$

Polynomial Arithmetic with Modulo Coefficients

- $Z_n[x]$ arithmetic
- In $Z_2[x]$
  - Let $f(x) = x^3 + x^2$ and $g(x) = x^2 + x + 1$
  - $f(x) + g(x) = x^3 + x + 1$
  - $f(x) * g(x) = x^5 + x^2$

Modular Polynomial Arithmetic

- Can we generalize $a \equiv b \ (mod\ n)$ to $a(x) \equiv b(x) \ (mod\ n(x))$?
- We can consider modular +, * on polynomials too.
  - If $f(x) = q(x) * g(x) + r(x)$
  - Interpret $r(x)$ as a remainder
  - $r(x) \equiv f(x) \ mod\ g(x)$
- If have $r(x)=0$, we say $g(x)$ divides $f(x)$.
- The set of all polynomials $R[x]$ modulo a fixed polynomial $g(x)$ also forms a ring.
- We call this ring the **quotient ring**, denoted by $R[x]/g(x)$ or $R[x] \ mod\ g(x)$.

Quotient Rings

- Z_p is actually a quotient ring too.
- $Z_p = Z/p$ or $Z \; mod \; p$.
  - c.f. $R[x]/g(x)$
- If $p \in Z$ is prime, then Z/p is a field.
- If $p(x) \in R[x]$ is a prime (what does this mean???), then $R[x]/p(x)$ is a field???

Irreducible Polynomials

- $g(x)$ is **irreducible** iff it has no divisors other than itself & 1.
- If $p(x) \in R[x]$ is irreducible, then $R[x]/p(x)$ is a field.
- We can find the multiplicative inverse of any polynomial by running the extended Euclid's algorithm just like what we did earlier with integers.

Euclid's Algorithm on Polynomials

- An efficient way to find the $GCD(f(x),g(x))$
- Based on the lemma that:
  - $GCD(f(x),g(x)) = GCD(g(x), f(x) \bmod g(x))$
- **Euclid's Algorithm** to compute $GCD(f,g)$:
  - A=f, B=g
  - while B>0
    - R = A mod B
    - A = B, B = R
  - return A

Finite Field Construction

- To construct $GF(p^n)$
  - Find an irreducible polynomial $p(x) \in Z_p[x]$
  - $GF(p^n)$ can be constructed as $Z_p[x]/p(x)$
- This is just one of many equivalent constructions.
- Multiplicative inverses always exist. Why?

Example of GF(2^3)

- Find an irreducible polynomial $p(x)=x^3+x+1 \in Z_2[x]$
- $GF(8)=Z_2[x]/p(x)$
- From now on, we use 1011 to represent $1x^3+0x^2+1x^1+1x^0$
- Everything is calculated mod 1011 (not modulo a number!!)
- **Never regard these bit-strings as binary numbers and perform these operations on numbers !!!**
- Example: 10001 = 111 mod 1011 because 10001=10*1011+111
- We only need 3 bits to represent each element. Why?

Example of GF(2^3)

Table 4.6 Polynomial Arithmetic Modulo ($x^3 + x + 1$)

	+	000 0	001 1	010 x	011 $x+1$	100 x^2	101 x^2+1	110 x^2+x	111 x^2+x+1
000	0	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
001	1	1	0	$x+1$	x	x^2+1	x^2	x^2+x+1	x^2+x
010	x	x	$x+1$	0	1	x^2+x	x^2+x+1	x^2	x^2+1
011	$x+1$	$x+1$	x	1	0	x^2+x+1	x^2+x	x^2+1	x^2
100	x^2	x^2	x^2+1	x^2+x	x^2+x+1	0	1	x	$x+1$
101	x^2+1	x^2+1	x^2	x^2+x+1	x^2+x	1	0	$x+1$	x
110	x^2+x	x^2+x	x^2+x+1	x^2	x^2+1	x	$x+1$	0	1
111	x^2+x+1	x^2+x+1	x^2+x	x^2+1	x^2	$x+1$	x	1	0

(a) Addition

	×	000 0	001 1	010 x	011 $x+1$	100 x^2	101 x^2+1	110 x^2+x	111 x^2+x+1
000	0	0	0	0	0	0	0	0	0
001	1	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
010	x	0	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
011	$x+1$	0	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
100	x^2	0	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
101	x^2+1	0	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
110	x^2+x	0	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
111	x^2+x+1	0	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$

(b) Multiplication

Computational Considerations

- Since coefficients are 0 or 1, we can always represent any polynomial as a bit string.
- Addition becomes XOR of these bit strings
- Multiplication can be done more easily
  - Shift & XOR (to be explained in lec 4)
- Modulo reduction can be done by repeatedly substituting highest power with remainder of an irreducible polynomial (also shift & XOR)