COM 5335
Lecture 8
Primality Testing

Scott CH Huang

Definition

- A prime number is a positive integer p having exactly two positive divisors, 1 and p.
- A composite number is a positive integer $n > 1$ which is not prime.

Primality Test vs Factorization

- Factorization's outputs are non-trivial divisors.
- Primality test's output is binary: either PRIME or COMPOSITE

Naïve Primality Test

Input: Integer $n > 2$

Output: PRIME or COMPOSITE

for (i from 2 to n-1){

    if (i divides n)

        return COMPOSITE;

}

return PRIME;

Still Naïve Primality Test

Input/Output: same as the naïve test

for (*i* from 1 to $\sqrt{n}$){
    if (*i* divides *n*)
        return COMPOSITE;
}
return PRIME;

Sieve of Eratosthenes

Input/Output: same as the naïve test

Let A be an arry of length *n*

Set all but the first element of A to TRUE

for (*i* from 2 to $\sqrt{n}$){

    if (A[i]=TRUE)

        Set all multiples of *i* to FALSE

}

if (A[i]=TRUE)   **return** PRIME

else return COMPOSITE

Primality Testing

Two categories of primality tests

- Probablistic
  - Miller-Rabin Probabilistic Primality Test
  - Cyclotomic Probabilistic Primality Test
  - Elliptic Curve Probabilistic Primality Test
- Deterministic
  - Miller-Rabin Deterministic Primality Test
  - Cyclotomic Deterministic Primality Test
  - Agrawal-Kayal-Saxena (AKS) Primality Test

Running Time of Primality Tests

- Miller-Rabin Primality Test
  - Polynomial Time
- Cyclotomic Primality Test
  - Exponential Time, but *almost* poly-time
- Elliptic Curve Primality Test
  - Don’t know. Hard to Estimate, but *looks* like poly-time.
- AKS Primality Test
  - Poly-time, but only asymptotically good.

Fermat's Primality Test

- It's more of a "compositeness test" than a primality test.
- Fermat's Little Theorem:

  If *p* is prime and $a \nmid p$, then $a^{p-1} \equiv 1 \pmod{p}$
- If we can find an *a* s.t. $\gcd(a, n-1),\ a^{n-1} \not\equiv 1 \pmod{n}$

  , then *n* must be a composite.
- If, for some *a*, *n* passes the test, we cannot conclude *n* is prime. Such n is a *pseudoprime*. If this pseudoprime *n* is not prime, then this *a* is called a **Fermat liar**.
- If, for all $1 \leq a \leq n-1, \text{s.t. } \gcd(a, n-1) = 1$

  we have $a^{n-1} \not\equiv 1 \pmod{n}$ can we conclude *n* is prime?
- No. Such *n* is called a **Carmichael number**.

Some Small Carmichael Numbers

Carmichael Numbers	Corresponding Factorizations
561	3*11*17
41041	7*11*13*14
825265	5*7*17*19*73
321197185	5*19*23*29*37*137

Carmichael numbers < 100,000
561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, 41041, 46657, 52633, 62745, 63973, and 75361.

Pseudocode of Fermat's Primality Test

```
FERMAT(n,t){
INPUT: odd integer n ≥ 3, # of repetition t
OUTPUT: PRIME or COMPOSITE
    for (i from 1 to t){
        Choose a random integer a s.t. 2 ≤ a ≤ n-2
        Compute r = a^(n-1) mod n
        if ( r ≠ 1 )    return COMPOSITE
    }
    return PRIME
}
```

Miller-Rabin Probabilistic Primality Test


- It's more of a "compositeness test" than a primality test.
- It does not give *proof* that a number *n* is prime, it only tells us that, with high probability, *n* is prime.
- It's a randomized algorithm of *Las Vegas* type.

A Motivating Observation

FACT:

- Let *p* be an odd prime. $x \in \mathbb{Z}_p^*$. If $x^2 = 1$ then $x = \pm 1$
- Moreover, if $n - 1 = m2^k$, and *m* is odd.

  Let $a \in \mathbb{N},\ \text{s.t. } \gcd(a, n) = 1$. Then either $a^m \equiv 1 \pmod n$

  or $a^{m2^i} \equiv -1 \pmod n$ for some $0 \le i \le k-1$

Miller-Rabin

- If $a^m \not\equiv 1$ and $a^{m2^i} \not\equiv -1 \pmod{n}, \quad \forall\, 0 \le i \le k-1$
  Then *a* is a **strong witness** for the compositeness of *n*.
- If $a^m \equiv 1 \pmod{n}$ or $a^{m2^i} \equiv -1 \pmod{n}$ for some $0 \le i \le k-1$ then *n* is called a *pseudoprime w.r.t. base a*, and *a* is called a **strong liar**.

Miller-Rabin: Algorithm Pseudocode

MILLER-RABIN(*n*,*t*){

INPUT: odd integer *n* ≥ 3, # of repetition *t*

Compute *k* & odd *m* s.t. $n-1=m2^k$

for (*i* from 1 to *t*){

Choose a random integer *a* s.t. $2 \leq a \leq n-2$

Compute $y = a^m \bmod n$

if ($y \neq 1$ and $y \neq n-1$){

Set $j \leftarrow 1$

while($j \leq k-1$ and $y \neq n-1$){

Set $y \leftarrow y^2 \bmod n$

if ($y = 1$) return COMPOSITE

$j \leftarrow j+1$

}

if ($y \neq n-1$) return COMPOSITE

}

}

return PRIME

}

Miller-Rabin: Example

- n = 2465=5*17*29 (a Carmichael number)
- n-1=2464=2^5*7*11
- a^{m2^i} values shown as below

	i = 0	1	2	3	4	5
a=2	1902	1449	1886	1	1	1
a=3	2018	144	1016	1886	1	1
a=5	2145	1335	30	900	1480	1480
a=7	2437	784	871	1886	1	1
a=11	1061	1681	871	1886	1	1
a=13	608	2379	1	1	1	1
a=47	302	-1	1	1	1	1

Miller-Rabin: Main Theorem

Theorem:

Given $n > 9$. Let B be the number of strong liars. Then $\frac{B}{\varphi(n)} \leq \frac{1}{4}$

If the **Generalized Riemann Hypothesis** is true, then

Miller-Rabin primality test can be made deterministic by running

MILLER-RABIN(n, $2\log^2 n$)