Chapter 3

VLSI Physical Design

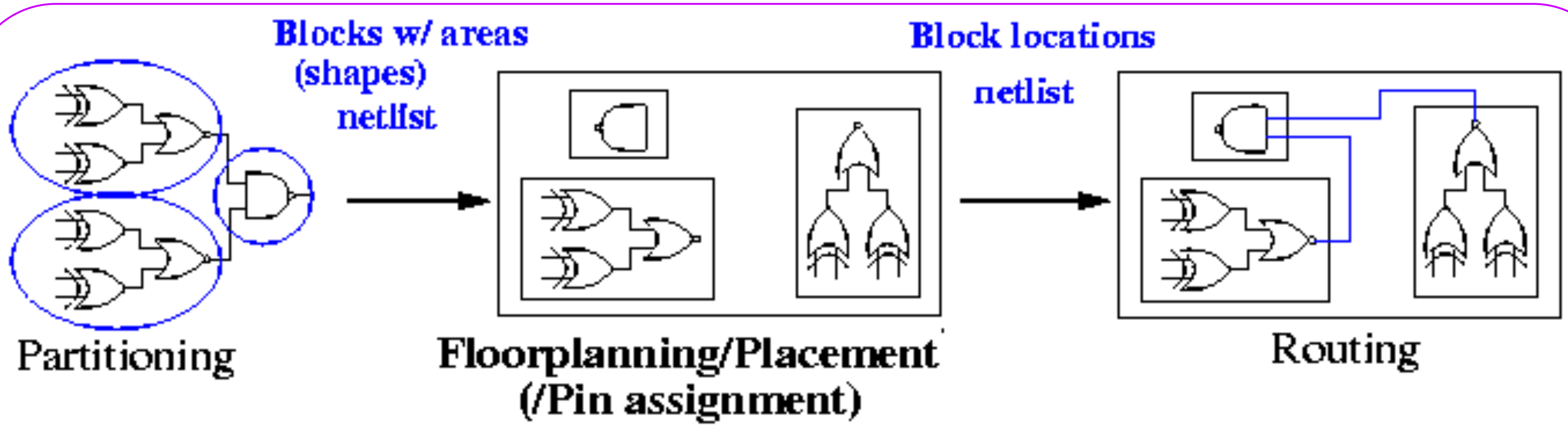
何宗易
Tsung-Yi Ho
tyho@cs.nthu.edu.tw
http://theta.cs.nthu.edu.tw
Department of Computer Science
National Tsing Hua University
Hsinchu, Taiwan

Basic Concept

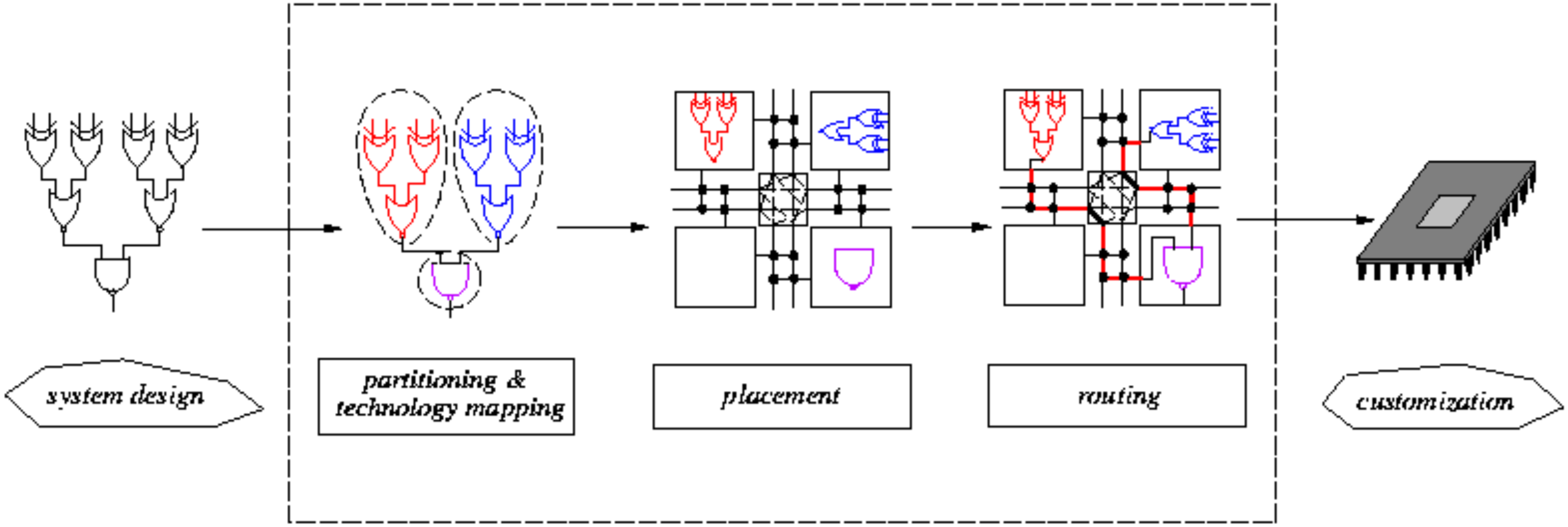
- Physical design
  - Creating circuits on silicon.
  - Schematic diagrams are translated into sets of geometric patterns.
  - Every layer is defined by a distinct pattern.

- The topology of the transistor network establishes the logic function.

Physical Design Flow



Cell-Based



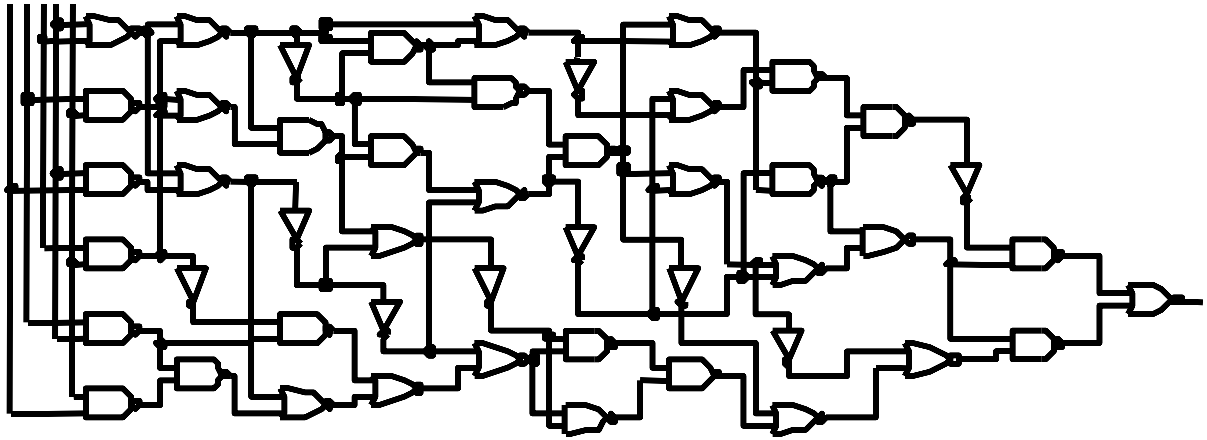
FPGA

Partitioning

- Decomposition of a complex system into smaller subsystems
  - Done hierarchically
  - Partitioning done until each subsystem has manageable size
  - Each subsystem can be designed independently
- Interconnections between partitions minimized
  - Less hassle interfacing the subsystems
  - Communication between subsystems usually costly

Example: Partitioning of a Circuit

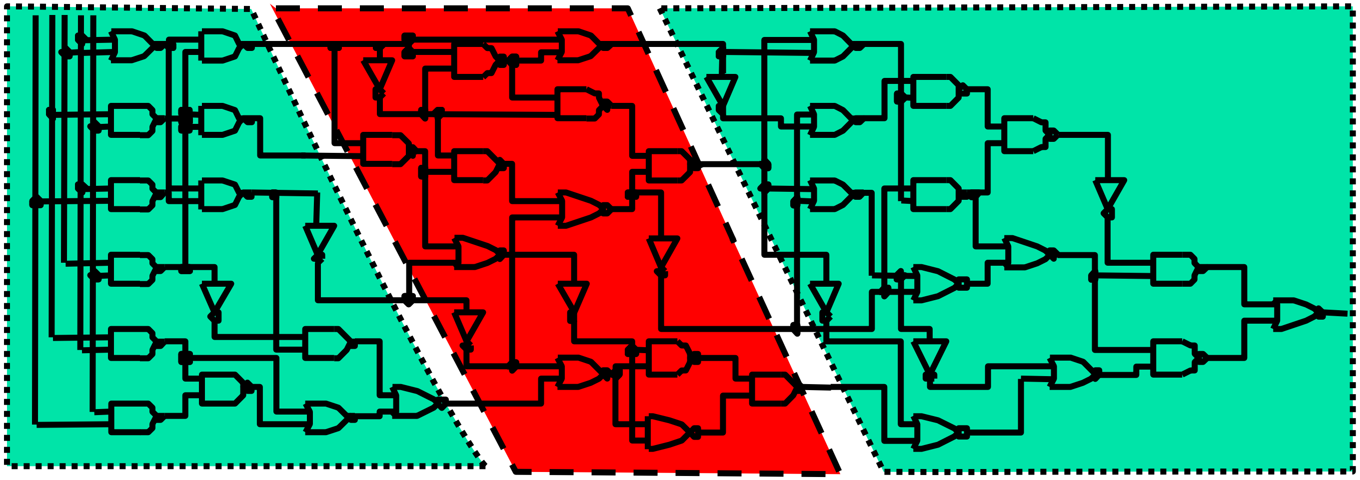
Input size: 48



Cut 1=4
Size 1=15

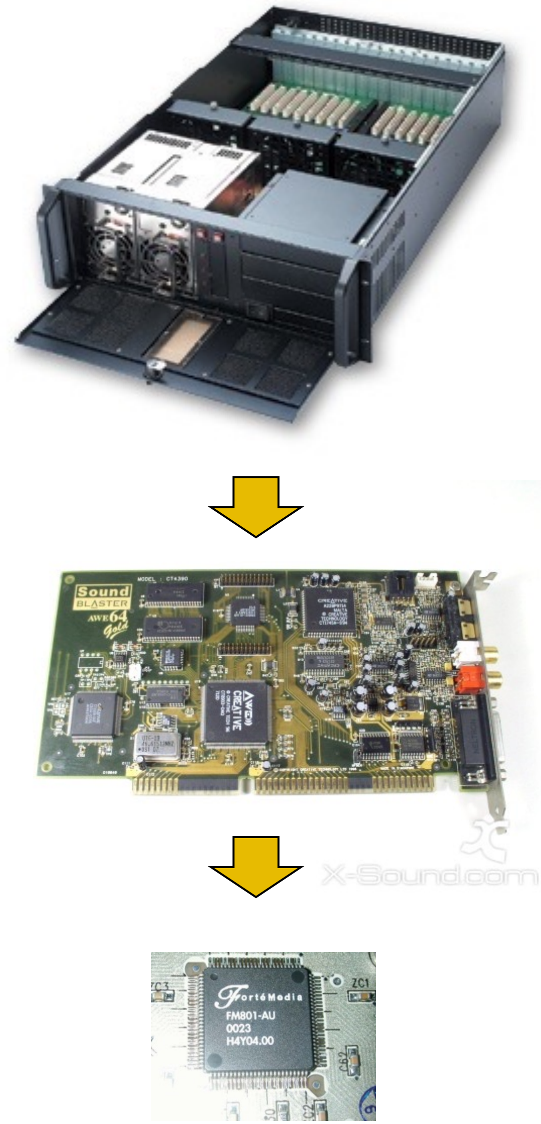
Cut 2=4
Size 2=16

Size 3=17

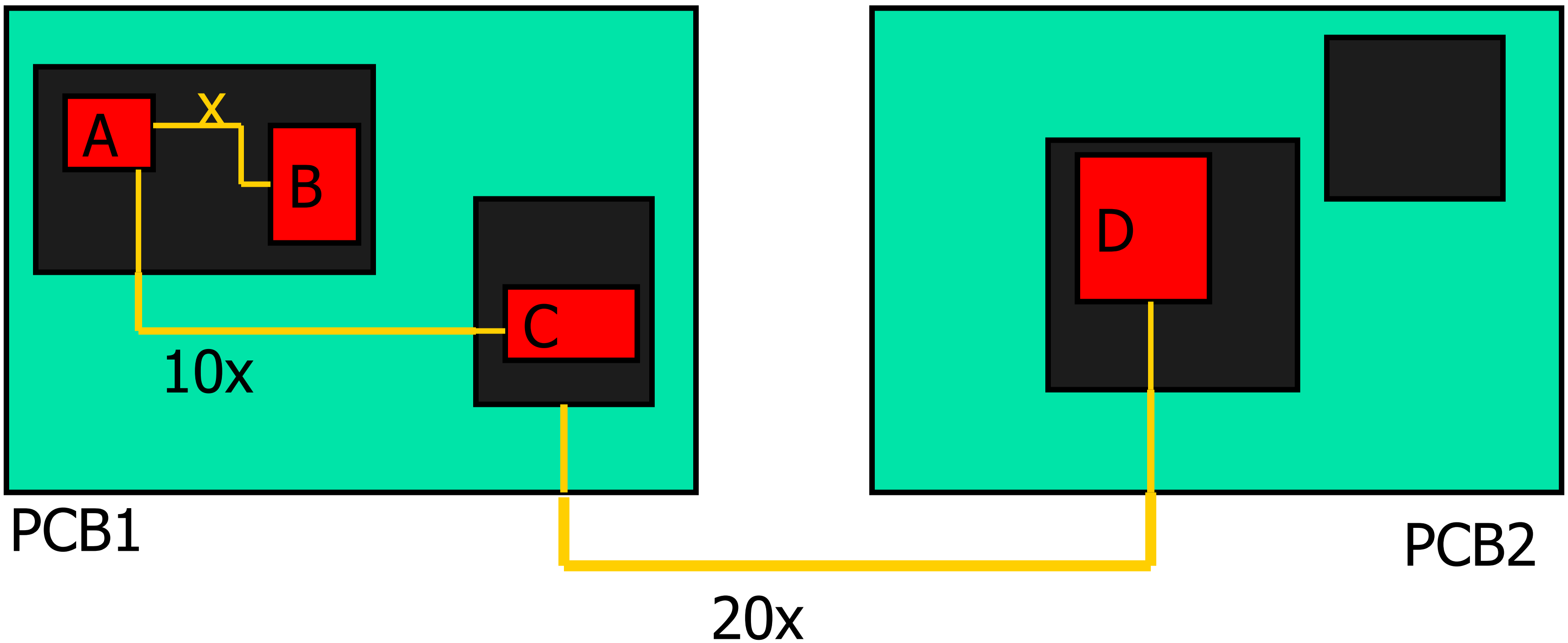


Hierarchical Partitioning

- Levels of partitioning:
  - System-level partitioning:
    Each sub-system can be designed as a single printed circuit board (PCB)
  - Board-level partitioning:
    Circuit assigned to a PCB is partitioned sub-circuits
    each fabricated as a VLSI chip
  - Chip-level partitioning:
    Circuit assigned to the chip is divided into manageable sub-circuits
    NOTE: physically not necessary



Delay at Different Levels of Partitions



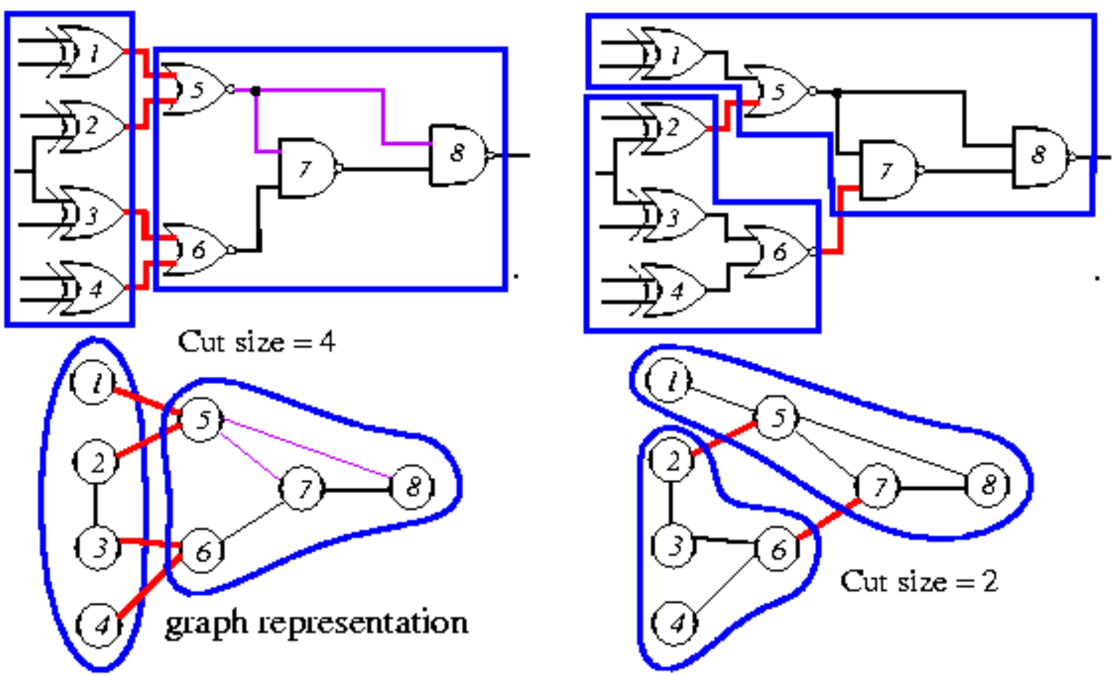
PCB = Printed Circuit Board

Partitioning: Formal Definition

- Input:
  - Graph or hypergraph
  - Usually with vertex weights (sizes)
  - Usually weighted edges
- Constraints
  - Number of partitions (K-way partitioning)
  - Maximum capacity of each partition
    OR
    maximum allowable difference between partitions
- Objective
  - Assign nodes to partitions subject to constraints
    s.t. the cutsize is minimized
- Tractability
  - Is NP-complete ☹

Circuit Partitioning

- **Objective:** Partition a circuit into parts such that every component is within a prescribed range and the # of connections among the components is minimized.
  - More constraints are possible for some applications.
- Cutset? Cut size? Size of a component?



Problem Definition: Partitioning

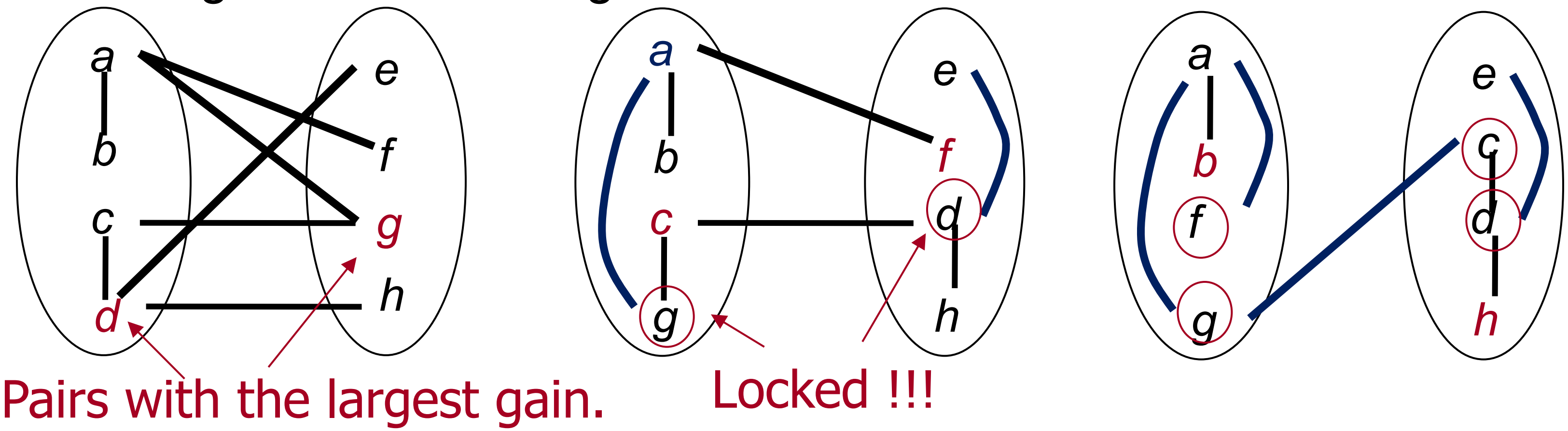
- ***k*-way partitioning:** Given a graph $G(V, E)$, where each vertex $v \in V$ has a **size** $s(v)$ and each edge $e \in E$ has a **weight** $w(e)$, the problem is to divide the set V into k **disjoint subsets** $V_1, V_2, \ldots, V_k$, such that an objective function is optimized, subject to certain constraints.
- **Bounded size constraint:** The size of the i-th subset is bounded by B_i ($\sum_{v \in V_i} s(v) \leq B_i$).
  - Is the partition balanced?
- **Min-cut cost between two subsets:** Minimize $\sum_{\forall e=(u,v) \wedge p(u) \neq p(v)} w(e)$, where $p(u)$ is the partition # of node u.
- The 2-way, balanced partitioning problem is NP-complete, even in its simple form with identical vertex sizes and unit edge weights.

Kernighan-Lin Heuristic

- Kernighan and Lin, "An efficient heuristic procedure for partitioning graphs," *The Bell System Technical Journal*, vol. 49, no. 2, Feb. 1970.
- An **iterative**, **2-way**, **balanced** partitioning (bi-sectioning) heuristic.
- Till the cut size keeps decreasing
  - Vertex pairs which give the largest decrease **or the smallest increase** in cut size are exchanged.
  - These vertices are then **locked** (and thus are prohibited from participating in any further exchanges).
  - This process continues until all the vertices are locked.
  - Find the set with the **largest partial sum** for swapping.
  - Unlock all vertices.

Kernighan-Lin Heuristic: A Simple Example

- Each edge has a unit weight.



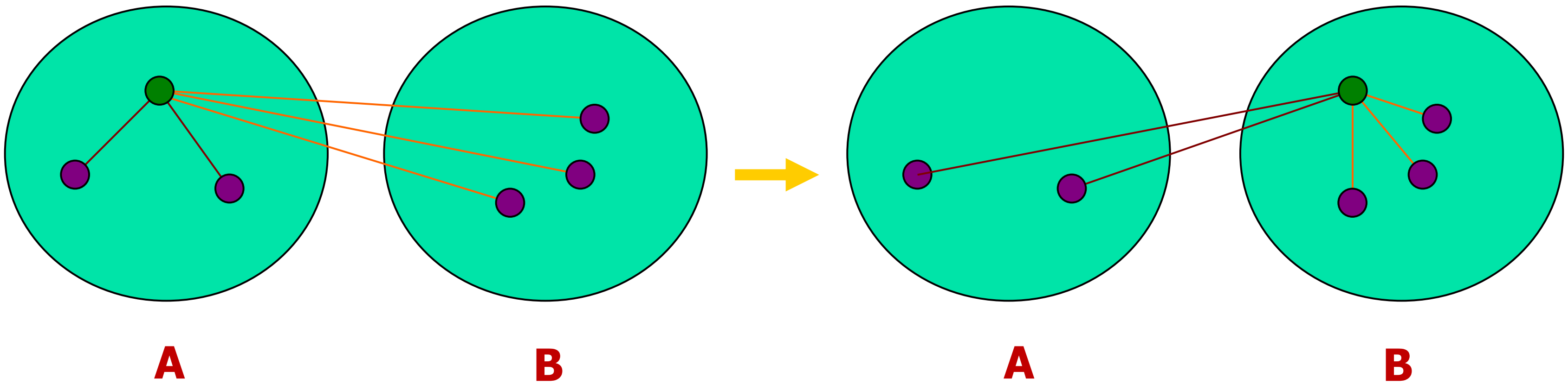
Step #	Vertex pair	Cost reduction	Cut cost
0	-	0	5
1	{*d*, *g*}	3	2
2	{*c*, *f* }	1	1
3	{*b*, *h*}	-2	3
4	{*a*, *e*}	-2	5

- **Questions:** How to compute cost reduction? What pairs to be swapped?
  - Consider the change of internal & external connections.

Observation

Move one node

Reduced cost = 1

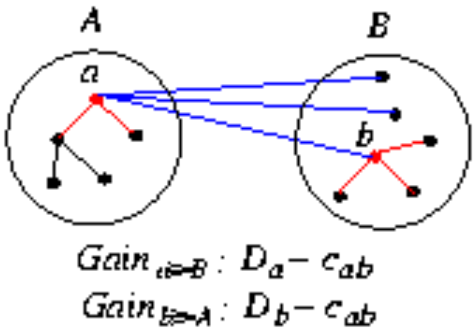


- Two sets A and B such that $|A| = n = |B|$ and $A \cap B = \varnothing$.
- **External cost** of $a \in A$: $E_a = \sum_{v \in B} c_{av}$.
- **Internal cost** of $a \in A$: $I_a = \sum_{v \in A} c_{av}$.
- ***D*-value** of a vertex a: $D_a = E_a - I_a$ (cost reduction for moving a).

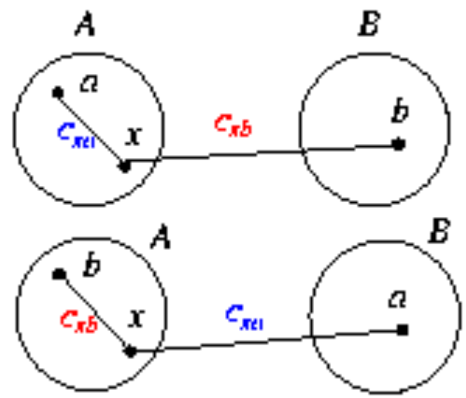
Properties

- Cost reduction (gain) for swapping *a* and *b*:
  $g_{ab} = D_a + D_b - 2c_{ab}$
- If $a \in A$ and $b \in B$ are interchanged, then the new *D*-values, *D*', are given by

$$
\begin{aligned}
D'_x &= D_x + 2c_{xa} - 2c_{xb}, \forall x \in A - \{a\} \\
D'_y &= D_y + 2c_{yb} - 2c_{ya}, \forall y \in B - \{b\}.
\end{aligned}
$$



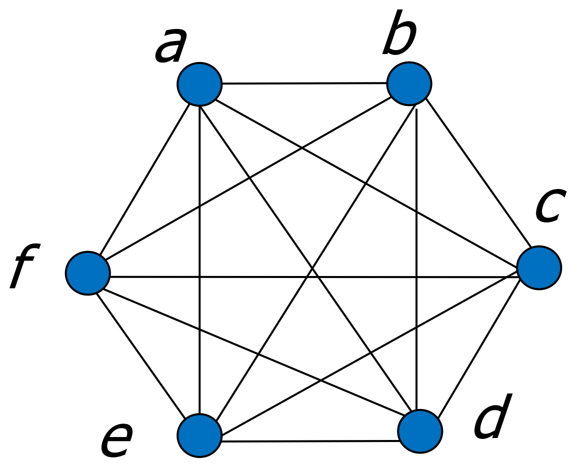
Internal cost vs. External cost



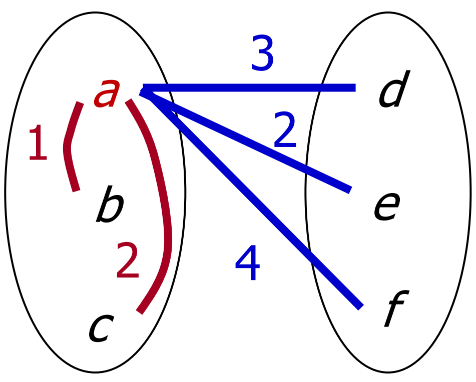
before swap	after swap	ΔC
$-c_{xa}$	$+c_{xa}$	$+2c_{xa}$
$+c_{xb}$	$-c_{xb}$	$-2c_{xb}$

updating D−values

Kernighan-Lin Heuristic: A Weighted Example



	a	b	c	d	e	f
a	0	1	2	3	2	4
b	1	0	1	4	2	1
c	2	1	0	3	2	1
d	3	4	3	0	4	3
e	2	2	2	4	0	2
f	4	1	1	3	2	0



cost associated with *a*

Initial cut cost = (3+2+4) +(4+2+1)+(3+2+1) = 22

- Iteration 1:

$$
\begin{array}{lll}
I_a = 1 + 2 = 3; & E_a = 3 + 2 + 4 = 9; & D_a = E_a - I_a = 9 - 3 = 6 \\
I_b = 1 + 1 = 2; & E_b = 4 + 2 + 1 = 7; & D_b = E_b - I_b = 7 - 2 = 5 \\
I_c = 2 + 1 = 3; & E_c = 3 + 2 + 1 = 6; & D_c = E_c - I_c = 6 - 3 = 3 \\
I_d = 4 + 3 = 7; & E_d = 3 + 4 + 3 = 10; & D_d = E_d - I_d = 10 - 7 = 3 \\
I_e = 4 + 2 = 6; & E_e = 2 + 2 + 2 = 6; & D_e = E_e - I_e = 6 - 6 = 0 \\
I_f = 3 + 2 = 5; & E_f = 4 + 1 + 1 = 6; & D_f = E_f - I_f = 6 - 5 = 1
\end{array}
$$

g-Value Computation

- Iteration 1:

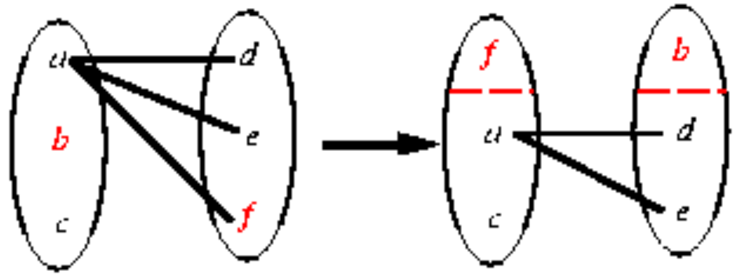
$$
\begin{array}{lll}
I_a = 1 + 2 = 3; & E_a = 3 + 2 + 4 = 9; & D_a = E_a - I_a = 9 - 3 = 6 \\
I_b = 1 + 1 = 2; & E_b = 4 + 2 + 1 = 7; & D_b = E_b - I_b = 7 - 2 = 5 \\
I_c = 2 + 1 = 3; & E_c = 3 + 2 + 1 = 6; & D_c = E_c - I_c = 6 - 3 = 3 \\
I_d = 4 + 3 = 7; & E_d = 3 + 4 + 3 = 10; & D_d = E_d - I_d = 10 - 7 = 3 \\
I_e = 4 + 2 = 6; & E_e = 2 + 2 + 2 = 6; & D_e = E_e - I_e = 6 - 6 = 0 \\
I_f = 3 + 2 = 5; & E_f = 4 + 1 + 1 = 6; & D_f = E_f - I_f = 6 - 5 = 1
\end{array}
$$

- $g_{xy} = D_x + D_y - 2c_{xy}$.

$$
\begin{aligned}
g_{ad} &= D_a + D_d - 2c_{ad} = 6 + 3 - 2 \times 3 = 3 \\
g_{ae} &= 6 + 0 - 2 \times 2 = 2 \\
g_{af} &= 6 + 1 - 2 \times 4 = -1 \\
g_{bd} &= 5 + 3 - 2 \times 4 = 0 \\
g_{be} &= 5 + 0 - 2 \times 2 = 1 \\
g_{bf} &= 5 + 1 - 2 \times 1 = 4 \ (maximum) \\
g_{cd} &= 3 + 3 - 2 \times 3 = 0 \\
g_{ce} &= 3 + 0 - 2 \times 2 = -1 \\
g_{cf} &= 3 + 1 - 2 \times 1 = 2
\end{aligned}
$$

- Swap *b* and *f* ! (g_1'=4)

D-Value Computation



- $D'_x = D_x + 2\,c_{xp} - 2\,c_{xq}$, $\forall\, x \in A - \{p\}$ (swap p and q, $p \in A$, $q \in B$)

$$\begin{aligned}
D'_a &= D_a + 2c_{ab} - 2c_{af} = 6 + 2 \times 1 - 2 \times 4 = 0 \\
D'_c &= D_c + 2c_{cb} - 2c_{cf} = 3 + 2 \times 1 - 2 \times 1 = 3
\end{aligned}$$

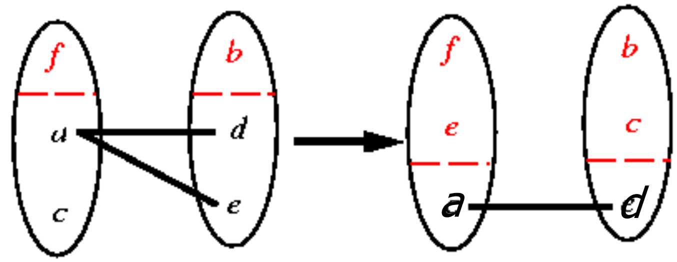
$$\boxed{\begin{aligned}
D'_d &= D_d + 2c_{df} - 2c_{db} = 3 + 2 \times 3 - 2 \times 4 = 1 \\
D'_e &= D_e + 2c_{ef} - 2c_{eb} = 0 + 2 \times 2 - 2 \times 2 = 0
\end{aligned}}$$

- $g_{xy} = D'_x + D'_y - 2c_{xy}$.

$$\begin{aligned}
g_{ad} &= D'_a + D'_d - 2c_{ad} = 0 + 1 - 2 \times 3 = -5 \\
g_{ae} &= D'_a + D'_e - 2c_{ae} = 0 + 0 - 2 \times 2 = -4 \\
g_{cd} &= D'_c + D'_d - 2c_{cd} = 3 + 1 - 2 \times 3 = -2 \\
g_{ce} &= D'_c + D'_e - 2c_{ce} = 3 + 0 - 2 \times 2 = -1 \ (maximum)
\end{aligned}$$

- Swap c and e! $(\widehat{g_2} = -1)$

Swapping Pair Determination



- $D''_x = D'_x + 2\ c_{xp} - 2\ c_{xq}$, $\forall\ x \in A - \{p\}$

$$
\begin{aligned}
D''_a &= D'_a + 2c_{ac} - 2c_{ae} = 0 + 2 \times 2 - 2 \times 2 = 0 \\
D''_d &= D'_d + 2c_{de} - 2c_{dc} = 1 + 2 \times 4 - 2 \times 3 = 3
\end{aligned}
$$

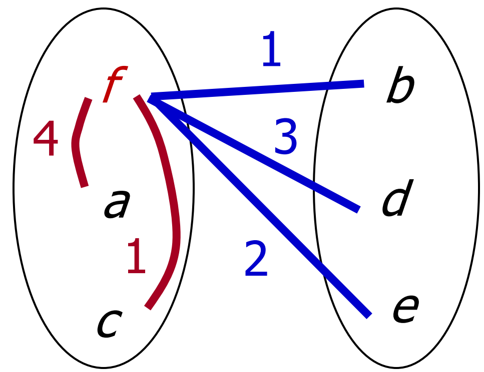
- $g_{xy} = D''_x + D''_y - 2c_{xy}$.

$$
g_{ad} = D''_a + D''_d - 2c_{ad} = 0 + 3 - 2 \times 3 = -3 (\hat{g_3} = -3)
$$

- Note that this step is redundant $(\sum_{i=1}^{n} \hat{g_i} = 0)$.
- Summary: $\hat{g_1} = g_{bf} = 4$, $\hat{g_2} = g_{ce} = -1$, $\hat{g_3} = g_{ad} = -3$.
- Largest partial sum $\max \sum_{i=1}^{k} \hat{g_i} = 4$ ($k = 1$) $\Rightarrow$ Swap b and f.

Next Iteration

	a	b	c	d	e	f
a	0	1	2	3	2	4
b	1	0	1	4	2	1
c	2	1	0	3	2	1
d	3	4	3	0	4	3
e	2	2	2	4	0	2
f	4	1	1	3	2	0



- Iteration 2: Repeat what we did at Iteration 1 (Initial cost = 22-4 =18).
- Summary: $g_1' = g_{ce} = -1$, $g_2' = g_{ab} = -3$, $g_3' = g_{fd} = 4$

❖ Largest partial sum = $\max \sum_{i=1}^{k} g_i' = 0$ (k=3) $\Rightarrow$ Stop!

Kernighan-Lin Heuristic

Algorithm: Kernighan-Lin(G)
Input: $G = (V, E)$, $|V| = 2n$.
Output: Balanced bi-partition A and B with "small" cut cost.

begin
Bipartition G into A and B such that $|V_A| = |V_B|$, $V_A \cap V_B = \emptyset$, and $V_A \cup V_B = V$.
repeat
  Compute D_v, $\forall\ v \in V$.
  for i =1 **to** n **do**
    Find a pair of unlocked vertices $v_{ai} \in V_A$ and $v_{bi} \in V_B$ whose exchange makes the largest decrease or smallest increase in cut cost;
    Mark v_{ai} and v_{bi} as locked, store the gain g', and compute the new D_v, for all unlocked $v \in V$;
  Find k, such that $G_k = \sum_{i=1}^{k} g_i{}'$ is maximized;
  if $G_k > 0$ **then**
    Move $v_{a1}, \ldots, v_{ak}$ from V_A to V_B and $v_{b1}, \ldots, v_{bk}$ from V_B to V_A;
  Unlock v, $\forall\ v \in V$.
until $G_k \leq 0$;
end

Time Complexity

- Line 4: Initial computation of D: $O(n^2)$
- Line 5: The **for**-loop: $O(n)$
- The body of the loop: $O(n^2)$.
  - Lines 6--7: Step i takes $(n-i+1)^2$ time.
- Lines 4--11: Each pass of the repeat loop: $O(n^3)$.
- Suppose the repeat loop terminates after r passes.
- The total running time: $O(rn^3)$.
  - Polynomial-time algorithm?

Extensions of K-L Heuristic

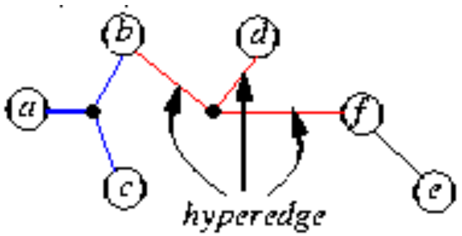
- **Unequal sized subsets** (assume $n_1 < n_2$)
  1. Partition: $|\boldsymbol{A}| = n_1$ and $|\boldsymbol{B}| = n_2$.
  2. Add n_2-n_1 dummy vertices to set ***A***. Dummy vertices have no connections to the original graph.
  3. Apply the Kernighan-Lin algorithm.
  4. Remove all dummy vertices.
- **Unequal sized "vertices"**
  1. Assume that the smallest "vertex" has unit size.
  2. Replace each vertex of size *s* with *s* vertices which are fully connected with edges of infinite weight.
  3. Apply the Kernighan-Lin algorithm.
- ***k*-way partition**
  1. Partition the graph into *k* equal-sized sets.
  2. Apply the Kernighan-Lin algorithm for each pair of subsets.
  3. Time complexity? Can be reduced by recursive bi-partition.

Drawbacks of the Kernighan-Lin Heuristic

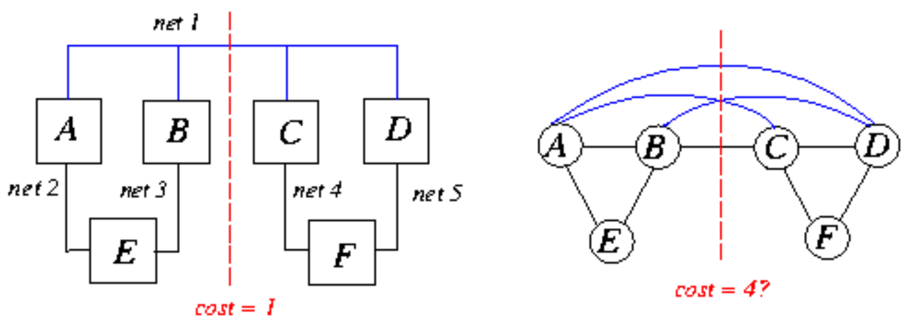
- The K-L heuristic **handles only unit vertex weights**.
  - Vertex weights might represent block sizes, different from blocks to blocks.
  - Reducing a vertex with weight $w(v)$ into a clique with $w(v)$ vertices and edges with a high cost increases the size of the graph substantially.
- The K-L heuristic **handles only exact bisections**.
  - Need dummy vertices to handle the unbalanced problem.
- The K-L heuristic **cannot handle hypergraphs.**
  - Need to handle multi-terminal nets directly.
- The **time complexity of a pass is high**, $O(rn^3)$.

Coping with Hypergraph

- A hypergraph $H=(N, L)$ consists of a set N of vertices and a set L of hyperedges, where each hyperedge corresponds to a **subset** N_i of distinct vertices with $|N_i| \geq 2$.

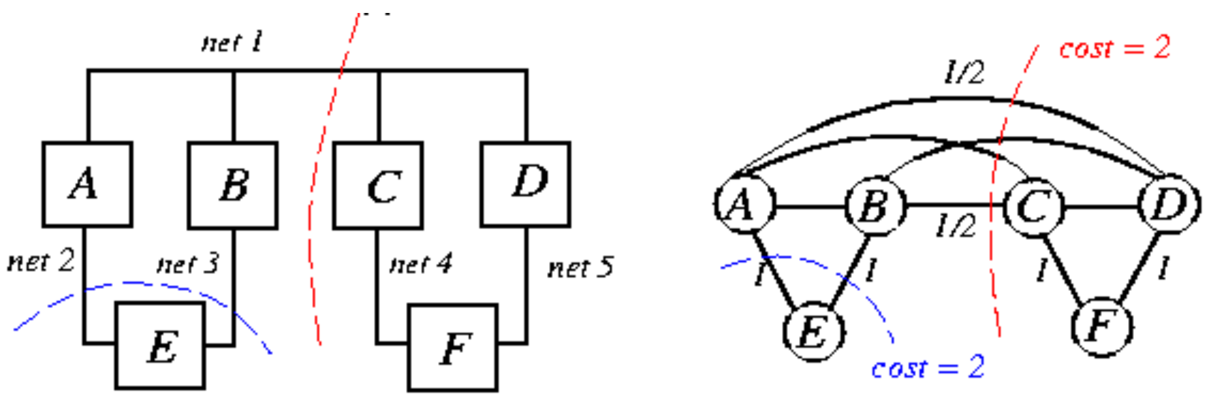


- Schweikert and Kernighan, "A proper model for the partitioning of electrical circuits," 9th Design Automation Workshop, 1972.
- For multi-terminal nets, **net cut** is a more accurate measurement for cut cost (i.e., deal with hyperedges).
  - {A, B, E}, {C, D, F} is a good partition.
  - Should not assign the same weight for all edges.

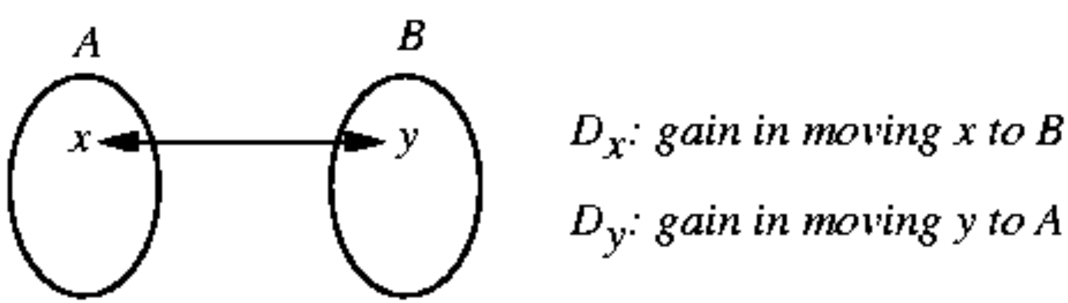


Net-Cut Model

- Let $n(i)$ = # of cells associated with Net i.
- Edge weight $w_{xy} = \frac{2}{n(i)}$ for an edge connecting cells x and y.



- Easy modification of the K-L heuristic.



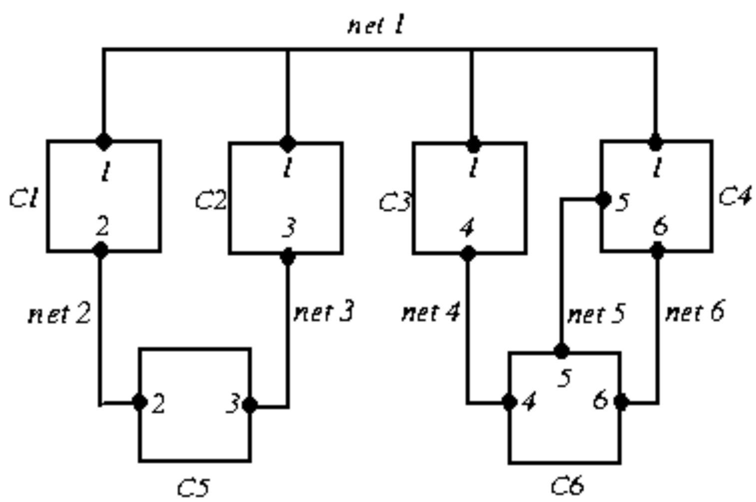
$$g_{xy} = D_x + D_y - Correction(x, y)$$

Fiduccia-Mattheyses Heuristic

- Fiduccia and Mattheyses, “A linear time heuristic for improving network partitions,” DAC-82.
- New features to the K-L heuristic:
  - Aims at **reducing net-cut costs**; the concept of cutsize is extended to hypergraphs.
  - Only a **single vertex is moved** across the cut in a single move.
  - Vertices are weighted.
  - Can handle “unbalanced” partitions; a balance factor is introduced.
  - A special data structure is used to select vertices to be moved across the cut to improve running time.
  - **Time complexity $O(P)$,** where P is the total # of pins.

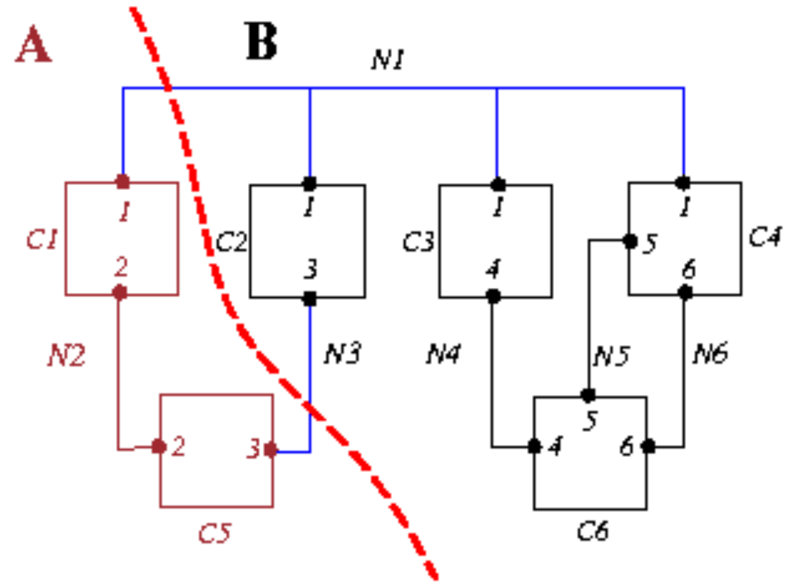
F-M Heuristic: Notation

- $n(i)$: # of cells in Net i; e.g., $n(1) = 4$.
- $s(i)$: size of Cell i.
- $p(i)$: # of pin terminals in Cell i; e.g., $p(6)=3$.
- C: total # of cells; e.g., $C=6$.
- N: total # of nets; e.g., $N=6$.
- P: total # of pins; $P = p(1) + \ldots + p(C) = n(1) + \ldots + n(N)$.

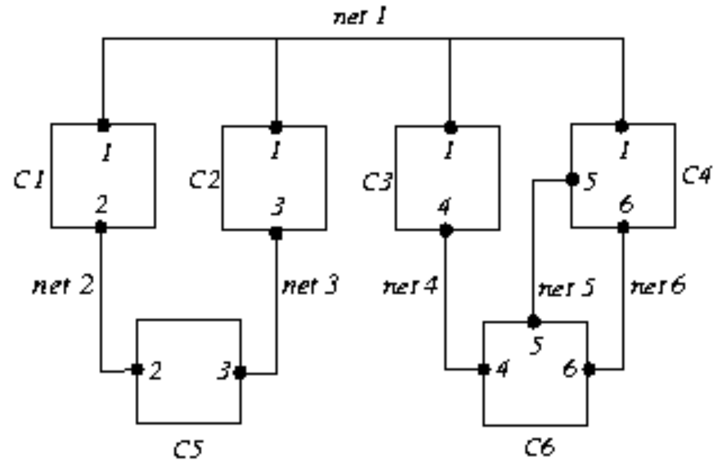


Cut

- **Cutstate** of a net:
  - Net 1 and Net 3 are **cut** by the partition.
  - Net 2, Net 4, Net 5, and Net 6 are **uncut**.
- **Cutset** = {Net 1, Net 3}.
- $|A|$ = size of A = $s(1)+s(5)$;
  $|B| = s(2)+s(3)+s(4)+s(6)$.
- **Balanced 2-way partition:** Given a fraction r, $0 < r < 1$, partition a graph into two sets A and B such that
  - $\frac{|A|}{|A|+|B|} \approx r$.
  - Size of the cutset is minimized.



Input Data Structures



Cell array		Net array	
C1	Nets 1, 2	Net 1	C1, C2, C3, C4
C2	Nets 1, 3	Net 2	C1, C5
C3	Nets 1, 4	Net 3	C2, C5
C4	Nets 1, 5, 6	Net 4	C3, C6
C5	Nets 2, 3	Net 5	C4, C6
C6	Nets 4, 5, 6	Net 6	C4, C6

- Size of the network: $P = \sum_{i=1}^{6} n(i) = 14$
- Construction of the two arrays takes $O(P)$ time.

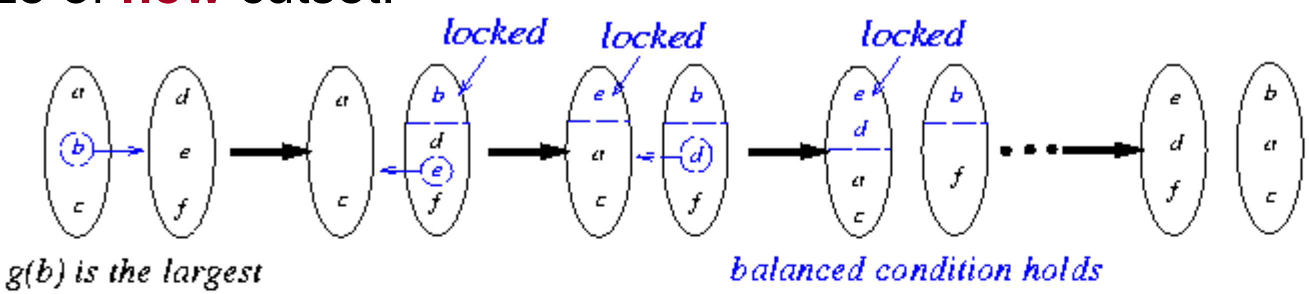
Basic Ideas: Balance and Movement

- Only move a cell at a time, preserving "balance."

$$\frac{|A|}{|A|+|B|} \approx r$$
$$rW - S_{max} \leq |A| \leq rW + S_{max},$$

where $W=|A|+|B|$; $S_{max}=\max_i s(i)$.

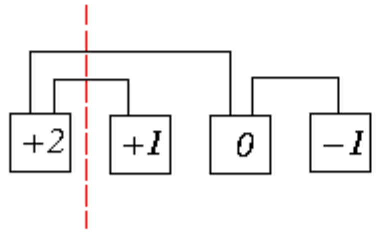
- $g(i)$: gain in moving cell i to the other set, i.e., size of **old** cutset - size of **new** cutset.

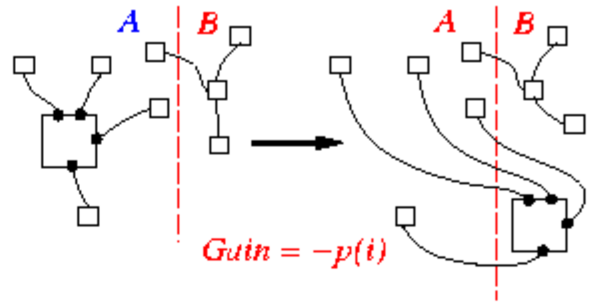


- Suppose $\hat{g}_i$'s: $g(b)$, $g(e)$, $g(d)$, $g(a)$, $g(f)$, $g(c)$ and the largest partial sum is $g(b)+g(e)+g(d)$. Then we should move b, e, d □ two resulting sets: $\{a, c, e, d\}$, $\{b, f\}$.

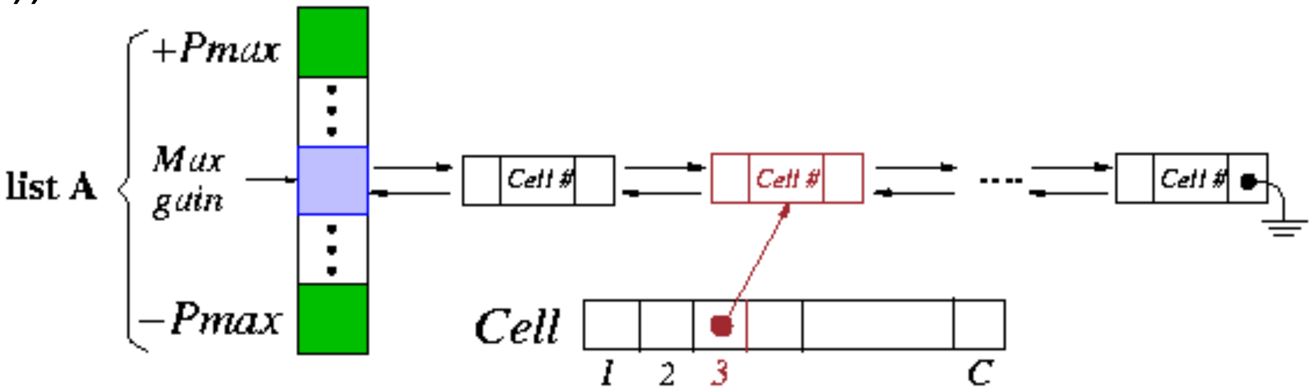
Cell Gains and Data Structure Manipulation

- $-p(i) \leq g(i) \leq p(i)$





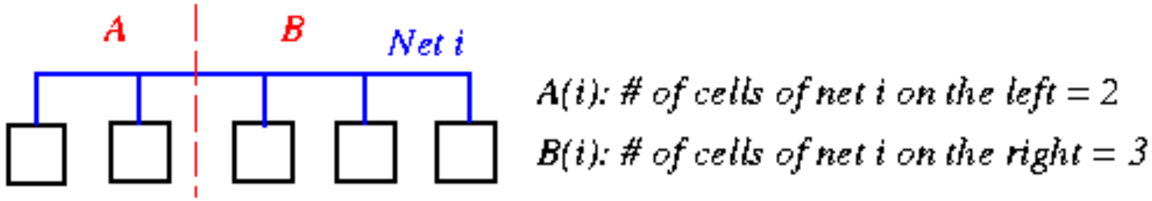
- Two “bucket list” structures, one for set *A* and one for set *B* ($P_{max} = \max_i p(i)$).



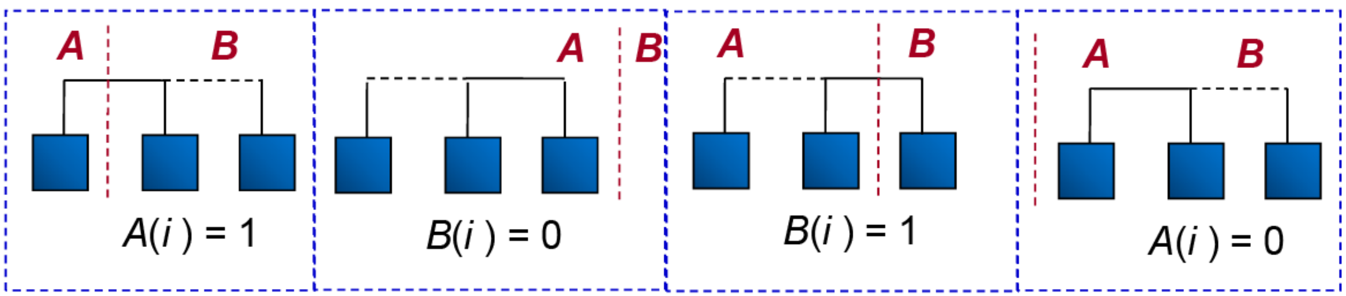
- ***O*****(1)-time operations:** find a cell with Max Gain, remove Cell *i* from the structure, insert Cell *i* into the structure, update $g(i)$ to $g(i) + \Delta$, and update the Max Gain pointer.

Net Distribution and Critical Nets

- Distribution of Net *i*: (*A*(*i*), *B*(*i*)) = (2, 3).
  - (*A*(*i*), *B*(*i*)) for all *i* can be computed in *O*(*P*) time.



- **Critical Nets**: A net is critical if it has a cell which if moved will change its cutstate.
  - For cells in *left*: *A*(*i*) = 1 or *B*(*i*) = 0.
  - For cells in *right*: *B*(*i*) = 1 or *A*(*i*) = 0.



- **Gain of a cell depends only on its critical nets.**

Computing Cell Gains

- Initialization of all cell gains requires $O(P)$ time:

  $g(i) \leftarrow 0$;

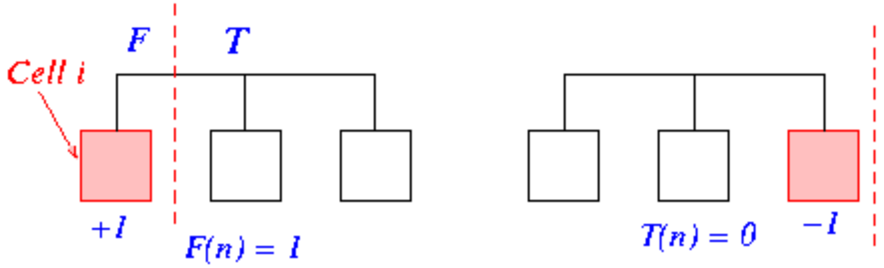
  $F \leftarrow$ the "from block" of Cell i;

  $T \leftarrow$ the "to block" of Cell i;

  for each net n on Cell i **do**

      if $F(n)=1$ **then** $g(i) \leftarrow g(i)+1$;

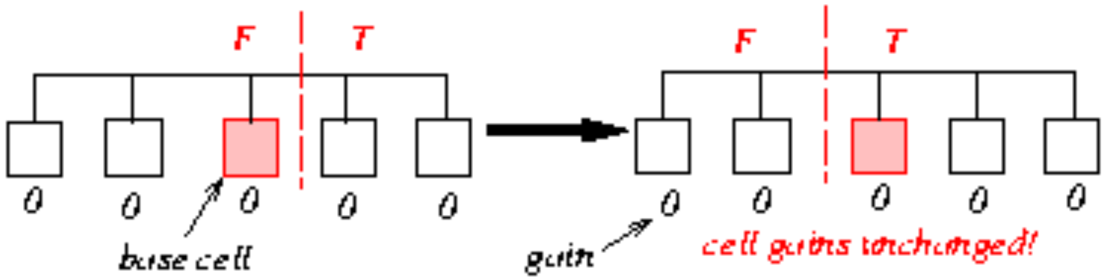
      if $T(n)=0$ **then** $g(i) \leftarrow g(i)-1$;



- Will show: Only need $O(P)$ time to maintain all cell gains in one pass.

Updating Cell Gains

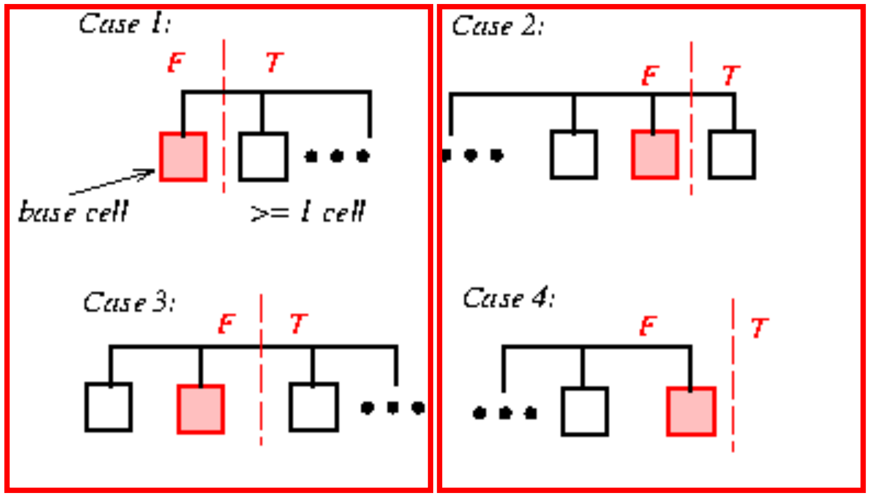
- To update the gains, we only need to look at those nets, connected to the base cell, which are critical **before** or **after** the move.
- **Base cell**: The cell selected for movement from one set to the other.



- Consider only the case where the base cell is in the left partition. The other case is similar.

After move

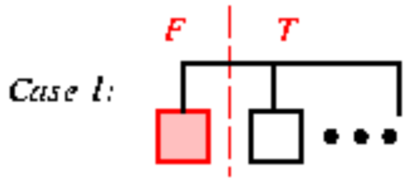
F(n) = 0 or 1



Before move

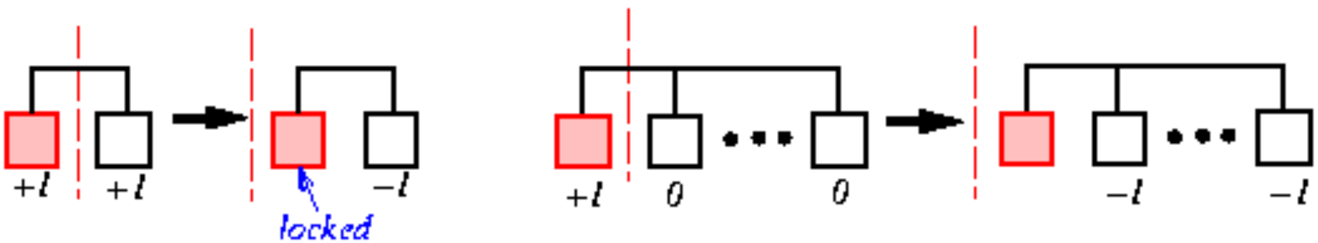
T(n) = 0 or 1

Updating Cell Gains (cont'd)

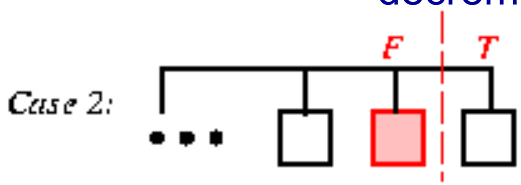


After move

F(n) = 0

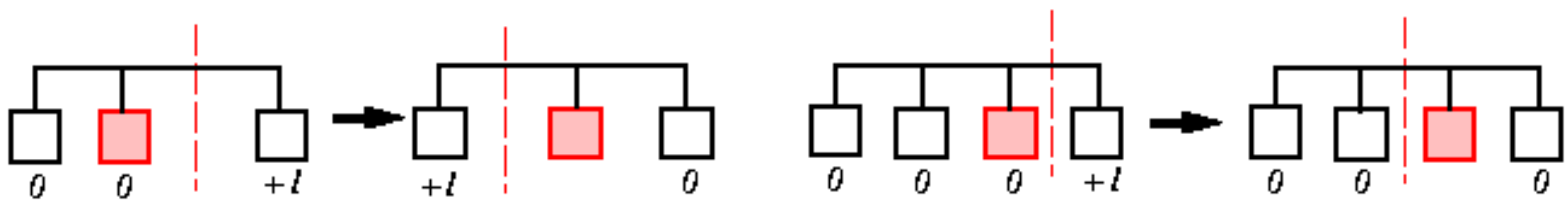


decrement gains of all free cells on *n*



Before move

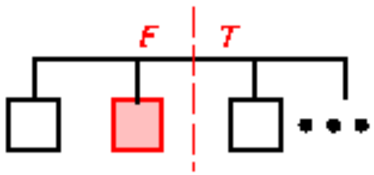
T(n) = 1



decrement gain of the only *T* cell on *n*

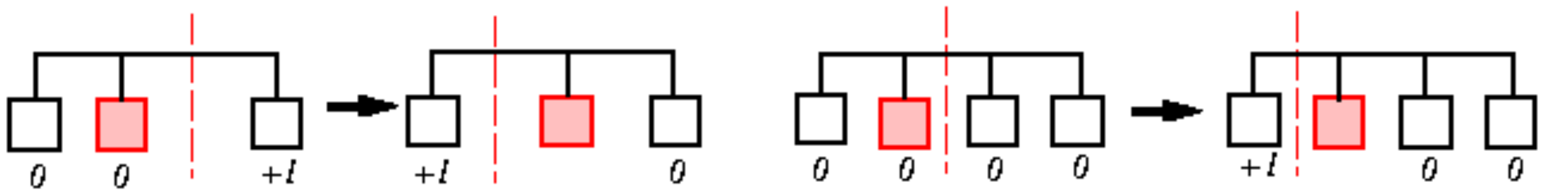
Updating Cell Gains (cont'd)

Case 3:



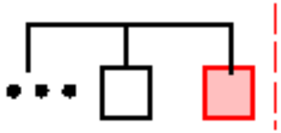
After move

F(n) = 1



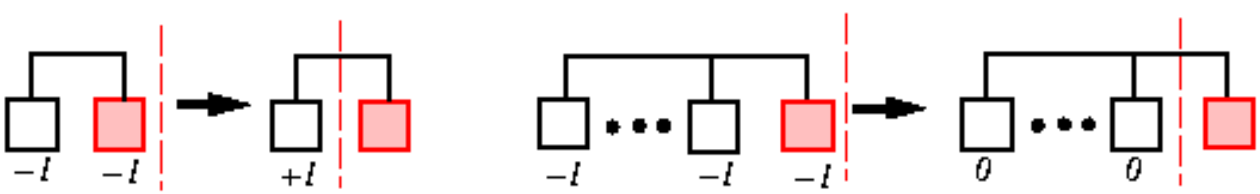
increment gain of the only *F* cell on *n*

Case 4:



Before move

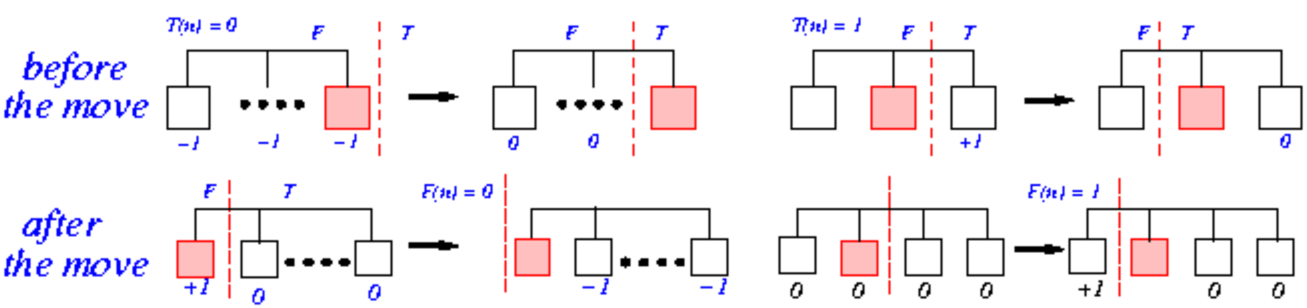
T(n) = 0



increment gains of all free cells on *n*

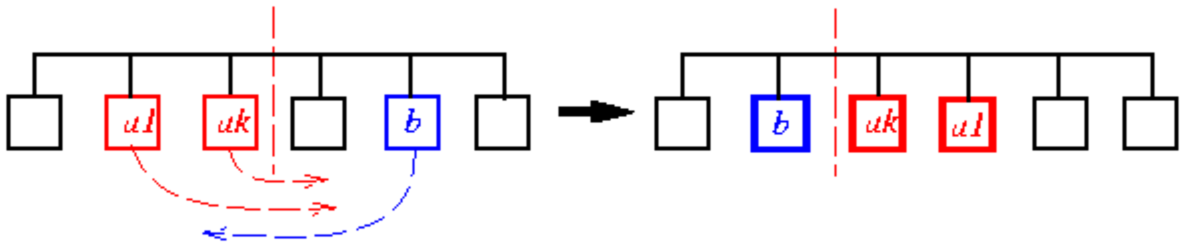
Algorithm for Updating Cell Gains

Algorithm: Update_Gain
1 begin ***/* move base cells and update neighbors' gains */***
2 *F* ← the *From Block* of the base cell;
3 *T* ← the *To Block* of the base cell;
4 Lock the base cell and complement its block;
5 for each net *n* on the base cell do
/* check critical nets before the move */
6 if *T*(*n*) = 0 then increment gains of all free cells on *n*
else if *T*(*n*)=1 then decrement gain of the only *T* cell on *n*, if it is free
/* change *F*(*n*) and *T*(*n*) to reflect the move */
7 *F*(*n*) ← *F*(*n*) - 1; *T*(*n*) ← *T*(*n*)+1;
/* check for critical nets after the move */
8 if *F*(*n*)=0 then decrement gains of all free cells on *n*
else if *F*(*n*) = 1 then increment gain of the only *F* cell on *n*, if it is free
9 end



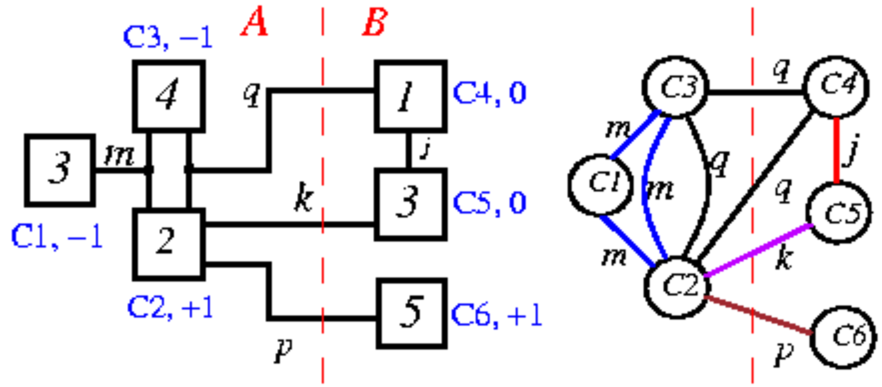
Complexity of Updating Cell Gains

- Once a net has "locked" cells at both sides, the net will remain cut from now on.
- Suppose we move a_1, a_2, …, a_k from left to right, and then move b from right to left ⍰ At most only moving a_1, a_2, …, a_k and b need updating!



- To update the cell gains, it takes $O(n(i))$ work for Net i.
- Total time = $n(1)+n(2)+\ldots+n(N) = O(P)$.

F-M Heuristic: An Example

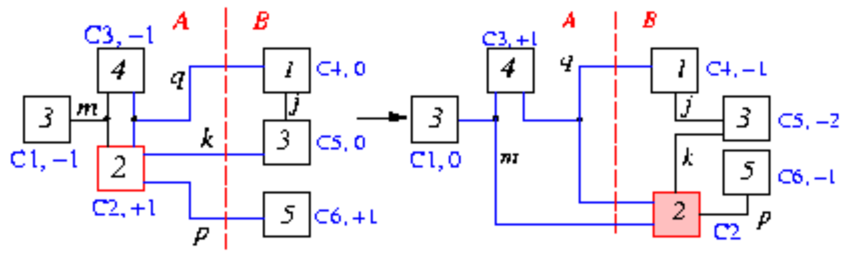


- Computing cell gains: $F(n) = 1 \Rightarrow g(i) + 1$; $T(n)=0 \Rightarrow g(i) - 1$

Cell	m		q		k		p		j		$g(i)$
	F	T	F	T	F	T	F	T	F	T	
c1	0	−1									−1
c2	0	−1	0	0	+1	0	+1	0			+1
c3	0	−1	0	0							−1
c4			+1	0					0	−1	0
c5					+1	0			0	−1	0
c6							+1	0			+1

- Balanced criterion: $r|V| - S_{max} \leq |A| \leq r|V| + S_{max}$. Let $r = 0.4 \Rightarrow |A| = 9$, $|V| = 18$, $S_{max} = 5$, $r|V| = 7.2 \Rightarrow$ Balanced: $2.2 \leq 9 \leq 12.2$!
- maximum gain: c_2 and balanced: $2.2 \leq 9-2 \leq 12.2 \Rightarrow$ Move c_2 from A to B (use size criterion if there is a tie).

F-M Heuristic: An Example (cont'd)



- Changes in net distribution:

Net	Before move		After move	
	F	T	F'	T'
k	1	1	0	2
m	3	0	2	1
q	2	1	1	2
p	1	1	0	2

- Updating cell gains on critical nets (run Algorithm Update_Gain):

Cells	Gains due to $T(n)$				Gain due to $F(n)$				Gain changes	
	k	m	q	p	k	m	q	p	Old	New
c_1		$+1$							-1	0
c_3		$+1$					$+1$		-1	$+1$
c_4			-1						0	-1
c_5	-1				-1				0	-2
c_6				-1				-1	$+1$	-1

- Maximum gain: c_3 and balanced! ($2.2 \le 7\text{-}4 \le 12.2$) $\rightarrow$ Move c_3 from *A* to *B* (use size criterion if there is a tie).

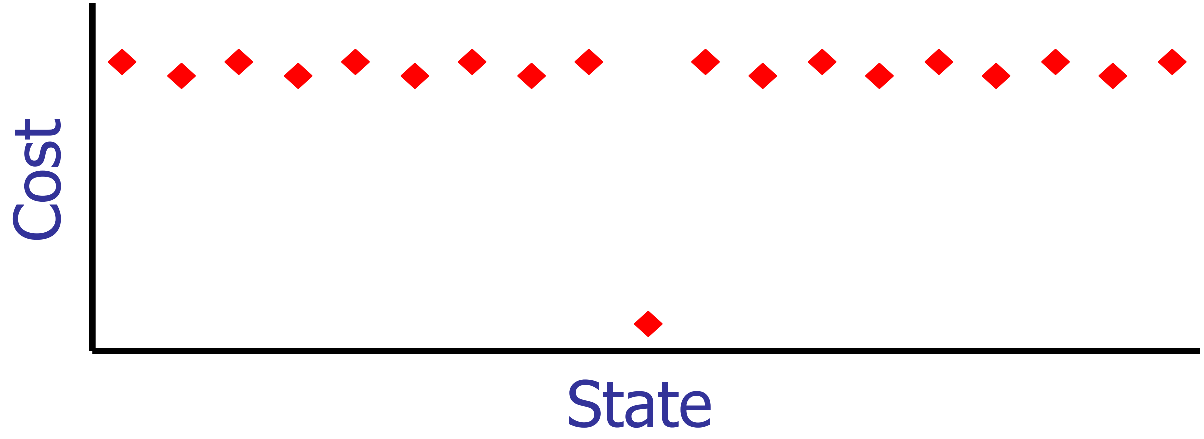
Summary of the Example

Step	Cell	Max gain	$\lvert A \rvert$	Balanced?	Locked cell	A	B
0	-	-	9	-	$\emptyset$	1, 2, 3	4, 5, 6
1	c_2	$+1$	7	yes	c_2	1, 3	2, 4, 5, 6
2	c_3	$+1$	3	yes	c_2, c_3	1	2, 3, 4, 5, 6
3	c_1	$+1$	0	no	-	-	-
$3'$	c_6	-1	8	yes	c_2, c_3, c_6	1, 6	2, 3, 4, 5
4	c_1	$+1$	5	yes	c_1, c_2, c_3, c_6	6	1, 2, 3, 4, 5
5	c_5	-2	8	yes	c_1, c_2, c_3, c_5, c_6	5, 6	1, 2, 3, 4
6	c_4	0	9	yes	all cells	4, 5, 6	1, 2, 3

- $\widehat{g_1} = 1, \widehat{g_2} = 1, \widehat{g_3} = -1, \widehat{g_4} = 1, \widehat{g_5} = -2, \widehat{g_6} = 0$ □ Maximum partial sum G_k = +2, k = 2 or 4.
- Since k=4 results in a better balanced partition □ Move c_1, c_2, c_3, c_6 □ A={6}, B={1, 2, 3, 4, 5}.
- **Repeat the whole process until new $G_k \leq 0$.**

Problem with Greedy Algorithms

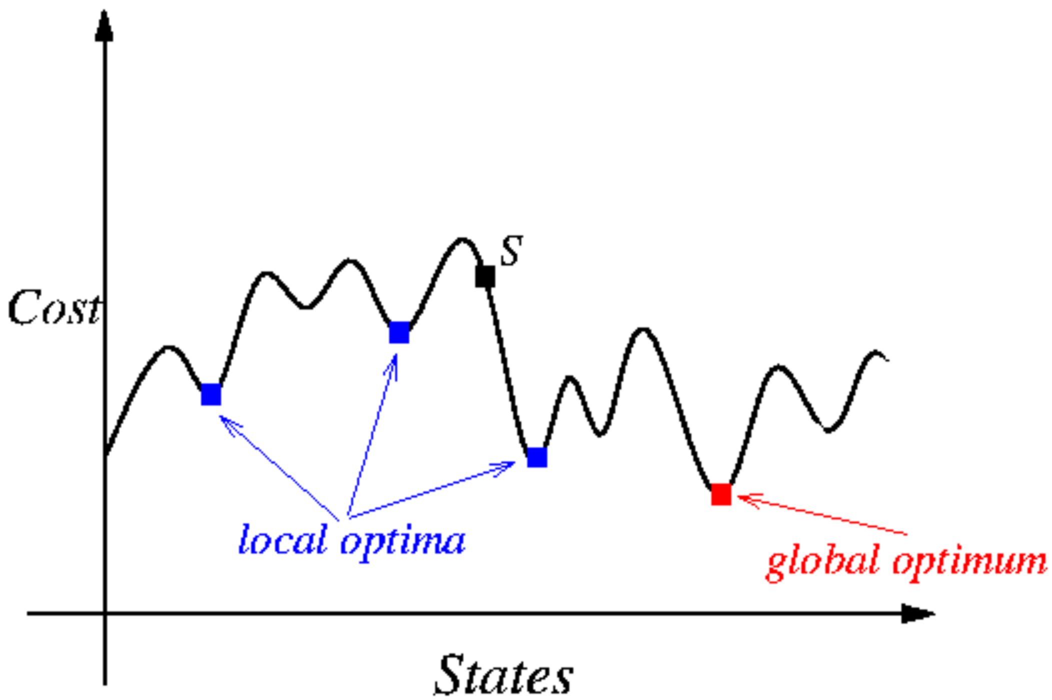
- Easily get stuck at local minimum.
- Will obtain non-optimal solutions.



- Optimal only for convex (or concave for maximization) funtions.

Simulated Annealing

- Kirkpatrick, Gelatt, and Vecchi, “Optimization by simulated annealing,” *Science*, May 1983.



Simulated Annealing Basics

- Non-zero probability for "up-hill" moves.
- Probability depends on
  1. magnitude of the "up-hill" movement
  2. total search time

$$Prob(S \rightarrow S') = \begin{cases} 1 & \text{if } \Delta C \leq 0 \quad /* \text{ ``}down - hill\text{'' } moves */ \\ e^{-\frac{\Delta C}{T}} & \text{if } \Delta C > 0 \quad /* \text{ ``}up - hill\text{'' } moves */ \end{cases}$$

- $\Delta C = cost(S') - Cost(S)$
- T: Control parameter (temperature)
- Annealing schedule: $T = T_0, T_1, T_2, \ldots$, where $T_i = r^i \, T_0$, $r < 1$.

Generic Simulated Annealing Algorithm

begin
Get an initial solution S;
Get an initial temperature $T > 0$;
while not yet “frozen” **do**
    for $1 \le i \le P$ **do**
        Pick a random neighbor S' of S;
        $\Delta \leftarrow cost(S') - cost(S)$;
        /* downhill move */
        if $\Delta \le 0$ **then** $S \leftarrow S'$
        /* uphill move */
        if $\Delta > 0$ **then** $S \leftarrow S'$ with probability $e^{-\frac{\Delta}{T}}$;
$T \leftarrow rT$; /* reduce temperature */
return S
end

Basic Ingredients for Simulated Annealing

- Analogy:

Physical system	Optimization problem
state	configuration
energy	cost function
ground state	optimal solution
quenching	iterative improvement
careful annealing	simulated annealing

- Basic Ingredients for Simulated Annealing:
  - **Solution space**
  - **Neighborhood structure**
  - **Cost function**
  - **Annealing schedule**